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Characteristic Polynomials

Sándor Kovács, Szilvia György and Noémi Gyúró

Abstract

In this chapter, we provide a short overview of the stability properties of polynomials and quasi-polynomials. They appear typically in stability investigations of equilibria of ordinary and retarded differential equations. In the case of ordinary differential equations we discuss the Hurwitz criterion, and its simplified version, the Lineard-Chippart criterion, furthermore the Mikhailov criterion and we show how one can prove the change of stability via the knowledge of the coefficients of the characteristic polynomial of the Jacobian of the given autonomous system. In the case of the retarded differential equation we use the Mikhailov criterion in order to estimate the length of the delay for which no stability switching occurs. These results are applied to the stability and Hopf bifurcation of an equilibrium solution of a system of ordinary differential equations as well as of retarded dynamical systems.

Keywords: Hurwitz stability, Schur stability, Mikhailov criterion, Hopf bifurcation

1. Introduction

As it is well-known, many systems of applied mathematics are modeled by retarded functional differential equations of type

$$\dot{x} = f(x, x(\cdot - \tau)) \quad (1)$$

(cf. [1]), where $f : \Omega \times \Omega \rightarrow \mathbb{R}^d$ is continuously differentiable and $\Omega \subset \mathbb{R}^d$ is an open set. Here $\tau \geq 0$ represents the so-called delay or time lag. In order to have a solution in some interval $(0, r)$, $r > 0$ one has to know the solution on $[-\tau, 0]$, which means that one has to attach a continuous initial function $\phi : [-\tau, 0] \rightarrow \mathbb{R}^d$ as an initial condition to the system (cf. [2]). Clearly, in the case of $\tau = 0$ one has to deal with the initial value problem for ordinary differential equations.

In order to examine the stability of an equilibrium $a \in \Omega$ of system (1), i.e. the equilibrium solution

$$\hat{a} : \mathbb{R} \rightarrow \mathbb{R}, \quad \hat{a}(t) := a \quad (2)$$

for which $f(a, a) = 0$ holds, one has to discuss the spectral properties of the linearized system

$$\dot{u} = Au + Bu(\cdot - \tau) \quad (3)$$

where

$$A := \partial_1 f(a, a) \in \mathbb{R}^{d \times d} \quad \text{and} \quad B := \partial_2 f(a, a) \in \mathbb{R}^{d \times d}. \quad (4)$$

It may be supposed that system (3) has a solution of the form $\mathbb{R} \ni t \mapsto e^{\lambda t} \cdot s$ where $0 \neq s \in \mathbb{R}^d$, that is

$$(\lambda I_d - A - Be^{-\lambda \tau})s = 0. \quad (5)$$

This can only happen if and only if $\Delta(\lambda; \tau) = 0$ holds, where

$$\Delta(z; \tau) := \det(zI_d - A - Be^{-z\tau}) \quad (z \in \mathbb{C}) \quad (6)$$

is called characteristic quasi-polynomial of the linear delay system (3).

The organization of the chapter is as follows. In the next section, we introduce and prove two criteria regarding stability of $\Delta := \Delta(\cdot; 0)$, i.e. we give conditions for which the zeros of Δ have negative real parts. Concretely, we deal with the Hurwitz criterion and with its simplified version, the Liénard-Chippart criterion, and the Mikhailov criterion. Furthermore, we show how one can check the conditions of Hopf bifurcation via knowledge of the coefficients of the characteristic polynomial. In the section that follows we examine the case when the delay τ is positive. We show how the Mikhailov criterion can be extended for quasi-polynomials and how the length of the delay can be estimated in order to have stability. Finally, we present a criterion for Hopf bifurcation.

2. The undelayed case: $\tau = 0$

If there is no delay present, i.e. $\tau = 0$ holds then we have to deal with the characteristic polynomial

$$\chi(z) := (-1)^d \det(A - zI_d) = z^d + a_{d-1}z^{d-1} + a_{d-2}z^{d-2} + \dots + a_1z + a_0 \quad (z \in \mathbb{C}) \quad (7)$$

where the coefficients of χ_A are determined recursively by the Faddeev–LeVerrier algorithm (cf. [3, 4]) as follows

$$\left. \begin{aligned} N_0 &:= O & a_d &:= 1 & (k=0), \\ N_k &:= AN_{k-1} + a_{d-k+1}I_d & a_{d-k} &:= -\frac{1}{k} \text{Tr}(AN_k) & (k \in \{1, \dots, d\}). \end{aligned} \right\} \quad (8)$$

2.1 The stability of the characteristic polynomial χ_A

The asymptotic stability of (3) is determined by the stability of the matrix A , i.e. by the stability of its characteristic polynomial χ_A . We are now supplying some criteria for the stability of the characteristic polynomial χ_A . Under stability, we mean the so-called Hurwitz stability, i.e. the zeros of χ_A lie in the open left half of the complex plane. In this case χ_A is called also Hurwitz polynomial.

There is a very simple but very important necessary condition for χ_A to be Hurwitz (cf. [5, 6]).

Theorem 1.1 (Stodola). If the characteristic polynomial χ_A in (7) is stable then all of its coefficients are positive, i.e. $a_k > 0$ holds where $k \in \{0 \dots, d-1\}$.

Proof: The real and complex zeros of χ_A may be written as λ_k , resp. $\alpha_l \pm i\beta_l$, where α_l and β_l are both real. If the multiplicity of the real, resp. complex zeros are denoted by σ_k , resp. τ_l , where $k \in \{1, \dots, r\}$, resp. $l \in \{1, \dots, s\}$, then

$$\sum_{k=1}^r \sigma_k + 2 \sum_{l=1}^s \tau_l = d, \quad (9)$$

and we can split χ_A into linear, resp. quadratic factors according to the real, resp. complex zeros as follows

$$\begin{aligned} \chi_A(z) &= a_0 + a_1 z + \dots + a_{d-1} z^{d-1} + z^d \\ &= \prod_{k=1}^r (z - \lambda_k)^{\sigma_k} \cdot \prod_{l=1}^s (z - \alpha_l - i\beta_l)^{\tau_l} (z - \alpha_l + i\beta_l)^{\tau_l} \\ &= \prod_{k=1}^r (z - \lambda_k)^{\sigma_k} \cdot \prod_{l=1}^s (z^2 - 2\alpha_l z + \alpha_l^2 + \beta_l^2)^{\tau_l}. \end{aligned} \quad (10)$$

Thus, the stability of χ_A implies the sign conditions

$$\lambda_k < 0, \quad \alpha_l < 0 \quad (k \in \{1, \dots, r\}; \quad l \in \{1, \dots, s\}). \quad (11)$$

This means that all coefficients of all factors in the product above are positive. By performing the multiplications one can see that the coefficients of χ_A are positive. ■

In the case of $d = 1$ and $d = 2$ this criterion is sufficient and necessary. Indeed, in case of $d = 1$ the characteristic polynomial has the form

$$\chi_A(z) = z + a_0 \quad (z \in \mathbb{C}), \quad (12)$$

furthermore in case of $d = 2$ we have

$$a_1 = -\text{Tr}(A), \quad a_0 = -\frac{1}{2} \left\{ \text{Tr}(A^2) - \text{Tr}(A)^2 \right\} = \det(A) \quad (13)$$

(cf. (8)), thus

$$\chi_A(\xi_{\pm}) = 0 \iff \xi_{\pm} = \frac{\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}}{2} \quad (14)$$

and χ_A is stable if and only if $\text{Tr}(A) < 0$ and $\det(A) > 0$ hold, because if

- $\text{Tr}(A)^2 - 4\det(A) > 0$, then we have $\xi_{-} < 0$,

$$\xi_{+} < 0 \iff \text{Tr}(A)^2 - 4\det(A) < \text{Tr}(A)^2 \iff \det(A) > 0; \quad (15)$$

- $\text{Tr}(A)^2 - 4\det(A) = 0$, then the zeros ξ_{-} and ξ_{+} are equal and real:

$$\xi_{-} = \xi_{+} = -\frac{\text{Tr}(A)}{2} < 0 \iff \text{Tr}(A) > 0; \quad (16)$$

- $\text{Tr}(A)^2 - 4\det(A) < 0$, then there are only complex zeros:

$$\Re(\xi_{\pm}) = -\frac{\text{Tr}(A)}{2} < 0 \iff \text{Tr}(A) > 0. \quad (17)$$

Unfortunately the criterion is for $d > 2$ not sufficient. For example, the polynomial

$$p(z) := z^4 + 3z^3 + 3z^2 + 3z + 3 = (z^2 + 1)(z + 1)(z + 2) \quad (z \in \mathbb{C}) \quad (18)$$

has positive coefficients, but two of its zero, namely $\pm i$ are not in the open left half-plane. In case of $d = 3$ there is a result which can be proved in several ways. Pontryagin (cf. [7]) proves it in a circumstantial way. He uses that the zeros of a polynomial are continuous functions of the coefficients (cf. [8, 9]). Our presentation is based on the results of Suter (cf. [10]).

Theorem 1.2 In case of $d = 3$ the characteristic polynomial of A has the form

$$\chi_A(z) := z^3 - \text{Tr}(A)z^2 + \text{Tr}(\text{adj}(A))z - \det(A) \quad (z \in \mathbb{C}) \quad (19)$$

(cf. (8)) and χ_A is stable if and only if

$$\text{Tr}(A) < 0, \quad \det(A) < 0, \quad \text{Tr}(A) \cdot \text{Tr}(\text{adj}(A)) < -\det(A) \quad (20)$$

hold

Proof: Using the Faddeev-LeVerrier-algorithm (cf. (8)) we have

$$\chi_A(z) = z^3 + az^2 + bz + c \quad (z \in \mathbb{C}), \quad (21)$$

where

$$a := -\text{Tr}(A), \quad b := \frac{\text{Tr}(A)^2 - \text{Tr}(A^2)}{2} = \text{Tr}(\text{adj}(A)), \quad c := -\det(A). \quad (22)$$

As a consequence of the fundamental theorem of algebra, we can split χ_A into a linear and a quadratic factor

$$\chi_A(z) = (z - \alpha)(z^2 + \beta z + \gamma) = z^3 + (\beta - \alpha)z^2 + (\gamma - \alpha\beta)z - \alpha\gamma \quad (z \in \mathbb{C}). \quad (23)$$

In view of the above considerations for the first and the second-order polynomials we see that χ_A is stable if and only if $\alpha < 0$, $\beta > 0$ and $\gamma > 0$ hold. Thus, it is enough to show that the equivalence

$$\alpha < 0, \quad \beta > 0, \quad \gamma > 0 \iff a > 0, \quad c > 0, \quad ab - c > 0. \quad (24)$$

holds. We prove this statement in two steps.

Step 1. We prove that the positivity of the coefficients a, b, c entails

$$\alpha < 0, \quad \gamma > 0 \quad \text{and} \quad \text{sgn}(\beta) = \text{sgn}(ab - c). \quad (25)$$

Indeed,

- from $-\alpha\gamma = c > 0$ it follows that $\alpha \neq 0$, $\gamma \neq 0$. Hence the equivalence

$$\alpha > 0 \iff \gamma < 0 \quad (26)$$

holds. The case $\alpha > 0$, $\gamma < 0$ cannot happen, because $\gamma - \alpha\beta = b > 0$ would imply $\beta < 0$, which is not possible due to $\beta - \alpha = a > 0$.

$$\alpha^2 + b > 0 \quad (27)$$

$$\begin{aligned} ab - c &= (\beta - \alpha)(\gamma - \alpha\beta) + \alpha\gamma = \alpha^2\beta - \alpha\beta^2 + \beta\gamma = \beta(\alpha^2 - \alpha\beta + \gamma) = \\ &= \beta(\alpha^2 + b). \end{aligned} \quad (28)$$

Step 2. It remains to prove that the equivalence (24) holds.

