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## Chapter

# Solving and Algorithm for Least-Norm General Solution to Constrained Sylvester Matrix Equation 

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#### Abstract

Keeping in view that a lot of physical systems with inverse problems can be written by matrix equations, the least-norm of the solution to a general Sylvester matrix equation with restrictions $A_{1} X_{1}=C_{1}, X_{1} B_{1}=C_{2}, A_{2} X_{2}=C_{3}, X_{2} B_{2}=C_{4}, A_{3} X_{1} B_{3}+$ $A_{4} X_{2} B_{4}=C_{c}$, is researched in this chapter. A novel expression of the general solution to this system is established and necessary and sufficient conditions for its existence are constituted. The novelty of the proposed results is not only obtaining a formal representation of the solution in terms of generalized inverses but the construction of an algorithm to find its explicit expression as well. To conduct an algorithm and numerical example, it is used the determinantal representations of the Moore-Penrose inverse previously obtained by one of the authors.


Keywords: linear matrix equation, generalized Sylvester matrix equation, MoorePenrose inverse

## 1. Introduction

Standardly, we state $\mathbb{C}$ and $\mathbb{R}$, respectively, for the complex and real numbers. Let $\mathbb{C}^{m \times n}$ denote the set of all $m \times n$ matrices over $\mathbb{C}$, and $\mathbb{C}_{r}^{m \times n}$ stay for a subset of $m \times n$ complex matrices with rank $r$. The rank of $A$ is denoted by both symbols $r(A)$ and rank $A$. The (complex) conjugate transpose matrix of $A \in \mathbb{C}^{m \times n}$ is written by $A^{*}$ and a matrix $A \in \mathbb{C}^{n \times n}$ is said to be Hermitian if $A^{*}=A$. An identity matrix with feasible shape is denoted by $I$.

Definition 1.1. The Moore-Penrose (MP-) inverse of $A \in \mathbb{C}^{m \times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution $X$ to the following four Penrose equations

$$
\begin{align*}
A X A & =A,  \tag{1}\\
X A X & =X,  \tag{2}\\
(A X)^{*} & =A X,  \tag{3}\\
(X A)^{*} & =X A . \tag{4}
\end{align*}
$$

Matrices satisfying the eqs. (1) and (2) are known as reflexive inverses, denoted by $A^{+}$.

In addition, $L_{A}=I-A^{\dagger} A$ and $R_{A}=I-A A^{\dagger}$ represent a pair of orthogonal projectors onto the kernels of $A$ and $A^{*}$, respectively.

Mathematical models of physical systems with inverse problems especially those has a finite number of model parameters can be written by matrix equations. In particular, the Sylvester-type matrix equations have far-reaching applications in singular system control [1], system design [2], robust control [3], feedback [4], perturbation theory [5], linear descriptor systems [6], neural networks [7] and theory of orbits [8], etc.

Some recent work on generalized Sylvester matrix equations and their systems can be observed in [9-21]. In 2014, Bao [22] examined the least-norm and extremal ranks of the least square solution to the quaternion matrix equations

$$
\begin{equation*}
A_{1} X=C_{1}, X B_{1}=C_{2}, \quad A_{3} X B_{3}=C_{c} . \tag{5}
\end{equation*}
$$

Wang et al. [23] examined the expression of the general solution to the system

$$
\begin{equation*}
A_{1} X_{1}=C_{1}, \quad A_{2} X_{2}=C_{3}, A_{3} X_{1} \quad B_{3}+A_{4} X_{2} B_{4}=C_{c} \tag{6}
\end{equation*}
$$

and as an application, the $P$-symmetric and $P$-skew-symmetric solution to

$$
A_{a} X=C_{a}, A_{b} X B_{b}=C_{b}
$$

has been established. Li et al. [24] established a novel expression of the general solution of the system (6) and they computed the least-norm of general solution to (6). In 2009, Wang et al. [25] constituted the expression of the general solution to

$$
\begin{array}{ll}
A_{1} & X_{1}=C_{1}, X_{1} B_{1}=C_{2}, \\
A_{2} & X_{2}=C_{3}, X_{2} B_{2}=C_{4},  \tag{7}\\
A_{3} & X_{1} B_{3}+A_{4} X_{2} B_{4}=C_{c},
\end{array}
$$

and as an application, they explored the $(P, Q)$-symmetric solution to the system

$$
A_{a} X=C_{a}, X B_{b}=C_{b}, A_{c} X B_{c}=C_{c} .
$$

Some latest findings on the least-norm of matrix equations and $(P, Q)$-symmetric matrices can be consulted in [26-30]. Furthermore, our main system (7) is a special case of the following system

$$
\begin{align*}
& A_{1} X_{1}=C_{1}, X_{2} B_{1}=D_{1}, \\
& A_{2} X_{3}=C_{2}, X_{3} B_{2}=D_{2}, \\
& A_{3} X_{4}=C_{3}, X_{4} B_{3}=D_{3},  \tag{8}\\
A_{4} X_{1}+X_{2} \quad & B_{4}+C_{4} X_{3} D_{4}+C_{5} X_{4} D_{5}=C_{c},
\end{align*}
$$

which has been investigated by Zhang in 2014.
Motivated by the latest interest of least-norm of matrix equations, we construct a novel expression of the general solution to the system (7) and apply this to investigate the least-norm of the general solution to the system (7) in this chapter. Observing that

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systems (5) and (6) are particular cases of our system (7), solving system (7) will encourage the least-norm to a wide class of problems.

We commence with the following lemmas which have crucial function in the construction of the chief outcomes of the following sections.

Lemma 1.2. [31]. Let $A, B$, and $C$ be given matrices over $\mathbb{C}$ with agreeable dimensions. Then.

1. $r(A)+r\left(R_{A} B\right)=r(B)+r\left(R_{B} A\right)=r\left[\begin{array}{ll}A & B\end{array}\right]$.
2. $r(A)+r\left(C L_{A}\right)=r(C)+r\left(A L_{C}\right)=r\left[\begin{array}{l}A \\ C\end{array}\right]$.
3. $r(B)+r(C)+r\left(R_{B} A L_{C}\right)=r\left[\begin{array}{ll}A & B \\ C & 0\end{array}\right]$.

Lemma 1.3. [32]. Let $A, B$, and $C$ be known matrices over $\mathbb{C}$ with right sizes. Then

1. $A^{\dagger}=\left(A^{*} A\right)^{\dagger} A^{*}=A^{*}\left(A A^{*}\right)^{\dagger}$.
2. $L_{A}=L_{A}^{2}=L_{A}^{*}, R_{A}=R_{A}^{2}=R_{A}^{*}$.
3. $L_{A}\left(B L_{A}\right)^{\dagger}=\left(B L_{A}\right)^{\dagger},\left(R_{A} C\right)^{\dagger} R_{A}=\left(R_{A} C\right)^{\dagger}$.

