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Chapter

Solving and Algorithm for Least-Norm General Solution to Constrained Sylvester Matrix Equation

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Abstract

Keeping in view that a lot of physical systems with inverse problems can be written by matrix equations, the least-norm of the solution to a general Sylvester matrix equation with restrictions $A_1X_1 = C_1, X_1B_1 = C_2, A_2X_2 = C_3, X_2B_2 = C_4, A_3X_1B_3 + A_4X_2B_4 = C_c$, is researched in this chapter. A novel expression of the general solution to this system is established and necessary and sufficient conditions for its existence are constituted. The novelty of the proposed results is not only obtaining a formal representation of the solution in terms of generalized inverses but the construction of an algorithm to find its explicit expression as well. To conduct an algorithm and numerical example, it is used the determinantal representations of the Moore–Penrose inverse previously obtained by one of the authors.

Keywords: linear matrix equation, generalized Sylvester matrix equation, Moore-Penrose inverse

1. Introduction

Standardly, we state \mathbb{C} and \mathbb{R} , respectively, for the complex and real numbers. Let $\mathbb{C}^{m \times n}$ denote the set of all $m \times n$ matrices over \mathbb{C} , and $\mathbb{C}_r^{m \times n}$ stay for a subset of $m \times n$ complex matrices with rank r. The rank of A is denoted by both symbols r(A) and *rankA*. The (complex) conjugate transpose matrix of $A \in \mathbb{C}^{m \times n}$ is written by A^* and a matrix $A \in \mathbb{C}^{n \times n}$ is said to be Hermitian if $A^* = A$. An identity matrix with feasible shape is denoted by I.

Definition 1.1. The Moore–Penrose (MP-) inverse of $A \in \mathbb{C}^{m \times n}$, denoted by A^{\dagger} , is defined to be the unique solution *X* to the following four Penrose equations

$$AXA = A, \tag{1}$$

$$XAX = X,$$
 (2)

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA. \tag{4}$$

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Matrices satisfying the eqs. (1) and (2) are known as reflexive inverses, denoted by A^+ .

In addition, $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$ represent a pair of orthogonal projectors onto the kernels of A and A^* , respectively.

Mathematical models of physical systems with inverse problems especially those has a finite number of model parameters can be written by matrix equations. In particular, the Sylvester-type matrix equations have far-reaching applications in singular system control [1], system design [2], robust control [3], feedback [4], perturbation theory [5], linear descriptor systems [6], neural networks [7] and theory of orbits [8], etc.

Some recent work on generalized Sylvester matrix equations and their systems can be observed in [9–21]. In 2014, Bao [22] examined the least-norm and extremal ranks of the least square solution to the quaternion matrix equations

$$A_1 X = C_1, X B_1 = C_2, \ A_3 X B_3 = C_c.$$
(5)

Wang et al. [23] examined the expression of the general solution to the system

$$A_1X_1 = C_1, \quad A_2X_2 = C_3, A_3X_1 \quad B_3 + A_4X_2B_4 = C_c,$$
 (6)

and as an application, the P-symmetric and P-skew-symmetric solution to

$$A_a X = C_a, A_b X B_b = C_b.$$

has been established. Li et al. [24] established a novel expression of the general solution of the system (6) and they computed the least-norm of general solution to (6). In 2009, Wang et al. [25] constituted the expression of the general solution to

$$A_1 \quad X_1 = C_1, X_1 B_1 = C_2, A_2 \quad X_2 = C_3, X_2 B_2 = C_4, A_3 \quad X_1 B_3 + A_4 X_2 B_4 = C_c,$$
(7)

and as an application, they explored the (P, Q)-symmetric solution to the system

 $A_a X = C_a, X B_b = C_b, A_c X B_c = C_c.$

Some latest findings on the least-norm of matrix equations and (P, Q)-symmetric matrices can be consulted in [26–30]. Furthermore, our main system (7) is a special case of the following system

$$A_{1}X_{1} = C_{1}, X_{2}B_{1} = D_{1},$$

$$A_{2}X_{3} = C_{2}, X_{3}B_{2} = D_{2},$$

$$A_{3}X_{4} = C_{3}, X_{4}B_{3} = D_{3},$$

$$A_{4}X_{1} + X_{2} \quad B_{4} + C_{4}X_{3}D_{4} + C_{5}X_{4}D_{5} = C_{c},$$
(8)

which has been investigated by Zhang in 2014.

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Motivated by the latest interest of least-norm of matrix equations, we construct a novel expression of the general solution to the system (7) and apply this to investigate the least-norm of the general solution to the system (7) in this chapter. Observing that

systems (5) and (6) are particular cases of our system (7), solving system (7) will encourage the least-norm to a wide class of problems.

We commence with the following lemmas which have crucial function in the construction of the chief outcomes of the following sections.

Lemma 1.2. [31]. Let A, B, and C be given matrices over \mathbb{C} with agreeable dimensions. Then.

$$1.r(A) + r(R_A B) = r(B) + r(R_B A) = r[A \quad B].$$

$$2.r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{bmatrix} A \\ C \end{bmatrix}.$$

$$3.r(B) + r(C) + r(R_B A L_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$

Lemma 1.3. [32]. Let A, B, and C be known matrices over \mathbb{C} with right sizes. Then

$$1.A^{\dagger} = (A^{*}A)^{\dagger}A^{*} = A^{*}(AA^{*})^{\dagger}.$$

$$2.L_{A} = L_{A}^{2} = L_{A}^{*}, R_{A} = R_{A}^{2} = R_{A}^{*}.$$

$$3.L_{A}(BL_{A})^{\dagger} = (BL_{A})^{\dagger}, (R_{A}C)^{\dagger}R_{A} = (R_{A}C)^{\dagger}.$$

Lemma 1.4. [33]. Let Φ , Ω be matrices over \mathbb{C} and

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \ \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ & \end{bmatrix}, \ F = \Phi_2 L_{\Phi_1}, \ T = R_{\Omega_1} \Omega_2$$

Then

$$L_{\Phi} = L_{\Phi_1} L_F, \quad L_{\Omega} = \begin{bmatrix} L_{\Omega_1} & -\Omega_1^{\dagger} \Omega_2 L_T \\ 0 & L_T \end{bmatrix},$$
$$R_{\Omega} = R_T R_{\Omega_1}, \quad R_{\Phi} = \begin{bmatrix} R_{\Phi_1} & 0 \\ -R_F \Phi_2 \Phi_1^{\dagger} & R_F \end{bmatrix},$$

where Φ_1^+ , Ω_1^+ are any fixed reflexive inverses, L_{Φ_1} and R_{Ω_1} stand for the projectors $L_{\Phi_1} = I - \Phi_1^+ \Phi_1$, $R_{\Omega_1} = I - \Omega_1 \Omega_1^+$ induced by Φ_1 , Ω_1 , respectively.