- If χ_A is stable, i.e. $\alpha < 0$, $\beta > 0$ and $\gamma > 0$ hold, then $a > 0$, $b > 0$ and clearly $c > 0$, even by Step 2 $\text{sgn}(ab - c) = \text{sgn}(\beta) = 1$, i.e. $ab - c > 0$.
- If inequalities $a > 0$, $c > 0$ and $ab - c > 0$ hold, then $b > 0$, hence by Step 2 we have $\alpha < 0$, $\gamma > 0$ and $\text{sgn}(\beta) = \text{sgn}(ab - c) = 1$, i.e. $\beta > 0$ which completes the proof. ■

Example 1. If the matrix $A \in \mathbb{R}^{3 \times 3}$ is antisymmetric, i.e. for suitable $a, b, c \in \mathbb{R}$

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \quad (29)$$

holds, then its characteristic polynomial has the form

$$\chi_A(z) = z^3 + (a^2 + b^2 + c^2)z = z[z^2 + (a^2 + b^2 + c^2)] \quad (z \in \mathbb{C}). \quad (30)$$

This means that χ_A and hence A is unstable.

In order to formulate the necessary and sufficient stability condition for the polynomial χ_A with arbitrary degree $d \in \mathbb{N}$, we shall first fix our terminology. Let us define the Hurwitz matrix of the characteristic polynomial χ_A by

$$\mathcal{H}_{\chi_A} := [h_{ij}], \quad \text{where} \quad h_{ij} := \begin{cases} a_{d-(2j-i)} & (0 \leq 2j - i \leq d), \\ 0 & (\text{elsewhere}) \end{cases} \quad (i, j \in \{1, \dots, d\}), \quad (31)$$

i.e.

$$\mathcal{H}_{\chi_A} := \begin{bmatrix} a_{d-1} & a_{d-3} & a_{d-5} & \dots & a_{d-2d+3} & a_{d-2d+1} \\ a_d & a_{d-2} & a_{d-4} & \dots & a_{d-2d+4} & a_{d-2d+2} \\ 0 & a_{d-1} & a_{d-3} & \dots & a_{d-2d+5} & a_{d-2d+3} \\ 0 & a_d & a_{d-2} & \dots & a_{d-2d+6} & a_{d-2d+4} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & a_1 & 0 \\ 0 & 0 & \dots & \dots & a_2 & a_0 \end{bmatrix} \in \mathbb{R}^{d \times d} \quad (32)$$

where $a_d := 1$ and $a_{-m} := 0$ if $m > 0$.

Theorem 1.3 (Routh-Hurwitz criterion). The characteristic polynomial χ_A in (7) is stable if and only if all leading principal minors

$$\Delta_k := \det \begin{bmatrix} h_{11} & \dots & h_{1k} \\ \vdots & \ddots & \vdots \\ h_{k1} & \dots & h_{kk} \end{bmatrix} \quad (k \in \{1, \dots, d\}) \quad (33)$$

of $\mathcal{H}_{\chi_A}$ are positive, i.e.

$$\Delta_1 = a_{d-1} > 0, \quad \Delta_2 = \det \begin{bmatrix} a_{d-1} & a_{d-3} \\ a_d & a_{d-2} \end{bmatrix} > 0, \quad \dots, \quad \Delta_d = a_0 \Delta_{d-1} > 0 \quad (34)$$

hold. $\square$

For

- two-dimensional system we have

$$\mathcal{H}_{\chi_A} := \begin{bmatrix} a_1 & a_{-1} \\ 1 & a_0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 1 & a_0 \end{bmatrix}. \quad (35)$$

Thus, this criterion can be stated as

$$\Delta_1 = a_1 > 0, \quad \Delta_2 = a_1 - a_0 > 0, \quad (36)$$

or

$$a_0 > 0, \quad a_1 > 0. \quad (37)$$

- third-dimensional system we have

$$\mathcal{H}_{\chi_A} := \begin{bmatrix} a_2 & a_0 & 0 \\ 1 & a_1 & 0 \\ 0 & a_2 & a_0 \end{bmatrix}. \quad (38)$$

Thus, this criterion is

$$\Delta_1 = a_2 > 0, \quad \Delta_2 = a_2 a_1 - a_0 > 0, \quad \Delta_3 = a_0 \Delta_2 > 0, \quad (39)$$

or

$$a_0 > 0, \quad a_2 > 0, \quad a_2 a_1 > a_0. \quad (40)$$

- fourth-dimensional system we have

$$\mathcal{H}_{\chi_A} := \begin{bmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{bmatrix}. \quad (41)$$

Thus, this criterion can be stated as $\Delta_1 = a_3 > 0$,

$$\Delta_2 = a_3 a_2 - a_1 > 0, \quad \Delta_3 = a_3 a_2 a_1 - a_3^2 a_0 - a_1^2 > 0, \quad \Delta_4 = a_0 \Delta_3 > 0 \quad (42)$$

or

$$a_0 > 0, \quad a_3 > 0, \quad a_3 a_2 > a_1, \quad a_1 a_2 a_3 > a_0 a_3^2 + a_1^2. \quad (43)$$

As the application of the above theorem, we mention the Orlando formula (cf. [11]) which establishes the useful relation between the Hurwitz determinants and the polynomial whose roots are sums of the roots of a given polynomial and which can be proved by mathematical induction (cf. [12]).

Theorem 1.4 (Orlando-formula) If $\xi_1, \dots, \xi_d$ are the roots of the characteristic polynomial χ_A then the $(d - 1)$ -th principal minor of the Hurwitz matrix can be expressed as

$$\Delta_{d-1} = (-1)^{d(d-1)/2} \cdot \prod_{\substack{i,j=1 \\ i < j}}^d (\lambda_i + \lambda_j). \quad (44)$$

In case of

- $d = 2$ this formula reduces to the well known Vieta formula in the quadratic equation

$$a_1 = \Delta_1 = -(\lambda_1 + \lambda_2); \quad (45)$$

- $d = 3$ the formula in (44) reduces to

$$a_2 a_1 - a_0 = \Delta_2 = -(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3). \quad (46)$$

We remark (cf. [12]) if criterion (34) is satisfied then χ_A is stable which due to the form (10) has the consequence that all coefficients of χ_A are positive:

$$a_0 > 0, \quad a_1 > 0, \quad \dots, \quad a_{d-1} > 0. \quad (47)$$

Clearly, if (47) holds then condition (34) is redundant: many of inequalities in (34) are unnecessary. For χ_A to be a Hurwitz stable, a necessary and sufficient condition can be established which requires about half amount of computations needed in the criterion of Routh-Hurwitz (cf. [12, 13]).

Theorem 1.5 (Liénard-Chipart) The following statements are equivalent:

1. the characteristic polynomial χ_A in (7) is Hurwitz stable;
2. $a_0 > 0, a_2 > 0, \dots; \Delta_1 > 0, \Delta_3 > 0, \dots;$
3. $a_0 > 0, a_2 > 0, \dots; \Delta_2 > 0, \Delta_4 > 0, \dots;$
4. $a_0 > 0, a_1 > 0, a_3 > 0, \dots; \Delta_1 > 0, \Delta_3 > 0, \dots;$
5. $a_0 > 0, a_1 > 0, a_3 > 0, \dots; \Delta_2 > 0, \Delta_4 > 0, \dots.$

Example 2. For $\alpha, \beta \in \mathbb{R}$ the matrix

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \alpha\beta - 1 & -2 & -3 & -2 & -1 \end{bmatrix} \quad (48)$$

has the characteristic polynomial

$$\chi_A(z) := 1 - \alpha\beta + 2z + 3z^2 + 2z^3 + z^4 \quad (z \in \mathbb{C}). \quad (49)$$

It suffices to calculate

$$\begin{aligned}\Delta_4 = \det(\mathcal{H}_{\chi_A}) &= (1 - \alpha\beta)\Delta_3 = (1 - \alpha\beta) \cdot \det \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 1 - \alpha\beta \\ 0 & 2 & 2 \end{bmatrix} \\ &= (1 - \alpha\beta) \cdot \{12 - 4(1 - \alpha\beta) - 4\} = 4(1 - \alpha\beta)(1 + \alpha\beta).\end{aligned}\quad (50)$$

Thus, χ_A and hence A is stable if and only if

$$(1 - \alpha\beta > 0 \text{ and } 1 + \alpha\beta > 0), \quad \text{i.e.} \quad -1 < \alpha\beta < 1 \quad (51)$$

holds (cf. **Figure 1**).

There is a criterion of geometric character which is useful for the study of the stability of χ_A .

Definition 1. *The curve*

$$\Gamma_A := \{\chi_A(i\omega) = (\Re(\chi_A(i\omega)), \Im(\chi_A(i\omega))) \in \mathbb{C} : \omega \in \mathbb{R}\} \quad (52)$$

is called Mikhailov hodograph or amplitude-phase curve (cf. [14]).

Some geometrical properties of the hodograph Γ_A are in strong relationship to the stability of the characteristic polynomial χ_A . This polynomial has no zero on the imaginary axis if and only if the origin does not lie on the curve Γ_A . In this case the function

$$\varphi_{\chi_A}(\omega) := \arg(\chi_A(i\omega)) = \Im(\log(\chi_A(i\omega))) \quad (\omega \in \mathbb{R}) \quad (53)$$

is continuous in every point of the real line. Moreover, we deal with the change

$$\Delta_{\chi_A} := \Delta_{\omega \in (-\infty, \infty)} \varphi_{\chi_A}(\omega) \quad (54)$$

Figure 1.
Stability chart of the polynomial $\mathbb{C} \ni z \mapsto z^4 + 2z^3 + 3z^2 + 2z + 1 - \alpha\beta$.

where Δ_{χ_A} denotes the change of argument of the vector $\varphi_{\chi_A}(\omega)$ in the complex plane as ω increases from $-\infty$ to $+\infty$. Because

$$\chi_A(-i\omega) = \overline{\chi_A(i\omega)}, \quad (55)$$

therefore we have

$$\Delta_{\omega \in [0, \infty)} \varphi_{\chi_A}(\omega) = \frac{1}{2} \cdot \Delta_{\omega \in (-\infty, \infty)} \varphi_{\chi_A}(\omega). \quad (56)$$

This means that it is enough to know the behavior of the vector $\varphi_{\chi_A}(\omega)$ for $0 \leq \omega < \infty$. The next theorem which is known as the Mikhailov criterion of stability is based on the principle of argument. Because the form as it is in the next theorem is a special case of the one formulated in the next section we omit its proof now.

Theorem 1.6 (Mikhailov). The polynomial χ_A is stable if and only if the following two conditions are fulfilled:

1. the curve Γ_A does not cross the origin, i.e. the implication

$$\chi_A(z) = 0 \Rightarrow \Re(z) \neq 0$$

is true, which means that χ_A has no zeros on the imaginary axis;

2. the curve Γ_A encircles the origin anticlockwise at an angle $d\pi/2$ while ω changes from 0 to $+\infty$, i.e.

$$\Delta_{\omega \in [0, \infty)} \arg(\chi_A(i\omega)) = \frac{d\pi}{2}$$

holds.