Lemma 1.4. [33]. Let $\Phi, \Omega$ be matrices over $\mathbb{C}$ and

$$
\Phi=\left[\begin{array}{c}
\Phi_{1} \\
\Phi_{2}
\end{array}\right], \Omega=\left[\begin{array}{ll}
\Omega_{1} & \Omega_{2} \\
&
\end{array}\right], \quad F=\Phi_{2} L_{\Phi_{1}}, \quad T=R_{\Omega_{1}} \Omega_{2}
$$

Then

$$
\begin{aligned}
& L_{\Phi}=L_{\Phi_{1}} L_{F}, \quad L_{\Omega}=\left[\begin{array}{cc}
L_{\Omega_{1}} & -\Omega_{1}^{\dagger} \Omega_{2} L_{T} \\
0 & L_{T}
\end{array}\right], \\
& R_{\Omega}=R_{T} R_{\Omega_{1}}, \quad R_{\Phi}=\left[\begin{array}{cc}
R_{\Phi_{1}} & 0 \\
-R_{F} \Phi_{2} \Phi_{1}^{\dagger} & R_{F}
\end{array}\right],
\end{aligned}
$$

where $\Phi_{1}^{+}, \Omega_{1}^{+}$are any fixed reflexive inverses, $L_{\Phi_{1}}$ and $R_{\Omega_{1}}$ stand for the projectors $L_{\Phi_{1}}=I-\Phi_{1}^{+} \Phi_{1}, R_{\Omega_{1}}=I-\Omega_{1} \Omega_{1}^{+}$induced by $\Phi_{1}, \Omega_{1}$, respectively.

Remark 1.5. Since the Moore-Penrose inverse is a reflexive inverse, this lemma can be used for the MP-inverse without any changes. It has taken place in ([32], Lemma 2.4).

Lemma 1.6. [34]. Suppose that

$$
\begin{equation*}
B_{1} X C_{1}+B_{2} Y C_{2}=A \tag{9}
\end{equation*}
$$

is consistent linear matrix equation. Then.

1. The general solution of the homogeneous equation

$$
B_{1} X C_{1}+B_{2} Y C_{2}=0
$$

can be expressed by

$$
X=X_{1} X_{2}+X_{3}, \quad Y=Y_{1} Y_{2}+Y_{3}
$$

where $X_{1}-X_{3}$ and $Y_{1}-Y_{3}$ are general solution to the system

$$
B_{1} X_{1}=-B_{2} Y_{1}, \quad X_{2} C_{1}=Y_{2} C_{2}, \quad B_{1} X_{3} C_{1}=0, \quad B_{2} Y_{3} C_{2}=0 .
$$

By computing the value of unknowns in above and using them in $X$ and $Y$, we have

$$
\begin{aligned}
X & =S_{1} L_{G} U R_{H} T_{1}+L_{B_{1}} V_{1}+V_{2} R_{C_{1}}, \\
Y & =S_{2} L_{G} U R_{H} T_{2}+L_{B_{2}} W_{1}+W_{2} R_{C_{2}},
\end{aligned}
$$

where $S_{1}=\left[\begin{array}{ll}I_{p}, & 0\end{array}\right], S_{2}=\left[0, I_{s}\right], T_{1}=\left[\begin{array}{c}I_{q} \\ 0\end{array}\right], T_{2}=\left[\begin{array}{l}0 \\ I_{t}\end{array}\right], G=\left[\begin{array}{ll}B_{1}, & B_{2}\end{array}\right]$, and $H=\left[\begin{array}{c}C_{1} \\ -C_{2}\end{array}\right]$; the matrices $U, V_{1}, V_{2}, W_{1}$ and $W_{2}$ are free to vary over $\mathbb{C}$.
2. Assume that Eq. (9) is solvable, then its general solution can be expressed as

$$
X=X_{0}+X_{1} X_{2}+X_{3}, \quad Y=Y_{0}+Y_{1} Y_{2}+Y_{3}
$$

where $X_{0}$ and $Y_{0}$ are any pair of particular solutions to (9).
It can also be written as

$$
\begin{aligned}
X & =X_{0}+S_{1} L_{G} U R_{H} T_{1}+L_{B_{1}} V_{1}+V_{2} R_{C_{1}} \\
Y & =Y_{0}+S_{2} L_{G} U R_{H} T_{2}+L_{B_{2}} W_{1}+W_{2} R_{C_{2}} .
\end{aligned}
$$

Lemma 1.7. [35]. Let $A_{1}, B_{1}, C_{1}, C_{2}$ be given matrices over $\mathbb{C}$ with agreeable sizes and $X_{1}$ to be determined. Then the system

$$
\begin{equation*}
A_{1} X_{1}=C_{1}, X_{1} B_{1}=C_{2}, \tag{10}
\end{equation*}
$$

is consistent if and only if

$$
\begin{equation*}
R_{A_{1}} C_{1}=0, \quad C_{2} L_{B_{1}}=0, \quad A_{1} C_{2}=C_{1} B_{1} \tag{11}
\end{equation*}
$$

Under these conditions, the general solution to (10) can be established as

$$
X_{1}=A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+L_{A_{1}} U_{1} R_{B_{1}},
$$

where $U_{1}$ is a free matrix over $\mathbb{C}$ with accordant dimension.
Lemma 1.8. [36]. Let $A, B$, and $C$ be known matrices over $\mathbb{C}$ with agreeable dimensions, and $X$ be unknown. Then the matrix equation

$$
\begin{equation*}
A X B=C \tag{12}
\end{equation*}
$$

is consistent if and only if $A A^{\dagger} C B^{\dagger} B=C$. In this case, its general solution can be expressed as

$$
\begin{equation*}
X=A^{\dagger} C B^{\dagger}+L_{A} V+W R_{B}, \tag{13}
\end{equation*}
$$

where $V, W$ are arbitrary matrices over $\mathbb{C}$ with appropriate dimensions.
In [37], it is proved that (13) is the least squares solution to (12), and its minimum norm least squares solution is $X_{L S}=A^{\dagger} C B^{\dagger}$.

Lemma 1.9. [25]. Let $A_{i}, B_{i}, C_{i},(i=1, \ldots, 4)$, and $C_{c}$ be given matrices over $\mathbb{C}$ with agreeable dimensions, and $X_{1}, X_{2}$ to be determined. Denote

$$
\begin{aligned}
& A=A_{3} L_{A_{1}}, B=R_{B_{1}} B_{3}, C=A_{4} L_{A_{2}}, D=R_{B_{2}} B_{4}, \\
& N=D L_{B}, M=R_{A} C, S=C L_{M}, \\
& E=C_{c}-A_{3} A_{1}^{\dagger} C_{1} B_{3}-A C_{2} B_{1}^{\dagger} B_{3}-A_{4} A_{2}^{\dagger} C_{3} B_{4}-C C_{4} B_{2}^{\dagger} B_{4} .
\end{aligned}
$$

Then the following conditions are tantamount:
1.System (7) is resolvable.
2. The conditions in (11) are met and

$$
\begin{align*}
& R_{A_{2}} C_{3}=0, \quad C_{4} L_{B_{2}}=0, \quad A_{2} C_{4}=C_{3} B_{2}, \\
& R_{M} R_{A} E=0, R_{A} E L_{D}=0, E L_{B} L_{N}=0, R_{C} E L_{B}=0 . \tag{14}
\end{align*}
$$

3. The equalities in (11) and (14) are satisfied and

$$
M M^{\dagger} R_{A} D^{\dagger} D=R_{A} E, \quad C C^{\dagger} E L_{B} N^{\dagger} N=E L_{B} .
$$