Remark 1.5. Since the Moore-Penrose inverse is a reflexive inverse, this lemma can be used for the MP-inverse without any changes. It has taken place in ([32], Lemma 2.4).

Lemma 1.6. [34]. Suppose that

$$B_1 X C_1 + B_2 Y C_2 = A \tag{9}$$

is consistent linear matrix equation. Then.

1. The general solution of the homogeneous equation

$$B_1XC_1+B_2YC_2=0,$$

can be expressed by

$$X = X_1 X_2 + X_3, \quad Y = Y_1 Y_2 + Y_3,$$

where $X_1 - X_3$ and $Y_1 - Y_3$ are general solution to the system

$$B_1X_1 = -B_2Y_1, \ X_2C_1 = Y_2C_2, \ B_1X_3C_1 = 0, \ B_2Y_3C_2 = 0.$$

By computing the value of unknowns in above and using them in X and Y, we have

$$X = S_1 L_G U R_H T_1 + L_{B_1} V_1 + V_2 R_{C_1},$$

$$Y = S_2 L_G U R_H T_2 + L_{B_2} W_1 + W_2 R_{C_2},$$

where
$$S_1 = \begin{bmatrix} I_p, & 0 \end{bmatrix}$$
, $S_2 = \begin{bmatrix} 0, & I_s \end{bmatrix}$, $T_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 \\ I_t \end{bmatrix}$, $G = \begin{bmatrix} B_1, & B_2 \end{bmatrix}$, and $H = \begin{bmatrix} C_1 \\ -C_2 \end{bmatrix}$; the matrices U, V_1, V_2, W_1 and W_2 are free to vary over \mathbb{C} .

2. Assume that Eq. (9) is solvable, then its general solution can be expressed as

$$X = X_0 + X_1 X_2 + X_3, \quad Y = Y_0 + Y_1 Y_2 + Y_3,$$

where X_0 and Y_0 are any pair of particular solutions to (9).

It can also be written as

$$X = X_0 + S_1 L_G U R_H T_1 + L_{B_1} V_1 + V_2 R_{C_1},$$

$$Y = Y_0 + S_2 L_G U R_H T_2 + L_{B_2} W_1 + W_2 R_{C_2}.$$

Lemma 1.7. [35]. Let A_1, B_1, C_1, C_2 be given matrices over \mathbb{C} with agreeable sizes and X_1 to be determined. Then the system



Under these conditions, the general solution to (10) can be established as

$$X_1 = A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + L_{A_1} U_1 R_{B_1},$$

where U_1 is a free matrix over \mathbb{C} with accordant dimension.

Lemma 1.8. [36]. Let A, B, and C be known matrices over \mathbb{C} with agreeable dimensions, and X be unknown. Then the matrix equation

$$AXB = C \tag{12}$$

is consistent if and only if $AA^{\dagger}CB^{\dagger}B = C$. In this case, its general solution can be expressed as

$$X = A^{\dagger}CB^{\dagger} + L_AV + WR_B, \tag{13}$$

where V, W are arbitrary matrices over \mathbb{C} with appropriate dimensions.

In [37], it is proved that (13) is the least squares solution to (12), and its minimum norm least squares solution is $X_{LS} = A^{\dagger}CB^{\dagger}$.

Lemma 1.9. [25]. Let $A_i, B_i, C_i, (i = 1, ..., 4)$, and C_c be given matrices over \mathbb{C} with agreeable dimensions, and X_1, X_2 to be determined. Denote

$$egin{aligned} &A = A_3 L_{A_1}, B = R_{B_1} B_3, C = A_4 L_{A_2}, D = R_{B_2} B_4, \ &N = D L_B, M = R_A C, S = C L_M, \ &E = C_c - A_3 A_1^\dagger C_1 B_3 - A C_2 B_1^\dagger B_3 - A_4 A_2^\dagger C_3 B_4 - C C_4 B_2^\dagger B_4. \end{aligned}$$

Then the following conditions are tantamount:

1. System (7) is resolvable.

2. The conditions in (11) are met and

$$R_{A_2}C_3 = 0, \quad C_4L_{B_2} = 0, \quad A_2C_4 = C_3B_2,$$

$$R_MR_AE = 0, R_AEL_D = 0, EL_BL_N = 0, R_CEL_B = 0.$$
(14)

3. The equalities in (11) and (14) are satisfied and

$$MM^{\dagger}R_{A}D^{\dagger}D=R_{A}E, \ \ CC^{\dagger}EL_{B}N^{\dagger}N=EL_{B}N^{\dagger}N$$

In these conditions, the general solution to the system (7) can be written as

$$X_{1} = A_{1}^{\dagger}C_{1} + L_{A_{1}}C_{2}B_{1}^{\dagger} + L_{A_{1}}A^{\dagger}EB^{\dagger}R_{B_{1}} - L_{A_{1}}A^{\dagger}CM^{\dagger}EB^{\dagger}R_{B_{1}} - -L_{A_{1}}A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger}R_{B_{1}} - L_{A_{1}}A^{\dagger}SV_{1}R_{N}DB^{\dagger}R_{B_{1}} + L_{A_{1}}(L_{A}U_{1} + Z_{1}R_{B})R_{B_{1}},$$

$$X_{2} = A_{2}^{\dagger}C_{3} + L_{A_{2}}C_{4}B_{2}^{\dagger} + L_{A_{2}}M^{\dagger}R_{A}ED^{\dagger}R_{B_{2}} + L_{A_{2}}L_{M_{b}}S^{\dagger}SC^{\dagger}EN^{\dagger}R_{B_{2}} + L_{A_{2}}L_{M}(V_{1} - S^{\dagger}SV_{1}NN^{\dagger})R_{B_{2}} + L_{A_{2}}W_{1}R_{D}R_{B_{2}},$$
(16)

where U_1, V_1, W_1 and Z_1 are free matrices over \mathbb{C} with agreeable dimensions.

