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Algebraic Approximations to Partial Group Structures

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Abstract

In this work, we use ‘Partial Group’ notion and we do further investigations about partial groups. We define ‘Partial Normal Subgroup’ using partial conjugation criteria and we prove few results about partial normal subgroups analogous to normal groups. Also, we define congruence relation for partial groups and via this relation, we state ‘The Quotient of Partial Group or Factor Group’. We give isomorphism theorems for partial groups. Explicitly, this is an analogous concept to group theory and our main is where differences partial groups from groups.

Keywords: partial group, partial normal subgroup, partial quotient group, isomorphism theorems for partial groups

1. Introduction

It is defined that a group is a set equipped with an operation described on it such that it has some properties as associated elements, an identity element, and inverse elements. Another definition can also be given as algebraic that the group is the set of all the permutations for algebraic expression’s roots that displays the typical that the assembly of the permutations pertains to the set.

If questions are “how was group theory developed?, What is the importance of group theory in science or real life? investigated for the mathematical topic group theory, then we can understand easily why we work on the structures of the theory of the many types of groups.

As we know from the literature, some primary sources are determined in the development of group theory such as Algebra (Lagrange in the 17. century), Number Theory (Gauss in the 18. century) (Euler’s product formula, Combinatorics, Fermat’s Last Theorem, Class group, Regular primes, Burnside’s lemma), Geometry (Klein, 1874), and Analysis (Lie, Poincaré, Klein in the 18. Century). It seems that three main areas have been described as Number Theory and Algebra (Galois theory, equation with degree 5, Class field theory), Geometry (Torus, Elliptic curves, Toric varieties, Resolution of singularities), and analysis in mathematics. Topology (((co)homology groups, homotopy groups) and algebraic part of it (Eilenberg–MacLane spaces, Torsion subgroups, Topological spaces), the Theory of Manifolds (manifolds with a metric), Algebra, Dynamical systems, Engineering (to create digital holograms), Combinatorial Number Theory, Mathematical Logic, Geometry in Riemannian Space, and Lie Algebra also belongs these three subjects.

Group theory is used not just in mathematics but also in computer science, physics, chemistry, engineering, and other sciences. Especially symmetry has a big potential property in the group theory. That is why it is considered as representation theory in physics. For example; mathematical works on quantum mechanics were done by von Neumann, Molecular Orbital Theory. Also, the Standard Model of particle physics, the equations of motion, or the energy eigenfunctions use group theory for their orbitals, classify crystal structures, Raman and infrared spectroscopy, circular dichroism spectroscopy, magnetic circular dichroism spectroscopy or getting periodic tables-gauge theory, the Lorentz group-the Poincaré's group in modern chemistry or physics. Also, group theory is defined as representation theory in physics. A lot of groups with prime caliber built-in cryptography for elliptic curve do a service for public-key cryptography and Diffie–Hellman key exchange takes advantage of cyclic groups (especially finite) too. Additionally, cryptographic protocols also consider infinite nonabelian groups.

We can state the applications of the group theory also in real life as follows:

1. Shopping online (we use our credit card with encryption which is obtained by group structure in RSA algorithm)
2. Music (Elementary group theory is used for the 12-periodicity in the circle of fifths in musical set theory. Transformational theory patterns musical transformations as if they are elements of a mathematical group, cyclic groups create octave and other notions, the musical actions of the dihedral groups.)
3. Medical science (to find out breast cancer) and computer science (robotics computer vision and computer graphics) and material sciences.
4. Machine learning, communication network, signal processing, etc. ...
5. Pipeline system, which is described as the Application - Business Object - Network Node Layer, is patterned and investigated by the theory of group. These systems are also related with vectors, matrices determined by group structures, and so on.

Thus, tools of the group theory are useful for working on applications in many different sciences and also real life as mentioned above.

Basic and simple examples can be given for usual groups as follows:

- Vector spaces $(V, +)$ have group structures under the addition of vectors with some properties of the scalar multiplication $*$.
- For p primes, elements of $\mathbb{Z}_p^* \subseteq \mathbb{Z}_p$ have algebraic structures under multiplication with unit 1.
- Assume that F be a number field and m is a natural number. Then $S = SL(m, F)$ can be determined to be the set of all regular $m \times m$ matrices with logins in F . This is a usual group with $a * b$ described by the multiplication of matrices.

We can ask readers “whether or not these examples are partial group”?

S_m is demonstrated by symmetric groups such that it includes $m!$ permutations where m objects are taken from a set A . As an illustration, we can give the symmetric group S_3 . Supposing that $A = \{a, b, c\}$ and S_3 contains following objects; identity element.

$$\text{Identity element} = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \text{ and others are;} \\ \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & a \\ c & a & b \end{pmatrix}$$

There are some properties in finite or infinite usual group theory. Some of them can be seen as follows:

- Each of the elements in the finite group has finite order.
- If H is a finite group, then it is satisfied for each of the subgroups of H .
- Assuming that X, Y are groups. Then, the product of them is defined by $X \times Y = \{(a, b) | a \in X, b \in Y\}$ with unit element $e := (e_X, e_Y)$, where we write e_X, e_Y are identity element and inverse elements are in the form of (a^{-1}, b^{-1}) of in X, Y , respectively.
- Suppose that X be a usual group and a in X . So, the cyclic group $\langle a \rangle$ becomes a subgroup of X . As an illustration, we can say that $(\mathbb{Z}_m, +)$ is a cyclic group that satisfies $\langle 1 \rangle = \mathbb{Z}_m$.
- If X be a usual group with order p (p is prime), X is also a cyclic group.
- X is abelian usual group iff center of X equals to X .
- Let us consider S_m and its two permutations. These are conjugate iff their cycle types are equal/same according to their ordering.
- ...
-

If literature (briefly, references [1–47]) is investigated, then it is easily seen that partial groups are considered as topological structures more than algebraic structures. It is tried to prepare some new algebraic perspectives/approximations for the partial group. As we know, there are many algebraic infrastructures such as finite-infinite group, abelian-nonabelian group, quaternion group, symmetric group, cyclic group, simple group, free group, orbits and stabilizers of the group, Lie group, and various kinds of theorems such as Sylow theorems, Cauchy theorem, Lagrange theorem, Cayley theorem, Isomorphism theorems as well as actions of groups for usual group theory.