In these conditions, the general solution to the system (7) can be written as

$$
\begin{align*}
X_{1}= & A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+L_{A_{1}} A^{\dagger} E B^{\dagger} R_{B_{1}}-L_{A_{1}} A^{\dagger} C M^{\dagger} E B^{\dagger} R_{B_{1}}- \\
& -L_{A_{1}} A^{\dagger} S C^{\dagger} E N^{\dagger} D B^{\dagger} R_{B_{1}}-L_{A_{1}} A^{\dagger} S V_{1} R_{N} D B^{\dagger} R_{B_{1}}+  \tag{15}\\
& +L_{A_{1}}\left(L_{A} U_{1}+Z_{1} R_{B}\right) R_{B_{1}}, \\
X_{2}= & A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+L_{A_{2}} M^{\dagger} R_{A} E D^{\dagger} R_{B_{2}}+L_{A_{2}} L_{M_{b}} S^{\dagger} S C^{\dagger} E N^{\dagger} R_{B_{2}}  \tag{16}\\
+ & L_{A_{2}} L_{M}\left(V_{1}-S^{\dagger} S V_{1} N N^{\dagger}\right) R_{B_{2}}+L_{A_{2}} W_{1} R_{D} R_{B_{2}},
\end{align*}
$$

where $U_{1}, V_{1}, W_{1}$ and $Z_{1}$ are free matrices over $\mathbb{C}$ with agreeable dimensions.
Since the general solutions of considered systems are expressed in terms of generalized inverses, another goal of the paper is to give determinantal representations of the least-norm of the general solution to the system (7) based on determinantal representations of generalized inverses.

Due to the important role of generalized inverses in many application fields, considerable effort has been exerted toward the numerical algorithms for fast and accurate calculation of matrix generalized inverse. In general, most existing methods for their obtaining are iterative algorithms for approximating generalized inverses of complex matrices (some recent papers, see, e.g. [38-40]). There are only several direct methods for finding MP-inverse for an arbitrary complex matrix $A \in \mathbb{C}^{m \times n}$. The most famous is method based on singular value decomposition (SVD), i.e. if $A=U \Sigma V^{*}$, then $A^{\dagger}=V \Sigma^{\dagger} U^{*}$. The computational cost of this method is dominated by the cost of computing the SVD, which is several times higher than matrix-matrix
multiplication. Another approach is constructing determinantal representations of the MP-inverse $A^{\dagger}$. A well-known determinantal representation of an ordinary inverse is the adjugate matrix with the cofactors in entries. It has an important theoretical significance and brings forth Cramer's rule for systems of linear equations. The same is desirable to have for the generalized inverses. Due to looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses (for the MP-inverse, see, e.g. [41, 42]). Because of the complexity of the previously obtained expressions of determinantal representations of the MP-inverse, they have little applicability.

In this chapter, we will use the determinantal representations of the MP-inverse recently obtained in [43].

Lemma 1.10. [43, Theorem 2.2] If $A \in \mathbb{C}^{m \times n}$ with rank $A=r$, then the Moore-Penrose inverse $A^{\dagger}=\left(a_{i j}^{\dagger}\right) \in \mathbb{C}^{n \times m}$ possess the following determinantal representations

$$
\begin{equation*}
a_{i j}^{\dagger}=\frac{\sum_{\beta \in J_{r, n}\{i\}}\left|\left(A^{*} A\right)_{. i}\left(a_{j}^{*}\right)\right|_{\beta}^{\beta}}{\sum_{\beta \in J_{r, n}}\left|A^{*} A\right|_{\beta}^{\beta}}=\frac{\sum_{\left.\left.\alpha \in I_{r, m}\right\}\right\}}\left|\left(A A^{*}\right)_{j .}\left(a_{i .}^{*}\right)\right|_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r, m}}\left|A A^{*}\right|_{\alpha}^{\alpha}} . \tag{17}
\end{equation*}
$$

Here $|A|_{\alpha}^{\alpha}$ denote a principal minor of $A$ whose rows and columns are indexed by $\alpha:=\left\{\alpha_{1}, \ldots, \alpha_{k}\right\} \subseteq\{1, \ldots, m\}$,

$$
L_{k, m}:=\left\{\alpha: 1 \leq \alpha_{1}<\cdots<\alpha_{k} \leq m\right\} \text {, and } I_{r, m}\{i\}:=\left\{\alpha: \alpha \in L_{r, m}, i \in \alpha\right\} .
$$

Also, $a_{j}^{*}$ and $a_{i .}^{*}$ denote the $j$ th column and the $i$ th row of $A^{*}$, and $A_{i .}(b)$ and $A_{j}(c)$ stand for the matrices obtained from $A$ by replacing its $i$ th row with the row vector $b \in \mathbb{C}^{1 \times n}$ and its $j$ th column with the column vector $c \in \mathbb{C}^{m}$, respectively.

The formulas (17) give very simple and elegant determinantal representations of the MP-inverse. So, for any $A \in \mathbb{C}_{r}^{m \times n}$, we have sum of all principal minors of $r$ order of the matrices $A^{*} A$ or $A A^{*}$ in denominators and sum of principal minors of $r$ order of the matrices $\left(A^{*} A\right)_{. i}\left(a_{j}^{*}\right)$ or $\left(A A^{*}\right)_{j .}\left(a_{i .}^{*}\right)$ that contain the $i$ th column or the $j$ th row, respectively, in numerators into (17).

Note that for an arbitrary full-rank matrix $A$, Lemma 1.10 gives a new way of finding an inverse matrix.

Corollary 1.11. If $A \in \mathbb{C}^{m \times n}$ with rank $A=\min \{m, n\}$, then the inverse $A^{-1}=$ $\left(a_{i j}^{-1}\right) \in \mathbb{C}^{n \times m}$ possess the following determinantal representations:

$$
a_{i j}^{-1}=\left\{\begin{array}{l}
\frac{\left|\left(A^{*} A\right)_{. i}\left(a_{j}^{*}\right)\right|}{\left|A^{*} A\right|} \text { if } \operatorname{rank} A=n, \\
\frac{\left|\left(A A^{*}\right)_{j .}\left(a_{i .}^{*}\right)\right|}{\left|A A^{*}\right|} \text { if rank } A=m .
\end{array}\right.
$$

These new determinantal representations of the Moore-Penrose inverse have been obtained by the developed novel limit-rank method in the case of quaternion matrices [44] as well. This method was successfully applied for constructing determinantal
representations of other generalized inverses in both cases for complex and quaternion matrices (see e.g. [45-47]). It also yields Cramer's rules of various matrix equations [48-54].

The remainder of our chapter is directed as follows. In Section 2, we provide a new expression of the general solution to our system (7) and discuss its least-norm. The algorithm and numerical example of finding the anti-Hermitian solution to (7) are presented in Section 3. (7). Finally, in Section 4, the conclusions are drawn.