Since the general solutions of considered systems are expressed in terms of generalized inverses, another goal of the paper is to give determinantal representations of the least-norm of the general solution to the system (7) based on determinantal representations of generalized inverses.

Due to the important role of generalized inverses in many application fields, considerable effort has been exerted toward the numerical algorithms for fast and accurate calculation of matrix generalized inverse. In general, most existing methods for their obtaining are iterative algorithms for approximating generalized inverses of complex matrices (some recent papers, see, e.g. [38–40]). There are only several direct methods for finding MP-inverse for an arbitrary complex matrix $A \in \mathbb{C}^{m \times n}$. The most famous is method based on singular value decomposition (SVD), i.e. if $A = U\Sigma V^*$, then $A^{\dagger} = V\Sigma^{\dagger}U^*$. The computational cost of this method is dominated by the cost of computing the SVD, which is several times higher than matrix–matrix multiplication. Another approach is constructing determinantal representations of the MP-inverse A^{\dagger} . A well-known determinantal representation of an ordinary inverse is the adjugate matrix with the cofactors in entries. It has an important theoretical significance and brings forth Cramer's rule for systems of linear equations. The same is desirable to have for the generalized inverses. Due to looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses (for the MP-inverse, see, e.g. [41, 42]). Because of the complexity of the previously obtained expressions of determinantal representations of the MP-inverse, they have little applicability.

In this chapter, we will use the determinantal representations of the MP-inverse recently obtained in [43].

Lemma 1.10. [43, Theorem 2.2] If $A \in \mathbb{C}^{m \times n}$ with rank A = r, then the Moore-Penrose inverse $A^{\dagger} = \left(a_{ij}^{\dagger}\right) \in \mathbb{C}^{n \times m}$ possess the following determinantal representations

$$a_{ij}^{\dagger} = \frac{\sum_{\beta \in J_{r,n}\{i\}} \left| (A^*A)_{.i} \left(a_{.j}^* \right) \right|_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |A^*A|_{\beta}^{\beta}} = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \left| (AA^*)_{j.} \left(a_{i.}^* \right) \right|_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,m}} |AA^*|_{\alpha}^{\alpha}}.$$
 (17)

Here $|A|^{\alpha}_{\alpha}$ denote a principal minor of A whose rows and columns are indexed by $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\},\$

$$L_{k,m} := \{ \alpha : 1 \le \alpha_1 < \cdots < \alpha_k \le m \}, \text{ and } I_{r,m}\{i\} := \{ \alpha : \alpha \in L_{r,m}, i \in \alpha \}$$

Also, a_j^* and $a_{i.}^*$ denote the *j*th column and the *i*th row of A^* , and $A_{i.}(b)$ and $A_{j.}(c)$ stand for the matrices obtained from A by replacing its *i*th row with the row vector $b \in \mathbb{C}^{1 \times n}$ and its *j*th column with the column vector $c \in \mathbb{C}^m$, respectively.

The formulas (17) give very simple and elegant determinantal representations of the MP-inverse. So, for any $A \in \mathbb{C}_r^{m \times n}$, we have sum of all principal minors of r order of the matrices A^*A or AA^* in denominators and sum of principal minors of r order of the matrices $(A^*A)_i(a_j^*)$ or $(AA^*)_j(a_i^*)$ that contain the *i*th column or the *j*th row, respectively, in numerators into (17).

Note that for an arbitrary full-rank matrix *A*, Lemma 1.10 gives a new way of finding an inverse matrix.

Corollary 1.11. If $A \in \mathbb{C}^{m \times n}$ with rank $A = \min\{m, n\}$, then the inverse $A^{-1} = (a_{ij}^{-1}) \in \mathbb{C}^{n \times m}$ possess the following determinantal representations:

$$a_{ij}^{-1} = \begin{cases} \frac{\left| (A^*A)_{.i} \left(a_j^* \right) \right|}{|A^*A|} & \text{if } rankA = n, \\ \frac{\left| (AA^*)_{j.} \left(a_{i.}^* \right) \right|}{|AA^*|} & \text{if } rankA = m. \end{cases}$$

These new determinantal representations of the Moore-Penrose inverse have been obtained by the developed novel limit-rank method in the case of quaternion matrices [44] as well. This method was successfully applied for constructing determinantal

representations of other generalized inverses in both cases for complex and quaternion matrices (see e.g. [45–47]). It also yields Cramer's rules of various matrix equations [48–54].

The remainder of our chapter is directed as follows. In Section 2, we provide a new expression of the general solution to our system (7) and discuss its least-norm. The algorithm and numerical example of finding the anti-Hermitian solution to (7) are presented in Section 3. (7). Finally, in Section 4, the conclusions are drawn.

2. A new expression of the general solution to the system

Now we demonstrate the principal theorem of this section (7).

Theorem 2.1. Assume that $S_1 = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$, $T_1 = \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}$,

 $G = \begin{bmatrix} A & C \end{bmatrix}, H = \begin{bmatrix} B \\ -D \end{bmatrix}, H_1 = L_{A_1}L_A, H_2 = L_{A_1}S_1L_G, H_3 = R_HT_1R_{B_1}, H_4 = L_{A_2}L_C, H_5 = L_{A_2}S_2L_G, H_6 = R_HT_2R_{B_2}$ and the system (7) is solvable, then the general solution to our

$$X_{1} = A_{1}^{\dagger}C_{1} + L_{A_{1}}C_{2}B_{1}^{\dagger} + L_{A_{1}}A^{\dagger}EB^{\dagger}R_{B_{1}} - L_{A_{1}}A^{\dagger}CM^{\dagger}EB^{\dagger}R_{B_{1}} - L_{A_{1}}A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger}R_{B_{1}} + H_{1}V_{1}R_{B_{1}} + H_{2}UH_{3} + L_{A_{1}}V_{2}R_{B}R_{B_{1}},$$
(18)

$$X_{2} = A_{2}^{\dagger}C_{3} + L_{A_{2}}C_{4}B_{2}^{\dagger} + L_{A_{2}}M^{\dagger}R_{A}ED^{\dagger}R_{B_{2}} + L_{A_{2}}L_{M}S^{\dagger}SC^{\dagger}EN^{\dagger}R_{B_{2}} + H_{4}W_{1}R_{B_{2}} + H_{5}UH_{6} + L_{A_{2}}W_{2}R_{D}R_{B_{2}},$$
(19)

where U, V_1, V_2, W_1 , and W_2 are free matrices over \mathbb{C} with allowable dimensions.