An effect algebra is introduced in the foundations of mechanics [1]. Furthermore, effect algebra subjects are fundamental in fuzzy probability theory [2, 3]. Also, partial group is defined by [4] and used for topological and homological investigations. A pregroup can be defined as following:

A pregroup, [5], of a set P containing an element 1, each element $p \in P$ has a unique inverse p^{-1} and to each pair of elements $p, t \in P$ there is defined at most one product $pt \in P$ so that;

- a. $1 * p = p * 1 = p$ is always defined,
- b. $p * p^{-1} = p^{-1} * p = 1$ is always defined,

- c. If $p * t$ is defined then $t^{-1} * p^{-1}$ is defined and equal to $(p * t)^{-1}$.
- d. If $r * p$ and $p * t$ are defined then either $r * (p * t)$ is defined if and only if $(r * p) * t$ is defined in which case two are equal.
- e. If $q * r, r * p$ and $p * t$ are defined then either $q * (r * p)$ is defined or $r * (p * t)$ is defined.

Every pregroup is a partial group, but the converse is not true in general. In that meaning a partial group definition can be stated as follows:

A set P is a partial group in the meaning of ([6], Lemma 4.2.5) if each associated pair $(x, y) \in P \times P$ there is at most one product $x.y$ so that:

1. There is an element $1 \in P$ satisfying $x.1 = 1.x = x$ for each $x \in P$.
2. For each $x \in P$ there exists an element x^{-1} so that $x.x^{-1} = x^{-1}.x = 1$
3. If $x.y = z$ is defined so is $y^{-1}.x^{-1} = z^{-1}$.

Inspiring by the groupoid and effect algebra [7] gave an alternative partial group definition as an algebraic style. They introduced partial subgroup, partial group homomorphism, etc. as an analog investigation for group theory. Moreover, readers can learn/consider a lot more structural results on the subject from others [8–47].

In this work, some (remained ones can be considered from readers using our works) fundamental results are given for partial groups. Similarities and differences have been noticed between usual groups and partial groups. Several of them also are described in this work.

Also, we do further investigations for partial groups in algebraic style. We define partial normal subgroup and give isomorphism theorems for partial groups. This work is important because it has both topological and algebraic applications. It can be expanded to rings or other algebraic structures.

2. Preliminary results

Recently, an algebraic structure named as partial group (also known as Clifford Semigroup is isomorphic to an explicit partial group of partial mappings and it is a semigroup with central idempotents) is investigated with new structures in the literature.

A partial group (Clifford semigroup) is a regular semigroup (it means if $\mathcal{M}$ is a semigroup of group G then idempotent elements of $\mathcal{M}$ exchange with H 's all elements). Another definition can be given for the partial group as “A regular semigroup with its central idempotents is named by Clifford semigroup.”

Additionally, several partial algebras such as partial monoid, partial ring, partial group ring, partial quasigroup etc. ... have been worked. For example, Jordan Holder Theorem of composition series is known to hold in every abelian category. The classical theory of subnormal series, refinements, and composition series in groups is extended to the class of partial groups which is known to be precisely the classes of Clifford semigroups, or equivalently semilattices of groups. Also, relations among the language theory, words, partial groups, universal group, and homology theory have been considered with an arrow diagram of the partial group.

In this chapter, we first state some basic properties of partial groups which are mentioned in several references.

Definition 2.1 [7] Suppose G^* is a nonempty set: G^* is called as a partial group if the following conditions hold for all x, y , and $z \in G^*$:

(G₁) If xy , $(xy)z$, yz and $x(yz)$ are defined, then the equality $(xy)z = x(yz)$ is valid.

(G₂) For each $x \in G^*$, there exists an $e \in G^*$ such that xe and ex are defined and the equality $xe = ex = x$ is valid.

(G₃) For each $x \in G^*$, there exists an $x' \in G^*$ such that xx' and $x'x$ are defined and the equality $xx' = x'x = e$ is valid.

The $e \in G^*$ satisfies (G₂) is called the identity element of G^* and the $x' \in G^*$ satisfies (G₃) is called the inverse of x and denoted by x^{-1} .

Another way, we can give partial definition as follows:

Definition 2.2. Let $\mathcal{M}$ be a semigroup. It is called a partial group if the followings are held.

- i. Every $k \in \mathcal{M}$ has a partial identity e_k
- ii. Every $k \in \mathcal{M}$ has a partial inverse k^{-1}
- iii. Mapping $e_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}, k \mapsto e_k$ is a semigroup homomorphism
- iv. Mapping $\beta : \mathcal{M} \rightarrow \mathcal{M}, k \mapsto k$ is a semigroup antihomomorphism.

Note:

- i. Let $\mathcal{M}$ be a semigroup. A partial identity of k , when exists, is unique and idempotent such that denoted by e_k .
- ii. Let $\mathcal{M}$ be a semigroup. A partial inverse of $k \in \mathcal{M}$, when exists, is uniquely denoted by k^{-1} .