## 2. A new expression of the general solution to the system

Now we demonstrate the principal theorem of this section (7).
Theorem 2.1. Assume that $S_{1}=\left[\begin{array}{ll}I_{p_{1}} & 0\end{array}\right], S_{2}=\left[\begin{array}{ll}0 & I_{p_{2}}\end{array}\right], T_{1}=\left[\begin{array}{c}I_{q_{1}} \\ 0\end{array}\right], T_{2}=\left[\begin{array}{c}I_{q_{2}} \\ 0\end{array}\right]$, $G=\left[\begin{array}{ll}A & C\end{array}\right], H=\left[\begin{array}{c}B \\ -D\end{array}\right], H_{1}=L_{A_{1}} L_{A}, H_{2}=L_{A_{1}} S_{1} L_{G}, H_{3}=R_{H} T_{1} R_{B_{1}}, H_{4}=L_{A_{2}} L_{C}, H_{5}=$ $L_{A_{2}} S_{2} L_{G}, H_{6}=R_{H} T_{2} R_{B_{2}}$ and the system (7) is solvable, then the general solution to our system can be formed as

$$
\begin{align*}
X_{1}= & A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+L_{A_{1}} A^{\dagger} E B^{\dagger} R_{B_{1}}-L_{A_{1}} A^{\dagger} C M^{\dagger} E B^{\dagger} R_{B_{1}}-  \tag{18}\\
& -L_{A_{1}} A^{\dagger} S C^{\dagger} E N^{\dagger} D B^{\dagger} R_{B_{1}}+H_{1} V_{1} R_{B_{1}}+H_{2} U H_{3}+L_{A_{1}} V_{2} R_{B} R_{B_{1}}, \\
X_{2}= & A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+L_{A_{2}} M^{\dagger} R_{A} E D^{\dagger} R_{B_{2}}+L_{A_{2}} L_{M} S^{\dagger} S C^{\dagger} E N^{\dagger} R_{B_{2}}+  \tag{19}\\
& +H_{4} W_{1} R_{B_{2}}+H_{5} U H_{6}+L_{A_{2}} W_{2} R_{D} R_{B_{2}},
\end{align*}
$$

where $U, V_{1}, V_{2}, W_{1}$, and $W_{2}$ are free matrices over $\mathbb{C}$ with allowable dimensions.
Proof. Our proof contains three parts. At the first step, we show that the matrices $X_{1}$ and $X_{2}$ have the forms of

$$
\begin{gather*}
X_{1}=\phi_{0}+H_{1} V_{1} R_{B_{1}}+L_{A_{1}} V_{2} R_{B} R_{B_{1}}+H_{2} U H_{3}  \tag{20}\\
X_{2}=\psi_{0}+H_{4} W_{1} R_{B_{2}}+L_{A_{2}} W_{2} R_{D} R_{B_{2}}+H_{5} U H_{6} \tag{21}
\end{gather*}
$$

where $\phi_{0}$ and $\psi_{0}$ are any pair of particular solution to the system (7), $V_{1}, V_{2}, W_{1}$, $W_{2}$, and $U$ are free matrices of able shapes over $\mathbb{C}$, are solutions to the system (7). In the second step, we display that any couple of solutions $\mu_{0}$ and $\nu_{0}$ to the system (7) can be established as (20) and (21), respectively. In the end, we confirm that

$$
\begin{aligned}
& \mu=A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} C M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} D B^{\dagger}, \\
& \nu=A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+L_{A_{2}} M^{\dagger} R_{A} E D^{\dagger}+L_{A_{2}} L_{M} S^{\dagger} S C^{\dagger} E N^{\dagger} R_{B_{2}}
\end{aligned}
$$

are a couple of particular solutions to the system (7).
Now we prove that a couple of matrices $X_{1}$ and $X_{2}$ having the shape of (20) and (21), respectively, are solutions to the system (7). Observe that

$$
\begin{aligned}
& A_{1}^{\dagger} C_{1} B_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger} B_{1}=A_{1}^{\dagger} A_{1} C_{2}+L_{A_{1}} C_{2}=C_{2}, \\
& A_{2}^{\dagger} C_{3} B_{2}+L_{A_{2}} C_{4} B_{2}^{\dagger} B_{2}=A_{2}^{\dagger} A_{2} C_{4}+L_{A_{2}} C_{4}=C_{4}
\end{aligned}
$$

It is evident that $X_{1}$ having the form (20) is a solution of $A_{1} X_{1}=C_{1}$, and $X_{1} B_{1}=$ $C_{2}$ and $X_{2}$ having the form (21) is a solution to $A_{2} X_{2}=C_{3}, X_{2} B_{2}=C_{4}$. Now we are left to show that $A_{3} X_{1} B_{3}+A_{4} X_{2} B_{4}=C_{c}$ is satisfied by $X_{1}$ and $X_{2}$ given in (20) and (21). By Lemma 1.4, we have

$$
\begin{align*}
A S_{1} L_{G} & =A\left[\begin{array}{ll}
I_{p_{1}} & 0
\end{array}\right]\left[\begin{array}{cc}
L_{A} & -A^{\dagger} C L_{M} \\
0 & L_{M}
\end{array}\right]=A\left[\begin{array}{ll}
L_{A} & -A^{\dagger} C L_{M}
\end{array}\right]  \tag{22}\\
& =\left[\begin{array}{cc}
0 & -A A^{\dagger} C L_{M}
\end{array}\right]=\left[\begin{array}{ll}
0 & -(C-M) L_{M}
\end{array}\right]=\left[\begin{array}{ll}
0 & -C L_{M}
\end{array}\right] \\
& =-\left[\begin{array}{ll}
0 & S
\end{array}\right]=-C S_{2} L_{G},
\end{align*}
$$

and

$$
\begin{align*}
R_{H} T_{1} B & =\left[\begin{array}{cc}
R_{B} & 0 \\
R_{N} D B^{\dagger} & R_{N}
\end{array}\right]\left[\begin{array}{c}
I_{q_{1}} \\
0
\end{array}\right] B=\left[\begin{array}{c}
R_{B} \\
R_{N} D B^{\dagger}
\end{array}\right] B \\
& =\left[\begin{array}{c}
0 \\
R_{N} D B^{\dagger} B
\end{array}\right]=\left[\begin{array}{c}
0 \\
R_{N} D\left(I-L_{B}\right)
\end{array}\right]=\left[\begin{array}{c}
0 \\
R_{N} D
\end{array}\right]  \tag{23}\\
& =R_{H} T_{2} D
\end{align*}
$$

Observe that $A L_{A}=0$ and by using (22) and (23), we arrive that

$$
A_{3} X_{1} B_{3}+A_{4} X_{2} B_{4}=C_{c} .
$$

Conversely, assume that $\mu_{0}$ and $\nu_{0}$ are any couple of solutions to our system (7). By Lemma 1.7, we have

$$
\begin{aligned}
& A_{1} A_{1}^{\dagger} C_{1}=C_{1}, C_{2} B_{1}^{\dagger} B_{1}=C_{2}, A_{2} A_{2}^{\dagger} C_{3}=C_{3}, \\
& C_{4} B_{2}^{\dagger} B_{2}=C_{4}, A_{1} C_{2}=C_{1} B_{1}, A_{2} C_{4}=C_{3} B_{2} .
\end{aligned}
$$

Observe that

$$
\begin{aligned}
L_{A_{1}} \mu_{0} R_{B_{1}} & =\left(I-A_{1}^{\dagger} A_{1}\right) \mu_{0}\left(I-B_{1} B_{1}^{\dagger}\right) \\
& =\mu_{0}-\mu_{0} B_{1} B_{1}^{\dagger}-A_{1}^{\dagger} A_{1} \mu_{0}+A_{1}^{\dagger} A_{1} \mu_{0} B_{1} B_{1}^{\dagger} \\
& =\mu_{0}-C_{2} B_{1}^{\dagger}-A_{1}^{\dagger} C_{1}+A_{1}^{\dagger} A_{1} C_{2} B_{1}^{\dagger} \\
& =\mu_{0}-L_{A_{1}} C_{2} B_{1}^{\dagger}-A_{1}^{\dagger} C_{1}
\end{aligned}
$$

produces

$$
\begin{equation*}
\mu_{0}=L_{A_{1}} C_{2} B_{1}^{\dagger}+A_{1}^{\dagger} C_{1}+L_{A_{1}} \mu_{0} R_{B_{1}} . \tag{24}
\end{equation*}
$$