Example 3. In case of $d = 2$ we have for the characteristic polynomial

$$\chi_A(z) := a_0 + a_1 z + z^2 \quad (z \in \mathbb{C}) : \quad (57)$$

$$\Re(\chi_A(i\omega)) = a_0 - \omega^2, \quad \Im(\chi_A(i\omega)) = a_1 \omega, \quad (58)$$

from which

$$\sin(\arg(\chi_A(i\omega))) = \frac{\Im(\chi_A(i\omega))}{\sqrt{[\Re(\chi_A(i\omega))]^2 + [\Im(\chi_A(i\omega))]^2}} = \frac{a_1 \omega}{\sqrt{[a_0 - \omega^2]^2 + a_1^2 \omega^2}} \quad (59)$$

$$\rightarrow \operatorname{sgn}(a_1) 0 \quad (\omega \rightarrow +\infty),$$

resp. as $\omega \rightarrow +\infty$

$$\cos(\arg(\chi_A(i\omega))) = \frac{\Re(\chi_A(i\omega))}{\sqrt{[\Re(\chi_A(i\omega))]^2 + [\Im(\chi_A(i\omega))]^2}} = \frac{a_0 - \omega^2}{\sqrt{[a_0 - \omega^2]^2 + a_1^2 \omega^2}} \rightarrow -1 \quad (60)$$

follows. Thus,

$$\Delta_{\omega \in [0, +\infty)} \arg(\chi_A(i\omega)) = \begin{cases} 0 & (a_1 < 0 \wedge a_0 < 0), \\ -\pi & (a_1 < 0 \wedge a_0 > 0), \\ 0 & (a_1 > 0 \wedge a_0 < 0), \\ \pi & (a_1 > 0 \wedge a_0 > 0) \end{cases} \quad (61)$$

which means that χ_A is stable if and only if $a_0 > 0$ and $a_1 > 0$ hold.

Often what is to be checked is not the stability of the characteristic polynomial χ_A but the question as to whether every zero of the polynomial lies in the interior of the unit circle around the origin of the complex plane. In this case χ_A is called Schur stable polynomial or simply Schur polynomial. This phenomenon plays a crucial role in the stability of discrete dynamical systems and in the asymptotic stability of periodic linear systems (cf. [15]). Regarding this problem there are two main treatments. The first way to investigate the Schur stability of χ_A is to introduce the Möbius-transformation

$$w := \frac{z+1}{z-1} \quad (-1 \neq z \in \mathbb{C}) \quad /z := \frac{w+1}{w-1} \quad (1 \neq w \in \mathbb{C})/ \quad (62)$$

which takes the interior of the unit circle of the complex plane $\{z \in \mathbb{C} : |z| < 1\}$ into the interior of the left half-plane $\{w \in \mathbb{C} : \Re(w) < 0\}$. Thus, if we want to know whether the polynomial χ_A is Schur stable we perform the transformation

$$\psi_A(w) := (w-1)^d \cdot \chi_A\left(\frac{w+1}{w-1}\right) \quad (1 \neq w \in \mathbb{C}). \quad (63)$$

It is clear that ψ_A is also a polynomial of degree d and χ_A is Schur stable if and only if ψ_A is Hurwitz stable. It is not difficult to calculate (cf. [2]) that in case of

- $d = 2$ the polynomial χ_A is Schur stable if and only if

$$-1 + |a_1| < a_0 < 1; \quad (64)$$

- $d = 3$ the polynomial χ_A is Schur stable if and only if the inequalities

$$1 + a_1 > |a_0 + a_2|, \quad 3 - a_1 > |3a_0 - a_2|, \quad 1 - a_1 > a_0(a_0 + a_2) \quad (65)$$

hold.

The second way is the application of the so called Jury test which proof is based on the Rouché theorem (cf. [16]).

2.2 Hopf bifurcation

In what follows we shall examine the situation when system (1) with $\tau := 0$ exhibits Hopf bifurcation. In order to have this, we rewrite the version of (1) without delay in the parameter-dependent form

$$\dot{x} = f \circ (x, p) \quad (66)$$

where p represents a parameter of the given system. Hopf bifurcation occurs if and only if for the eigenvalues $\mu(p) \pm i\nu(p)$ of the Jacobi matrix A of f at the critical value p_*

- the eigenvalue crossing condition holds:

$$\mu(p_*) = 0, \quad \nu(p_*) \neq 0, \quad (\sigma(A) \setminus \{\pm \nu(p_*)\}) \cap i\mathbb{R} = \emptyset; \quad (67)$$

- the transversality condition $\mu'(p_*) \neq 0$ is fulfilled.

In the case of the two-dimensional system there is a result about the fulfillment of the above two conditions.

Lemma 1. Let $I \subset \mathbb{R}$ be an open interval and $\beta, \gamma : I \rightarrow \mathbb{R}$ smooth functions. The roots of the characteristic polynomial

$$\chi_A(z) := z^2 + \beta z + \gamma \quad (z \in \mathbb{C}) \quad (68)$$

fulfill the eigenvalue crossing condition and the transversality condition if and only if at the critical value $p = p_* \in I$

$$\beta(p_*) = 0, \quad \gamma(p_*) > 0 \quad \text{and} \quad \beta'(p_*) \neq 0 \quad (69)$$

hold.

Proof:

Step 1. The polynomial χ_A has purely imaginary zeros $\pm \omega i$ with $\omega \neq 0$ if and only if

$$z^2 + \beta z + \gamma = (z - \omega i)(z + \omega i) = z^2 + \omega^2 \quad (z \in \mathbb{C}). \quad (70)$$

Thus, at the critical value $p = p_* \in I$ the eigenvalue crossing condition holds exactly in case

$$\beta(p_*) = 0 \quad \text{and} \quad \gamma(p_*) > 0. \quad (71)$$

Step 2. Let denote by ρ the root of the equation

$$z^2 + \beta z + \gamma = 0 \quad (72)$$

which at p_* takes the value ωi : $\rho(p_*) = \omega i$, and let us introduce the following function

$$\mathcal{F}(z, p) := z^2 + \beta(p)z + \gamma(p). \quad (73)$$

Since

$$\mathcal{F}(\rho(p_*), p_*) = 0 \quad \text{and} \quad \partial_1 \mathcal{F}(\rho(p_*), p_*) = 2\omega i + \beta(p_*) = 2\omega i \neq 0, \quad (74)$$

therefore we have

$$\begin{aligned} \rho'(p_*) &= -\frac{\partial_2 \mathcal{F}(\omega i, p_*)}{\partial_1 \mathcal{F}(\omega i, p_*)} = -\frac{\beta'(h)z + \gamma'(h)}{2z + \beta(h)} \Big|_{\substack{z = \omega i \\ h = p_*}} \\ &= -\frac{\beta'(p_*)\omega i + \gamma'(p_*)}{2\omega i + \beta(p_*)} =: \frac{A + Bi}{C + Di}. \end{aligned} \quad (75)$$

Using the well known calculation

$$\frac{A + Bi}{C + Di} = \frac{A + Bi}{C + Di} \cdot \frac{C - Di}{C - Di} = \frac{(AC + BD) - (AD - BC)i}{C^2 + D^2} = \frac{AC + BD}{C^2 + D^2} + \frac{BC - AD}{C^2 + D^2}i, \quad (76)$$

furthermore the first and third part of (69) the formula

$$\frac{d\Re(\rho(p_*))}{dh} = \Re(\rho'(p_*)) = \frac{2\omega^2\beta'(p_*) + \gamma'(p_*)\beta(p_*)}{4\omega^2 + (\beta(p_*))^2} = -\frac{\beta'(p_*)}{2} \neq 0 \quad (77)$$

proves the lemma. ■

Because a matrix of order two can have no other eigenvalues besides the critical eigenvalues the crossing can happen only if for suitable $r > 0$

$$\beta^2(h) - 4\gamma(h) < 0 \quad (h \in (p_* - r, p_* + r)) \quad (78)$$

holds.

We have to remark that there are two forms of Hopf bifurcation: the standard one and the so called non-standard. Under standard Hopf bifurcation we mean the phenomenon when the critical eigenvalues of the Jacobian matrix A cross the imaginary axis from left to right and all other eigenvalues remain in the open left complex plane, whereas non-standard Hopf bifurcation means that the critical eigenvalues cross the imaginary axis from the right to the left and there is no restriction for the location of the other eigenvalues.

Theorem 1.7. If (78) holds then crossing can only happen

- From the left half-plane to the right, if the function β changes at the critical values p_* its sign from positive to negative;
- From the right half-plane to the left, if the function β changes at the critical values p_* its sign from negative to positive.

Proof: If condition (78) holds, then χ_A has a pair of conjugate roots. From elementary mathematics, we know that

- if function β changes its sign from positive to negative at the critical values p_* , then the sign of the real parts of the roots change their signs from the negative to positive;
- if function β changes at the critical values p_* its sign from negative to positive, then the sign of the real parts of the roots change their signs from the positive to negative. ■

Example 4. Let be $0 < a, b \in \mathbb{R}$ and consider the activator-inhibitor system of Schnackenberg-type

$$\dot{x} = a - x + x^2y, \quad \dot{y} = b - x^2y \quad (79)$$

(cf. [17]). System (79) has the unique equilibrium point

$$(x_*, y_*) = \left(a + b, \frac{b}{(a + b)^2} \right). \quad (80)$$

The Jacobian of (79) at (x_*, y_*) takes the form

$$A := J(x_*, y_*) = \begin{bmatrix} 2x_* y_* - 1 & x_*^2 \\ -2x_* y_* & -x_*^2 \end{bmatrix} \quad (81)$$

whose eigenvalues are the zeros of its characteristic polynomial

$$\begin{aligned} \chi_A(z) &= \det(zI_2 - A) = z^2 - \text{Tr}(A)z + \det(A) = z^2 + (1 - 2x_* y_* + x_*^2)z + x_*^2 \\ &= z^2 + \frac{a - b + (a + b)^3}{a + b}z + (a + b)^2 =: z^2 + \beta z + \gamma \quad (z \in \mathbb{C}). \end{aligned} \quad (82)$$

It is easy to see that if we choose $b =: p$ as parameter by fixed a then for every $p(>0)$ we have $\gamma(p) > 0$ and

$$\beta'(p) = \frac{[-1 + 3(a + p)^2](a + p) - a + p - (a + p)^3}{(a + p)^2} = \frac{2(a + p)^3 - 2a}{(a + p)^2} \neq 0. \quad (83)$$

This means that Hopf bifurcation occurs at the critical value p_* if and only if p_* is a positive real solution of the equation

$$\kappa(p) := (a + p) \cdot \beta(h) = a - p + (a + p)^3 = 0 \quad (84)$$

extended in p to the whole real axis. For example,

- In case of $a = 1$ the polynomial κ has one real root:

$$\kappa(p) = 0 \iff p \in \left\{ 0, \frac{-3 - \sqrt{7}i}{2}, \frac{-3 + \sqrt{7}i}{2} \right\}. \quad (85)$$

Clearly, $\kappa(1) = 8 > 0$, therefore κ and so β assume on the positive half line positive values, which has a consequence that the characteristic polynomial and hence the equilibrium point (x_*, y_*) is stable, since a second order characteristic polynomial is stable if and only if its coefficients have the same (positive) sign.

- in case of $a = 0.1$ the polynomial κ has three real roots:

$$\kappa(h) = 0 \iff h \in \{p_1 := -1.18803 \dots, p_2 := 0.109149 \dots, p_3 := 0.778885 \dots\}. \quad (86)$$

Because

$$\kappa(0) = a + a^3 > 0, \quad \kappa(0.5) = 0.184 < 0, \quad \kappa(1) = a(a^2 + 3a + 4) > 0, \quad (87)$$

the polynomial κ and so β changes its sign at p_2 from positive to negative, and at p_3 from negative to positive. This means in the light of the above that at the parameter value $p = p_2$ standard Hopf bifurcation occurs: the roots migrate from the left open half plane to the right, (x_*, y_*) loses its stability; furthermore at the parameter value $p = p_3$ non-standard Hop bifurcation takes place, i.e. the roots migrate from the right half plane to the left and as a consequence (x_*, y_*) becomes stable.

