**Chapter**

# Decoupling of Attributes and Aggregation for Fuzzy Number Ranking

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## Abstract

Intuition, expressed as verbal arguments or axiom formulations, has often been used as a guiding principle for fuzzy number ranking (FNR). This chapter adopts the multi-attribute decision making (MADM) framework to analyze such intuition with three results. First, intuition in FNR should have involved multiple attributes, which are often implicated in the existing ranking methods. Then, we suggest three attributes (i.e., representative x-value, x-value range, overall membership ratio), which can be used to characterize the FNR intuition. Second, we decouple two issues in FNR: selection of attributes and aggregation of values, where aggregation is concerned with the trade-off among attributes to determine a single index for FNR. Then, the discount factors are proposed for the attributes of range and membership ratio to model the trade-off and formulate a ranking index. Third, the decoupling of attributes and aggregation reveals a fundamental tension between information content and the satisfaction of the FNR axioms. That is, if we can consider more information (in terms of attributes) as relevant to FNR, the ranking method will likely violate some FNR axioms. However, if we consider less information, the ranking method will be less sensitive to distinguish some fuzzy numbers for ranking. In the end, the proposed multi-attribute approach can provide a practical aspect to analyze and address the FNR problems.


**Keywords:** fuzzy number ranking, multi-attribute decision making, aggregation, decoupling, overall membership ratio

## 1. Introduction

Intuition has been a criterion for researchers to evaluate and comment the ranking results from a set of fuzzy numbers. As a pattern described by Wang and Kerre [1], a ranking method can be criticized by yielding “counter-intuitive” results from some examples, and thus it is motivated to develop new ranking methods (e.g., [2, 3]). Despite of its common use, the meaning of “intuition for ranking” is somewhat unclear. It should be related to the ranking of real numbers, which is fundamental in our intuition. However, this alone is not sufficient for fuzzy number ranking (FNR). Why? When we compare two real numbers: 3 and 5, we can state 5 > 3 because these real numbers can be ordered on a single dimension, i.e., the real line. In the context of fuzzy sets, the membership information is added. For example, consider two ordered pairs: (3, 0.9) and (5, 0.4), where the second elements are the membership values. Here, we cannot straightforwardly state (5, 0.4) $\succ$ (3, 0.9) due to the presence of the second dimension, membership, in the ranking consideration. Notably, in this paper, the symbol "$>$" is used to compare two real numbers, while the symbol "$\succ$" or "$\succcurlyeq$" is used to represent the ranking relation.

Consider that a fuzzy number contains a set of such ordered pairs. We argue that the problem structure of FNR should contain multiple dimensions to explain "intuition" properly. Then, we employ the classical framework of multi-attribute decision making (MADM) [4] for the analysis of ranking intuition. The framework of MADM distinguishes the concepts of attributes and aggregation. Attributes are used to evaluate the properties of options, and they are subject to the selection by decision makers, who determine what properties (or information) are deemed relevant to the decision problems. On the other hand, aggregation captures the weighting strategies (e.g., weighted sum) to address the trade-off consideration among the option's properties (or information).

In FNR, each attribute represents a single dimension for ranking consideration. In literature, numerous attributes have been implied in the formulations of ranking indices. For example, the approach of the maximizing and minimizing sets [3, 5, 6] implicates the attributes that articulate the optimistic and pessimistic aspects of a fuzzy number for ranking. In the centroid-based approach [7, 8], centroid can be interpreted as an attribute that focuses on the "middle" aspect over the geometry of a fuzzy number. Notably, each notion of attribute can be quantified in multiple ways. For example, we may express the notion of "average" via the formulations of "value" by Delgado et al. [9] or "median" by Bodjanova [10]. In addition, new ranking methods have been proposed by adding attributes to the ranking indices. For example, to address some non-distinguishable results from Abbasbandy and Hajjari [2], Asady [11] and Ezzati et al. [12] formulated additional attributes (namely, the epsilon-neighborhood and $Mag'(u)$, respectively) in their ranking indices.

The consideration of multiple attributes for FNR is not new. In literature, some approaches have explicitly considered multiple measures (or attributes) to describe a fuzzy number such as value and ambiguity [9, 13, 14], mean and standard deviation [15], average value and degree of deviation [16], expected value (in transfer coefficient) and deviation degree [17, 18], general concepts of area/mode/spreads/weights [19–21] and extensions from the centroid concept [21–25].

Aggregation is a separate issue from the selection of attributes. It aims to handle the given information of attributes for decision making. In literature, different aggregation approaches over the same attributes have been reported. For example, aggregation over the x- and y-coordinates of a centroid can be done via a distance measure [26] or an area measure [8, 27]. The weighted sum approach has been used to aggregate two attributes such as the right/left utility values [3] and the average and deviation values [16]. In addition to closed-form equations, aggregation can also be done by rules and procedures. For example, Asady [11] and Chi and Yu [23] determine the ranking of fuzzy numbers based on the priority of two or three attributes, which basically is a lexicographical ordering procedure ([4], pp. 77–79).

The aggregation approach can influence the ranking results since it controls the trade-off among attributes. To illustrate, consider the earlier ordered pairs (3, 0.9) and (5, 0.4). Suppose that two attributes are considered for ranking: real number and membership value, and we assume "higher value ➔ higher rank" for both attributes. Then, we can have multiple ways to aggregate these two values such as 3 + 0.9 and

$3 \times 0.9$, which are consistent with the "higher-the-better" direction. However, different aggregation functions can lead to different ranking results, e.g., $(3 + 0.9) < (5 + 0.4)$ and $(3 \times 0.9) > (5 \times 0.4)$. Different results can be explained by the trade-off approach implied in the aggregation functions. For example, in this case, addition tends to give an advantage to real number, whereas multiplication allows more influence from membership value.

Based on the above discussion, the theme of this chapter is to adopt the MADM framework, which purposely decouples attributes and aggregation for FNR. In this way, we can compare ranking methods in view of their selections of attributes and the formulations of aggregation functions independently. In addition, the multi-attribute aspect can help explain the axiomatic properties of ranking methods. To avoid the reliance on the "intuition criterion", Wang and Kerre [1] suggested seven axioms as reasonable properties to specify the meaning of intuition more clearly. Ban and Coroianu [28] derived a class of ranking functions that can satisfy six of these axioms with literature examples that can belong to this class under some conditions (e.g., [2, 29, 30]).

Despite of the formal work by Ban and Coroianu [28], new ranking methods emerge continually as researchers considered this class of ranking functions did not address two aspects. First, the development by Ban and Coroianu [28] was intended for normalized fuzzy numbers, and some work has been developed for the non-normalized cases (e.g., [31]). Second, their class of ranking functions cannot distinguish two symmetric fuzzy numbers with different spreads (e.g., for cases in Ezzati et al., [12]). In some recent work, Dombi and Jónás [32] applied the probability-based preference intensity index, and Van Hop [33] developed the dominant interval measure (namely relative dominant degree) for fuzzy number ranking. Their approaches basically generalized the numerical techniques of intervals for fuzzy number ranking without decomposing or analyzing the ranking attributes.

More fundamentally, it seems to us that if a ranking function is designed to satisfy the axioms by Wang and Kerre [1], this ranking function will be less sensitive to the distribution of membership values and the spreads of fuzzy numbers to determine the ranking results. In other words, the satisfaction of these axioms is strongly influenced by the type of information (or attributes) that is selected for FNR but it is less relevant to the aggregation approach. The distinction between information selection and aggregation has not been investigated for fuzzy number ranking in literature. This chapter will use the multi-attribute aspect to analyze this issue.

After the preliminaries in Section 2, this chapter will discuss and illustrate our selection of three attributes for FNR in Section 3 and then our aggregation approach using the discount factors in Section 4. Section 5 will suggest some guidance for the application of the proposed multi-attribute ranking method. Section 6 will discuss the relation between the information content for FNR and the axiomatic properties of ranking methods. This chapter is concluded in Section 7.