Definition 2.3. The regular element of the $\mathcal{M}$ semigroup is defined if there exists $s \in \mathcal{M}$ such that $ysy = y$. Each element of $\mathcal{M}$ is regular element also $\mathcal{M}$ is named by regular semigroup.

Definition 2.4. $\mathcal{M}$ semigroup is called as completely regular semigroup for every element $s \in \mathcal{M}$ $ysy = y$ and $ys = sy$ are satisfied.

Note. Unions of groups give us completely regular semigroups which are named by Clifford semigroups.

Definition 2.5. Let $\mathcal{M}$ be a semigroup. Elements k and s of a semigroup $\mathcal{M}$ are said to be inverse of each other if and only if $sk = s$ and $ks = k$.

Then following theorem can be given from the literature.

Theorem 2.1. The following results are equivalent to each other for a semigroup $\mathcal{M}$:

- i. $\mathcal{M}$ is a Clifford semigroup,
- ii. There exists $r \in \mathcal{M}$ such that $wrw = w$ and $wr = rw$ for every $w \in \mathcal{M}$,
- iii. $\mathcal{M}$ is a semilattice of groups,
- iv. $\mathcal{M}$ is a completely regular inverse semigroup.

Proposition 2.1. $\mathcal{M}$ is a completely regular semigroup iff there are e_k and k^{-1} for every $k \in \mathcal{M}$.

Proposition 2.2 [7] Every group is a partial group and every partial group which is closed under its partial group operation is a group.

Proposition 2.3. Assuming that $\mathcal{M}$ is a partial group. Then, the following are given:

- Every idempotent element in partial group $\mathcal{M}$ is its own partial identity and partial inverse,
- Let $k \in \mathcal{M}$. Then, $e_k^{-1} = e_k = (e_k)^{-1}$ is hold.
- Suppose that $k \in \mathcal{M}$. $(k^{-1})^{-1} = k$ is satisfied.

Example 2.1 [7] Following sets with the given operation, can be seen as an example to the partial groups:

1. Let $G = \{0, \pm 1, \dots, \pm n\}$ where $n \in \mathbb{Z}^+$ and $+$ be known addition operation on $\mathbb{Z}$. Then it is easily seen that G is a partial group but is not a group.
2. Let $G = \mathbb{Z}^* \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. So it is obvious that G is a partial group but is not a group by the known multiplication on $\mathbb{R}$.
3. Let $G = [-r, r]$ where $r \in \mathbb{R}^+$ and $+$ be known addition operations on $\mathbb{R}$. Then it is obvious G is a partial group but is not a group.

Definition 2.6 [7] Suppose G^* be a partial group, $m \in \mathbb{Z}^+$ and $a \in G^*$. If a^m is defined and m is the least integer such that $a^m = e$, the number m is called the order of a . In this case, it is called that a has a finite order element. If there does not exist an $m \in \mathbb{Z}^+$ such that $a^m = e$, (if only $a^0 = e$); then it is called that a has infinite order. The order of a is denoted by $|a|$.

Example 2.2 [7] $G = \{1, -1, i, -i, 2i, -\frac{i}{2}\}$ with the multiplication operation on $\mathbb{C}$ is a partial group and $|i| = 4, |2i| = \infty$.

Definition 2.7 [7]. Suppose G^* be a partial group. $Z(G^*) = \{x \in G^* | \text{If } ax \text{ and } xa \text{ are defined for all } a \in A; ax = xa\}$ is called the center of G^* .

Lemma 2.1 [7] A partial group is called *centerless* if $Z(G^*)$ is trivial i.e., consists of only the identity element. If G^* is commutative then $G^* = Z(G^*)$.

Definition 2.8. Supposing that $\mathcal{M} = G_p$ be a partial group and $\mathbb{T} = H_p$ be a subset of $\mathcal{M}$. $\mathbb{T}$ is called by sub partial group of $\mathcal{M}$ if $\mathbb{T}$ is a sub semigroup of $\mathcal{M}$ and e_k, k^{-1} are in $\mathbb{T}$ for all $k \in \mathbb{T}$.

Especially, $\mathcal{M}$ and the set of idempotents elements of $\mathcal{M}$ are sub partial groups of $\mathcal{M}$.

Definition 2.9 [7] Let G^* be a partial group and H^* be a nonempty subset of G^* . If H^* is a partial group with the operation in G^* then H^* is called a partial subgroup of G^* .

Example 2.3 [7] In Example 2.2 the set $G^* = \{0, \pm 1, \dots, \pm n\}$ where $n \in \mathbb{Z}^+$ and $+$ be known addition operation on $\mathbb{Z}$ is a partial group and let $H^* = \{0, \pm 1, \dots, \pm k\}$ where $0 \leq k \leq n$ and $k \in \mathbb{Z}$. Then H^* is a partial subgroup of G^* .

Lemma 2.2 [7] Let G^* be a partial group and H^* be a nonempty subset of G^* . H^* is a partial subgroup of G^* if and only if the following conditions hold:

- i. $e \in H^*$;
- ii. $a^{-1} \in H^*$ for all $a \in H^*$.

Moreover, Let G^* be a partial group and let a be an element of G^* such that the elements $\{a^k \text{ for all } k \in \mathbb{Z}\}$ are defined. Denote $\{a^k; k \in \mathbb{Z}\} = \langle a \rangle$. It is clear that the

.	<i>e</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>e</i>	<i>e</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>e</i>
<i>b</i>	<i>b</i>	<i>c</i>	<i>e</i>	<i>a</i>
<i>c</i>	<i>c</i>	<i>e</i>	<i>d</i>	<i>b</i>

Table 1.
(G) is a partial group even it has not a group structure.

set $\langle a \rangle$ is a partial subgroup of G^* . The partial subgroup $\langle a \rangle$ of G^* is called the cyclic partial subgroup generated by a . If there exists an element a in G^* such that $\langle a \rangle = G^*$, then G^* is called a cyclic partial group.