In the case of the three-dimensional system we post a result (cf. [18]), the proof of which is similar to the one in [19].

Lemma 2. Let be $I \subset \mathbb{R}$ an open interval and $\alpha, \beta, \gamma : I \rightarrow \mathbb{R}$ smooth functions. The roots of the characteristic polynomial

$$\chi_A(z) := z^3 + \alpha z^2 + \beta z + \gamma \quad (z \in \mathbb{C}) \quad (88)$$

fulfill the crossing and the transversality conditions if and only if at the critical value $p = p_* \in I$

$$\beta(p_*) > 0, \quad \alpha(p_*) \neq 0, \quad \gamma(p_*) = \alpha(p_*)\beta(p_*) \quad (89)$$

furthermore

$$\frac{d}{dh} \{ \alpha(h)\beta(h) - \gamma(h) \}_{|_{h=p_*}} \neq 0 \quad (90)$$

hold.

Proof:

Step 1. We show that the characteristic polynomial χ_A has purely imaginary roots $\pm \omega i$ ($\omega \neq 0$), if $\alpha\beta - \gamma = 0$ and $\beta > 0$ hold. Indeed, if ξ, η, ζ denote the roots of the polynomial χ_A , then using Orlando formula we have

$$\beta = \xi\eta + \xi\zeta + \eta\zeta, \quad \text{resp.} \quad \alpha\beta - \gamma = -(\xi + \eta)(\xi + \zeta)(\eta + \zeta). \quad (91)$$

This means that

- if for some $0 \neq \omega \in \mathbb{R}$ the equalities

$$\chi_A(\omega i) = 0 = \chi_A(-\omega i) \quad (92)$$

hold, then one of the three zeros, like ζ is real, furthermore

$$\xi = \omega i = -\eta, \quad \text{resp.} \quad \beta = \omega^2 + \zeta\omega i - \zeta\omega i = \omega^2 > 0; \quad (93)$$

- χ_A has exactly zeros with opposite sign but the same absolute value, if $\alpha\beta - \gamma = 0$ holds, furthermore χ_A has a complex root, since if $\xi = -\eta$ then

$$\beta = -\xi^2 + \xi\zeta - \xi\zeta = -\xi^2 \leq 0, \quad (94)$$

which contradicts the fact that $\beta > 0$.

It is clear that from conditions (89) it follows that the zeros of χ_A are

$$-\alpha \quad \text{and} \quad \pm \sqrt{\beta}i = \pm \sqrt{\frac{\gamma}{\alpha}}i, \quad (95)$$

because

$$\begin{aligned} (z + \alpha)(z - \sqrt{\beta}i)(z + \sqrt{\beta}i) &\equiv (z + \alpha)(z^2 + \beta) \equiv z^3 + \alpha z^2 + \beta z + \alpha\beta \\ &\equiv z^3 + \alpha z^2 + \beta z + \gamma. \end{aligned} \quad (96)$$

Step 2. Let denote the roots of χ_A by ρ which assumes at p_* the value ωi : $\rho(p_*) = \omega i$, furthermore let define

$$\mathcal{F}(z, h) := z^3 + \alpha(h)z^2 + \beta(h)z + \gamma(h). \quad (97)$$

Because

$$\mathcal{F}(\rho(p_*), p_*) = 0, \text{ and } \partial_1 \mathcal{F}(\omega i, p_*) = \beta(p_*) - 3\omega^2 + 2\alpha(p_*)\omega i \neq 0, \quad (98)$$

it follows for the derivative of the implicit function ρ at p_* that

$$\begin{aligned} \rho'(p_*) &= -\frac{\partial_2 \mathcal{F}(\omega i, p_*)}{\partial_1 \mathcal{F}(\omega i, p_*)} = -\frac{\alpha'(h)z^2 + \beta'(h)z + \gamma'(h)}{3z^2 + 2\alpha(h)z + \beta(h)} \bigg|_{\substack{z = \omega i \\ h = p_*}} \\ &= \frac{\alpha'(p_*)\omega^2 - \beta'(p_*)\omega i - \gamma'(p_*)}{-3\omega^2 + 2\alpha(p_*)\omega i + \beta(p_*)}. \end{aligned} \quad (99)$$

Using (89), resp. (90), furthermore $\beta(p_*) = \omega^2$ we have

$$\begin{aligned} \frac{d\Re(\rho(p_*))}{dh} &= \Re(\rho'(p_*)) \\ &= \frac{\{\alpha'(p_*)\omega^2 - \gamma'(p_*)\}\{\beta(p_*) - 3\omega^2\} - 2\alpha(p_*)\beta'(p_*)\omega^2}{[\beta(p_*) - 3\omega^2]^2 + 4\alpha(p_*)^2\omega^2} \\ &= \frac{\{\alpha'(p_*)\beta(p_*) - \gamma'(p_*)\}\{-2\beta(p_*)\} - 2\alpha(p_*)\beta'(p_*)\beta(p_*)}{[-2\beta(p_*)]^2 + 4\alpha(p_*)^2\beta(p_*)} \\ &= \frac{2\beta(p_*) \cdot \{\gamma'(p_*) - \alpha'(p_*)\beta(p_*) - \alpha(p_*)\beta'(p_*)\}}{4\beta(p_*)\{\beta(p_*) + \alpha(p_*)^2\}} = \\ &= \frac{(\gamma - \alpha\beta)'(p_*)}{2\{\beta(p_*) + \alpha(p_*)^2\}} \neq 0. \quad \blacksquare \end{aligned} \quad (100)$$

As a consequence of the Stodola criterion the necessary condition for the stability of the polynomial χ_A is the positivity of its coefficients α, β, γ . In the case when its zeros cross the imaginary axis from the right to the left, one of the coefficients should be local negative.

Theorem 1.8. If conditions of Lemma 2., i.e. (89) and (90) hold, furthermore, for every $\varepsilon > 0$, resp. arbitrary $p_* \neq h \in (p_* - \varepsilon, p_* + \varepsilon)$ the equalities

$$\beta(h) = \alpha^2(h) \text{ and } \gamma(h) \notin \{0, \zeta(h)\} \quad (101)$$

are valid, where

$$\zeta(h) := \frac{3\alpha(h)\beta(h) - 2\alpha^2(h)}{9} \quad (h \in I), \quad (102)$$

then the roots of χ_A cross the imaginary axis

- from left to right, if

$$3\sqrt[3]{\gamma(h) - \zeta(h)} < 2\alpha(h) \quad (103)$$

- from right to left, if

$$3\sqrt[3]{\gamma(h) - \zeta(h)} > 2\alpha(h) \quad (104)$$

hold.

Proof: Using the notations

$$a := \frac{\alpha}{3}, \quad b := \frac{\beta}{3}, \quad \text{resp.} \quad A := a^2 - b, \quad B := 2a^2 - 3ab + \gamma \quad (105)$$

one can see that if $A = 0$ and $0 \neq B \in \mathbb{R}$ hold then the zeros of the polynomial χ_A are as follows (cf. [17]):

$$\xi = -\sqrt[3]{B} - a, \quad \eta = \frac{\sqrt[3]{B}}{2} - a + \frac{3\sqrt[3]{B^2}}{4}i, \quad \zeta = \frac{\sqrt[3]{B}}{2} - a + \frac{3\sqrt[3]{B^2}}{4}i. \quad (106)$$

Thus, from $B \neq 0$ and $\sqrt[3]{B} \neq 2a$ it follows that χ_A has a pair of complex conjugate roots. Because of condition $\gamma \neq 0$ the third root could not be zero, furthermore this pair of complex conjugate zeros lies

- in the left half-plane if and only if $\sqrt[3]{B} < 2a$, i.e. (103) holds;
- in the right half-plane if and only if $\sqrt[3]{B} > 2a$, i.e. (104) holds. ■

Example 5. In [20] Liao, Zhou, and Tang proposed the following autonomous system of ordinary differential equations

$$\dot{x} = a(y - x), \quad \dot{y} = dx + cy - xz, \quad \dot{z} = -bz + xy \quad (107)$$

where $a, b, c, d \in \mathbb{R}$. If (a, b, c, d) belongs to the set

$$\{(a, b, -1, d), \quad (a, b, c, c - a), \quad (a, b, c, 0)\} \quad (108)$$

then we get the Lorenz system (cf. [21]), the Chen system (cf. [22]) and the Lü system (cf. [23]). Yan showed (cf. [24]) that in the system (107) Hopf bifurcation may occur. In what follows we show that his calculations can be simplified as we know the coefficients of the characteristic polynomials of the Jacobian of system (107). If $b(c + d) > 0$ holds then system (107) has three equilibria:

$$E_0 := (0, 0, 0), \quad E_{\pm} := (\pm\sqrt{b(c+d)}, \pm\sqrt{b(c+d)}, c+d). \quad (109)$$

The Jacobian of (107) takes the form

$$J(x, y, z) := \begin{bmatrix} -a & a & 0 \\ d - z & c & -x \\ y & x & -b \end{bmatrix} \quad ((x, y, z) \in \mathbb{R}^3). \quad (110)$$

Hence the corresponding Jacobians are:

$$A_0 := J(E_0) := \begin{bmatrix} -a & a & 0 \\ d & c & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad (111)$$

resp.

$$A_{\pm} := J(E_{\pm}) := \begin{bmatrix} -a & a & 0 \\ d - (c+d) & c & \mp\sqrt{b(c+d)} \\ \pm\sqrt{b(c+d)} & \pm\sqrt{b(c+d)} & -b \end{bmatrix}. \quad (112)$$

The eigenvalues of A_0 are the roots of the characteristic polynomial

$$\chi_{A_0}(\xi) := \det(\xi I_3 - J_0) = \xi^3 + \alpha \xi^2 + \beta \xi + \gamma \quad (\xi \in \mathbb{C}), \quad (113)$$

where $\gamma := -\det(A_0) = -ab(c+d)$, $\beta := \text{Tr}(\text{adj}(A_0)) = ab - bc - ac - ad$ and $\alpha := -\text{Tr}(A_0) = a + b - c$.

We remark that only a special parameter configuration was investigated in [24], namely $a = c, d = -2c$ ($c > 0$). On the other hand one can observe more due to Lemma 2:

1. it is easy to see from the characteristic polynomial that in case $c = a + b$ Hopf bifurcation may not occur because no matter what will be chosen as a bifurcation parameter, the coefficient of the first-order term of the polynomial vanishes, which has the consequence (cf. Lemma 2) that there can be a pair of complex conjugate zeros on the imaginary axis only in case if the constant term of the polynomial vanishes. This contradicts the crossing condition.

2. the parameter d cannot be considered as a bifurcation parameter, because as its value is changed Hopf bifurcation may not occur. This can be explained as follows. If $a \neq 0$ then

$$(\gamma - \alpha\beta)(d_*) = 0 \iff d_* = \frac{ab + b^2 - ac - bc}{a}$$

(in case of $a = 0$ the system would be two dimensional) and $\beta(d_*) = -b^2$, this contradicts the first condition in (89).

3. the parameter c can be chosen as a bifurcation parameter only under some restrictions, because in the case of $a \neq -b$

$$(\gamma - \alpha\beta)(c_*) = 0 \iff c_* \in \left\{ a, \frac{ab + b^2 - ad}{a + b} \right\}$$

and

$$\beta(c_*) \in \{-a(a+d), -b^2\}$$

will be positive only in case if a and $a + d$ have opposite signs as it is in case $a = c, d = -2c$ ($c > 0$) proposed in [24]. If $a = -b$ and $a \neq 0$ then

$$(\gamma - \alpha\beta)(c_*) = 0 \iff c_* = \frac{ab + b^2 - ac - bc}{a}$$

(in case of $a = 0$ the system would be two dimensional) and $\beta(c_*) = -b^2$ which contradicts the first condition in (89). If $a = c$ then

$$(\gamma - \alpha\beta)(b_*) = 0 \iff d = 0.$$

In this case we have

$$\frac{d}{db}(\gamma - \alpha\beta)(b_*) = 0,$$

which contradicts the transversality condition.