On the same lines, we can get

$$
\begin{equation*}
\nu_{0}=L_{A_{2}} C_{4} B_{2}^{\dagger}+A_{2}^{\dagger} C_{3}+L_{A_{2}} \nu_{0} R_{B_{2}} . \tag{25}
\end{equation*}
$$

It is manifest that $\mu_{0}$ and $\nu_{0}$ defined in (24)-(25) are also solution pair of

$$
\begin{equation*}
A X_{1} B+C X_{2} D=E . \tag{26}
\end{equation*}
$$

Since

$$
\begin{aligned}
A X_{1} B+C X_{2} D= & A_{3} L_{A_{1}} \mu_{0} R_{B_{1}} B_{3}+A_{4} L_{A_{2}} \nu_{0} R_{B_{2}} B_{4} \\
= & A_{3}\left(\mu_{0}-L_{A_{1}} C_{2} B_{1}^{\dagger}-A_{1}^{\dagger} C_{1}\right) B_{3}+A_{4}\left(\nu_{0}-L_{A_{2}} C_{4} B_{2}^{\dagger}-A_{2}^{\dagger} C_{3}\right) B_{4} \\
= & A_{3} \mu_{0} B_{3}-A_{3} L_{A_{1}} C_{2} B_{1}^{\dagger} B_{3}-A_{1}^{\dagger} C_{1} B_{3}+A_{4} \nu_{0} B_{4} \\
& -A_{4} L_{A_{2}} C_{4} B_{2}^{\dagger} B_{4}-A_{4} A_{2}^{\dagger} C_{3} B_{4} \\
= & A_{3} \mu_{0} B_{3}+A_{4} \nu_{0} B_{4}-A C_{2} B_{1}^{\dagger} B_{3}-A_{1}^{\dagger} C_{1} B_{3}-C C_{4} B_{2}^{\dagger} B_{4}-A_{4} A_{2}^{\dagger} C_{3} B_{4} \\
= & C_{c}-A C_{2} B_{1}^{\dagger} B_{3}-A_{1}^{\dagger} C_{1} B_{3}-C C_{4} B_{2}^{\dagger} B_{4}-A_{4} A_{2}^{\dagger} C_{3} B_{4} \\
= & E .
\end{aligned}
$$

Hence by Lemma 1.6, $\mu_{0}$ and $\nu_{0}$ can be written as

$$
\begin{gather*}
\mu_{0}=X_{01}+S_{1} L_{G} U R_{H} T_{1}+L_{A} V_{1}+V_{2} R_{B},  \tag{27}\\
\nu_{0}=X_{02}+S_{2} L_{G} U R_{H} T_{2}+L_{C} W_{1}+W_{2} R_{D}, \tag{28}
\end{gather*}
$$

where $X_{01}$ and $X_{02}$ are a couple of special solutions to (26) and $U, V_{1}, V_{2}, W_{1}$ and $W_{2}$ are free matrices with agreeable dimensions. Using (27) and (28) in (24) and (25), respectively, we get

$$
\begin{aligned}
& \mu_{0}=X_{10}+H_{2} U H_{3}+H_{1} V_{1} R_{B_{1}}+L_{A_{1}} V_{2} R_{B} R_{B_{1}}, \\
& \nu_{0}=X_{20}+H_{5} U H_{6}+H_{4} W_{1} R_{B_{2}}+L_{A_{2}} W_{2} R_{D} R_{B_{2}},
\end{aligned}
$$

where $X_{10}=A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+L_{A_{1}} X_{01} R_{B_{1}}$ and $X_{20}=A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+$ $L_{A_{2}} X_{02} R_{B_{2}}$. It is evident that $X_{10}$ and $X_{20}$ are a couple of solutions to the system (7). It is clear that $\mu_{0}$ and $\nu_{0}$ can be represented by (20) and (21), respectively. Lastly, by putting $U_{1}, V_{1}, W_{1}$, and $Z_{1}$ equal to zero in (15) and (16), we conclude that $\mu$ and $\nu$ are special solutions to the system (7). Hence the expressions (18) and (19) represent the general solution to the system (7) and the theorem is completed.

Remark 2.2. Due to Lemma 1.3 and taking into account $L_{A_{2}} L_{M}=L_{M} L_{A_{2}}$, we have the following simplification of the solution pair to the system (7) that is identical for (15)-(16) and (18)-(19) when $U, U_{1}, V_{1}, V_{2}, Z_{1}, W_{1}$, and $W_{2}$ disappear,

$$
\begin{aligned}
& X_{1}=A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger} \\
& X_{2}=A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+M^{\dagger} E D^{\dagger}+S^{\dagger} S C^{\dagger} E N^{\dagger}
\end{aligned}
$$

Comment 2.3. We have established a novel expression of the general solution to the system (7) in Theorem 2.1 which is different from one created in [25]. With the help of this novel expression, we can explore the least-norm of the general solution which can not be studied with the help of the expression given in [25], which is one of the advantage of our new expression.

Now we discuss some special cases of our system.
If $B_{1}, B_{2}, C_{2}$ and $C_{4}$ disappear in Theorem 2.1, then we gain the following conclusion.
Corollary 2.4. Denote $S_{1}=\left[\begin{array}{ll}I_{p_{1}} & 0\end{array}\right], S_{2}=\left[\begin{array}{ll}0 & I_{p_{2}}\end{array}\right], T_{1}=\left[\begin{array}{c}I_{q_{1}} \\ 0\end{array}\right], T_{2}=\left[\begin{array}{c}I_{q_{2}} \\ 0\end{array}\right]$, $G=\left[\begin{array}{ll}A & C\end{array}\right], H=\left[\begin{array}{c}B_{3} \\ -B_{4}\end{array}\right], H_{1}=L_{A_{1}} L_{A}, H_{2}=L_{A_{1}} S_{1} L_{G}, H_{3}=R_{H} T_{1}, H_{4}=L_{A_{2}} L_{C}, H_{5}=$
$L_{A_{2}} S_{2} L_{G}, H_{6}=R_{H} T_{2}$ and the system (6) is solvable, then the general solution to (6) can be formed as

$$
\begin{gathered}
X_{1}=A_{1}^{\dagger} C_{1}+A^{\dagger} E B_{3}^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B_{3}^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B_{3}^{\dagger}-H_{1} Y_{1}+ \\
\quad+H_{2} V H_{3}+L_{A_{1}} Y_{2} R_{B_{3}}, \\
X_{2}=A_{2}^{\dagger} C_{3}+M^{\dagger} E B_{4}^{\dagger}+S^{\dagger} S C^{\dagger} E N^{\dagger}+H_{4} Z_{1}+H_{5} V H_{6}+L_{A_{2}} Z_{2} R_{B_{4}},
\end{gathered}
$$

where $A, C, N, M, S$ are the same as in Lemma 1.6, $E=C_{c}-A_{3} A_{1}^{\dagger} C_{1} B_{3}-A_{4} A_{2}^{\dagger} C_{3} B_{4}$, $V, Y_{1}, Y_{2}, Z_{1}$, and $Z_{2}$ are free matrices over $\mathbb{C}$ obeying agreeable dimensions.