## 2. Preliminaries

Fuzzy number is described as a fuzzy subset of the real line $\mathbb{R}$ [34]. This work considers trapezoidal fuzzy number (TrFN) as a special case of fuzzy number. Let $F_A$ denote a TrFN with a maximum membership equal to $h_A$, as illustrated in **Figure 1**. Let $x$ be any element of the real line, and its membership according to TrFN, denoted as $\mu_{F_A}(x)$, can be expressed in the following formulation where $a_1, a_2, a_3, a_4$ are real

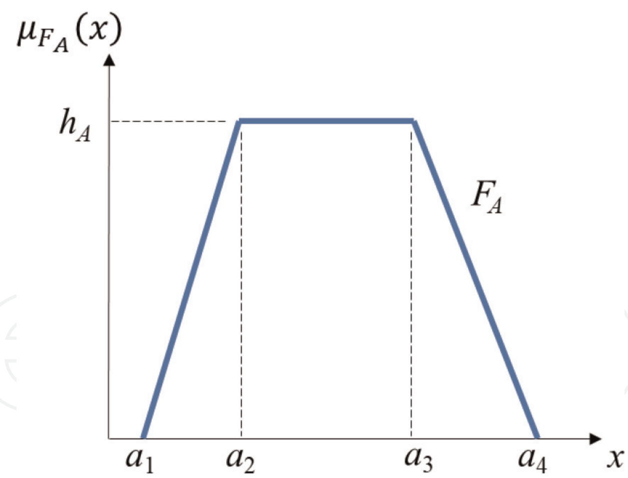



**Figure 1.**
*A trapezoidal fuzzy number.*

numbers to specify $F_A$. As a convenient notation, $F_A$ can be expressed as a 5-tuple, where $F_A = (a_1, a_2, a_3, a_4; h_A)$.

$$\mu_{F_A}(x) = \begin{cases} 0 & x < a_1 \\ \left(\dfrac{x - a_1}{a_2 - a_1}\right) h_A & a_1 \le x < a_2 \\ h_A & a_2 \le x < a_3 \\ \left(\dfrac{x - a_4}{a_3 - a_4}\right) h_A & a_3 \le x < a_4 \\ 0 & x > a_4 \end{cases} \tag{1}$$

Let $supp(F_A)$ be the support of $F_A$, and we have $supp(F_A) = \{x \in \mathbb{R} \mid a_1 \le x \le a_4\}$. Then we have the infimum and supremum of $supp(F_A)$ as $\inf supp(F_A) = a_1$ and $\sup supp(F_A) = a_4$, respectively. Also, let $I_{F_A}(\alpha) = [l_{F_A}(\alpha), r_{F_A}(\alpha)]$ be the α-cut interval of $F_A$. For $\alpha \le h_A$, the left and right bounds of the α-cut interval can be formulated as follows.

$$l_{F_A}(\alpha) = a_1 + \left(\frac{\alpha}{h_A}\right)(a_2 - a_1) \tag{2}$$

$$r_{F_A}(\alpha) = a_4 + \left(\frac{\alpha}{h_A}\right)(a_3 - a_4) \tag{3}$$

Suppose we have two fuzzy numbers: $F_A = (a_1, a_2, a_3, a_4, h_A)$ and $F_B = (b_1, b_2, b_3, b_4, h_B)$, and a constant, denoted as $\lambda$ (i.e., $\lambda \in \mathbb{R}$). We can have fuzzy number addition and multiplication with a constant as follows [17, 18, 34].

$$F_A \oplus F_B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; \min\{h_A, h_B\}) \tag{4}$$

$$\lambda \cdot F_A = (\lambda \cdot a_1, \lambda \cdot a_2, \lambda \cdot a_3, \lambda \cdot a_4; h_A) \tag{5}$$

To describe some reasonable properties of ranking methods, Wang and Kerre [1] have proposed seven axioms. Ban and Coroianu [28] have dropped one axiom by considering a ranking (or an ordering) over a given set of fuzzy numbers. This chapter follows the choice made by Ban and Coroianu [28]. Let $\boldsymbol{F}$ be a set of fuzzy numbers, and a ranking method determines the binary relation $\succcurlyeq$ over $\boldsymbol{F}$. Then, their six axioms are summarized (without elaborating their variants) below.

**Axiom 1**: $F_A \succcurlyeq F_A$. for any $F_A \in \boldsymbol{F}$.

**Axiom 2**: For any $F_A$, $F_B \in \boldsymbol{F}$, if $F_A \succcurlyeq F_B$ and $F_B \succcurlyeq F_A$, then $F_A \sim F_B$.

**Axiom 3**: For any $F_A$, $F_B$, $F_C \in \boldsymbol{F}$, if $F_A \succcurlyeq F_B$ and $F_B \succcurlyeq F_C$, then $F_A \succcurlyeq F_C$.

**Axiom 4**: For any $F_A$, $F_B \in \boldsymbol{F}$, if $\inf supp(F_A) \geq \sup supp(F_B)$, then $F_A \succcurlyeq F_B$.

**Axiom 5**: Suppose that $F_A$, $F_B$, $F_A \oplus F_C$, $F_B \oplus F_C$ are elements of $\boldsymbol{F}$. If $F_A \succcurlyeq F_B$, then $F_a \oplus F_c \succcurlyeq F_b \oplus F_c$.

**Axiom 6**: Suppose that $\lambda \in \mathbb{R}$ and $F_A$, $F_B$, $\lambda \cdot F_A$, $\lambda \cdot F_B$ are elements of $\boldsymbol{F}$. If $F_A \succcurlyeq F_B$ and $\lambda \geq 0$, then $\lambda \cdot F_A \succcurlyeq \lambda \cdot F_B$. If $F_A \succcurlyeq F_B$ and $\lambda \leq 0$, then $\lambda \cdot F_A \preccurlyeq \lambda \cdot F_B$.

Axioms 1 and 3 are referred to as the reflexive and transitive properties of binary relations, respectively, for a total pre-order on $\boldsymbol{F}$ [28]. Axiom 2 defines the conditions for the equality "$\sim$". Axiom 4 specifies that $F_A$ is larger than or equal to $F_B$ if the lower bound of the support of $F_A$ is larger than the upper bound of the support of $F_B$. Axioms 5 and 6 generally imply that the ordering of $F_A \succcurlyeq F_B$ should be preserved if they are added by the same fuzzy number $F_C$ or multiplied by the same positive quantity $\lambda$. Notably, index-based ranking methods will satisfy Axioms 1 to 3 [1], and this chapter will focus more on Axioms 4 to 6.

## 3. Three attributes for fuzzy number ranking

In this section, we characterize the comparison of fuzzy numbers through one primary attribute and two secondary attributes. The primary measure is concerned with the representative value of a fuzzy number on the real line, which is a common intuition for ranking. One secondary attribute checks the range of real numbers enclosed by a fuzzy number, which information is independent of the representative value but can be relevant for ranking. Another secondary attribute is associated with membership, which is concerned with the shape of a fuzzy number.

### 3.1 Representative x-value

Since the real line of a fuzzy number is often expressed on the x-axis, we use "x-value" to label the values associated with the real line. As a fuzzy number encloses a range of possible x-values, one common intuition is to identify a representative x-value of a fuzzy number for comparison. There can be several options that are aligned with this intuition such as the expected value [34, 35], the x-coordinate of a centroid [36] and median [10]. In this chapter, we adopt the class of ranking indices derived by Ban and Coroianu [28]. Let $rep(F_A, w)$ be the function to evaluate the representative x-value of the fuzzy number $F_A$, and its formulation is given as follows.

$$rep(F_A, w) = w \cdot a_1 + \left(\frac{1}{2} - w\right)a_2 + \left(\frac{1}{2} - w\right)a_3 + w \cdot a_4 \tag{6}$$

where $w$ is a weighting constant with $0 \leq w \leq 1$. As proven by Ban and Coroianu [28] (Theorem 39), if this function is used as a ranking index, it satisfies the six axioms discussed in the preliminaries section. Beyond this theorem result, we can interpret this formulation as a weighted function of a fuzzy number's core values ($a_2$ and $a_3$) with a weight (1/2-$w$) and boundary values ($a_1$ and $a_4$) with a weight $w$. When $w$ = 0, only the core values are considered. Alternately, when $w$ = 1/2, only the boundary values are considered. To emphasize the importance of core values (i.e., $a_2$ and $a_3$) through weighting, we set (1/2-$w$) ≥ $w$, and then we have $0 \leq w \leq 1/4$.

Derived from Eq. (6), we have $rep(F_A, w) \geq rep(F_B, w)$ if the following condition is satisfied.

$$w[(a_1 + a_4) - (b_1 + b_4)] + \left(\frac{1}{2} - w\right)[(a_2 + a_3) - (b_2 + b_3)] \geq 0 \tag{7}$$

This condition implies a weighted comparison between core and boundary values of $F_A$ and $F_B$. Apparently, we cannot guarantee the satisfaction of this condition if $F_A$ and $F_B$ are partially overlapped (i.e., $supp(F_A) \cap supp(F_B) \neq \varnothing$). Then, the value of $w$ can influence the ordering of $rep(F_A, w)$ and $rep(F_B, w)$. Following the discussion in Ban and Coroianu [28], we consider the presence of $w$ as a generalization of some existing indices, which have implicitly pre-defined weighting factors for core and boundary values of a fuzzy number. For example, the ranking index developed by Abbasbandy and Hajjari [2] is an instance by setting $w$ = 1/12. Given a ranking problem, decision makers can consider some sensitivity analysis (e.g., evaluate the value of $w$ that makes $rep(F_A, w)$ = $rep(F_B, w)$) to define the value of $w$ for their ranking problems.