Proof. Our proof contains three parts. At the first step, we show that the matrices X_1 and X_2 have the forms of

$$X_1 = \phi_0 + H_1 V_1 R_{B_1} + L_{A_1} V_2 R_B R_{B_1} + H_2 U H_3, \tag{20}$$

$$X_2 = \psi_0 + H_4 W_1 R_{B_2} + L_{A_2} W_2 R_D R_{B_2} + H_5 U H_6, \tag{21}$$

where ϕ_0 and ψ_0 are any pair of particular solution to the system (7), V_1 , V_2 , W_1 , W_2 , and U are free matrices of able shapes over \mathbb{C} , are solutions to the system (7). In the second step, we display that any couple of solutions μ_0 and ν_0 to the system (7) can be established as (20) and (21), respectively. In the end, we confirm that

$$\begin{split} \mu &= A_{1}^{\dagger}C_{1} + L_{A_{1}}C_{2}B_{1}^{\dagger} + A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger}, \\ \nu &= A_{2}^{\dagger}C_{3} + L_{A_{2}}C_{4}B_{2}^{\dagger} + L_{A_{2}}M^{\dagger}R_{A}ED^{\dagger} + L_{A_{2}}L_{M}S^{\dagger}SC^{\dagger}EN^{\dagger}R_{B_{2}} \end{split}$$

are a couple of particular solutions to the system (7).

Now we prove that a couple of matrices X_1 and X_2 having the shape of (20) and (21), respectively, are solutions to the system (7). Observe that

$$A_1^{\dagger}C_1B_1 + L_{A_1}C_2B_1^{\dagger}B_1 = A_1^{\dagger}A_1C_2 + L_{A_1}C_2 = C_2,$$

$$A_2^{\dagger}C_3B_2 + L_{A_2}C_4B_2^{\dagger}B_2 = A_2^{\dagger}A_2C_4 + L_{A_2}C_4 = C_4.$$

It is evident that X_1 having the form (20) is a solution of $A_1X_1 = C_1$, and $X_1B_1 = C_2$ and X_2 having the form (21) is a solution to $A_2X_2 = C_3$, $X_2B_2 = C_4$. Now we are left to show that $A_3X_1B_3 + A_4X_2B_4 = C_c$ is satisfied by X_1 and X_2 given in (20) and (21). By Lemma 1.4, we have

$$AS_{1}L_{G} = A \begin{bmatrix} I_{p_{1}} & 0 \end{bmatrix} \begin{bmatrix} L_{A} & -A^{\dagger}CL_{M} \\ 0 & L_{M} \end{bmatrix} = A \begin{bmatrix} L_{A} & -A^{\dagger}CL_{M} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -AA^{\dagger}CL_{M} \end{bmatrix} = \begin{bmatrix} 0 & -(C-M)L_{M} \end{bmatrix} = \begin{bmatrix} 0 & -CL_{M} \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & S \end{bmatrix} = -CS_{2}L_{G},$$
(22)

and

$$R_{H}T_{1}B = \begin{bmatrix} R_{B} & 0 \\ R_{N}DB^{\dagger} & R_{N} \end{bmatrix} \begin{bmatrix} I_{q_{1}} \\ 0 \end{bmatrix} B = \begin{bmatrix} R_{B} \\ R_{N}DB^{\dagger} \end{bmatrix} B$$
$$= \begin{bmatrix} 0 \\ R_{N}DB^{\dagger}B \end{bmatrix} = \begin{bmatrix} 0 \\ R_{N}D(I-L_{B}) \end{bmatrix} = \begin{bmatrix} 0 \\ R_{N}D \end{bmatrix}$$
(23)
$$= R_{H}T_{2}D.$$

Observe that $AL_A = 0$ and by using (22) and (23), we arrive that

$$A_3 X_1 B_3 + A_4 X_2 B_4 = C_c$$

Conversely, assume that μ_0 and ν_0 are any couple of solutions to our system (7). By Lemma 1.7, we have

$$A_1A_1^{\dagger}C_1 = C_1, C_2B_1^{\dagger}B_1 = C_2, A_2A_2^{\dagger}C_3 = C_3, \ C_4B_2^{\dagger}B_2 = C_4, A_1C_2 = C_1B_1, A_2C_4 = C_3B_2.$$

Observe that

$$L_{A_1}\mu_0R_{B_1} = (I - A_1^{\dagger}A_1)\mu_0(I - B_1B_1^{\dagger}) \ = \mu_0 - \mu_0B_1B_1^{\dagger} - A_1^{\dagger}A_1\mu_0 + A_1^{\dagger}A_1\mu_0B_1B_1^{\dagger} \ = \mu_0 - C_2B_1^{\dagger} - A_1^{\dagger}C_1 + A_1^{\dagger}A_1C_2B_1^{\dagger} \ = \mu_0 - L_{A_1}C_2B_1^{\dagger} - A_1^{\dagger}C_1$$

produces

$$\mu_0 = L_{A_1} C_2 B_1^{\dagger} + A_1^{\dagger} C_1 + L_{A_1} \mu_0 R_{B_1}.$$
(24)

On the same lines, we can get

$$\nu_0 = L_{A_2} C_4 B_2^{\dagger} + A_2^{\dagger} C_3 + L_{A_2} \nu_0 R_{B_2}.$$
⁽²⁵⁾