Example 2.4 [7] Let $G = \{e, a, b, c\} \subset S = \{e, a, b, c, d\}$ and “.” be a partially defined operation on G as in **Table 1**.

Remark. Note that $c.b$ is undefined. Then G is not a group but it is a partial group. Additionally, in this partial group, $\langle a \rangle = G$, G is the cyclic partial group. But all partial subgroups of a cyclic partial group can not be cyclic. For instance, the partial subgroup $H = \{e, a, c\}$ is not cyclic. But in group theory, if a group is cyclic, all subgroups of it are also cyclic. The partial groups are different from groups in that meaning.

Definition 2.10. Assume that $\mathcal{M} = G_p$ be a partial group and $k \in \mathcal{M}$. Then, we define $\mathcal{M}_k = \{s \in \mathcal{M} : e_k = e_s\}$.

Theorem 2.2. Suppose that $\mathcal{M}$ is a partial group and $k \in \mathcal{M}$. Then, $\mathcal{M}_k$ is a maximal subgroup $\mathcal{M}$ of which has identity e_k and $\mathcal{M} = \bigcup \{\mathcal{M}_k : k \in \mathcal{M}\}$.

Definition 1.11 [7] Let $\mathcal{M}$ and $\mathfrak{N}$ be partial groups. A function $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ is called a partial group homomorphism if for all $a, b \in \mathcal{M}$ such that ab is defined in $\mathcal{M}$, $\sigma(a)\sigma(b)$ is defined in $\mathfrak{N}$ and

$$\sigma(ab) = \sigma(a)\sigma(b).$$

If σ is injective as a map of sets, σ is said to be a monomorphism. If σ is surjective, σ is called an epimorphism.

Definition 2.12. For a partial group homomorphism $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ it is defined $\ker \sigma = \{k \in \mathcal{M} : \sigma(k) = e_{\sigma(k)}\}$ and $Im \sigma = \{\sigma(k) : k \in \mathcal{M}\}$. Also, $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ is named isomorphism if it is bijective.

As a consequence of the definition the following lemmas can be given:

Definition 2.13. Suppose that $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ be a partial group homomorphism. Then, we define $\ker \sigma = \{k \in \mathcal{M} : \sigma(k) = e_{\sigma(k)}\}$ and $Im \sigma = \{\sigma(k) : k \in \mathcal{M}\}$.

Theorem 2.3. Assuming that $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ be a partial group homomorphism and $k \in \mathcal{M}$. Then, the following are given.

- i. $\sigma(e_k) = e_{\sigma(k)}$.
- ii. $\sigma(k^{-1}) = \sigma(k)^{-1}$.
- iii. $\ker \sigma$ is a subpartial group of $\mathcal{M}$.
- iv. $Im \sigma$ is a subpartial group of $\mathfrak{N}$.
- v. $\sigma(\mathcal{M}_k)$ is a subpartial group of $\mathfrak{N}_{\sigma(k)}$.
- vi. $\sigma^{-1}(\mathfrak{N}_{e_k})$ is a subpartial group of $\mathcal{M}$.

Proposition 2.4 [7] Suppose $\mathcal{M}, \mathfrak{N}$ be partial groups and $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ be a homomorphism of partial groups. Then the following conditions are satisfied:

- i. If A is a partial subgroup of $\mathcal{M}$, then $\sigma(A)$ is a partial subgroup of $\mathfrak{N}$.
- ii. If B is a partial subgroup of $\mathfrak{N}$, then $\sigma^{-1}(B)$ is a partial subgroup of $\mathcal{M}$.

Proposition 2.5. Let $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ be a homomorphism of partial groups. Then, it is obtained that

$$\sigma(e_k) = e_{\sigma(k)}, \sigma(k^{-1}) = (\sigma(k))^{-1} \text{ for all } k \in \mathcal{M}.$$

Definition 2.14. A sub partial group $\mathbb{T} = H_p$ of $\mathcal{M} = G_p$ is named wide if the set of idempotent elements of $\mathcal{M}$ is a subset of $\mathbb{T} = H_p$ and normal, written $\triangleleft \mathcal{M}$. (if it is wide and $k\mathbb{T}k^{-1} \subseteq \mathbb{T}$ for all $k \in \mathcal{M}$). It is also trivial that the set of idempotents elements of $\mathcal{M}$ is a normal subgroup of $\mathcal{M}$ and it is called the set of idempotents elements of $\mathcal{M}$ the trivial normal subpartial group of $\mathcal{M}$.

Theorem 2.4. Assuming that $\mathcal{M}$ be a partial group and $k \in \mathcal{M}$, then

- i. $\mathcal{M}_k$ is a maximal subgroup of $\mathcal{M}$ with identity e_k .
- ii. $\mathcal{M} = \bigcup \{\mathcal{M}_k : k \in \mathcal{M}\} = \bigcup \{\mathcal{M}_{e_k} : e_k \text{ is in the set of idempotents elements of } \mathcal{M}\}.$

Theorem 2.5. If $\mathcal{M}$ is a partial group, then the set of idempotents elements of $\mathcal{M}$ is commutative and central.

Definition 2.15. For a partial group homomorphism $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ it is defined $\ker \sigma = \{k \in \mathcal{M} : \sigma(k) = e_{\sigma(k)}\}$ and $\text{Im } \sigma = \{\sigma(k) : k \in \mathcal{M}\}$. Also, $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$ is named isomorphism if it is bijective.