Comment 2.5. The above consequence is a chief result of [32].
If $A_{2}, B_{2}, C_{3}, A_{4}, B_{4}$ and $C_{4}$ vanish in our system (7), then we get the following outcome.

Corollary 2.6. Suppose that $A_{1}, B_{1}, C_{1}, C_{2}, A_{3}, B_{3}$ and $C_{c}$ are given. Then the general solution to system (5) is established by

$$
\begin{aligned}
X_{1}= & A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+\left(A_{3} L_{A_{1}}\right)^{\dagger}\left[C_{c}-A_{3} A_{1}^{\dagger} C_{1} B_{3}-A_{3} L_{A_{1}} C_{2} B_{1}^{\dagger} B_{3}\right]\left(R_{B_{1}} B_{3}\right)^{\dagger}+ \\
& +L_{A_{1}} L_{A_{3} L_{A_{1}}} W_{1} R_{B_{1}}+L_{A_{1}} W_{2} R_{R_{B_{1}} B_{3}} R_{B_{1}},
\end{aligned}
$$

where $W_{1}$ and $W_{2}$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.
We experience the least-norm to the system (7) in this section. By the definition and [55], we can get the following result easily.

Lemma 2.7. Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m}$. Then we have.
(1) $\|A+B\|^{2}=\|A\|^{2}+\|B\|^{2}+2 \operatorname{Re}\left[\operatorname{tr}\left(B^{*} A\right)\right]$.
(2) $\operatorname{Re}[\operatorname{tr}(A B)]=\operatorname{Re}[\operatorname{tr}(B A)]$.

Theorem 2.8. Assume that system (7) is solvable, then the least-norm of the solution pair $X_{1}$ and $X_{2}$ to system (7) can be extracted as follows:

$$
\begin{align*}
\left\|X_{1}\right\|_{\text {min }}= & A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-  \tag{29}\\
& -A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}, \\
\left\|X_{2}\right\|_{\text {min }}= & A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+M^{\dagger} E D^{\dagger}+S^{\dagger} S C^{\dagger} E N^{\dagger} . \tag{30}
\end{align*}
$$

Proof. By Theorem 2.1 and Remark 2.2, the general solution to (7) can be formed as

$$
\begin{aligned}
X_{1}= & A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger} \\
& -H_{1} V_{1} R_{B_{1}}+H_{2} U H_{3}+L_{A_{1}} V_{2} R_{B} R_{B_{1}}, \\
X_{2}= & A_{2}^{\dagger} C_{3}+L_{A_{2}} C_{4} B_{2}^{\dagger}+M^{\dagger} E D^{\dagger}+S^{\dagger} S C^{\dagger} E N^{\dagger} \\
& +H_{4} W_{1} R_{B_{2}}+H_{5} U H_{6}+L_{A_{2}} W_{2} R_{D} R_{B_{2}},
\end{aligned}
$$

where $U, V_{1}, V_{2}, W_{1}$, and $W_{2}$ are free matrices over $\mathbb{C}$ having executable dimensions. By Lemma 2.7, the norm of $X_{1}$ can be established as

$$
\begin{align*}
\left\|X_{1}\right\|^{2}= & \| A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}- \\
& -A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}-H_{1} V_{1} R_{B_{1}}+H_{2} U H_{3}+L_{A_{1}} V_{2} R_{B} R_{B_{1}} \|^{2}  \tag{31}\\
& =\left\|A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right\|^{2} \\
& +\left\|H_{1} V_{1} R_{B_{1}}+H_{2} U H_{3}+L_{A_{1}} V_{2} R_{B} R_{B_{1}}\right\|^{2}+J,
\end{align*}
$$

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DOI: http://dx.doi.org/10.5772/intechopen. 109749
where

$$
\begin{align*}
J=2 \operatorname{Re} & {\left[\operatorname { t r } \left(\left(H_{1} V_{1} R_{B_{1}}+H_{2} U H_{3}+L_{A_{1}} V_{2} R_{B} R_{B_{1}}\right)^{*}\right.\right.} \\
& \left.\left.\left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right] . \tag{32}
\end{align*}
$$

Now we want to show that $J=0$. Applying Lemmas 1.3, 1.4 and 2.7, we have

$$
\begin{align*}
& \operatorname{Re}\left[\operatorname { t r } \left(( H _ { 1 } V _ { 1 } R _ { B _ { 1 } } ) ^ { * } \left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}\right.\right.\right. \\
& \left.\left.\left.-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname { t r } \left(R _ { B _ { 1 } } V _ { 1 } ^ { * } H _ { 1 } ^ { * } \left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}\right.\right.\right. \\
& \left.\left.\left.\quad-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname { t r } \left(R _ { B _ { 1 } } V _ { 1 } ^ { * } L _ { A } L _ { A _ { 1 } } \left(A_{1}^{\dagger} C_{1}\right.\right.\right. \\
& \left.\left.\left.\quad+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right] \\
& \quad=\operatorname{Re}\left[\operatorname{tr}\left(R_{B_{1}} V_{1}^{*} L_{A} L_{A_{1}}\left(L_{A_{1}} C_{2} B_{1}^{\dagger}\right)\right)\right] \\
& \quad=\operatorname{Re}\left[\operatorname{tr}\left(V_{1}^{*} L_{A} L_{A_{1}}\left(L_{A_{1}} C_{2} B_{1}^{\dagger}\right) R_{B_{1}}\right)\right]=0,  \tag{33}\\
& \operatorname{Re}\left[\operatorname { t r } \left(( L _ { A _ { 1 } } V _ { 2 } R _ { B } R _ { B _ { 1 } } ) ^ { * } \left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}\right.\right.\right. \\
& \left.\left.\left.\quad-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=R e\left[\operatorname { t r } \left(R _ { B _ { 1 } } R _ { B } V _ { 2 } ^ { * } L _ { A _ { 1 } } ^ { * } \left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}\right.\right.\right. \\
& \left.\left.\left.\quad+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right] \\
& \quad=\operatorname{Re}\left[\operatorname { t r } \left(R _ { B _ { 1 } } R _ { B } V _ { 2 } ^ { * } L _ { A _ { 1 } } \left(L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}\right.\right.\right. \\
& \left.\left.\left.\quad-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname { t r } \left(V _ { 2 } ^ { * } L _ { A _ { 1 } } \left(L_{A_{1}} C_{2} B_{1}^{\dagger}\right.\right.\right. \\
& \left.\left.\left.\quad+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right) R_{B_{1}} R_{B}\right)\right] \\
& \quad=\operatorname{Re}\left[\operatorname{tr}\left(V_{2}^{*} L_{A_{1}}\left(A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right) R_{B}\right)\right]=0, \tag{34}
\end{align*}
$$

$$
\begin{aligned}
& \operatorname{Re}\left[\operatorname { t r } \left(( H _ { 2 } U H _ { 3 } ) ^ { * } \left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}\right.\right.\right. \\
&\left.\left.\left.-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname { t r } \left(H _ { 3 } ^ { * } U ^ { * } H _ { 2 } ^ { * } \left(A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}\right.\right.\right. \\
&\left.\left.\left.-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname { t r } \left(H _ { 3 } ^ { * } U ^ { * } L _ { G } S _ { 1 } ^ { * } L _ { A _ { 1 } } \left(L_{A_{1}} C_{2} B_{1}^{\dagger}\right.\right.\right.
\end{aligned}
$$