### 3.2 X-value range

Another attribute is associated with the range of possible x-values of a fuzzy number. Fuzzy numbers can have the same representative x-values with different ranges (e.g., symmetric triangular fuzzy numbers with the same core value but different boundary values). Some argue that the information of range should be considered for ranking (e.g., [11]). There can be several options to quantify this intuition such as ambiguity value [9, 13], standard deviation [15] and deviation degree [16–18]. In this chapter, we adopt the range (or size) of the α-cut interval (denoted as $rng(F_A, \alpha)$), and it is formulated as follows.

$$rng(F_A, \alpha) = r_{F_A}(\alpha) - l_{F_A}(\alpha) \tag{8}$$

**Figure 2** illustrates the α-cut interval of a trapezoidal fuzzy number $F_A$., where the lower (left) and upper (right) bounds of the α-cut interval are denoted as $l_{F_A}(\alpha)$ and

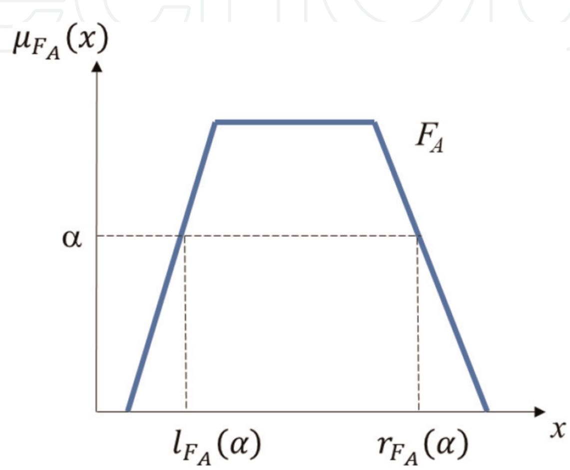



**Figure 2.**
*Illustration of the α-cut interval.*

*DOI: http://dx.doi.org/10.5772/intechopen.109992*

$r_{F_A}(\alpha)$, respectively. The formulations of $l_{F_A}(\alpha)$ and $r_{F_A}(\alpha)$ can be found in Eqs. (2) and (3), respectively. The value of α can be interpreted as the minimum membership value that is deemed relevant for the ranking analysis. For example, if we set α at a lower value, we will receive a wider interval.

Here, we suppose that a large range of possible x-values tends to yield a lower rank because decision makers do not want high uncertainty associated with a large range. This stated intuition of "larger range ➔ lower rank" is aligned with Wang and Luo [6] and Nasseri et al. [37]. Also, we classify range as a secondary attribute because some decision makers may find this attribute not necessary to their ranking problems (e.g., ranking a set of triangular fuzzy numbers with a similar size of support). Then, using the measure of representative x-value only could be sufficient for ranking. In contrast, if decision makers find the information of range relevant to their ranking problems, our suggested approach is to take the range information as a modifier to the representative x-value. This approach will be discussed in Section 5.

### 3.3 Overall membership ratio

The notion of overall membership is associated with the shape of a fuzzy number, regardless of where this shape is placed on the real line. To illustrate, consider two comparisons in **Figure 3**. In **Figure 3a**, while $F_A$ and $F_B$ have different representative x-values, their overall membership values should be the same due to the common shape. In contrast, $F_C$ in **Figure 3b** should have higher overall membership than $F_D$ as $F_C$'s membership values are higher than or equal to those of $F_D$ over the common support (note: the common support is not necessary; it just makes the comparison easier to observe).

To capture the above idea of the overall membership of a fuzzy number, we formulate the ratio using two areas: the shape's area and the full membership area over the same support. Also, we keep the concept of α-cut interval so that the decision maker can identify the minimum level of membership that is relevant for their ranking problem. **Figure 4** is used to illustrate the concept of both types of area. First, the shape's area is considered as the area under the fuzzy number and enclosed by the α-cut interval, as shaded by gray lines in **Figure 4**. Then, the full membership area is based on the rectangle with the width of the α-cut interval and the height of 1 (i.e., maximum membership). Accordingly, the shape's area (denoted as $area_{shape}$) and the full membership area (denoted as $area_{full}$) can be formulated as follows.

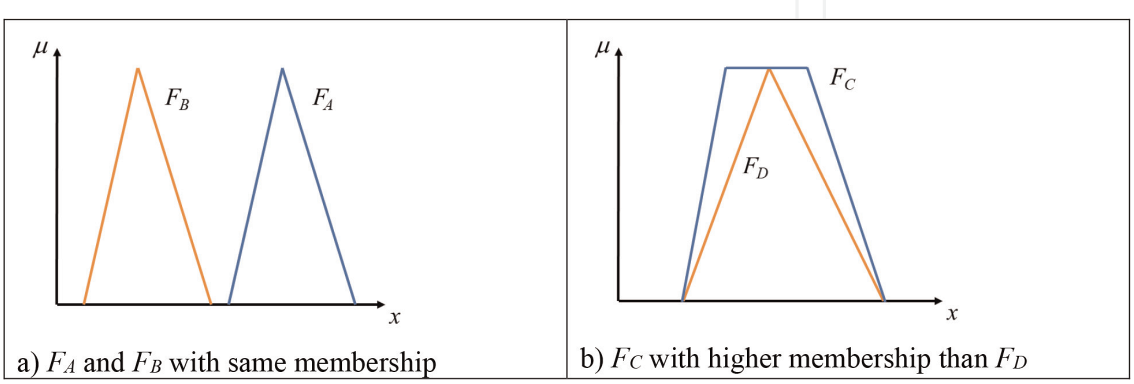



**Figure 3.**
*Illustration of the concept for overall membership a)* $F_A$ *and* $F_B$ *with same membership b)* $F_C$ *with higher membership than* $F_D$.

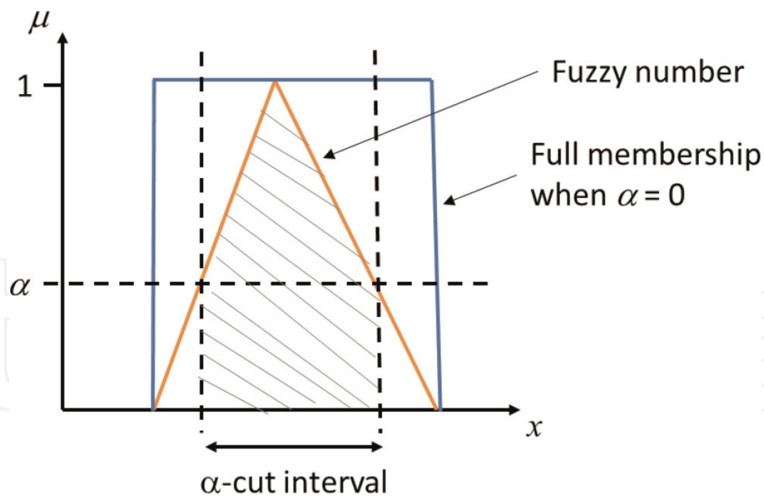



**Figure 4.**
*Illustration of the shape's area and full membership area.*

$$area_{shape}(F_A, \alpha) = \int_{l_{F_A}(\alpha)}^{r_{F_A}(\alpha)} \mu_{F_A}(x)dx \tag{9}$$

$$area_{full}(F_A, \alpha) = [r_{F_A}(\alpha) - l_{F_A}(\alpha)] \times 1 \tag{10}$$

The overall membership ratio of a fuzzy number (denoted as $mem(F_A, \alpha)$) can be expressed as follows.

$$mem(F_A, \alpha) = \frac{area_{shape}(F_A, \alpha)}{area_{full}(F_A, \alpha)} \tag{11}$$

Here, we suppose that higher overall membership ratio tends to yield a higher rank. We classify (overall) membership ratio as another secondary attribute because it may not be necessary for ranking problems with normal fuzzy numbers (e.g., if $F_A$ is a normal triangular fuzzy number, $mem(F_A, 0)$ is always equal to 0.5). Yet, if this information is considered relevant, Section 5 will suggest one approach to use it as a modifying factor for ranking.