Definition 2.16. σ is named as idempotent separating if $\sigma(e_k) = \sigma(e_s)$ implies that $e_k = e_s$, where e_k, e_s are in the set of idempotent elements of $\mathcal{M}$ for a partial group homomorphism $\sigma : \mathcal{M} \longrightarrow \mathfrak{N}$.

3. Partial normal subgroups

In group theory, normal subgroup plays an important role in the classification of groups and gives lots of algebraic results. Now, we will construct an analog definition for partial groups. Throughout G_p will denote the partial group. In this chapter, we should notice that if G is a group, then G is a partial group with $\text{fect}\{e\}$. Also, [8] in a group every element has a unique inverse, but in partial groups [7] for every element $a \in G$, we have $\text{Inv}(a) \neq 0$ because of that reason the identity element of the group differs from the identity element of the partial group. We can continue to work under these assumptions. From here on in, we will use the notation G_p for partial groups.

Definition 3.1 (Partial Conjugation Criteria). Let G_p be a partial group, the element $g_p \cdot x_p \cdot g_p^{-1}$ (or $g_p^{-1} \cdot x_p \cdot g_p$) is called partial conjugate of x_p by g_p for fixed $g_p, x_p \in G_p$.

Theorem 3.1. Let G_p be a partial group and N_p be a partial subgroup of G_p then following conditions are satisfied:

- i. N_p is normal in G_p if and only if for all $x_p \in N_p$ and $g_p \in G_p$ we have $g_p^{-1} \cdot x_p \cdot g_p \in N_p$

- ii. N_p is normal in G_p if and only if for every element of N_p all partial conjugates of that element also lie in N_p .

Proof:

- i. It comes from the definition of partial normal subgroup.

- ii. ($\Rightarrow$): Let N_p is partial normal subgroup in G_p . Then we need to show for every $x_p \in N_p$ and fixed $g_p \in N_p$ the partial conjugates $g_p^{-1} \cdot x_p \cdot g_p$ lies in N_p . Since $g_p \in N_p$ then $g_p \in G_p$. Also, since N_p is a partial normal subgroup of G_p $g_p^{-1} \cdot x_p \cdot g_p \in N_p$, this gives the proof.

($\Leftarrow$): Conversely, let for every element of N_p all partial conjugates of that element lie in N_p : Then it comes directly from partial subgroup definition Conclusion(s). It is preferable to include a Conclusion(s) section which will summarize the content of the book chapter.

Remark 3.1. Any partial subgroup $H_p \subseteq G_p$ has right and left congruence (equivalence) class that cannot be the same. But if left and right congruence classes are the same (i.e., for any $x \in G_p$, $H_p \cdot x = x \cdot H_p$) then H_p is called as normal partial subgroup.

Theorem 3.2. Let G_p be a partial group and N_p be a partial subgroup of a partial group G_p and so the following conditions are coincided:

Proposition 3.1.

- i. N_p is a partial normal subgroup of a partial group G_p .
 ii. $g_p \cdot N_p = N_p \cdot g_p$ for all $g_p \in G_p$,
 iii. $g_p \cdot N_p \cdot g_p^{-1} \subseteq N_p$ for all $g_p \in G_p$.

Proof:

(i) $\Rightarrow$ (ii) If N_p is a partial normal subgroup of G_p then it is easy that for all $x_p \in N_p$ and $g_p \in G_p$ we have $g_p \cdot x_p \cdot g_p^{-1} \in N_p$ and from just before the theorem $g_p \cdot N_p \cdot g_p^{-1} \in N_p$.

(iii) $\Rightarrow$ (ii) Let g_p be an element of G_p : We need to see $g_p \cdot N_p = N_p \cdot g_p$. Assume that $x_p \in g_p \cdot N_p$. Then $x_p = g_p \cdot n_1$ is satisfied for $n_1 \in N_p$. Since $x_p \cdot g_p^{-1} = g_p \cdot n_1 \cdot g_p^{-1}$ and $g_p \cdot N_p \cdot g_p^{-1} \subseteq N_p$ we have $x_p \cdot g_p^{-1} \in N_p$ and so that there exists an $n_2 \in N_p$ such that $x_p \cdot g_p^{-1} = n_2$. If we product from right with g_p then we have the $(x_p \cdot g_p^{-1}) \cdot g_p = n_2 \cdot g_p$ equality.

Using the associativity property the equality becomes $x_p \cdot (g_p^{-1} \cdot g_p) = n_2 \cdot g_p$ and using identity element property and converse element property we get $x_p = n_2$. $g_p \in N_p \cdot g_p$. It implies that $g_p \cdot N_p \subseteq N_p \cdot g_p$. In a similar way $g_p^{-1} \cdot N_p \cdot g_p \subseteq N_p$ and we have $N_p \cdot g_p \subseteq g_p \cdot N_p$. Therefore we obtain $g_p N_p = N_p \cdot g_p$.

(ii) $\Rightarrow$ (i) Supposing that $g_p \in G_p$ we have to prove $g_p \cdot n_p \cdot g_p^{-1}$ is contained in N_p for all $n_p \in N_p$. Since $g_p \cdot N_p = N_p \cdot g_p$, we can say that $g_p \cdot n_p \in N_p \cdot g_p$ for all $n_p \in N_p$. Then, by associativity $g_p \cdot n_p \cdot g_p^{-1}$ is contained in $N_p \cdot g_p \cdot g_p^{-1}$ and then for all $n_p \in N_p$, $g_p \cdot n_p \cdot g_p^{-1} \in N_p$.