It is manifest that μ_0 and ν_0 defined in (24)–(25) are also solution pair of

$$AX_1B + CX_2D = E. (26)$$

Since

$$\begin{aligned} AX_{1}B + CX_{2}D &= A_{3}L_{A_{1}}\mu_{0}R_{B_{1}}B_{3} + A_{4}L_{A_{2}}\nu_{0}R_{B_{2}}B_{4} \\ &= A_{3}(\mu_{0} - L_{A_{1}}C_{2}B_{1}^{\dagger} - A_{1}^{\dagger}C_{1})B_{3} + A_{4}(\nu_{0} - L_{A_{2}}C_{4}B_{2}^{\dagger} - A_{2}^{\dagger}C_{3})B_{4} \\ &= A_{3}\mu_{0}B_{3} - A_{3}L_{A_{1}}C_{2}B_{1}^{\dagger}B_{3} - A_{1}^{\dagger}C_{1}B_{3} + A_{4}\nu_{0}B_{4} \\ &- A_{4}L_{A_{2}}C_{4}B_{2}^{\dagger}B_{4} - A_{4}A_{2}^{\dagger}C_{3}B_{4} \\ &= A_{3}\mu_{0}B_{3} + A_{4}\nu_{0}B_{4} - AC_{2}B_{1}^{\dagger}B_{3} - A_{1}^{\dagger}C_{1}B_{3} - CC_{4}B_{2}^{\dagger}B_{4} - A_{4}A_{2}^{\dagger}C_{3}B_{4} \\ &= C_{c} - AC_{2}B_{1}^{\dagger}B_{3} - A_{1}^{\dagger}C_{1}B_{3} - CC_{4}B_{2}^{\dagger}B_{4} - A_{4}A_{2}^{\dagger}C_{3}B_{4} \\ &= E. \end{aligned}$$

Hence by Lemma 1.6, μ_0 and ν_0 can be written as

$$\mu_0 = X_{01} + S_1 L_G U R_H T_1 + L_A V_1 + V_2 R_B, \tag{27}$$

$$\nu_0 = X_{02} + S_2 L_G U R_H T_2 + L_C W_1 + W_2 R_D, \tag{28}$$

where X_{01} and X_{02} are a couple of special solutions to (26) and U, V_1 , V_2 , W_1 and W_2 are free matrices with agreeable dimensions. Using (27) and (28) in (24) and (25), respectively, we get

$$\mu_0 = X_{10} + H_2 U H_3 + H_1 V_1 R_{B_1} + L_{A_1} V_2 R_B R_{B_1},$$

 $u_0 = X_{20} + H_5 U H_6 + H_4 W_1 R_{B_2} + L_{A_2} W_2 R_D R_{B_2},$

where $X_{10} = A_1^{\dagger}C_1 + L_{A_1}C_2B_1^{\dagger} + L_{A_1}X_{01}R_{B_1}$ and $X_{20} = A_2^{\dagger}C_3 + L_{A_2}C_4B_2^{\dagger} + L_{A_2}X_{02}R_{B_2}$. It is evident that X_{10} and X_{20} are a couple of solutions to the system (7). It is clear that μ_0 and ν_0 can be represented by (20) and (21), respectively. Lastly, by putting U_1, V_1, W_1 , and Z_1 equal to zero in (15) and (16), we conclude that μ and ν are special solutions to the system (7). Hence the expressions (18) and (19) represent the general solution to the system (7) and the theorem is completed.

Remark 2.2. Due to Lemma 1.3 and taking into account $L_{A_2}L_M = L_ML_{A_2}$, we have the following simplification of the solution pair to the system (7) that is identical for (15)–(16) and (18)–(19) when $U, U_1, V_1, V_2, Z_1, W_1$, and W_2 disappear,

$$egin{aligned} X_1 &= A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + A^{\dagger} E B^{\dagger} - A^{\dagger} A_4 M^{\dagger} E B^{\dagger} - A^{\dagger} S C^{\dagger} E N^{\dagger} B_4 B^{\dagger}, \ X_2 &= A_2^{\dagger} C_3 + L_{A_2} C_4 B_2^{\dagger} + M^{\dagger} E D^{\dagger} + S^{\dagger} S C^{\dagger} E N^{\dagger}. \end{aligned}$$

Comment 2.3. We have established a novel expression of the general solution to the system (7) in Theorem 2.1 which is different from one created in [25]. With the help of this novel expression, we can explore the least-norm of the general solution which can not be studied with the help of the expression given in [25], which is one of the advantage of our new expression.

Now we discuss some special cases of our system.

If B_1, B_2, C_2 and C_4 disappear in Theorem 2.1, then we gain the following conclusion.

Corollary 2.4. Denote
$$S_1 = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix}$$
, $S_2 = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$, $T_1 = \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}$,
 $G = \begin{bmatrix} A & C \end{bmatrix}$, $H = \begin{bmatrix} B_3 \\ -B_4 \end{bmatrix}$, $H_1 = L_{A_1}L_A$, $H_2 = L_{A_1}S_1L_G$, $H_3 = R_HT_1$, $H_4 = L_{A_2}L_C$, $H_5 =$

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 $L_{A_2}S_2L_G$, $H_6 = R_HT_2$ and the system (6) is solvable, then the general solution to (6) can be formed as

$$\begin{split} X_{1} &= A_{1}^{\dagger}C_{1} + A^{\dagger}EB_{3}^{\dagger} - A^{\dagger}A_{4}M^{\dagger}EB_{3}^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B_{3}^{\dagger} - H_{1}Y_{1} + \\ &+ H_{2}VH_{3} + L_{A_{1}}Y_{2}R_{B_{3}}, \\ X_{2} &= A_{2}^{\dagger}C_{3} + M^{\dagger}EB_{4}^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + H_{4}Z_{1} + H_{5}VH_{6} + L_{A_{2}}Z_{2}R_{B_{4}} \end{split}$$

where A, C, N, M, S are the same as in Lemma 1.6, $E = C_c - A_3 A_1^{\dagger} C_1 B_3 - A_4 A_2^{\dagger} C_3 B_4$, V, Y_1, Y_2, Z_1 , and Z_2 are free matrices over \mathbb{C} obeying agreeable dimensions.

Comment 2.5. The above consequence is a chief result of [32].

If A_2, B_2, C_3, A_4, B_4 and C_4 vanish in our system (7), then we get the following outcome.