Notably, it is probably more common to apply two measures (instead of three) for FNR in literature (e.g., [value, ambiguity] and [average value, degree of deviation] as mentioned in Introduction). From there, they tend to integrate the information of range and membership ratio into one measure. We choose to handle such information in terms of two separate attributes for two reasons. First, the concepts of range and membership ratio are relatively direct for decision makers to visualize and interpret (thus supporting their intuition) in the comparison of fuzzy numbers. Second, range and membership ratio can indicate independent information. For example, consider two normal fuzzy numbers: one triangle and one trapezoid. While the trapezoid shape always yields a higher membership ratio, the ranges of both shapes can be changed arbitrarily, thus explaining the independence of range and membership ratio.

To demonstrate the evaluation of the three attributes, consider a fuzzy number: $F_A$ = (1, 2, 3, 4; 1), which has a lower bound of 1 and an upper bound of 4. Its maximum membership value is 1, which covers the range between 2 and 3 (check **Figure 1** for an illustrative reference). Suppose that α = 0 (i.e., we consider the whole fuzzy number) and $w$ = 1/12 (i.e., according to Abbasbandy and Hajjari [2]), we can evaluate the values of the three attributes according to the following:

- Representative x-value using Eq. (6): $rep(F_A, w)$ = (1/12) × 1 + (1/2–1/12) × 2 + (1/2–1/12) × 3 + (1/12) × 4 = 2.5

- X-value range using Eq. (8): $rng(F_A, \alpha) = r_{F_A}(\alpha) - l_{F_A}(\alpha)$ = 3

  - From Eq. (2): $l_{F_A}(\alpha)$ = 1 + (0/1) × (2–1) = 1

  - From Eq. (3): $r_{F_A}(\alpha)$ = 4 + (0/1) × (3–4) = 4

- Overall membership ratio using Eq. (11): $mem(F_A, \alpha) = \frac{area_{shape}(F_A, \alpha)}{area_{full}(F_A, \alpha)} = \frac{2}{3}$

  - From Eq. (9) = $area_{shape}(F_A, \alpha)$ = trapezoid's area = (1 + 3) × 1/2 = 2

  - From Eq. (10) = $area_{full}(F_A, \alpha) = [4–1] \times 1 = 3$

### 3.4 Pareto optimality

After defining three attributes, we can rank fuzzy numbers for some cases using the Pareto optimality principle [4]. In a less formal expression, we have $F_A \succcurlyeq F_B$ if $rep(F_A, w) \geq rep(F_B, w)$, $rng(F_A, \alpha) \leq rng(F_B, \alpha)$ and $mem(F_A, \alpha) \geq mem(F_B, \alpha)$. To examine how well these attributes can speak for the ranking intuition, numerical examples are used in the next sub-section to check the following situations.

- If two fuzzy numbers can be ranked based on Pareto optimality, this ranking order should be considered "obvious" to the ranking intuition with less room for arguments.

- If two fuzzy numbers cannot be ranked based on Pareto optimality, decision makers can effectively use the selected attribute to explain their arguments.

### 3.5 Numerical examples

The numerical cases from Bortolan and Degani [38] are employed for demonstration, and they can illustrate systematically how the selected attributes are changed with different fuzzy numbers. While we keep the case labels from Bortolan and Degani [38] for cross checking, we classify these cases into five groups for discussion. Also, we follow Abbasbandy and Hajjari [2] by setting $w$ = 1/12 to evaluate $rep(F_A, w)$. Also we set $\alpha$ = 0 for $rng(F_A, \alpha)$ and $mem(F_A, \alpha)$ in this numerical demonstration.

*Group 1: Non-overlapping, triangular fuzzy numbers*

This group covers the cases of a, b, c, d and e from Bortolan and Degani [38], and the results are shown in **Table 1**. By examining the Pareto optimality with the three attributes, we can first pass the membership ratio because $mem(F_A, 0)$ is always equal to 0.5 if $F_A$ is normal and triangular. The rankings of fuzzy numbers in cases a to d are obvious as the fuzzy numbers with higher representative x-values have the same (i.e., cases a, b, d) or smaller (i.e., case c) ranges. In case e, while the fuzzy number $F_{E3}$ is ranked highest, we cannot immediately rank $F_{E2}$ higher than $F_{E1}$ based on Pareto optimality only since $F_{E2}$ has a larger range. Through these five cases, we want to note

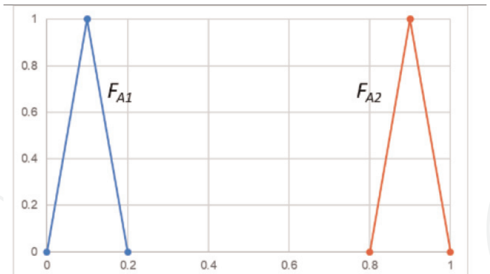
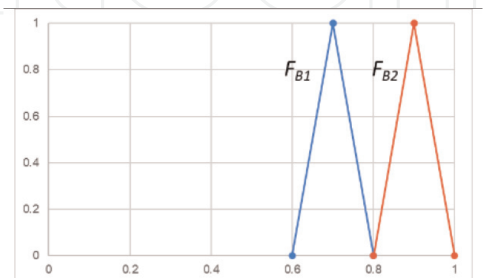
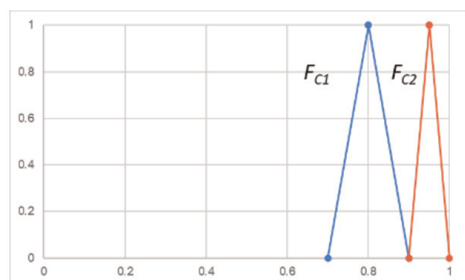
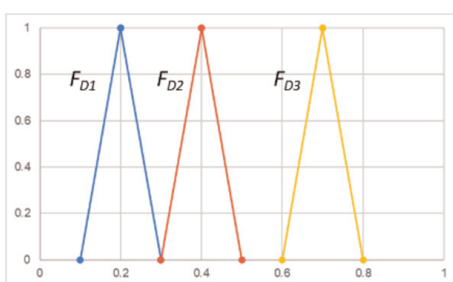
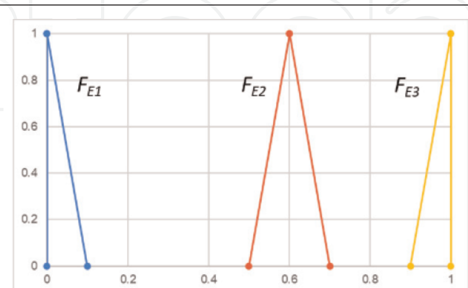
| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case a |  | $F_{A1}$ | 0.1 | 0.2 | 0.5 |
| | | $F_{A2}$ | 0.9 | 0.2 | 0.5 |
| Case b |  | $F_{B1}$ | 0.7 | 0.2 | 0.5 |
| | | $F_{B2}$ | 0.9 | 0.2 | 0.5 |
| Case c |  | $F_{C1}$ | 0.8 | 0.2 | 0.5 |
| | | $F_{C2}$ | 0.95 | 0.1 | 0.5 |
| Case d |  | $F_{D1}$ | 0.2 | 0.2 | 0.5 |
| | | $F_{D2}$ | 0.4 | 0.2 | 0.5 |
| | | $F_{D3}$ | 0.7 | 0.2 | 0.5 |
| Case e |  | $F_{E1}$ | 0.0083 | 0.1 | 0.5 |
| | | $F_{E2}$ | 0.6 | 0.2 | 0.5 |
| | | $F_{E3}$ | 0.9917 | 0.1 | 0.5 |

**Table 1.**
*Results of comparing non-overlapping, triangular fuzzy numbers.*

that the proposed attributes vary according to our "intuition" to interpret and rank fuzzy numbers (e.g., check how representative x-values and ranges vary independently in these cases).

*Group 2: Overlapping, triangular fuzzy numbers*

This group covers the cases of f, i and l from Bortolan and Degani [38], and the results are shown in **Table 2**. In case f, while $F_{F2}$ should be ranked higher than $F_{F1}$ due to higher representative x-value shown in **Table 2**, we should note that this ranking is sensitive to the pre-set value of $w$. If $w < 1/6$ (i.e., more emphasis to the core values), we have $rep(F_{F2}) > rep(F_{F1})$. If $w \geq 1/6$ (i.e., more emphasis to the boundary values), we have $rep(F_{F1}) \geq rep(F_{F2})$.