Lemma 3.1. The center of a partial group G_p is a partial normal subgroup of G_p .

Proof: The center of a partial group is defined as below:

$Z(G_p) = \{x_p \in G_p \mid \text{If for every } g_p \in G_p \text{ } x_p \cdot g_p \text{ and } g_p \cdot x_p \text{ are defined, } x_p \cdot g_p = g_p \cdot x_p\}$.

Let $g_p \in G_p$ and $x_p \in Z(G_p)$ then we need to show $g_p \cdot x_p \cdot g_p^{-1}$ is contained in $Z(G_p)$. Since $x_p \in Z(G_p)$ for every $g_p \in G_p$ if $g_p \cdot x_p$ and $x_p \cdot g_p$ is defined then $x_p \cdot g_p = g_p \cdot x_p$. Using this argument $g_p \cdot x_p \cdot g_p^{-1} = x_p \cdot g_p \cdot g_p^{-1} = x_p \in Z(G_p)$ and then we have $Z(G_p)$ is a partial normal subgroup of G_p .

Proposition 3.2. Let $\varphi: G_p \rightarrow H_p$ be a partial group homomorphism. Then the kernel of φ is partial normal subgroup of G_p .

Proof: Let $K = \text{Ker}(\varphi)$. We know that $\text{Ker}(\varphi)$ is a partial subgroup of G_p . Suppose that $y \in K$ and $g_p \in G_p$. Then using the fact φ is a partial group homomorphism

$$\begin{aligned}\varphi(g_p \cdot y_p \cdot g_p^{-1}) &= \varphi(g_p) \cdot \varphi(y_p) \cdot \varphi(g_p^{-1}) \\ &= \varphi(g_p) \cdot e_{H_p} \cdot \varphi(g_p^{-1}) \\ &= \varphi(g_p) \cdot \varphi(g_p^{-1}) \\ &= e_{H_p} \text{ we have } \varphi(g_p \cdot y_p \cdot g_p^{-1}) = e_{H_p}\end{aligned}$$

and we get $g_p \cdot y_p \cdot g_p^{-1} \in K$. So $\text{Ker}(\varphi)$ is a partial normal subgroup of G_p .

Partial normal subgroups

Theorem 3.3. Suppose G_p is a partial group and H_p is a partial subgroup of G_p . H_p is a partial normal subgroup of G_p if and only if $(aH_p)(bH_p) = abH_p$ equality holds for all $a, b \in H_p$.

Proof:

($\Rightarrow$): Suppose H_p is a partial normal subgroup of G_p . We need to see the equality $(aH_p)(bH_p) = abH_p$ holds.

($\subseteq$): Let $x \in (aH_p)(bH_p)$ then $\exists h_1, h_2 \in H$ such that $x = (ah_1)(bh_2)$. By using associativity property, we get $x = (h_1bh_2)$. From the identity element, $x = abb^{-1}h_1bh_2$ is obtained. Considering associativity property we can write $x = ab(b^{-1}h_1b)h_2$. Since H_p is a partial normal subgroup of G_p , $b^{-1}h_1b \in H_p$. Then $x \in abH_p$ and this implies that $(aH_p)(bH_p) \subseteq abH_p$.

($\supseteq$): Conversely, let $y \in abH_p$ then there exists an $h \in H_p$ such that $y = abh$. So we can write $y = abh$ as follows; $y = (ae)(bh) \in (aH_p)(bH_p)$. It implies that $abH_p \subseteq (aH_p)(bH_p)$. Therefore $(aH_p)(bH_p) = abH_p$.

($\Leftarrow$): Let us consider $(aH_p)(bH_p) = abH_p$ for all $a, b \in G_p$. If $h \in H_p$ and $g \in G_p$ then we must see whether or not $ghg^{-1} \in H$. Using associativity property and considering hypothesis; $ghg^{-1} = (gh)(g^{-1}e) \in (gH_p)(g^{-1}H_p) = gg^{-1}H_p = H_p$. This implies that $ghg^{-1} \in H_p$. So that H_p is a partial normal subgroup of G_p .

Theorem 3.4. Suppose G_p be a partial group and H, K are partial subgroups of G_p . If K is a partial normal subgroup of G_p , then the following cases are satisfied:

- i. $H \cap K$ is a partial normal subgroup of H .
- ii. If K and H are partial normal subgroups of G_p and $H \cap K = \{e\}$ then $hk = kh$ (or, $HK = KH$) for every $h \in H$ and every $k \in K$.

Proof:

- i. Since H and K are partial subgroups, $H \cap K$ is also a partial subgroup of G . By $H \cap K \subseteq K$ we can conclude that $H \cap K$ is also a partial subgroup of H . Let consider $a \in H \cap K$ and $h \in H$: Since $a, h \in H$, ha and hah^{-1} can be defined. It gives that $hah^{-1} \in H$. Also since K is a partial normal subgroup of G_p we have $hah^{-1} \in K$. It shows that $H \cap K$ is a partial normal subgroup of H .

- ii. Let H be a partial normal subgroup of G_p , $H \cap K = \{e\}$ and $h \in H$ and $k \in K$. Since H is a partial normal subgroup, we know that $k \cdot h \cdot k^{-1} \in H$. If $h \cdot k \cdot h^{-1} \cdot k^{-1} \in H$ then we get $(h \cdot k \cdot h^{-1}) \cdot k^{-1} \in K$ by K is a partial normal subgroup. So

we have, $h.k.h^{-1}.k^{-1} \in H \cap K = \{e\}$. Then $h.k.h^{-1}.k^{-1}=e$ and so, $hk = kh$; for all $h \in H, k \in K$. Thus, we prove that $HK = KH$.