Corollary 2.6. Suppose that $A_1, B_1, C_1, C_2, A_3, B_3$ and C_c are given. Then the general solution to system (5) is established by

$$\begin{split} X_1 &= A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + (A_3 L_{A_1})^{\dagger} \big[C_c - A_3 A_1^{\dagger} C_1 B_3 - A_3 L_{A_1} C_2 B_1^{\dagger} B_3 \big] (R_{B_1} B_3)^{\dagger} + \\ &+ L_{A_1} L_{A_3 L_{A_1}} W_1 R_{B_1} + L_{A_1} W_2 R_{R_{B_1} B_3} R_{B_1}, \end{split}$$

where W_1 and W_2 are arbitrary matrices over \mathbb{C} with appropriate sizes.

We experience the least-norm to the system (7) in this section. By the definition and [55], we can get the following result easily.

Lemma 2.7. Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times m}$. Then we have.

(1) $||A + B||^2 = ||A||^2 + ||B||^2 + 2Re[tr(B^*A)].$

(2) Re[tr(AB)] = Re[tr(BA)].

Theorem 2.8. Assume that system (7) is solvable, then the least-norm of the solution pair X_1 and X_2 to system (7) can be extracted as follows:

$$||X_1||_{min} = A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + A^{\dagger} E B^{\dagger} - A^{\dagger} A_4 M^{\dagger} E B^{\dagger} - A^{\dagger} S C^{\dagger} E N^{\dagger} B_4 B^{\dagger},$$
(29)

$$\|X_2\|_{min} = A_2^{\dagger}C_3 + L_{A_2}C_4B_2^{\dagger} + M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger}.$$
 (30)

Proof. By Theorem 2.1 and Remark 2.2, the general solution to (7) can be formed as

$$egin{aligned} X_1 &= A_1^{\dagger}C_1 + L_{A_1}C_2B_1^{\dagger} + A^{\dagger}EB^{\dagger} - A^{\dagger}A_4M^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B^{\dagger} \ &-H_1V_1R_{B_1} + H_2UH_3 + L_{A_1}V_2R_BR_{B_1}, \ X_2 &= A_2^{\dagger}C_3 + L_{A_2}C_4B_2^{\dagger} + M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} \ &+H_4W_1R_{B_2} + H_5UH_6 + L_{A_2}W_2R_DR_{B_2}, \end{aligned}$$

where U, V_1, V_2, W_1 , and W_2 are free matrices over \mathbb{C} having executable dimensions. By Lemma 2.7, the norm of X_1 can be established as

$$||X_{1}||^{2} = ||A_{1}^{\dagger}C_{1} + L_{A_{1}}C_{2}B_{1}^{\dagger} + A^{\dagger}EB^{\dagger} - A^{\dagger}A_{4}M^{\dagger}EB^{\dagger} - -A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger} - H_{1}V_{1}R_{B_{1}} + H_{2}UH_{3} + L_{A_{1}}V_{2}R_{B}R_{B_{1}}||^{2} = ||A_{1}^{\dagger}C_{1} + L_{A_{1}}C_{2}B_{1}^{\dagger} + A^{\dagger}EB^{\dagger} - A^{\dagger}A_{4}M^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}||^{2} + ||H_{1}V_{1}R_{B_{1}} + H_{2}UH_{3} + L_{A_{1}}V_{2}R_{B}R_{B_{1}}||^{2} + J,$$
(31)

where

$$J = 2 Re \left[tr((H_1 V_1 R_{B_1} + H_2 U H_3 + L_{A_1} V_2 R_B R_{B_1})^* \left(A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + A^{\dagger} E B^{\dagger} - A^{\dagger} A_4 M^{\dagger} E B^{\dagger} - A^{\dagger} S C^{\dagger} E N^{\dagger} B_4 B^{\dagger} \right) \right) \right].$$
(32)

Now we want to show that J = 0. Applying Lemmas 1.3, 1.4 and 2.7, we have

$$\begin{aligned} & Re\left[\operatorname{tr}\left((H_{1}V_{1}R_{B_{1}})^{*}\left(A_{1}^{\dagger}C_{1}+L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right) \\ & -A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right))\right] = Re\left[\operatorname{tr}\left(R_{B_{1}}V_{1}^{*}H_{1}^{*}\left(A_{1}^{\dagger}C_{1}+L_{A_{1}}C_{2}B_{1}^{*}+A^{\dagger}EB^{\dagger}\right) \\ & -A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right))\right] = Re\left[\operatorname{tr}\left(R_{B_{1}}V_{1}^{*}L_{A}L_{A_{1}}\left(A_{1}^{\dagger}C_{1}+L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)\right)\right] \\ & = Re\left[\operatorname{tr}\left(R_{B_{1}}V_{1}^{*}L_{A}L_{A_{1}}\left(L_{A_{1}}C_{2}B_{1}^{\dagger}\right)R_{B_{1}}\right)\right] = 0, \quad (33) \\ Re\left[\operatorname{tr}\left(\left(L_{A_{1}}V_{2}R_{B}R_{B_{1}}\right)^{*}\left(A_{1}^{\dagger}C_{1}+L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right)\right)\right] \\ & = Re\left[\operatorname{tr}\left(N_{1}^{*}L_{A}L_{A_{1}}\left(L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right)\right)\right] \\ & = Re\left[\operatorname{tr}\left(R_{B_{1}}R_{B}V_{2}^{*}L_{A_{1}}\left(L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right)\right] \\ & = Re\left[\operatorname{tr}\left(R_{B_{1}}R_{B}V_{2}^{*}L_{A_{1}}\left(L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right)\right] \\ & = Re\left[\operatorname{tr}\left(R_{B_{1}}R_{B}V_{2}^{*}L_{A_{1}}\left(L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right)\right] \\ & = Re\left[\operatorname{tr}\left(V_{2}^{*}L_{A_{1}}\left(L_{A}^{*}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)\right)\right] \\ & = Re\left[\operatorname{tr}\left(V_{2}^{*}L_{A_{1}}\left(A^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)R_{B_{1}}\right] \\ & = Re\left[\operatorname{tr}\left(V_{2}^{*}L_{A_{1}}\left(A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)R_{B}\right)\right] = 0, \\ (34) \\ Re\left[\operatorname{tr}\left(\left(H_{2}UH_{3}\right)^{*}\left(A_{1}^{\dagger}C_{1}+L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}\right) \\ & -A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)\right] = Re\left[\operatorname{tr}\left(H_{3}^{*}U^{*}H_{2}^{*}\left(A_{1}^{\dagger}C_{1}+L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}\right) \\ & -A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)\right] = Re\left[\operatorname{tr}\left(H_{3}^{*}U^{*}L_{C}S_{1}^{*}+A^{\dagger}EB^{\dagger}\right) \\ & -A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)\right] \\ = Re\left[\operatorname{tr}\left(H_{2}^{*}U^{*}\left[L_{A}^{*}-A^{\dagger}CL_{M}\right]\right]\left[I\right] \\ 0\left(L_{A_{1}}C_{2}B_{1}^{\dagger}+A^{\dagger}EB^{\dagger}\right) \\ & -A^{\dagger}A_{4}M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}\right)\right] \\ = Re\left[\operatorname{tr}\left(H_{3}^{*}U^{*}$$