In contrast, as the fuzzy numbers in case i share the same support, their ranking is not sensitive to the value of $w$. Finally, the ranking in case l depends on the information of range, and our intuition assumes that smaller range is better. Notably, our intuition here is not universal, and some decision maker can rank a fuzzy number of larger range higher for a positive likelihood of higher x-values. Here we are not arguing which "intuition" (or ranking rule) is right. Instead, we want to keep the intuition more transparent through explicit attributes so that researchers can argue their ranking intuitions on a common ground.

*Group 3: Triangular and trapezoidal fuzzy numbers*

This group covers the cases of g and h from [38], and the results are shown in **Table 3**. The trapezoid fuzzy numbers have a large shape, giving higher values of range and membership ratio. The triangular fuzzy numbers in both cases have higher

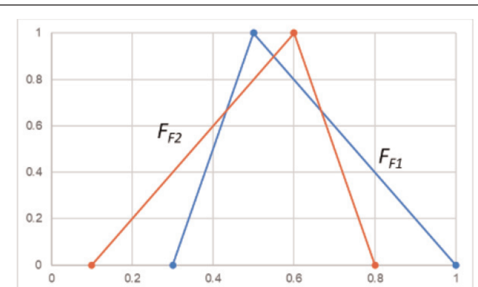
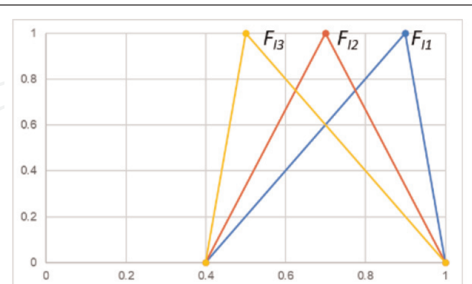
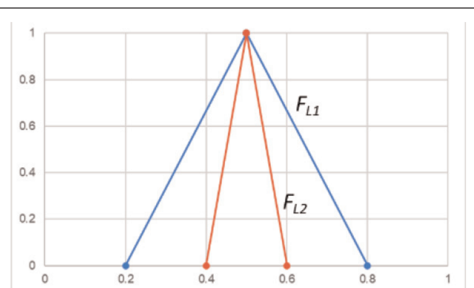
| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case f |   | $F_{F1}$ | 0.525 | 0.7 | 0.5 |
| | | $F_{F2}$ | 0.575 | 0.7 | 0.5 |
| Case i |   | $F_{I1}$ | 0.8667 | 0.6 | 0.5 |
| | | $F_{I2}$ | 0.7 | 0.6 | 0.5 |
| | | $F_{I3}$ | 0.5333 | 0.6 | 0.5 |
| Case l |   | $F_{L1}$ | 0.5 | 0.6 | 0.5 |
| | | $F_{L2}$ | 0.5 | 0.2 | 0.5 |

**Table 2.**
*Results of comparing overlapping, triangular fuzzy numbers.*

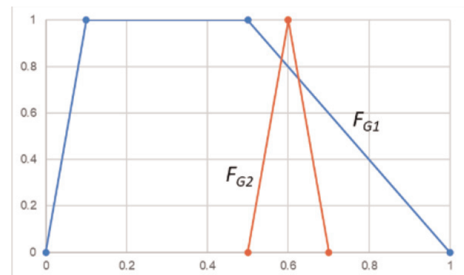
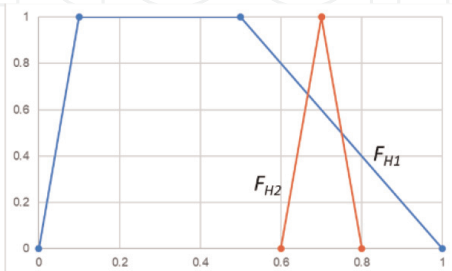
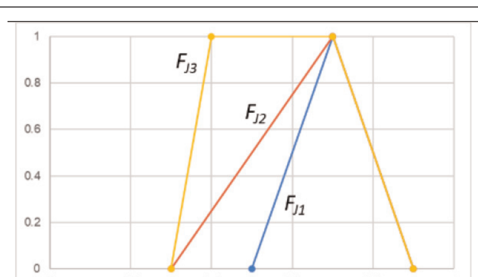
| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case g |   | $F_{G1}$ | 0.3333 | 1 | 0.7 |
| | | $F_{G2}$ | 0.6 | 0.2 | 0.5 |
| Case h |   | $F_{H1}$ | 0.3333 | 1 | 0.7 |
| | | $F_{H2}$ | 0.7 | 0.2 | 0.5 |

**Table 3.**
*Results of comparing triangular and trapezoidal fuzzy numbers.*

representative x-values. Their triangular shapes are the same, with a shift to the right side by 0.1 in case h. In view of Pareto optimality with three attributes, there is no dominant fuzzy number. Yet, we can note that if $F_{G2} \succcurlyeq F_{G1}$ in case g, we would have $F_{H2} \succcurlyeq F_{H1}$ in case h. It is because $F_{H2} \succcurlyeq F_{G2}$ due to Pareto optimality and $F_{G1} = F_{H1}$. This note should make sense when we observe the graphical shift of triangular fuzzy numbers from $F_{G2}$ to $F_{H2}$ in **Table 3**. This demonstrates how the three attributes can characterize some intuitive reasoning in FNR.

*Group 4: Nested fuzzy numbers.*

This group covers the cases of j and k from [38], and the results are shown in **Table 4**. In case j, $F_{J2}$ is created by shifting the lower bound of $F_{J1}$ to the left; $F_{J2}$ and $F_{J3}$ share the same support with a different shape. Fuzzy numbers in case k have a similar pattern in an opposite direction (see **Table 4**). By checking from the order $F_{J1} \rightarrow F_{J2} \rightarrow F_{J3}$ or $F_{K1} \rightarrow F_{K2} \rightarrow F_{K3}$, we argue that the three attributes can reasonably capture and quantify the characteristics of these fuzzy numbers.

| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case j |   | $F_{J1}$ | 0.7 | 0.4 | 0.5 |
| | | $F_{J2}$ | 0.6833 | 0.6 | 0.5 |
| | | $F_{J3}$ | 0.5583 | 0.6 | 0.75 |

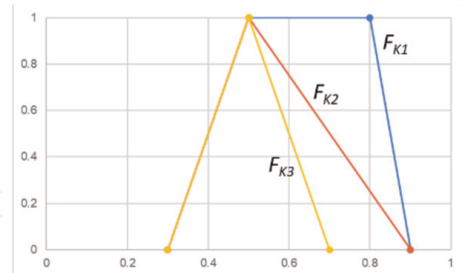
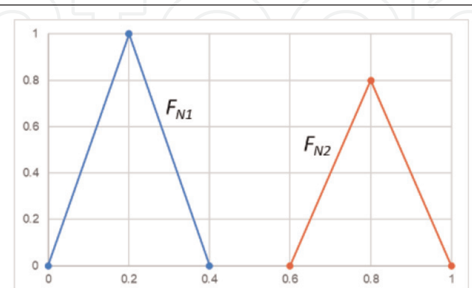
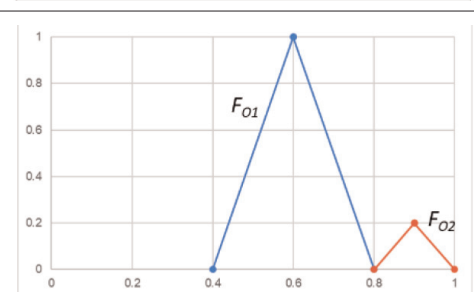
| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case k |  | $F_{K1}$ | 0.6417 | 0.6 | 0.75 |
| | | $F_{K2}$ | 0.5167 | 0.6 | 0.5 |
| | | $F_{K3}$ | 0.5 | 0.4 | 0.5 |

**Table 4.**
*Results of comparing nested fuzzy numbers.*

*Group 5: Non-normal fuzzy numbers.*

Non-normal fuzzy numbers have their maximum membership less than 1 (i.e., $h_A < 1$). Notably, the literature of FNR often assumes normal fuzzy numbers (e.g., [28]). By inspecting the earlier cases, we should note that the variations of membership ratio of normal fuzzy numbers do not change much (from 0.5 for triangular to 0.7 or 0.75 for trapezoidal). Thus, it is not unreasonable if one chooses not to consider membership ratio for comparing normal fuzzy numbers. Yet, non-normal fuzzy numbers will open other possibilities, where the membership ratio can be an important consideration.