$$
\left.\left.\left.+A^{\dagger} E B^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]
$$

$$
=R e\left[\operatorname { t r } \left(H _ { 3 } ^ { * } U ^ { * } [ \begin{array} { c c } 
{ L _ { A } } & { - A ^ { \dagger } C L _ { M } } \\
{ 0 } & { L _ { M } }
\end{array} ] [ \begin{array} { l } 
{ I } \\
{ 0 }
\end{array} ] \left(L_{A_{1}} C_{2} B_{1}^{\dagger}+A^{\dagger} E B^{\dagger}\right.\right.\right.
$$

$$
\left.\left.\left.-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname { t r } \left(H _ { 3 } ^ { * } U ^ { * } L _ { A } \left(A^{\dagger} E B^{\dagger}\right.\right.\right.
$$

$$
\left.\left.\left.-A^{\dagger} A_{4} M^{\dagger} E B^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B^{\dagger}\right)\right)\right]=\operatorname{Re}\left[\operatorname{tr}\left(H_{3}^{*} U^{*} L_{A} L_{A_{1}} C_{2} B_{1}^{\dagger}\right)\right]
$$

$$
\begin{equation*}
=\operatorname{Re}\left[\operatorname{tr}\left(R_{B_{1}} T_{1}^{*} R_{H} U^{*} L_{A} L_{A_{1}} C_{2} B_{1}^{\dagger}\right)\right] \tag{35}
\end{equation*}
$$

By using (33)-(35) in (32) produces $J=0$. Since $X_{1}$ is arbitrary, we get (29) from (31). In the same way, we can prove that (30) hold.

A special case of our system (7) is given below.
If $B_{1}, B_{2}, C_{2}$, and $C_{4}$ become zero matrices in Theorem 2.8, then again we get the principal result of [20].

Corollary 2.9. Assume that system (6) is solvable, then the least-norm of the solution pair $X_{1}$ and $X_{2}$ to system (6) can be furnished as

$$
\begin{aligned}
& \left\|X_{1}\right\|_{\text {min }}=A_{1}^{\dagger} C_{1}+A^{\dagger} E B_{3}^{\dagger}-A^{\dagger} A_{4} M^{\dagger} E B_{3}^{\dagger}-A^{\dagger} S C^{\dagger} E N^{\dagger} B_{4} B_{3}^{\dagger}, \\
& \left\|X_{2}\right\|_{\text {min }}=A_{2}^{\dagger} C_{3}+M^{\dagger} E B_{4}^{\dagger}+S^{\dagger} S C^{\dagger} E N^{\dagger} .
\end{aligned}
$$

If $A_{2}, B_{2}, C_{3}, A_{4}, B_{4}$ and $C_{4}$ vanish in our system, then we get the next consequence.
Corollary 2.10. Suppose that $A_{1}, B_{1}, C_{1}, C_{2}, A_{3}, B_{3}$ and $C_{c}$ are given. Then the leastnorm of the least square solution to system (5) is launched by

$$
\begin{aligned}
\left\|X_{1}\right\|_{\text {min }}= & A_{1}^{\dagger} C_{1}+L_{A_{1}} C_{2} B_{1}^{\dagger} \\
& +\left(A_{3} L_{A_{1}}\right)^{\dagger}\left[C_{c}-A_{3} A_{1}^{\dagger} C_{1} B_{3}-A_{3} L_{A_{1}} C_{2} B_{1}^{\dagger} B_{3}\right]\left(R_{B_{1}} B_{3}\right)^{\dagger} .
\end{aligned}
$$

Comment 2.11. Corollary 2.10 is the key result of [22].

## 3. Algorithm with example

In this section, we construct the algorithm for finding the least-norm of the solution to (7) that is inducted by Theorem 2.8.

Algorithm 1.

1. By Lemma 1.10 find the matrices $A_{i}^{\dagger}, B_{i}^{\dagger}$ for $i=1, \ldots, 4$, and $R_{A_{i}}=I-A_{i} A_{i}^{\dagger}$, $L_{A_{i}}=I-A_{i}^{\dagger} A_{i}, R_{B_{i}}=I-B_{i} B_{i}^{\dagger}$, and $L_{B_{i}}=I-B_{i}^{\dagger} B_{i}$ for $i=1,2$.
2. By Lemma 1.9 calculate the matrices $A, B, C, D, M, S$, and $E$, and by Lemma 1.10 find their MP-inverses and orthogonal projectors when it is needed.
3. Verify the consistence equalities (11) and (14). If these equalities are hold, then we find solutions by the next steps.
4. Finally, by (29) and (30), compute the least-norm of the solution pair $X_{1}$ and $X_{2}$.

The following example will be considered by using Algorithm 1 . Note that our goal is both to confirm correctness of main results from Theorems 2.1 and 2.8, and to demonstrate the technique of applying the determinantal representations of the MP-inverse from Lemma 1.10 by using a not too complicated and understandable example.

Example 1. Given the matrices:

$$
A_{1}=\left[\begin{array}{cccc}
1+i & 1-i & -1+i & -1-i \\
-1+i & 1+i & -1-i & 1-i \\
2 i & 2 & -2 & -2 i
\end{array}\right], B_{1}=\left[\begin{array}{ccc}
2 i & -1 & i+3 \\
-i & 1 & -3-i \\
-1 & i & 1-3 i \\
1 & -i & -1+3 i
\end{array}\right], A_{2}=\left[\begin{array}{ccc}
i & 1 & -1 \\
1 & -i & i \\
-1 & i & -i \\
-i & -1 & 1
\end{array}\right]
$$

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$$
\begin{align*}
B_{2}=\left[\begin{array}{ccc}
2-i & 2 i-1 & i+1 \\
2 i+1 & -i-2 & i-1 \\
-2 i+1 & i-2 & -i-1 \\
i+2 & -2 i-1 & -i+1
\end{array}\right], C_{1}=\left[\begin{array}{ccc}
8 i & -8 & -8 i \\
4 & 4 i & -4 \\
2+4 i & -4+2 i & 2-4 i \\
4-2 i
\end{array}\right], \\
C_{2}=\left[\begin{array}{ccc}
11 i & 44 i-11 & -44 \\
22 & 22 i+88 & 88 i \\
-11 i & 44 i+11 & 44 \\
-22 & -22 i-88 & -88 i
\end{array}\right], A_{3}=\left[\begin{array}{ccc}
5 i+2 & 5-2 i & -2+5 i \\
2 i-5 & 5 i+2 & -2 i-5 \\
4 i & 4 & -2+5 i \\
4 i & -4
\end{array}\right], \\
B_{3}=\left[\begin{array}{ccc}
-i & -i+2 & -1 \\
-2 & -2-4 i & 2 i \\
-2 i & 4-2 i & -2 \\
1 & 1+2 i & -i
\end{array}\right], A_{4}=\left[\begin{array}{ccc}
-2 i-3 & -3 i+2 & 2 i+3 \\
-i & 1 & i \\
-3 & -3 i & 3
\end{array}\right], \\
C_{3}=\left[\begin{array}{ccc}
3 i & 3 & -3 \\
-3 i \\
3 & -3 i & 3 i \\
-3 & 3 i & -3 i \\
-3 \\
-3 i & -3 & 3 \\
3 i
\end{array}\right], \\
B_{4}=\left[\begin{array}{ccc}
7 i & -i & -2 \\
-7 & -3 & 2 i \\
-7 i & i & 2 \\
7 & 3 & -2 i
\end{array}\right], C_{4}=\left[\begin{array}{ccc}
4-2 i & -2+4 i & 2+2 i \\
2+4 i & -4-2 i & -2+2 i \\
-2-4 i & 4+2 i & 2-2 i
\end{array}\right], \\
C_{c}=\frac{1}{21}\left[\begin{array}{ccc}
-1130-502 i & -1344+612 i & -2798-1250 i \\
-1808-688 & -1398+834 i & -2942-1538 i \\
-1154-946 i & -1488+624 i & -2654-1394 i
\end{array}\right] . \tag{36}
\end{align*}
$$