$$-A^{\dagger}A_{4}M^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}B_{4}B^{\dagger}))] = Re\left[tr(H_{3}^{*}U^{*}L_{A}L_{A_{1}}C_{2}B_{1}^{\dagger})\right]$$

= $Re\left[tr(R_{B_{1}}T_{1}^{*}R_{H}U^{*}L_{A}L_{A_{1}}C_{2}B_{1}^{\dagger})\right]$ (35)

By using (33)–(35) in (32) produces J = 0. Since X_1 is arbitrary, we get (29) from (31). In the same way, we can prove that (30) hold.

A special case of our system (7) is given below.

If B_1, B_2, C_{2} , and C_4 become zero matrices in Theorem 2.8, then again we get the principal result of [20].

Corollary 2.9. Assume that system (6) is solvable, then the least-norm of the solution pair X_1 and X_2 to system (6) can be furnished as

$$\begin{aligned} \|X_1\|_{min} &= A_1^{\dagger}C_1 + A^{\dagger}EB_3^{\dagger} - A^{\dagger}A_4M^{\dagger}EB_3^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B_3^{\dagger}, \\ \|X_2\|_{min} &= A_2^{\dagger}C_3 + M^{\dagger}EB_4^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger}. \end{aligned}$$

If A_2, B_2, C_3, A_4, B_4 and C_4 vanish in our system, then we get the next consequence. **Corollary 2.10.** Suppose that $A_1, B_1, C_1, C_2, A_3, B_3$ and C_c are given. Then the least-

norm of the least square solution to system (5) is launched by $\|X_1\|_{\mathcal{A}} = A_1^{\dagger}C_1 + L_4 C_2 B_1^{\dagger}$

$$\begin{aligned} \mathbf{x}_{1} \|_{min} &= A_{1} \mathbf{C}_{1} + L_{A_{1}} \mathbf{C}_{2} \mathbf{B}_{1} \\ &+ (A_{3} L_{A_{1}})^{\dagger} \big[C_{c} - A_{3} A_{1}^{\dagger} C_{1} B_{3} - A_{3} L_{A_{1}} C_{2} B_{1}^{\dagger} B_{3} \big] (R_{B_{1}} B_{3})^{\dagger}. \end{aligned}$$

Comment 2.11. Corollary 2.10 is the key result of [22].

3. Algorithm with example

In this section, we construct the algorithm for finding the least-norm of the solution to (7) that is inducted by Theorem 2.8. *Algorithm 1.*

- 1. By Lemma 1.10 find the matrices A_i^{\dagger} , B_i^{\dagger} for i = 1, ..., 4, and $R_{A_i} = I A_i A_i^{\dagger}$, $L_{A_i} = I A_i^{\dagger} A_i$, $R_{B_i} = I B_i B_i^{\dagger}$, and $L_{B_i} = I B_i^{\dagger} B_i$ for i = 1, 2.
- 2. By Lemma 1.9 calculate the matrices *A*, *B*, *C*, *D*, *M*, *S*, and *E*, and by Lemma 1.10 find their MP-inverses and orthogonal projectors when it is needed.
- 3. Verify the consistence equalities (11) and (14). If these equalities are hold, then we find solutions by the next steps.
- 4. Finally, by (29) and (30), compute the least-norm of the solution pair X_1 and X_2 .

The following example will be considered by using Algorithm 1. Note that our goal is both to confirm correctness of main results from Theorems 2.1 and 2.8, and to demonstrate the technique of applying the determinantal representations of the MP-inverse from Lemma 1.10 by using a not too complicated and understandable example.

Example 1. Given the matrices:

$$B_{2} = \begin{bmatrix} 2-i & 2i-1 & i+1\\ 2i+1 & -i-2 & i-1\\ -2i+1 & i-2 & -i-1\\ i+2 & -2i-1 & -i+1 \end{bmatrix}, C_{1} = \begin{bmatrix} 8i & -8 & -8i & 8\\ 4 & 4i & -4 & -4i\\ 2+4i & -4+2i & 2-4i & 4-2i \end{bmatrix},$$

$$C_{2} = \begin{bmatrix} 11i & 44i - 11 & -44\\ 22 & 22i + 88 & 88i\\ -11i & 44i + 11 & 44\\ -22 & -22i - 88 & -88i \end{bmatrix}, A_{3} = \begin{bmatrix} 5i+2 & 5-2i & -2+5i & 2i+5\\ 2i-5 & 5i+2 & -2i-5 & -2+5i \\ 4i & 4 & -4i & -4 \end{bmatrix},$$

$$B_{3} = \begin{bmatrix} -i & -i+2 & -1\\ -2 & -2-4i & 2i\\ -2i & 4-2i & -2\\ 1 & 1+2i & -i \end{bmatrix}, A_{4} = \begin{bmatrix} -2i-3 & -3i+2 & 2i+3\\ -i & 1 & i\\ -3 & -3i & 3 \end{bmatrix},$$

$$C_{3} = \begin{bmatrix} 3i & 3 & -3 & -3i\\ 3 & -3i & 3i & -3\\ -3 & 3i & -3i & 3\\ -3i & -3 & 3 & 3i \end{bmatrix},$$

$$B_{4} = \begin{bmatrix} 7i & -i-2\\ -7 & -3 & 2i\\ -7i & i & 2\\ 7 & 3 & -2i \end{bmatrix}, C_{4} = \begin{bmatrix} 4-2i & -2+4i & 2+2i\\ 2+4i & -4-2i & -2+2i\\ -2-4i & 4+2i & 2-2i \end{bmatrix},$$

$$C_{c} = \frac{1}{21} \begin{bmatrix} -1130 - 502i & -1344 + 612i & -2798 - 1250i\\ -1808 - 688 & -1398 + 834i & -2942 - 1538i\\ -1154 - 946i & -1488 + 624i & -2654 - 1394i \end{bmatrix}.$$
(36)

Let us find a solution to the system (7) with the given above matrices by Algorithm 1.