It is easy to see the following: by fixing the parameters b, c , resp. $d = -2c$ ($c > 0$) (cf. [24]) the parameter a will be chosen as bifurcation parameter then at value $a_* := c$ Hopf bifurcation takes place, because with substitution $d = -2c$ we have

$$\beta(a_*) = cb - bc - cc - c(-2c) = c^2 > 0, \quad \alpha(a_*) = b \neq 0, \quad (114)$$

resp.

$$(\gamma - \alpha\beta)(a_*) = [abc - (a + b - c)(ab - bc + ac)]_{a=c} = bc^2 - bc^2 = 0 \quad (115)$$

and

$$\begin{aligned} \frac{d}{da} \{\gamma - \alpha\beta\}(a_*) &= \frac{d}{da} (abc - (a + b - c)(ab - bc + ac)) \Big|_{a=c} \\ &= [bc - (ab - bc + ac) - (a + b - c)(b + c)]_{a=c} = bc - c^2 - b(b + c) = -c^2 - b^2 \neq 0. \end{aligned} \quad (116)$$

The eigenvalues of the matrix $A_{\pm}$ are the zeros of the characteristic polynomial

$$\chi_{A_{\pm}}(\xi) := \det(\xi I_3 - A_{\pm}) = \xi^3 + \alpha\xi^2 + \beta\xi + \gamma \quad (\xi \in \mathbb{C}) \quad (117)$$

where

$$\gamma := -\det(J) = 2ab(c + d), \quad \beta := \text{Tr}(\text{adj}(J)) = b(a + d), \quad \alpha := -\text{Tr}(J) = a + b - c. \quad (118)$$

If

$$a(3a + d)(a + b + d) \neq 0, \quad (119)$$

then at

$$c_* = \frac{a^2 + ab - ad + bd}{3a + d} \quad (120)$$

Hopf bifurcation takes place, because

$$\beta(c_*) = b(a + d) > 0, \quad \alpha(c_*) = \frac{2a(a + b + d)}{3a + d}, \quad (\gamma - \alpha\beta)(c_*) = 0, \quad (121)$$

resp.

$$\frac{d}{dc} (\gamma - \alpha\beta)(c_*) = -2ab - b(a + d) = -b(3a + d) \neq 0. \quad (122)$$

3. The delayed case: $\tau > 0$

When modeling and analyzing processes and behaviors which come from a natural environment, it often happens that we need a bit of distance in time to see the changes of the considered quantities (which are the variables of our model). For example, when we think about the epidemiological models, it is a well-founded thought that we need some time while susceptibles become infectious, and hence it is reasonable to assume that the migration of the individuals from the class of susceptibles into the infected is subject to delay.

Another expressive example is the modeling of the processes of the human body or the brain, like emotions: love, hate, etc. A bit of time has to pass for the brain to process the signals coming from various places, and only after this delay, the mood could change. These changes can be described and analyzed with delayed differential equations. One type of these systems is the so called Romeo and Juliet model, where the changes of Romeo's and Juliet's love and hate in time are described as a system of two linear ordinary differential equations. In this chapter we are going to consider this model with general coefficients and investigate the stability of the linear system.

We are going to consider the following linear system:

$$\begin{aligned}\dot{x} &= \alpha_1 x + A_1 x(\cdot - \tau) + \alpha_2 y + A_2 y(\cdot - \tau), \\ \dot{y} &= \alpha_3 x + A_3 x(\cdot - \tau) + \alpha_4 y + A_4 y(\cdot - \tau),\end{aligned}\quad (123)$$

where $A_i, \alpha_i \in \mathbb{R}$, ($i \in \{1, 2, 3, 4\}$), $\tau > 0$, with initial conditions $\Phi = (\Phi_1, \Phi_2)$ in the Banach space

$$\{\Phi \in C([- \tau, 0], \mathbb{R}_+^2) : \Phi_1(\theta) = x(\theta), \Phi_2(\theta) = y(\theta)\}, \quad (124)$$

where $\Phi_i(\theta) > 0$, ($\theta \in [- \tau, 0]$, $i \in \{1, 2\}$). Straightforward calculation shows that the characteristic function of the above system (with regard to the trivial equilibrium point) takes the form:

$$\begin{aligned}\Delta(z, \tau) &\equiv z^2 - (\alpha_1 + \alpha_4)z + \alpha_1\alpha_4 - \alpha_2\alpha_3 + e^{-z\tau} \cdot (-(A_1 + A_4)z + \alpha_1A_4 \\ &\quad + \alpha_4A_1 - \alpha_2A_3 - \alpha_3A_2) + e^{-2z\tau}(A_1A_4 - A_2A_3) \quad (z \in \mathbb{C}).\end{aligned}\quad (125)$$

In [25] the authors treat delay differential equations, which characteristic function for arbitrary $z \in \mathbb{C}$ has the form

$$\Delta(z, \tau) = z^2 + a_1z + a_0 + (b_1z + b_0)e^{-z\tau} + ce^{-2z\tau} \quad (126)$$

where a_0, a_1, b_0, b_1, c are arbitrary real constants. It can be seen that the characteristic functions (125) and (126) has the same form. In [25] the authors assume that $c = 0$ to simplify the analysis. Furthermore, they say that due to the continuous dependence of eigenvalues of the model parameters (cf. [26]) their results are valid for sufficiently small c parameters, too. Nevertheless, in the literature a lot of models and systems are investigated in which the coefficient of $e^{-2z\tau}$ of the characteristic function is not equal to zero, and maybe not sufficiently small. Therefore, the aim of this section is to show that we can analyze the stability of the system in the case where $c \neq 0$, too.

In this section we assume that $\tau > 0$ holds and investigate the qualitative behavior of the linearized system (3), more precisely we study the stability of characteristic function

$$\Delta(z, \tau) \equiv p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \quad (127)$$

where p, q and r are polynomials with real coefficients and $\deg(r) < \deg(q)$. Under stability of Δ we mean that the zeros of Δ lie in the open left half of the complex plane. Using the Mikhailov criterion we give for special p, q and r fulfilling the above condition an estimate on the length of delay τ for which no stability switching occurs. Then for special parameters we compare our results with other methods. It follows then a delay independent stability analysis. Finally, a formula for Hopf bifurcation is calculated in terms of p, q and r . If we assume that the characteristic function has the form as in (126), then we can give conditions easily

on the parameters a_1, a_0, b_1, b_0, c and an upper bound τ_1 such that with $\tau < \tau_1$ the system is asymptotically stable. In other words stability change may happen only for $\tau \geq \tau_1$.

In what follows, the Mikhailov stability criterion will be proved, which is the implication of the argument principle (cf. [27, 28]). The treatment is based on [29].

Theorem 1.9 (Mikhailov criterion). Consider the quasi-polynomial

$$M(z) := Q(z) + \sum_{k=1}^p R_k(z)e^{-s\tau_k} \quad (z \in \mathbb{C}), \quad (128)$$

where the order of the polynomials Q and R_k is less than or equal to $d \in \mathbb{N}$, and they are defined as

$$Q(z) := q_d z^d + \dots + q_0, \quad R_k(z) := r_{k,d} z^d + \dots + r_{k,0} \quad (z \in \mathbb{C}) \quad (129)$$

where $q_i, r_{k,i} \in \mathbb{R}$ for $i = 1, \dots, d, k = 1, \dots, p, q_d > 0$ and

$$\max_{k \in \{1, \dots, p\}} (\deg(R_k)) < d, \quad (130)$$

furthermore $\tau_k \geq 0$ for $k = 1, \dots, p$. If M defined by (128) has no zeros on the imaginary axis, then M is stable if and only if

$$\Delta := \Delta_{\omega \in [0, +\infty)} \arg(M(i\omega)) = \frac{d\pi}{2} \quad (131)$$

holds where Δ denotes the change of argument of the vector $M(i\omega)$ anticlockwise in the complex plane as ω increases from 0 to $+\infty$.

Proof: In order to prove the theorem, we will apply the argument principle (cf. [28]) to M on the Γ -contour (cf. **Figure 2**) where $\Gamma =: C_1 \cup C_2$ denotes the

Figure 2.
 Γ -contour on the complex plane.

positive oriented curve in the complex plane which consists of the interval $[-\rho, \rho]$ ($\rho > 0$) on the imaginary axis, i.e.

$$C_1 := \{is \in \mathbb{C} : s \in [-\rho, \rho]\} \quad (132)$$

and the semicircle C_2 of the radius ρ in the right-hand half-plane:

$$C_2 := \left\{ \rho e^{i\phi} \in \mathbb{C} : \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}. \quad (133)$$

Since

$$\max_{k \in \{1, \dots, p\}} (\deg(R_k)) < \deg(Q), \quad (134)$$

there is only a finite number of roots of M in the right-half plane.

On C_2 the characteristic equation can be written for every $z \in \mathbb{C}$ as follows:

$$\begin{aligned} M(z) &= q_d \rho^d (\cos(d\phi) + i \sin(d\phi)) + \dots + q_0 \\ &+ \sum_{k=1}^p \left\{ r_{k(d-1)} \rho^{d-1} (\cos((d-1)\phi) + i \sin((d-1)\phi)) e^{-\rho \tau_k e^{i\phi}} + \dots + r_{k_0} e^{-\rho \tau_k e^{i\phi}} \right\}. \end{aligned} \quad (135)$$

Now from the summation, we can write a typical term as

$$\begin{aligned} r_{k(d-1)} \rho^{d-1} e^{i(d-1)\phi} e^{-\rho \tau_k e^{i\phi}} &= r_{k(d-1)} \rho^{d-1} e^{i(d-1)\phi} e^{-\rho \tau_k \cos(\phi) - i \rho \tau_k \sin(\phi)} \\ &= r_{k(d-1)} \rho^{d-1} e^{-\rho \tau_k \cos(\phi)} e^{i((d-1)\phi - \rho \tau_k \sin(\phi))} \end{aligned} \quad (136)$$

Therefore,

$$\begin{aligned} M(z) &= q_d \rho^d (\cos(d\phi) + i \sin(d\phi)) + \dots + q_0 + \sum_{k=1}^p e^{-\rho \tau_k \cos(\phi)} \\ &\cdot \{ r_{k(d-1)} \rho^{d-1} [\cos((d-1)\phi - \rho \tau_k \sin(\phi)) + i \sin((d-1)\phi - \rho \tau_k \sin(\phi))] \\ &+ r_{k(d-2)} \rho^{d-2} [\cos((d-2)\phi - \rho \tau_k \sin(\phi)) + i \sin((d-2)\phi - \rho \tau_k \sin(\phi))] \\ &+ \dots + r_{k_0} [\cos(\rho \tau_k \sin(\phi)) - i \sin(\rho \tau_k \sin(\phi))] \}. \end{aligned} \quad (137)$$