This group covers the cases of n, o, p, q and r from Bortolan and Degani [38], and the results are provided in **Table 5**. As shown in **Table 5**, the values of membership ratio vary more significantly as some fuzzy numbers have smaller maximum membership. Consequently, the trade-off consideration can be more challenging. For example, how should we compare $F_{N1}$ and $F_{N2}$ in case n with the trade-off of

| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case n |  | $F_{N1}$ | 0.2 | 0.4 | 0.5 |
| | | $F_{N2}$ | 0.8 | 0.4 | 0.4 |
| Case o |  | $F_{O1}$ | 0.6 | 0.4 | 0.5 |
| | | $F_{O2}$ | 0.9 | 0.2 | 0.1 |

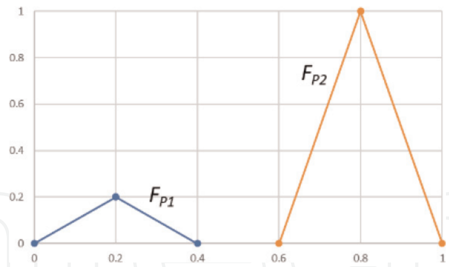
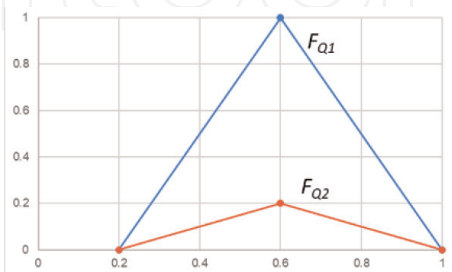
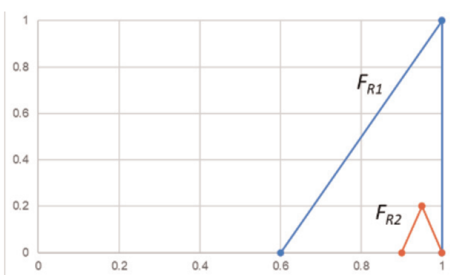
| | | | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) |
|---|---|---|---|---|---|
| Case p |  | $F_{P1}$ | 0.2 | 0.4 | 0.1 |
| | | $F_{P2}$ | 0.8 | 0.4 | 0.5 |
| Case q |  | $F_{Q1}$ | 0.6 | 0.8 | 0.5 |
| | | $F_{Q2}$ | 0.6 | 0.8 | 0.1 |
| Case r |  | $F_{R1}$ | 0.9667 | 0.4 | 0.5 |
| | | $F_{R2}$ | 0.95 | 0.1 | 0.1 |

**Table 5.**
*Results of comparing non-normal fuzzy numbers.*

representative x-value and membership ratio (similarly for case o)? While we see $F_{P2} \succcurlyeq F_{P1}$ in case p and $F_{Q1} \succcurlyeq F_{Q2}$ in case q due to Pareto optimality, the trade-off consideration is present in case r with different values of range.

The main theme of this section is that we need some attributes to characterize our intuition for FNR. Otherwise, it is difficult to get a common ground for constructive arguments. In this section, we choose three attributes to make clear our "intuition" for FNR. Aligned with the note in Keeney and Raiffa [4], we do not claim the uniqueness of this selection of attributes for FNR. Other researchers can propose other sets of attributes to characterize their intuition.

## 4. Aggregation: proposal of a ranking index

If the Pareto optimality principle cannot rank two fuzzy numbers, trade-off consideration is required to finalize the ranking decision. That is, a fuzzy number of a higher rank must have some "weaker" aspect in terms of the three attributes but its "stronger" aspect is sufficient to bring it to a higher rank overall. This ranking process should involve an aggregation that combines all aspects into an overall evaluation and then determines the ranking result. This section will propose a ranking index for aggregation along with numerical examples.

### 4.1 Discount factors and ranking index

As discussed in Section 3, representative x-values are used as the primary attribute to rank fuzzy numbers. Then, we view the information of range and membership ratio as secondary attributes that will "discount" the representative x-values. To illustrate, consider a crisp number, 5, which has the representative x-value of 5, range of 0 and membership ratio of 1. If a fuzzy number with the representative x-value of 5 has a range larger than 0 and a membership ratio less than 1, this fuzzy number should be ranked lower than the crisp number 5. The discount factors are intended to capture this idea. Let $I_{rank}(F_A)$ be the index as the discounted representative x-value of $F_A$ for ranking, and it can be formulated as follows.

$$I_{rank}(F_A) = d_{rng}(F_A) \cdot d_{mem}(F_A) \cdot rep(F_A, w) \tag{12}$$

where $d_{rng}(F_A)$ and $d_{mem}(F_A)$ are the discount factors associated with range and membership ratio, respectively. To quantify these discount factors, we consider the following conditions:

- $0 \leq d_{rng}(F_A) \leq 1$ and $0 \leq d_{mem}(F_A) \leq 1$
- If $rng(F_A, \alpha) \geq rng(F_B, \alpha)$, $d_{rng}(F_A) \leq d_{rng}(F_B)$.
- If $mem(F_A, \alpha) \geq mem(F_B, \alpha)$, $d_{mem}(F_A) \geq d_{mem}(F_B)$.

Apparently, many forms of formulations can be used for the discount factors and satisfy these conditions. In this chapter, we use a simple ratio with respect to some reference (or extreme) values. Let $rng_{min}$ be the minimum reference for range, and $mem_{max}$ be the maximum reference for membership ratio. We also set that $rng_{min} > 0$ and $0 < mem_{max} \leq 1$. Then, the discount factors for $F_A$ can be formulated as follows.

$$d_{rng}(F_A) = \frac{rng_{min}}{rng(F_A, \alpha)} \tag{13}$$

$$d_{mem}(F_A) = \frac{mem(F_A, \alpha)}{mem_{max}} \tag{14}$$

With these discount factors, if $F_A$ has a range equal to $rng_{min}$, its discount factor, $d_{rng}(F_A)$, is equal to 1 (i.e., no discount). A similar effect is also set for $d_{mem}(F_A)$. The selection of the values for $rng_{min}$ and $mem_{max}$ depends on how decision makers interpret the discount ratio for their ranking problems. One suggestion is to identify the minimum range and the maximum membership ratio from the set of fuzzy numbers to be ranked. That is, suppose that $\boldsymbol{FR} = \{F_A, F_B, F_C, \ldots\}$ be the set of fuzzy numbers that need to be ranked in a problem. We can select $rng_{min}$ and $mem_{max}$ according to the following equations. Then, we can interpret the discount ratio with respect to the "best values" among the set of fuzzy numbers in the problem.

$$rng_{min} = min\,\{rng(F_A, \alpha), rng(F_B, \alpha), rng(F_C, \alpha) \ldots\} \tag{15}$$

$$mem_{max} = max\,\{mem(F_A, \alpha), mem(F_B, \alpha), mem(F_C, \alpha) \ldots\} \tag{16}$$

### 4.2 Overview of the ranking method

After defining the attributes in Section 3 and the ranking index in Section 4.1, this sub-section will overview our proposed approach to rank fuzzy numbers. The procedure to determine the ranking index is illustrated in **Figure 5**. Given a fuzzy number $F_A$, we first determine the values of three attributes: representative x-value, x-value range and overall membership ratio. Then, we can evaluate the discount factors for x-value range and overall membership ratio. In the end, we can determine the ranking index for the given fuzzy number.

Suppose that we are tasked to rank a set of fuzzy numbers. We first determine the ranking index for each fuzzy number. Then, we can use the index, $I_{rank}$, for this ranking task. That is, if $I_{rank}(F_A) \geq I_{rank}(F_B)$, we rank $F_A$ higher than $F_B$, symbolically, $F_A \succcurlyeq F_B$.

### 4.3 Numerical examples

As a recall from Section 3, we set $w = 1/12$ and $\alpha = 0$ to evaluate representative x-value, range and membership ratio. We use Eqs. (15) and (16) to obtain $rng_{min}$ and $mem_{max}$ and then calculate the values of the discounts and the ranking index. We reuse the numerical examples from Section 3.5 with the cases where Pareto optimality cannot finalize the ranking. The results are presented in **Table 6**.

Case e comes from Group 1 (see **Table 1**), where $F_{E3}$ is ranked on the top per Pareto optimality (same result from the ranking index). Between $F_{E1}$ and $F_{E2}$, though $F_{E2}$ should be ranked higher intuitively, trade-off is involved logically because $F_{E2}$ has a large range (reflected in its range discount of 0.5 as well). Per the ranking index, we still have $F_{E2} \succcurlyeq F_{E1}$, which matches the general intuition.