1. Thanks to Lemma 1.10, we calculate the Moore-Penrose inverses. So,

$$A_{1}^{\dagger} = \frac{1}{32} \begin{bmatrix} 1-i & -1-i & -2i \\ 1+i & 1-i & 2 \\ -1-i & -1+i & -2i \\ -1+i & 1+i & 2i \end{bmatrix}, B_{1}^{\dagger} = \frac{1}{44} \begin{bmatrix} -11i & 11i & -11 & 11 \\ 39 & 41 & 20-i & 20+i \\ 7-i & 1+i & 5+3i & 3-3i \end{bmatrix},$$

$$A_{2}^{\dagger} = \frac{1}{12} \begin{bmatrix} -i & 1 & -1 & i \\ 1 & i & -i & -1 \\ -1 & -i & i & 1 \end{bmatrix}, B_{2}^{\dagger} = \frac{1}{12} \begin{bmatrix} 1 & -i & i & 1 \\ -i & -1 & -1 & i \\ 1-i & -1-i & i & 1 \end{bmatrix},$$

$$A_{3}^{\dagger} = \frac{1}{80} \begin{bmatrix} -2i & -2 & 2-5i \\ 2 & -2i & 5+2i \\ -2i & -2 & 2+5i \\ 2 & -2i & -5+2i \end{bmatrix}, B_{3}^{\dagger} = \frac{1}{70} \begin{bmatrix} i & -2 & 2i & 1 \\ 2+i & -2+4i & 4+2i & 1-2i \\ -1 & -2i & -2 & i \end{bmatrix},$$

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$$A_{4}^{\dagger} = \frac{1}{69} \begin{bmatrix} -3+2i & i & -3\\ 2+3i & 1 & 3i\\ 3-2i & -i & 1 \end{bmatrix}, B_{4}^{\dagger} = \frac{1}{792} \begin{bmatrix} -35i & -21 & 35i & 21\\ 47i & -51 & -47i & 51\\ -52 & -48i & 52 & 48i \end{bmatrix}.$$

Then,

$$A = \frac{1}{2} \begin{bmatrix} 2+5i & 5-2i & 1+8i & 12+9i \\ -5+2i & 2+5i & -8+i & -9+12i \\ 4i & 4 & 4-8i & -8+4i \end{bmatrix},$$

$$B = \frac{1}{22} \begin{bmatrix} -52 - 31i & 10 - 135i & -31 + 52i \\ 8+9i & -10+25i & 9-8i \\ -9+8i & -25-10i & 8+9i \\ 31 - 52i & 135+10i & -52-31i \end{bmatrix},$$

$$C = \frac{1}{3} \begin{bmatrix} -11 - 3i & 9-7i & 6+4i \\ -1 - 3i & 3+i & 2i \\ -9+3 & 3-9i & 6 \end{bmatrix}, D = \begin{bmatrix} 0 & -2i & -2 \\ 0 & -2 & 2i \\ 0 & 2 & -2i \end{bmatrix},$$

$$N = \frac{1}{7} \begin{bmatrix} 4+4i & -4-2i & -10-4i \\ 4-4i & -2+4i & -4+10i \\ -4-4i & 4+2i & 10+4i \\ -4+4i & 2-4i & 4-10i \end{bmatrix}, M = \frac{1}{3} \begin{bmatrix} -4-2i & 4-2i & 2+2i \\ -2+4i & -2-4i & 2-2i \\ 0 & 0 & 0 \end{bmatrix}, S = 0$$

$$E = \frac{1}{84} \begin{bmatrix} 19931 - 108289i & 236509 - 68427i & -108289 - 19931i \\ 110417 + 16211i & 77995 + 79015i & 16211 - 110417i \\ 74624 + 106424i & -138224 + 255672i & 106424 - 74624i \end{bmatrix}.$$

3. Finally, by (29) and (30), we find that the least-norm of the solution pair X_1 and X_2 to the system (7) is following

$$X_{1} = \frac{1}{365760} \begin{bmatrix} -11103239 + 18670545i & -9851419 + 14002307i & -5154373 + 3862099i & -4697553 + 10234559i \\ 26688873 + 4258681i & 29888893 + 5510501i & 12048461 + 4721147i & 17746081 + 5177967i \\ 6556168 + 9656066i & 5321848 + 2196342i & 4452786 + 10360112i & -6757414 + 7845632i \\ -17049264 - 2930378i & -26304464 - 11113378i & -10244698 - 3367816i & -7362609 - 13720296i \end{bmatrix},$$

$$X_{2} = \frac{1}{1344} \begin{bmatrix} 2052 - 963i & 233 - 1985i & -2159 + 3481i & -1465 - 367i \\ -792 + 2565i & 1901 - 205i & 317 + 445i & -221 + 317i \\ 171 + 585i & -146 + 28i & 868 - 1714i & 146 + 2884i \end{bmatrix}.$$

Note that Maple 2021 was used to perform the numerical experiment.

4. Conclusion

We have constructed a novel expression of the general solution to system (7) over \mathbb{C} and used this result to explore the least-norm of the general solution to this system when it is solvable. Some particular cases of our system are also discussed. Our results carry the principal results of [22, 32]. To give an algorithm finding the explicit numerical expression of the least-norm of the general solution, it is used the determinantal representations of the MP-inverse recently obtained by one of the authors. The novelty of the conducted research is obtaining necessary and sufficient conditions to exist a solution, its formal representation of by closed formula in terms of generalized inverses, and the construction of an algorithm to find its explicit expression. A numerical example is also given to interpret the results established in this paper.

Conflict of interest

The authors declare that they have not conflicts of interest.

Data Availability

The data used to support the findings of this study are included within the article titled "Solving and Algorithm for Least-Norm General Solution to Constrained Sylvester Matrix Equation". The prior studies (and datasets) are cited at relevant places within the text.

Classification

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