Proposition 3.3. Let G_p be a partial group and H_p be a partial subgroup of index 2 in G_p : Then H_p is a partial normal subgroup in G_p .

Proof: Let H_p be a partial subgroup of index 2 in G_p and g_p be an element of G_p . If $g_p \in H_p$, then $g_p H_p = H_p g_p$ is satisfied. If g_p is not in H_p , two left cosets must be as H_p and $g_p H_p$. Since left cosets are disjoint we know $g_p H_p = G_p - H_p$. Also, the right cosets are disjoint so we can write $H_p g_p = G_p - H_p$. Thus $g_p H_p = H_p g_p$ for all $g_p \in G_p$. So H_p is normal.

Example 3.1. Let G_p be an abelian partial group. Then any subgroup of G_p is a partial normal subgroup of G_p .

Proof:

If G_p is an abelian partial group and $x_p, y_p \in G_p$ then $x_p y_p = y_p x_p$ for every $x_p, y_p \in G_p$. If $g_p x_p g_p^{-1} \in N_p$ then N_p is a partial subgroup of G_p . Using the hypothesis we get

$$\begin{aligned} g_p x_p g_p^{-1} &= g_p g_p^{-1} x_p \\ &= e G_p \\ &= x_p \in N_p \end{aligned}$$

Therefore, the partial subgroup N_p is the partial normal subgroup of G_p .

Example 3.2. Let H_p and K_p be any partial normal subgroup of G_p . Then $H_p \times K_p$ is also a partial normal subgroup of $G_p \times G_p$.

Proof:

$H_p \times K_p = \{n_p = (h_p, k_p) \mid h_p \in H_p \text{ and } k_p \in K_p\}$. We have to show that for all g_p in G_p and n_p in $H_p \times K_p$ $g_p n_p g_p^{-1}$ is in $H_p \times K_p$.

$$\begin{aligned} g_p n_p &= g_p \cdot (h_p, k_p) \\ &= (g_p h_p, g_p k_p) \end{aligned}$$

and

$$\begin{aligned} g_p n_p g_p^{-1} &= (g_p h_p, g_p k_p) \cdot g_p^{-1} \\ &= (g_p h_p g_p^{-1}, g_p k_p g_p^{-1}) \end{aligned}$$

and since H_p and K_p are partial normal subgroups of G_p , then $g_p h_p g_p^{-1} \in H_p$, and $g_p k_p g_p^{-1} \in K_p$ and so that $(g_p h_p g_p^{-1}, g_p k_p g_p^{-1}) \in H_p \times K_p$. Then the Cartesian product of two partial normal subgroups is also a partial normal subgroup.

Theorem 3.5. Let be a partial group and N_p be a partial normal subgroup of G_p . The congruence modulo N_p is a congruence relation for the partial group operation “.”.

Proof. Let $x \mathcal{R} N_p y$ denote that x and y are in the same coset, that is;
 $x \mathcal{R} N_p y \iff x.N_p = y.N_p$

Let $x \mathcal{R} N_p x$ and $y \mathcal{R} N_p y$. To demonstrate that $\mathcal{R} N_p$ is a congruence relation for ., we need to show, reflexivity, symmetry, and transitivity. These axioms are obvious from the definition of relation.

Theorem 3.6. If N_p is a partial normal subgroup of a partial group G_p and G_p/N_p is the set of all cosets of N_p in G_p , then G_p/N_p is a partial group under the operation given by $(aN_p)(bN_p) = abN_p$.

Proof. Let $aN_p, bN_p, cN_p \in G_p/N_p$. We must see partial group axioms are satisfied:

(G1) If $(aN_p)(bN_p) = abN_p$, $(bN_p)(cN_p) = bcN_p$ and $a.(bcN_p)$ are defined then

$$\begin{aligned} a.(bcN_p) &= (aN_p)((bN_p)(cN_p)) \\ &= ((aN_p)(bN_p))(cN_p) \\ &= (ab)cN_p \end{aligned}$$

(G2) For any aN_p in G_p/N_p , eN_p is a candidate for identity element, i.e.,

$$\begin{aligned} (aN_p)(eN_p) &= aeN_p \\ &= aN_p \end{aligned}$$

and

$$\begin{aligned} (eN_p)(aN_p) &= eaN_p \\ &= aN_p \end{aligned}$$

(G3) Since is a partial group $a \in G_p$ has an inverse $a' \in G_p$. For every aN_p in G_p/N_p $a'N_p$ is a candidate for the inverse of aN_p .

$$\begin{aligned} (aN_p)(a'N_p) &= (aa')N_p \\ &= eN_p \\ &= N_p \end{aligned}$$

This completes the proof.

Definition 3.2. Let G_p be a partial group and N_p is a partial normal subgroup of G_p , then the partial group G_p/N_p is called the quotient of the partial group or factor group of G_p by N_p .

Proposition 3.4. Let $f : G_p \rightarrow H_p$ be a homomorphism of partial groups, then the kernel of f is a partial normal subgroup of G_p . Conversely, if N_p is a partial normal subgroup of G_p , then the map $\Pi : G_p \rightarrow G_p/N_p$ given by $\Pi(a_p) = aN_p$ is an epimorphism with kernel N_p .

Proof: $\text{Ker}f = \{x_p \in G_p \mid f(x_p) = eH_p\}$, we need to show that if $x_p \in \text{Ker}f$ and $a_p \in G_p$ whether or not $a_p x_p a_p^{-1} \in \text{Ker}f$ if and only if $f(a_p x_p a_p^{-1}) = f(a_p)f(x_p)f(a_p^{-1}) = eH_p$.