Hence the argument or phase θ of the vector $M(z)$ on C_2 may be written

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} =: \frac{A}{B}, \quad (138)$$

where

$$\begin{aligned} A := \sin(\theta) &= q_d \rho^d \sin(d\phi) + \dots + q_1 \rho \sin(\phi) + \sum_{k=1}^p e^{-\rho \tau_k \cos(\phi)} \\ &\cdot \{ r_{k(d-1)} \rho^{d-1} \sin((d-1)\phi - \rho \tau_k \sin(\phi)) \\ &+ \dots + r_{k_1} \rho \sin(\phi - \rho \tau_k \sin(\phi)) - r_{k_0} \sin(\rho \tau_k \sin(\phi)) \} \end{aligned} \quad (139)$$

and

$$B := \cos(\theta) = q_d \rho^d \cos(d\phi) + \dots + q_1 \rho \cos(\phi) + q_0 + \sum_{k=1}^p e^{-\rho \tau_k \cos(\phi)} \cdot \{r_{k(d-1)} \rho^{d-1} \cos((d-1)\phi - \rho \tau_k \sin(\phi)) + \dots + r_{k_0} \cos(\rho \tau_k \sin(\phi))\}. \quad (140)$$

Dividing the numerator and denominator by ρ^d gives

$$\tan(\theta) =: \frac{A_1}{B_1}, \quad (141)$$

where

$$A_1 := q_d \sin(d\phi) + \dots + \frac{q_1}{\rho^{d-1}} \sin(\phi) + \sum_{k=1}^p e^{-\rho \tau_k \cos(\phi)} \cdot \left\{ \frac{r_{k(d-1)}}{\rho} \sin((d-1)\phi - \rho \tau_k \sin(\phi)) + \dots + \frac{r_{k_1}}{\rho^{d-1}} \sin(\phi - \rho \tau_k \sin(\phi)) - \frac{r_{k_0}}{\rho^d} \sin(\rho \tau_k \sin(\phi)) \right\} \quad (142)$$

and

$$B_1 := q_d \cos(d\phi) + \dots + \frac{q_1}{\rho^{d-1}} \cos(\phi) + \frac{q_0}{\rho^d} + \sum_{k=1}^p e^{-\rho \tau_k \cos(\phi)} \cdot \left\{ \frac{r_{k(d-1)}}{\rho} \cos((d-1)\phi - \rho \tau_k \sin(\phi)) + \dots + \frac{r_{k_0}}{\rho^d} \cos(\rho \tau_k \sin(\phi)) \right\}. \quad (143)$$

Now since

$$|\cos(\alpha)| \leq 1 \text{ and } |\sin(\alpha)| \leq 1 \quad (\alpha \in \mathbb{R}) \quad (144)$$

and since

$$\cos(\alpha) \geq 0 \quad \left(\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right), \quad (145)$$

we have

$$\tan(\theta) = \frac{\sin(d\phi)}{\cos(n\phi)} = \tan(d\phi) \quad (146)$$

or

$$\theta = d\phi + m\pi, \quad m \in \mathbb{Z} \quad (147)$$

as $\rho \rightarrow \infty$. Therefore, the change in argument of $M(z)$ on C_2 is given by

$$\Delta_{C_2} \arg(M) = \frac{d\pi}{2} + m\pi - \left(-\frac{d\pi}{2} + m\pi \right) = d\pi. \quad (148)$$

Now from the argument principle we can write

$$\Delta_{C_1} \arg(M) + \Delta_{C_2} \arg(M) = 2\pi N, \quad (149)$$

where N is the total number of zeros of M inside Γ . Therefore

$$\Delta_{C_1} \arg(M) = 2\pi N - d\pi. \quad (150)$$

If we reverse the direction of integration along C_1 and note the symmetry about the real axis, we have

$$\Delta_{\omega \in [0, +\infty)} \arg(M(i\omega)) = \frac{1}{2}(d\pi - 2N) = \frac{d\pi}{2} - \pi N. \quad (151)$$

in case of stability we have $N = 0$. Hence, stability requires

$$\Delta_{\omega \in [0, +\infty)} \arg(M(i\omega)) = \frac{d\pi}{2}. \quad \blacksquare \quad (152)$$

As a consequence, we have the following.

Lemma 3. Let p , q and r be polynomials, with condition

$$\deg(p) > \max \{ \deg(q), \deg(r) \}, \quad (153)$$

and assume that the quasi-polynomial in (127) has no roots on the imaginary axis. Then $\Delta(\cdot, \tau)$ is stable, i.e. all of its roots have negative real part if and only if

$$[\arg \Delta(i\omega, \tau)]_{\omega=0}^{\omega=+\infty} = \frac{\pi}{2} \cdot \deg(p(i\omega)), \quad (154)$$

i.e. the argument of $\Delta(i\omega, \tau)$ increases $\pi/2 \cdot \deg(p(i\omega))$ as ω increases from 0 to $+\infty$. $\square$

Theorem 1.10 If for the delay parameter τ in the characteristic function (126)

$$\tau < \frac{a_1 - |b_1|}{|b_0| + 2|c|}, \quad a_0 + b_0 + c > 0 \quad (155)$$

hold, then the characteristic function, and hence the trivial equilibrium point of system (123) is asymptotically stable.

Proof: Substituting $z = i\omega$ ($\omega > 0$) into (126), we get

$$\Delta(i\omega, \tau) = p(i\omega) + q(i\omega)e^{-i\omega\tau} + r(i\omega)e^{-2i\omega\tau}. \quad (156)$$

Hence using the characteristic function (126), where

$$p(z) \equiv z^2 + a_1z + a_0, \quad q(z) \equiv b_1z + b_0, \quad r(z) \equiv c \quad (157)$$

we have for $\omega > 0$ that

$$\Delta_R(i\omega, \tau) \equiv -\omega^2 + a_0 + b_0 \cos(\omega\tau) + b_1\omega \sin(\omega\tau) + c \cos^2(\omega\tau) - c \sin^2(\omega\tau), \quad (158)$$

and also

$$\Delta_I(i\omega, \tau) \equiv a_1\omega + b_1\omega \cos(\omega\tau) - b_0 \sin(\omega\tau) - 2c \sin(\omega\tau) \cos(\omega\tau). \quad (159)$$

It could be seen that

$$\Delta_R(0, \tau) = a_0 + b_0 + c > 0 \quad \text{and} \quad \Delta_I(0, \tau) = 0, \quad (160)$$

furthermore

$$\lim_{\omega \rightarrow +\infty} \Delta_R(i\omega, \tau) = -\infty. \quad (161)$$

Therefore, we have to show that $\Delta_I(i\omega, \tau) > 0$ for each $\omega > 0$. If it holds, then

$$\arg(\Delta(i\omega, \tau)) = \pi, \quad (162)$$

and hence by using the Mikhailov criterion we have stability.

Substituting $w := \omega\tau$ into (159) and multiplying the result by τ , we get

$$\tau\Delta_I\left(i\frac{w}{\tau}, \tau\right) = a_1w + b_1w \cos(w) - \tau b_0 \sin(w) - 2\tau c \sin(w) \cos(w). \quad (163)$$

By using straightforward estimations we obtain that

$$\tau\Delta_I\left(i\frac{w}{\tau}, \tau\right) > (a_1 - |b_1| - \tau(|b_0| + 2|c|))w, \quad (164)$$

and hence $\Delta_I(i\omega, \tau) > 0$ fulfills for $\omega > 0$, if the first condition of (155) is satisfied, i.e. if

$$\tau < \frac{a_1 - |b_1|}{|b_0| + 2|c|} \quad (165)$$

fulfills. ■

We show now a simple example in order to demonstrate the above theorem and what the conditions say. First of all, we are going to see that the conditions of the previous theorem are sufficient, but not necessary.

Example 6. Let us consider the following system of two linear delay differential equations.

$$\dot{x} = -\frac{1+\sqrt{7}}{2}x + x(\cdot - \tau) + \frac{-4+\sqrt{7}}{2}y - \frac{1}{2}y(\cdot - \tau), \quad \dot{y} = x + x(\cdot - \tau) + \frac{-3+\sqrt{7}}{2}y. \quad (166)$$

The characteristic polynomial of (166) is

$$\Delta(z, \tau) = z^2 + 2z + 1 + (z-1)e^{-z\tau} + \frac{1}{2}e^{-2z\tau} \quad (z \in \mathbb{C}, \tau \geq 0). \quad (167)$$

Since $a_0 + b_0 + c = \frac{1}{2} > 0$, i.e. the second condition in Theorem 1.10 fulfills, we know from that theorem that if

$$\tau < \frac{a_1 - |b_1|}{|b_0| + 2|c|} = \frac{1}{2}, \quad (168)$$

then the trivial solution of (166) is asymptotically stable. As earlier mentioned, the conditions of Theorem 1.10 are sufficient, but not necessary, which can be easily seen, if we study the phase portrait of the system (166) with the following different values of the parameter τ : firstly with $\tau = 0.48$, then with $\tau = 0.6$ and finally with $\tau = 0.94$. The **Figures 3** and **4** represent the solutions of system (166) with different values of the parameter τ .

Figure 3.
Example: solutions with $\tau = 0.48$ and $\tau = 0.6$.

Figure 4.
Example: solutions with $\tau = 0.94$.

In **Figure 3** (above) the parameter τ is less than half, so the parameters fulfill the condition of Theorem 1.10, and the origin is asymptotically stable. In **Figure 3** (bottom) the parameter value shows that the theorem does not give a necessary condition, because here the value of the parameter is bigger than half, but the origin is still asymptotically stable. But if we increase more the value of the parameter τ , the quasi-polynomial and hence the origin changes to unstable.

The previous example shows that it would be useful to give the largest bound in the theorem, because if we have a larger bound, then we can guarantee the stability of the quasi-polynomial for higher value of the parameter τ . In this sense, we can compare our result in Theorem 1.10 with another result in the literature. In this chapter, we compare the conditions of the theorem coming from [30, 31]. In [31] Stépán considered the system

$$\dot{x} = -a_{11}x - a_{12}y + b_{11}x(\cdot - \tau), \quad \dot{y} = -a_{21}x - a_{22}y + b_{22}y(\cdot - \tau) \quad (169)$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{22} > 0$ and $\tau \geq 0$. The characteristic function of system (169) is the quasi-polynomial

$$\Delta_S(z, \tau) := z^2 + (a_{11} + a_{22})z + a_{11}a_{22} - a_{12}a_{21} - ((b_{11} + b_{22})z + a_{22}b_{11} + a_{11}b_{22}) \cdot e^{-z\tau} + b_{11}b_{22}e^{-2z\tau} \quad (z \in \mathbb{C}, \tau \geq 0). \quad (170)$$

In [30] Freedman and Rao worked with the system

$$\dot{x} = -D_1x - D_2y + B_1x(\cdot - \tau), \quad \dot{y} = -F_1x - F_2y + E_2y(\cdot - \tau) \quad (171)$$

where $B_1, D_1, D_2, E_2, F_1, F_2 > 0$ are constants and the characteristic function of (171) is the quasi-polynomial

$$\Delta_{FR}(z, \tau) := z^2 + (D_1 + F_2)z + D_1F_2 - D_2F_1 - ((B_1 + E_2)z + B_1F_2 + D_1E_2) \cdot e^{-z\tau} + B_1E_2e^{-2z\tau} \quad (z \in \mathbb{C}, \tau \geq 0). \quad (172)$$

Similarly to papers [30, 31] we gave an upper bound for τ under which the quasi-polynomial is Hurwitz stable.