Cases g and h come from Group 3 (see **Table 3**), where wide trapezoidal fuzzy numbers are compared with narrow triangular fuzzy numbers. Per the ranking index, the triangular fuzzy numbers are ranked higher mainly because of the large difference of the range discount (1 vs. 0.2). In contrast, the difference of the membership ratio discount is less substantial.

Cases j and k come from Group 4 (see **Table 4**), where two triangular fuzzy numbers are nested in a trapezoidal fuzzy number. In both cases, the wider triangular fuzzy numbers (i.e., $F_{J2}$ and $F_{K2}$) are ranked lowest as they receive both discounts

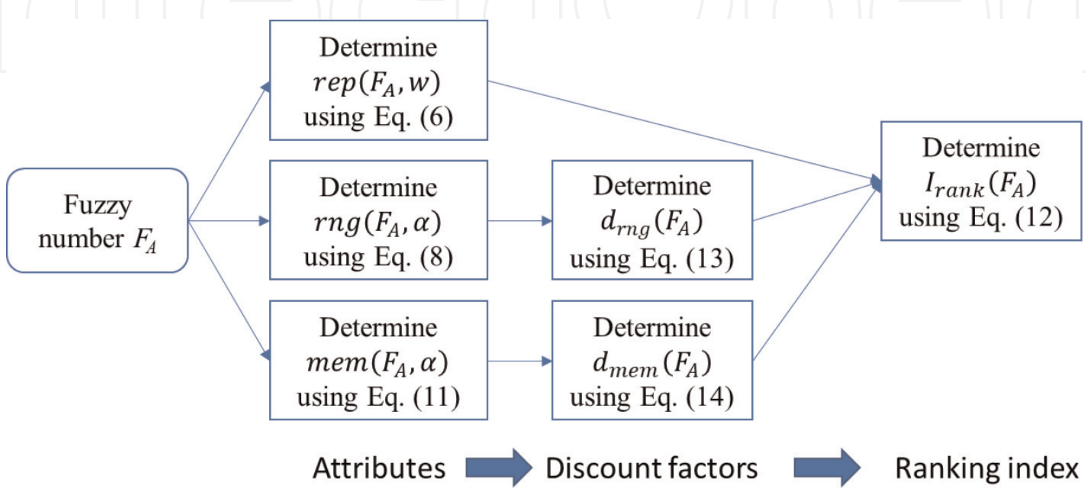



**Figure 5.**
*Procedure to determine the ranking index.*

| | Fuzzy number | Range discount ($d_{rng}$) | Mem. discount ($d_{mem}$) | Ranking index ($I_{rank}$) |
|---|---|---|---|---|
| Case e | $F_{E1}$ = (0, 0, 0, 0.1; 1) | 1 | 1 | 0.0083 |
| | $F_{E2}$ = (0.5, 0.6, 0.6, 0.7; 1) | 0.5 | 1 | 0.3 |
| | $F_{E3}$ = (0.9, 1, 1, 1; 1) | 1 | 1 | 0.9917 |
| Case g | $F_{G1}$ = (0, 0.1, 0.5, 1; 1) | 0.2 | 1 | 0.0667 |
| | $F_{G2}$ = (0.5, 0.6, 0.6, 0.7; 1) | 1 | 0.7143 | 0.4286 |
| Case h | $F_{H1}$ = (0, 0.1, 0.5, 1; 1) | 0.2 | 1 | 0.0667 |
| | $F_{H2}$ = (0.6, 0.7, 0.7, 0.8; 1) | 1 | 0.7143 | 0.5 |
| Case j | $F_{J1}$ = (0.5, 0.7, 0.7, 0.9; 1) | 1 | 0.6667 | 0.4667 |
| | $F_{J2}$ = (0.3; 0.7, 0.7, 0.9; 1) | 0.6667 | 0.6667 | 0.3037 |
| | $F_{J3}$ = (0.3, 0.4, 0.7, 0.9; 1) | 0.6667 | 1 | 0.3722 |
| Case k | $F_{K1}$ = (0.3, 0.5, 0.8, 0.9; 1) | 0.6667 | 1 | 0.4278 |
| | $F_{K2}$ = (0.3, 0.5, 0.5, 0.9; 1) | 0.6667 | 0.6667 | 0.2296 |
| | $F_{K3}$ = (0.3, 0.5, 0.5, 0.7; 1) | 1 | 0.6667 | 0.3333 |
| Case n | $F_{N1}$ = (0, 0.2, 0.2, 0.4; 1) | 1 | 1 | 0.2 |
| | $F_{N2}$ = (0.6, 0.8, 0.8, 1; 0.8) | 1 | 0.8 | 0.64 |
| Case o | $F_{O1}$ = (0.4, 0.6, 0.6, 0.8; 1) | 0.5 | 1 | 0.3 |
| | $F_{O2}$ = (0.8, 0.9, 0.9, 1; 0.2) | 1 | 0.2 | 0.18 |
| Case r | $F_{R1}$ = (0.6, 1, 1, 1; 1) | 0.25 | 1 | 0.2417 |
| | $F_{R2}$ = (0.9, 0.95, 0.95, 1; 0.2) | 1 | 0.2 | 0.19 |

**Table 6.**
*Ranking index results for cases with trade-off consideration.*

(i.e., $d_{rng}$ = $d_{mem}$ = 0.6667). In case j, the narrower triangular fuzzy number ($F_{J1}$) is ranked first because its representative x-value and range can "win" over its weaker membership ratio as compared to the trapezoidal fuzzy number ($F_{J3}$). In contrast, in case k, the trapezoidal fuzzy number ($F_{K1}$) "wins" because it has better representative x-value and membership ratio as compared to $F_{K3}$.

Cases n, o and r come from Group 5 (see **Table 5**). In case n, we have $F_{N2} \succcurlyeq F_{N1}$, as $F_{N2}$ has higher representative x-value despite lower membership ratio (associated with the discount $d_{mem}(F_{N2})$ = 0.8). In cases o, we have $F_{O1} \succcurlyeq F_{O2}$ because the membership ratio of $F_{O2}$ is substantially lower despite its higher representative x-value. In case r, the trade-off between range and membership ratio is relatively close. In the end, we have $F_{R1} \succcurlyeq F_{R2}$, as $F_{R2}$ has a lower value of the discount from membership ratio (i.e., $d_{mem}(F_{R2})$ = 0.2 vs. $d_{rng}(F_{R1})$ = 0.25).

Notably, the judgment for ranking with trade-off can become difficult when the trade-off among the three attributes is getting close. We argue that such difficulty is fundamentally embedded into the problem structure of FNR, which involves multiple dimensions of considerations. Thus, our solution strategy is not about providing the best ranking procedure. Instead, we emphasize the importance of defining attributes to quantify the "intuition". Then, decision makers can explicitly explain their trade-off considerations in the ranking process.

## 5. Multi-attribute ranking method in practice

### 5.1 General suggestions for application

As we consider that ranking methods should be dependent on a given set of fuzzy numbers to be ranked (i.e., context-dependent), we want to discuss two types of adjustable elements of our proposed method in practice. The first type is the selection of attributes. Among three attributes: representative x-value, range and membership ratio, representative x-value should be a default choice as it intuitively corresponds to the ranking of real numbers (e.g., compare representative x-values of different fuzzy numbers). We suggest the class of ranking indices by Ban and Coroianu [28] (i.e., in Eq. (6)) as it satisfies the six axioms. To determine the weight, $w$, of this attribute, decision makers may consider sensitivity analysis for their given set of fuzzy numbers (i.e., how sensitive of the value of $w$ can alter the ranking of two fuzzy numbers).

In contrast to representative x-value, the choice of range and membership ratio is optional. The attribute of x-value range is common in literature, and other formulations of this attribute (e.g., ambiguity and deviation degree as mentioned in Section 3.2) can be considered as a choice by decision makers for this attribute. If the ranking problem has fuzzy numbers of similar ranges (e.g., triangular fuzzy numbers with similar supports), we think it is legitimate not to consider range in FNR (in order to preserve some axiomatic properties, to be discussed in Section 6). The attribute of membership ratio is less common, and it should be more relevant for non-normal fuzzy numbers (e.g., normal triangular fuzzy numbers always have the same membership ratio equal to 0.5).