Let us find a solution to the system (7) with the given above matrices by Algorithm 1.
1.Thanks to Lemma 1.10, we calculate the Moore-Penrose inverses. So,

$$
\begin{aligned}
& A_{1}^{\dagger}=\frac{1}{32}\left[\begin{array}{ccc}
1-i & -1-i & -2 i \\
1+i & 1-i & 2 \\
-1-i & -1+i & -2 \\
-1+i & 1+i & 2 i
\end{array}\right], B_{1}^{\dagger}=\frac{1}{44}\left[\begin{array}{cccc}
-11 i & 11 i & -11 & 11 \\
39 & 41 & 20-i & 20+i \\
7-i & 1+i & 5+3 i & 3-3 i
\end{array}\right] \text {, } \\
& A_{2}^{\dagger}=\frac{1}{12}\left[\begin{array}{cccc}
-i & 1 & -1 & i \\
1 & i & -i & -1 \\
-1 & -i & i & 1
\end{array}\right], B_{2}^{\dagger}=\frac{1}{12}\left[\begin{array}{cccc}
1 & -i & i & 1 \\
-i & -1 & -1 & i \\
1-i & -1-i & -1+i & 1+i
\end{array}\right] \text {, } \\
& A_{3}^{\dagger}=\frac{1}{80}\left[\begin{array}{ccc}
-2 i & -2 & 2-5 i \\
2 & -2 i & 5+2 i \\
-2 i & -2 & 2+5 i \\
2 & -2 i & -5+2 i
\end{array}\right], B_{3}^{\dagger}=\frac{1}{70}\left[\begin{array}{cccc}
i & -2 & 2 i & 1 \\
2+i & -2+4 i & 4+2 i & 1-2 i \\
-1 & -2 i & -2 & i
\end{array}\right] \text {, }
\end{aligned}
$$

$$
A_{4}^{\dagger}=\frac{1}{69}\left[\begin{array}{ccc}
-3+2 i & i & -3 \\
2+3 i & 1 & 3 i \\
3-2 i & -i & 1
\end{array}\right], B_{4}^{\dagger}=\frac{1}{792}\left[\begin{array}{cccc}
-35 i & -21 & 35 i & 21 \\
47 i & -51 & -47 i & 51 \\
-52 & -48 i & 52 & 48 i
\end{array}\right]
$$

Then,
$A=\frac{1}{2}\left[\begin{array}{cccc}2+5 i & 5-2 i & 1+8 i & 12+9 i \\ -5+2 i & 2+5 i & -8+i & -9+12 i \\ 4 i & 4 & 4-8 i & -8+4 i\end{array}\right]$,
$B=\frac{1}{22}\left[\begin{array}{ccc}-52-31 i & 10-135 i & -31+52 i \\ 8+9 i & -10+25 i & 9-8 i \\ -9+8 i & -25-10 i & 8+9 i \\ 31-52 i & 135+10 i & -52-31 i\end{array}\right]$,
$C=\frac{1}{3}\left[\begin{array}{ccc}-11-3 i & 9-7 i & 6+4 i \\ -1-3 i & 3+i & 2 i \\ -9+3 & 3-9 i & 6\end{array}\right], D=\left[\begin{array}{ccc}0 & -2 i & -2 \\ 0 & -2 & 2 i \\ 0 & 2 i & 2 \\ 0 & 2 & -2 i\end{array}\right]$,
$N=\frac{1}{7}\left[\begin{array}{ccc}4+4 i & -4-2 i & -10-4 i \\ 4-4 i & -2+4 i & -4+10 i \\ -4-4 i & 4+2 i & 10+4 i \\ -4+4 i & 2-4 i & 4-10 i\end{array}\right], M=\frac{1}{3}\left[\begin{array}{ccc}-4-2 i & 4-2 i & 2+2 i \\ -2+4 i & -2-4 i & 2-2 i \\ 0 & 0 & 0\end{array}\right], S=0$
$E=\frac{1}{84}\left[\begin{array}{ccc}19931-108289 i & 236509-68427 i & -108289-19931 i \\ 110417+16211 i & 77995+79015 i & 16211-110417 i \\ 74624+106424 i & -138224+255672 i & 106424-74624 i\end{array}\right]$.
2. Confirm that (11) and (14) are true for given matrices.
3. Finally, by (29) and (30), we find that the least-norm of the solution pair $X_{1}$ and $X_{2}$ to the system (7) is following

$$
\begin{aligned}
& X_{1}=\frac{1}{365760}\left[\begin{array}{cccc}
-11103239+18670545 i & -9851419+14002307 i & -5154373+3862099 i & -4697553+10234559 i \\
26688873+4258681 i & 29888893+5510501 i & 12048461+4721147 i & 17746081+5177967 i \\
6556168+9656066 i & 5321848+2196342 i & 4452786+10360112 i & -6757414+7845632 i \\
-17049264-2930378 i & -26304464-11113378 i & -10244698-3367816 i & -7362609-13720296 i
\end{array}\right], \\
& X_{2}=\frac{1}{1344}\left[\begin{array}{cccc}
2052-963 i & 233-1985 i & -2159+3481 i & -1465-367 i \\
-792+2565 i & 1901-205 i & 317+445 i & -221+317 i \\
171+585 i & -146+28 i & 868-1714 i & 146+2884 i
\end{array}\right] .
\end{aligned}
$$

Note that Maple 2021 was used to perform the numerical experiment.

## 4. Conclusion

We have constructed a novel expression of the general solution to system (7) over $\mathbb{C}$ and used this result to explore the least-norm of the general solution to this system when it is solvable. Some particular cases of our system are also discussed. Our results carry the principal results of [22,32]. To give an algorithm finding the explicit numerical expression of the least-norm of the general solution, it is used the determinantal representations of the MP-inverse recently obtained by one of the authors. The novelty of the conducted research is obtaining necessary and sufficient conditions to exist a solution, its formal representation of by closed formula in terms of generalized inverses, and the construction of an algorithm to find its explicit expression. A numerical example is also given to interpret the results established in this paper.

## Conflict of interest

The authors declare that they have not conflicts of interest.

## Data Availability

The data used to support the findings of this study are included within the article titled "Solving and Algorithm for Least-Norm General Solution to Constrained Sylvester Matrix Equation". The prior studies (and datasets) are cited at relevant places within the text.

## Classification

2000 AMS subject classifications: 15A09, 15A15, 15A24.

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