Let us write these conditions for τ using the notations of Stépán. In [31] we can find the condition

$$\tau < \frac{a_{11} + a_{22} - b_{11} - b_{22}}{b_{11}(b_{22} + 0.22a_{22}) + b_{22}(b_{11} + 0.22a_{11})} =: C_S, \quad (173)$$

in [30] the condition

$$\tau < \frac{D_1 + F_2 - B_1 - E_2}{2(B_1F_2 + D_1E_2 + B_1E_2)} = \frac{a_{11} + a_{22} - b_{11} - b_{22}}{2(b_{11}b_{22} + a_{11}b_{22} + a_{22}b_{11})} =: C_{FR}. \quad (174)$$

Furthermore let us denote the right hand side of the condition in

Theorem 1.10 by C_{GyK}

$$\tau < \frac{a_1 - |b_1|}{|b_0| + 2|c|} = \frac{a_{11} + a_{22} - |b_{11} + b_{22}|}{2b_{11}b_{22} + |a_{11}b_{22} + a_{22}b_{11}|} =: C_{GyK}. \quad (175)$$

Since $a_{11}, a_{22}, b_{11}, b_{22} > 0$, the numerators of C_{FR} and C_{GyK} are equal and

$$2b_{11}b_{22} + |a_{11}b_{22} + a_{22}b_{11}| = 2b_{11}b_{22} + a_{11}b_{22} + a_{22}b_{11} < 2(b_{11}b_{22} + a_{11}b_{22} + a_{22}b_{11}), \quad (176)$$

because

$$a_{11}b_{22} + a_{22}b_{11} < 2(a_{11}b_{22} + a_{22}b_{11}) \quad (177)$$

is true following from the positivity of these constants. This means that $C_{GyK} > C_{FR}$.

Repeatedly, following from the positivity of the constants a_{11} , a_{22} , b_{11} and b_{22} we get that the numerators of C_{GyK} and C_S are equal, but

$$b_{11}(b_{22} + 0.22a_{22}) + b_{22}(b_{11} + 0.22a_{11}) < 2b_{11}b_{22} + a_{11}b_{22} + a_{22}b_{11}, \tag{178}$$

hence $C_{GyK} < C_S$. On one hand our result in Theorem 1.10 is applicable in general cases, because we have no additional constraints on the sign of the coefficients of the system (123), which means that in this sense our result is better. On the other hand we can increase the upper bound C_S a little bit in the following way. In [31] Stépán used the estimation

$$\sin(x) > -0.22x \quad (x > 0), \tag{179}$$

but actually this estimation is not sharp for positive x , cf. **Figure 5**. If we find a tangent line of the sine function at a certain point x_0 , such that this line passes through the origin, i.e. the equation of this line is $y = ax$ with a certain $a < 0$, then we can get a better estimation than (179), namely

$$\sin(x) \geq ax \quad (x > 0). \tag{180}$$

We can easily determine the constant $a < 0$ in the following way: the equation of the searched tangent line at x_0 is

$$y = f'(x_0)(x - x_0) + f(x_0) = f'(x_0)x + f(x_0) - f'(x_0)x_0, \tag{181}$$

where f is the sine function. We would like to find an x_0 such that

$$f(x_0) - f'(x_0)x_0 = \sin(x_0) - \cos(x_0) \cdot x_0 = 0 \tag{182}$$

Figure 5.
The estimation of the sine function on the interval $[0, 5.5]$ and $[4, 4.9]$.

fulfills, which is true if and only if x_0 is a solution of the equation $\tan(x) = x$. Let x_0 be the solution of this equation, then we find a better linear lower estimation for the function sine on the positive half-line

$$\sin(x) \geq \cos(x_0) \cdot x \quad (x > 0), \quad (183)$$

i.e. $a := \cos(x_0) \approx -0.21724$.

The proposition of the last theorem of this section is similar to one of Theorem 1.10. In the proof of Theorem 1.10 we used first-order Taylor polynomials to approximate the functions sine and cosine to obtain the estimation (164). But if we use higher-order polynomials, we can get a better result, i.e. a better estimation for τ , such that if τ satisfies the conditions, then the quasi-polynomial (126) is stable.

Theorem 1.11. If the coefficients of the characteristic function (125) fulfill the conditions

$$(2 - 6\sqrt{105})a_0 < 59b_0, \quad -\frac{4}{21}a_1 \leq b_1 < 0, \quad \frac{32}{3\sqrt{105} - 31}c < b_0 < -32c, \quad (184)$$

and for the delay parameter τ

$$\tau < -\kappa + \sqrt{\kappa^2 - \xi} =: C_2 \quad (185)$$

holds, where

$$\kappa := \frac{B}{2A}, \quad \xi := \frac{C}{4A} \quad (186)$$

with

$$A := -(b_0^2 + 124b_0c + 64c^2), \quad B := 6(a_1(b_0 + 32c) - 4b_1(b_0 + 2c)), \quad C := 45b_1^2 \quad (187)$$

then the quasi-polynomial (126) is Hurwitz-stable.

Proof: Firstly, let us make the same steps as in the proof of Theorem 1.10 and study the imaginary part of $\Delta(i\omega, \tau)$:

$$\Delta_I(i\omega, \tau) = a_1\omega + b_1\omega \cos(\omega\tau) - b_0 \sin(\omega\tau) - 2c \sin(\omega\tau) \cos(\omega\tau). \quad (188)$$

To show that $\Delta_I(i\omega, \tau) > 0$ for each $\omega > 0$ we apply the estimations

$$\sin(x) < x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \cos(x) > 1 - \frac{x^2}{2} \quad (x > 0). \quad (189)$$

Then we have

$$\Delta_I(i\omega, \tau) > \omega \cdot P_\tau(\omega), \quad (190)$$

with

$$P_\tau(\omega) := \omega \left(-\omega^4 \frac{\tau^5}{120} (b_0 + 32c) + \omega^2 \frac{\tau^2}{6} (\tau(b_0 + 8c) - 3b_1) + a_1 + b_1 - \tau(b_0 + 2c) \right). \quad (191)$$

If the conditions (184) are fulfilled, then the coefficient of ω^4 is positive in the polynomial P_τ , moreover if τ satisfies the condition (185) too, then the discriminant

of P_τ is negative. Hence with conditions (184) and (185) the inequality $\Delta_I(i\omega, \tau) > 0$ is valid for all $\omega > 0$. Similarly to the previous proof we get by applying the Mikhailov criterion that the quasi-polynomial is asymptotically stable.

In the following example, we are going to show that in some cases the result of Theorem 1.11 is better than the result in [31].

Example 7. Let us consider the following system of delay differential equations

$$\dot{x} = -0.7x - y + 0.01x(\cdot - \tau), \quad \dot{y} = -0.2x - 0.4y + 0.07y(\cdot - \tau). \quad (192)$$

The characteristic function of (192) is

$$\Delta(z; \tau) := z^2 + 1.1z + \frac{227}{10000} - e^{-z\tau} \left(\frac{2}{25}z + \frac{53}{1000} \right) + e^{2z\tau} \frac{7}{10000} \quad (z \in \mathbb{C}, \tau \geq 0). \quad (193)$$

Let us see what condition gives [31] for the parameter τ . The condition

$$(a_{11} - b_{11})(a_{22} - b_{22}) > a_{12}a_{21} \quad (194)$$

fulfills, hence if

$$\tau < \frac{a_{11} + a_{22} - b_{11} - b_{22}}{b_{11}(b_{22} + 0.22a_{22}) + b_{22}(b_{11} + 0.22a_{11})} \approx 78.1, \quad (195)$$

then the quasi-polynomial (193) is asymptotically stable. Furthermore simple calculations show that following from Theorem 1.11, the system is asymptotically stable for all $\tau \leq 170.07$, which is greater, than (195). In **Figure 6** the phase portrait of the system (192) could be seen with some values of the parameter τ .

3.1 Stability investigation, independently of the delay

We are going to consider the general form of the characteristic function (126):

$$\Delta(z, \tau) = p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \quad (z \in \mathbb{C}) \quad (196)$$

where $\deg(r) < \deg(q)$. We assume that if there is not any delay in the system, i.e. $\tau = 0$, then the trivial equilibrium point is asymptotically stable, which is equivalent to the assumption that the polynomial $\Delta(\cdot, 0)$ is Hurwitz stable. We

Figure 6.
The solutions of system (192) with $\tau = 0.5$ and $\tau = 170$.

know from [32] that with this assumption the system (and also its trivial equilibrium point) is delay-independently asymptotically stable if and only if for every $\tau > 0$ the quasi-polynomial $\Delta(\cdot, \tau)$ has no non-zero real root on the imaginary axis.

In this chapter we will add a condition to the polynomial p, q and r such that the mentioned property on the root of $\Delta(\cdot, \tau)$ fulfills. In the computations we will follow the idea of [33].

Firstly, let us multiply the equality $\Delta(i\omega, \tau) = 0$ by $e^{i\omega\tau}$, then we can see that the equivalence

$$\Delta(i\omega, \tau) = 0 \iff e^{i\omega\tau} \Delta(i\omega, \tau) = 0 \quad (197)$$

is valid. Let us introduce the notations

$$\begin{aligned} p(i\omega) &:= p_R(i\omega) + ip_I(i\omega), & q(i\omega) &:= q_R(i\omega) + iq_I(i\omega), \\ r(i\omega) &:= r_R(i\omega) + ir_I(i\omega), & \Delta(i\omega, \tau) &:= \Delta_R(i\omega, \tau) + i\Delta_I(i\omega, \tau). \end{aligned} \quad (198)$$

With these notations the characteristic function (196) can be written at $z = i\omega$ in the form

$$\begin{aligned} e^{i\omega\tau} \Delta(i\omega, \tau) &= e^{i\omega\tau} p(i\omega) + q(i\omega) + r(i\omega) e^{-i\omega\tau} = (p_R(i\omega) + ip_I(i\omega))(\cos(\omega\tau) \\ &\quad + i \sin(\omega\tau) + (q_R(i\omega) + iq_I(i\omega)) + (r_R(i\omega) + ir_I(i\omega))(\cos(\omega\tau) \\ &\quad - i \sin(\omega\tau) = \Delta_R(i\omega, \tau) + i\Delta_I(i\omega, \tau). \end{aligned} \quad (199)$$

Let

$$x := \cos\left(\frac{\omega\tau}{2}\right) \text{ and } y := \sin\left(\frac{\omega\tau}{2}\right), \quad (200)$$

then with straightforward calculations we can make the following transformations:

$$\begin{aligned} \Delta_R(i\omega, \tau) &= \cos(\omega\tau)(p_R + r_R) + q_R + \sin(\omega\tau)(r_I - p_I) \\ &= \left(\cos^2\left(\frac{\omega\tau}{2}\right) - \sin^2\left(\frac{\omega\tau}{2}\right)\right)(p_R + r_R) \\ &\quad + \left(\cos^2\left(\frac{\omega\tau}{2}\right) + \sin^2\left(\frac{\omega\tau}{2}\right)\right)q_R + 2\sin\left(\frac{\omega\tau}{2}\right) \cdot \cos\left(\frac{\omega\tau}{2}\right)(r_I - p_I) \\ &:= x(x(q_R + p_R + r_R) + y(r_I - p_I)) + y(x(q_R - p_R - r_R) + y(r_I - p_I)) \\ &:= x(A_x \cdot x + A_y \cdot y) + y(B_x \cdot x + B_y \cdot y) =: Ax + By. \end{aligned} \quad (201)$$

Furthermore we can write the imaginary part in the same way, too:

$$\begin{aligned} \Delta_I(i\omega, \tau) &= \cos(\omega\tau)(p_I + r_I) + q_I + \sin(\omega\tau)(r_R - p_R) \\ &= \left(\cos^2\left(\frac{\omega\tau}{2}\right) - \sin^2\left(\frac{\omega\tau}{2}\right)\right)(p_I + r_I) + \left(\cos^2\left(\frac{\omega\tau}{2}\right) + \sin^2\left(\frac{\omega\tau}{2}\right)\right)q_I \\ &\quad + 2\sin\left(\frac{\omega\tau}{2}\right) \cdot \cos\left(\frac{\omega\tau}{2}\right)(r_R - p_R) \\ &= x(x(q_I + p_I + r_I) + y(r_R - p_R)) + y(x(q_I - p_I - r_I) + y(r_R - p_R)) \\ &:= x(C_x \cdot x + C_y \cdot y) + y(D_x \cdot x + D_y \cdot y) =: Cx + Dy. \end{aligned} \quad (202)$$