The second type of adjustable elements of our proposed method is the specification of the reference values (i.e., $rng_{min}$ and $mem_{max}$) and the formulations of the discount factors. Notably, our formulations of discount factors (i.e., Eqs. (13) and (14)) are only one simple suggestion. One possible disadvantage of our discount factors is that they can be too sensitive to the reference values. For example, if two fuzzy numbers with the range values of 0.5 and 1 are compared, the range discount ($d_{rng}$) can be equal to 0.5 for one fuzzy number, cutting half of its representative x-value. Decision makers can consider adjusting the effects of discount factors through other formulations for their problems (e.g., additional scaling component).

### 5.2 Applicability to specific forms

The origin of fuzzy numbers can be viewed as a generalization of crisp numbers to describe approximate information. Consider the 5-tuple definition of a fuzzy number, $F_A = (a_1, a_2, a_3, a_4; h_A)$ as the generalized form of fuzzy numbers in this work. Accordingly, three specific forms can be considered as follows.

- Interval: if $a_1 = a_2, a_3 = a_4$ and $h_A = 1$;
- Ordered pair (an element of a fuzzy set): if $a_1 = a_2 = a_3 = a_4$ and $h_A < 1$;
- Crisp number: if $a_1 = a_2 = a_3 = a_4$ and $h_A = 1$.

Then, we want to investigate the reducibility property that whether a ranking method can still be applicable if the above specific forms are considered. **Table 7** shows that our proposed ranking method can be still used for these specific forms. First, a general fuzzy number can be evaluated using the equations as listed in the first

| | Representative x-value (*rep*) | Range (*rng*) | Membership ratio (*mem*) | Range discount ($d_{rng}$) | Membership discount ($d_{mem}$) |
|---|---|---|---|---|---|
| Fuzzy number | Eq. (6) | Eq. (8) | Eq. (11) | Eq. (13) | Eq. (14) |
| Interval | $(a_2 + a_3)/2$ | $(a_4 – a_1)$ | 1 | Eq. (13) | 1 |
| Ordered pair | $a_1$ $(= a_2 = a_3 = a_4)$ | 0 | $h_A$ | 1 | Eq. (14) |
| Crisp number | $a_1$ $(= a_2 = a_3 = a_4)$ | 0 | 1 | 1 | 1 |

**Table 7.**
*Overview of the reducibility property.*

row of **Table 7**. When these equations are applied to interval, ordered pair and crisp number, we can obtain the results that match our expectations. For example, the representative x-value of an interval will be the midpoint of $a_2$ and $a_3$, and a crisp number has no discount effect (i.e., $d_{rng}$ = 1 and $d_{mem}$ = 1). In this way, our proposed method can be used to compare fuzzy numbers with intervals or crisp numbers in the same methodical framework.

## 6. Information content and axiomatic properties

Since the pioneer work by Wang and Kerre [1], researchers have examined the axiomatic properties (i.e., the six axioms listed in Section 2) of fuzzy number ranking methods. The intent of this section is to discuss how the information content for ranking can influence the axiomatic properties in the context of our ranking approach. One key message is that the satisfaction of axioms depends on the selection of information that is deemed relevant to FNR. If more information is selected and considered for ranking, the ranking method is more likely to violate the axioms. This message is aligned with the topic of information basis in the analysis of the Arrow's Impossibility Theorem [39, 40].

### 6.1 Analysis of Axiom 4

Axiom 4 somewhat dictates the ranking of non-overlapping fuzzy numbers. If we only consider representative x-values for ranking (i.e., no range, membership ratio and discount factors), our ranking procedure will directly follow the results from Ban and Coroianu [28], and it will thus satisfy Axiom 4. However, if range and membership ratio are considered as relevant information for ranking, Axiom 4 can be violated, and the reason is given below.

Axiom 4 only focuses on the boundary values without considering any distributional information (e.g., range and membership). When multiple attributes are considered for ranking, Axiom 4 can be violated by strengthening the distributional aspect of the inferior fuzzy number (in view of Axiom 4). For example, we have $F_{O2} \succcurlyeq F_{O1}$ in case o (see **Table 5**) according to Axiom 4, no matter how small of membership ratio of $F_{O2}$. However, when we consider range and membership ratio, we obtain $F_{O1} \succcurlyeq F_{O2}$ (see **Table 6**), which violates Axiom 4. Notably, the logic of such violation can be held whenever we deem membership ratio as relevant information for FNR, regardless of the details of the ranking procedures.

Notably, this discussion is not about rejecting Axiom 4. Instead, we want to explain one logical tension with Axiom 4. That is, Axiom 4 dictates some ranking of fuzzy numbers based on their boundary values only, and this opens a chance for the information of range and membership ratio to violate Axiom 4. Alternately, if we choose the index class by [28], representative x-values will be the only information considered for ranking, and the information of range (for example) will become irrelevant for FNR. In other words, if we consider that FNR should involve trade-off with multiple attributes in addition to representative x-value, Axiom 4 could be violated in some situations.

### 6.2 Dependence of $rng_{min}$ and $mem_{max}$

As we use reference values (i.e., $rng_{min}$ and $mem_{max}$) to evaluate the discount factors (i.e., $d_{rng}$ and $d_{mem}$), the ranking index, $I_{rank}$, belongs to the second class of ranking indices according to the classification by [1]. In their axiomatic analysis, they have identified five indices of the second class, i.e., $J^K()$, $K()$, $CH^K()$, $W()$ and $KP^K()$, which all do not satisfy Axioms 5 and 6. Without listing counter-examples, $I_{rank}$ of the same class follows the same conclusion because they share a common feature of these ranking indices, i.e., use of reference values.

Why using reference values could violate Axioms 5 and 6? It is because the index values would depend on the information that is external to the fuzzy numbers themselves. For example, if a fuzzy number with a very small range is added to a set of fuzzy numbers for ranking (i.e., $\boldsymbol{FR}$), this newly added fuzzy number will decrease the reference value, $rng_{min}$, and thus generally decrease the range discount values ($d_{rng}$) for the original set of fuzzy numbers. Then, all values of $I_{rank}$ would change because of adding a new fuzzy number to the set (i.e., $\boldsymbol{FR}$).

While it seems undesirable by setting $rng_{min}$ and $mem_{max}$ per individual sets of fuzzy numbers, can we simply set these two values as universal numbers that are applicable to all ranking problems (e.g., simply set $mem_{max} = 1$)? Theoretically, it is a viable option. However, by doing so, we somehow lose our interpretation of "discount" factors that are relevant to a given set of fuzzy numbers that we want to rank in the problem. For example, it is not easy to interpret if the range of a fuzzy number, say 5, is large or small until we know a reference for comparison (e.g., if $rng_{min} = 1$, the range of 5 will quite large). In other words, the reference values, $rng_{min}$ and $mem_{max}$, provide a numerical context as relevant information for comparison.

To close this section, we want to make a note about the historical development of the Arrow's Impossibility Theorem, which proves that no voting method (or social welfare function) can satisfy a set of "reasonable" properties (or axioms) [41]. One famous "escaping route" is the information basis approach, which classifies the information content (or availability) for interpersonal comparisons with different axiomatic results [39, 40]. Back to our context, if representative x-value is taken as the only relevant information for FNR, the results by [28] are sufficient to design a ranking index that satisfies the six axioms in Section 2. However, if additional information is considered for FNR, the axiomatic properties cannot be guaranteed. To us, this tension seems fundamental.

## 7. Conclusion

The main theme of this chapter is to use the multi-attribute approach to analyze and address the problems of fuzzy number ranking (FNR) since numerous ranking methods in literature have implicated multiple attributes in their ranking intuition. The multi-attribute approach has two phases: the selection of attributes and the formulation of the aggregation function. The selection of attributes determines what information is deemed relevant for FNR, and the aggregation function controls the trade-off of the attribute values of fuzzy numbers. In this work, we propose three attributes (i.e., representative x-value, range and membership ratio) as three possible dimensions to evaluate a fuzzy number. In aggregation, we formulate the discount factors for range and membership ratio to modify the representative x-value of a fuzzy number for FNR. The proposed method has been illustrated via numerical examples to reveal the rationale of using multiple attributes to articulate the intuition behind FNR.

In future work, there can be two directions to consider: practice and theory. In the practice direction, we can develop more methodical guidance toward the selection and formulations of attributes and the aggregation procedures. In particular, we can formalize the ranking intuition in terms of the selected attributes and the trade-off rationale through the aggregation approach for different FNR problems. Along this effort, we can also compare the ranking results from this multi-attribute approach with other FNR approaches. In the theory direction, while this chapter has initially explored the tension between information content and the satisfaction of the FNR axioms. This tension should call for more mathematical analyses such as classification of information content for FNR and relaxation of axioms for expanded information basis.

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