Hence

$$e^{i\omega\tau}\Delta(i\omega, \tau) = 0 \iff \begin{bmatrix} A(\omega\tau) & B(\omega\tau) \\ C(\omega\tau) & D(\omega\tau) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0. \quad (203)$$

Since the coefficients A , B , C and D in the above matrix are expressed as the linear combination of x and y , we can write expressions $e^{i\omega\tau}\Delta_R(i\omega, \tau)$ and $e^{i\omega\tau}\Delta_I(i\omega, \tau)$ as follows

$$\begin{aligned} e^{i\omega\tau}\Delta_R(i\omega, \tau) &= A_x B_x x^2 + (A_x B_y + A_y B_x)xy + A_y B_y y^2 \\ &=: c_0 x^2 + c_1 xy + c_2 y^2 \end{aligned} \quad (204)$$

and

$$\begin{aligned} e^{i\omega\tau}\Delta_I(i\omega, \tau) &= C_x D_x x^2 + (C_x D_y + C_y D_x)xy + C_y D_y y^2 \\ &=: d_0 x^2 + d_1 xy + d_2 y^2. \end{aligned} \quad (205)$$

Then, by dividing the equalities $e^{i\omega\tau}\Delta_R(i\omega, \tau) = 0$ and $e^{i\omega\tau}\Delta_I(i\omega, \tau) = 0$ by y^2 , and introducing a new variable $u := x/y$, we obtain

$$\Delta_R(i\omega, \tau) = \frac{y^2(c_0 u^2 + c_1 u + c_2)}{e^{i\omega\tau}}, \quad \Delta_I(i\omega, \tau) = \frac{y^2(d_0 u^2 + d_1 u + d_2)}{e^{i\omega\tau}}. \quad (206)$$

Thus, since $e^{i\omega\tau}/y^2 \neq 0$, the equation $\Delta(i\omega, \tau) = 0$ has no real non-zero root for any given $\tau > 0$ if and only if the polynomials f and g have no common real non-zero root, where $f(u)$, resp. $g(u)$ denote the expressions for Δ_R , resp. Δ_I . This is equivalent to that $\text{res}(f, g) = \det(R[f, g]) \neq 0$ or if $\text{res}(f, g) = \det(R[f, g]) = 0$, then $\text{discr}[f] < 0$ and $\text{discr}[g] < 0$, where the resultant of the polynomials f and g is defined as

$$R[f, g] = \det \begin{bmatrix} c_0 & c_1 & c_2 & 0 \\ 0 & c_0 & c_1 & c_2 \\ d_2 & d_1 & d_0 & 0 \\ 0 & d_2 & d_1 & d_0 \end{bmatrix} = (c_0 d_0 - c_2 d_2)^2 + (c_2 d_1 - c_1 d_0)(c_0 d_1 - c_1 d_2), \quad (207)$$

and the discriminant of a polynomial $F(u) := au^2 + bu + c$ is $\text{discr}[F] := b^2 - 4ac$. Hence we have proved the following statement.

Theorem 1.12. The characteristic function (196) has not a non-zero root on the imaginary axis if and only if the polynomial $\Delta(\cdot, 0)$ is Hurwitz stable, and $\text{res}(f, g) = \det(R[f, g]) \neq 0$ or if $\text{res}(f, g) = \det(R[f, g]) = 0$, then $\text{discr}[f] < 0$ and $\text{discr}[g] < 0$, where

$$f(u) = c_0 u^2 + c_1 u + c_2 \quad \text{and} \quad g(u) = d_0 u^2 + d_1 u + d_0, \quad (208)$$

where c_0, c_1, c_2, d_0, d_1 and d_2 are defined in (204) and (205).

Example 8. Let us consider again system (166), i.e.

$$\dot{x} = Ax + x(\cdot - \tau) + By - \frac{1}{2}y(\cdot - \tau), \quad \dot{y} = x + x(\cdot - \tau) + Cy \quad (209)$$

with

$$A := -\left(1 + \sqrt{7}\right)/2, \quad B := \left(-4 + \sqrt{7}\right)/2, \quad C := \left(-3 + \sqrt{7}\right)/2. \quad (210)$$

Straightforward calculations show that for all $\omega \in \mathbb{R}$

$$\text{res}(f, g) = 8\omega^4(0.5 - \omega^2)(\omega^2 - 0.25) \cdot 0 = 0, \quad (211)$$

$$\text{discr}(f) = -16\omega^2 < 0, \quad \text{discr}(g) = 16\omega^2(\omega^2 - 0.5)^2 \geq 0. \quad (212)$$

Thus, the discriminant of g is not negative, therefore the stability of system (209) changes at some value τ^* of the delay (as we have seen in the previous example).

Figure 7 also shows the changing of the stability of system (209), with $\tau = 0$ the origin is asymptotically stable, but with $\tau = 2$ the origin changes to unstable. The solutions of system (209) with different values of the parameter τ can be seen on **Figure 7**. The stability of system (209) changes if the value of the parameter τ increases.

3.2 Hopf bifurcation

In this subsection we are going to see for which value of the delay τ could change the stability of the system (123). For this purpose we are going to give conditions on the coefficients p , q and r to obtain the value of the delay at which stability switch may occur. Let us assume that for $\omega^* > 0$ the conditions of Theorem 1.12 do not fulfill, i.e. for the polynomials f and g (defined in (206)) the resultant is equal to 0 and the discriminants of f or g is nonnegative. Furthermore let us assume that τ^* is

Figure 7.
Example: solutions with $\tau = 0$ and $\tau = 2$.

a solution of $\Delta(i\omega^*, \tau) = 0$. Let us denote by $z_3(\tau)$ the root of the quasi-polynomial (196) that assumes $z_3(\tau^*) = i\omega^*$ and the characteristic function $\Delta(\cdot; \tau)$ as a function of the parameter τ by

$$I(z, \tau) := p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \quad (z \in \mathbb{C}, \tau > 0). \quad (213)$$

Thus, we can determine the derivative of z_3 at τ^* by the Implicit Function Theorem (cf. [34]):

$$z'_3(\tau^*) = -\frac{\partial_\tau I(i\omega^*, \tau^*)}{\partial_z I(i\omega^*, \tau^*)} = \frac{\mathcal{E}(z, \tau)}{\mathcal{D}(z, \tau)} \Big|_{z=i\omega^*, \tau=\tau^*} \quad (214)$$

where $\mathcal{E}(z, \tau) := zq(z) + 2zr(z)e^{-z\tau}$ and $\mathcal{D}(z, \tau) := p'(z)e^{z\tau} + (q'(z) - \tau q(z)) + (r'(z) - 2\tau r(z))e^{-z\tau}$. To investigate and prove the occurrence of the Hopf bifurcation we have to see the sign of the real part of the above fraction. But since p, q and r are almost arbitrary polynomials, the fraction could be too complicated, that is why we introduce the following notation:

$$\frac{a + ib}{c + id} := \frac{\mathcal{E}(z, \tau)}{\mathcal{D}(z, \tau)} \Big|_{z=i\omega^*, \tau=\tau^*} \quad \text{with} \quad \Re\left(\frac{a + ib}{c + id}\right) = \frac{ac + bd}{c^2 + d^2}. \quad (215)$$

Since $c^2 + d^2 > 0$, it is enough to consider the sign of $ac + bd$. We are going to use the notations introduced in (198) and along the lines of these we introduce the following notations, too:

$$p'(i\omega) := p'_R(i\omega) + ip'_I(i\omega), \quad q'(i\omega) := q'_R(i\omega) + iq'_I(i\omega), \quad r'(i\omega) := r'_R(i\omega) + ir'_I(i\omega). \quad (216)$$

(For sake of simplicity replacing ω by ω^* we write $P(i\omega^*) := P$ for $P \in \{p_I, p_R, q_I, q_R, r_I, r_R\}$.)

Computing the exact value of a, b, c and d we have:

$$\begin{aligned} a &= 2\omega^* \left(r_R \sin(\omega^* \tau^*) - r_I \cos(\omega^* \tau^*) - \frac{1}{2} q_I \right), \\ b &= 2\omega^* \left(r_R \cos(\omega^* \tau^*) + r_I \sin(\omega^* \tau^*) + \frac{1}{2} q_R \right), \\ c &= (p'_R + r'_R - 2\tau^* r_R) \cos(\omega^* \tau^*) + (-p'_I + r'_I - 2\tau^* r_I) \sin(\omega^* \tau^*) + q'_R - \tau^* q_R, \\ d &= (p'_I + r'_I - 2\tau^* r_I) \cos(\omega^* \tau^*) + (p'_R - r'_R + 2\tau^* r_R) \sin(\omega^* \tau^*) + q'_I - \tau^* q_I. \end{aligned} \quad (217)$$

Thus,

$$\begin{aligned} ac + bd &= 2\omega^* \left\{ \frac{1}{2} (q_R q'_I - q_I q'_R) + \left[r_R q'_I - r_I q'_R + \frac{1}{2} (q_R (p'_I + r'_I) - q_I (p'_R + r'_R)) \right] \right. \\ &\quad \cdot \cos(\omega^* \tau^*) + \left[r_I q'_I + r_R q'_R + \frac{1}{2} (q_R (p'_R - r'_R) + q_I (p'_I - r'_I)) \right] \cdot \sin(\omega^* \tau^*) \\ &\quad + \cos^2(\omega^* \tau^*) [r_R (p'_I + r'_I) - r_I (p'_R + r'_R)] + \sin^2(\omega^* \tau^*) [r_I (p'_R - r'_R) - r_R (p'_I - r'_I)] \\ &\quad \left. + \sin(2\omega^* \tau^*) (r_R p'_R + r_I p'_I) \right\}. \end{aligned} \quad (218)$$

Using the elementary identities

$$\Re(z) \equiv \Re(\bar{z}) \quad \Im(z) \equiv -\Im(\bar{z}), \quad \Re(iz) \equiv -\Im(z) \quad \Im(iz) \equiv \Re(z), \quad (219)$$

furthermore the Euler formula $e^{\pm iz} \equiv \cos(z) \pm i \sin(z)$, we can simplify the numerator of $\Re(z'_3(\tau))$ as follows $ac + bd = \omega^* \cdot \Im(\mathfrak{A})$, where

$$\mathfrak{A} := \bar{q}q' + 2\bar{r}r' + (2\bar{r}q' + \bar{q}p')e^{i\omega^* \tau^*} + \bar{q}r'e^{-i\omega^* \tau^*} + 2\bar{r}p'e^{2i\omega^* \tau^*}. \quad (220)$$

Therefore, Hopf bifurcation occurs if $\operatorname{sgn}(\Im(\mathfrak{A})) = \pm 1$ holds.

4. Summary

The location of zeros of polynomials and quasi-polynomials as well is crucial in the point of view of the stability of ordinary and retarded differential equations. Namely, if the zeros of the characteristic polynomial of the linearized matrix lie in the open left half of the complex plane, then the constant solution of the particular equation is asymptotically stable. The main task of our work was to depict different methods which allow the investigation of the stability of characteristic (quasi-) polynomials, too. The second objective of this work was in the case of retarded differential equations to treat a method how to estimate the length of the delay for which no stability switching occurs. As an application, we showed a method to detect Hopf bifurcation in ordinary and retarded dynamical systems.

Author details

Sándor Kovács^{1*}, Szilvia György² and Noémi Gyúró²

¹ Department of Numerical Analysis, Eötvös Loránd University, Budapest, Hungary

² Eötvös Loránd University, Budapest, Hungary

*Address all correspondence to: alex@ludens.elte.hu

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