So, $a_p x_p a_p^{-1} \in \text{Ker}f$. Therefore, $\text{Ker}f$ is a partial normal subgroup of G_p . It is trivial that $\Pi : G_p \rightarrow G_p/N_p$ is surjective. $\text{Ker}(\Pi) = \{x_p \mid \Pi(x_p) = e_p N_p\} = N_p$.

Theorem 3.7. Let $f : G_p \rightarrow H_p$ is partial group homomorphism and N_p is a partial normal subgroup of G_p contained in the kernel of f , then there is exactly unique homomorphism $\bar{f} : G_p/N_p \rightarrow H_p$ such that $\bar{f}(aN_p) = f(a)$ for all $a \in G_p$. Besides $\text{Im}f = \text{Im}\bar{f}$ and $\text{ker}\bar{f} = (\text{ker}f)/N_p$, $\bar{f}$ an isomorphism if and only if f is an epimorphism and $N_p = \text{ker}f$.

Proof: $f : G_p \xrightarrow{f} H_p$, $G_p \rightarrow G_p/N_p$, $G_p/N_p \xrightarrow{\bar{f}} H_p$ diagram is commutative. If $b \in aN_p$, then $b = an_p$, $n_p \in N$ and also $f(b_p) = f(an_p) = f(a)f(n_p) = f(a)e = f(a)$, since $N_p \leq \text{ker}f$. Therefore, f has the same effect on every element of aN_p and the map $\bar{f} : G_p/N_p \rightarrow H_p$ given by $\bar{f}(aN_p) = f(a)$. It is easily seen that $\bar{f}$ is a well-defined function. Now we need to prove whether or not $\bar{f}$ is a homomorphism of partial groups.

$$f(aN_p.bN_p) = \bar{f}(abN_p) = f(ab)$$

$$\begin{aligned} &= f(a).f(b) \\ &= \bar{f}(aN_p)\bar{f}(bN_p) \end{aligned}$$

So, f is a partial group homomorphism. $Im\bar{f} = Imf$ and $aN_p \in ker\bar{f} \Leftrightarrow f(a) = e \Leftrightarrow a \in kerf$, whence

$$\begin{aligned} ker\bar{f} &= \{aN_p | a \in kerf\} \\ &= kerf/N_p \end{aligned}$$

$\bar{f}$ is unique since it is completely determined by f . Also, $\bar{f}$ is a partial group if and only if f is an epimorphism of partial groups $\bar{f}$ is a monomorphism if and only if for $ker\bar{f} = (kerf)/N_p$ equal to N_p .

Example 3.3. In Example 2.2, it is stated that $G = \{0, \pm 1, \pm 2, \dots, \pm n\}$ is a partial group with known addition operation on $\mathbb{Z}$. We can easily say that the subset $N = \{0, \pm 1, \pm 2, \dots, \pm n-1\}$ of G is a partial subgroup of G . Let us show whether or not N is a partial normal subgroup. If $g_p^{-1} + n_p + g_p \in N$ for all $g_p \in G_p$ then N is a partial normal subgroup. $g_p^{-1} = g_p$ in this group so that $g_p + n_p + g_p \in N_p$ i.e. $n_p \in N$ and then N is a partial normal subgroup of G .

Theorem 3.8 (First Isomorphism Theorem for Partial Groups). Let G_p, H_p be partial groups and $f : G_p \rightarrow H_p$ be partial group epimorphism then $G_p/Kerf$ is isomorphic to H_p .

Proof: We know that $kerf$ is a partial normal subgroup of G_p . Then $G_p/Kerf$ is defined. Let show $Kerf = K$ and $g : G_p/K \rightarrow H_p$ mapping defined as $g((aK)(bK)) = g(abK) = f(ab)$ for every $aK, bK \in G_p/K$.

Since $g(a.bK) = f(ab) = f(a)f(b) = g(aK)g(bK)$ then g is a homomorphism. Since f is onto there exists $a \in G_p$ such that $f(a) = h$ then $aK \in G_p/K$ and $g(aK) = f(a) = h$ and g is onto. For one-to-one conditions let $aK, bK \in G_p/K$ and

$$\begin{aligned} g(aK) &= g(bK) \text{ iff } f(a) = f(b) \\ f(b)^{-1}f(a) &= f(b)^{-1}f(b) \text{ iff } f(b^{-1}a) = eH_p \\ b^{-1}a &\in K \text{ or } kerf \text{ iff } aK = bK \end{aligned}$$

Then, g is an isomorphism and $G_p/Kerf \cong H_p$.

Lemma 3.2. Let G_p be a partial group and H_p be a partial subgroup of G_p and N_p be a partial normal subgroup of G_p . If H_p is a partial normal subgroup of G_p ; then H_pN_p is a partial normal subgroup of G_p .

Proof: Trivial.

Theorem 3.9 (Second Isomorphism Theorem for Partial Groups). Let G_p be a partial group. H_p be a partial subgroup of G_p and N_p be a partial normal subgroup of G_p , then the following isomorphism holds:

$$H_pN_p/N_p \cong H_p(H_p \cap N_p)$$

Proof: It can be easily seen using First Isomorphism Theorem. So, the proof is left to the reader.

Theorem 3.10 (Third Isomorphism Theorem for Partial Groups). Let G_p be a partial group and H_p, N_p partial normal subgroups of G_p with $H_p \leq N_p$ Then H_p is also a partial normal subgroup of N_p . N/H_p is a partial normal subgroup of G_p/H_p and also G_p/N_p is isomorphic to $(G_p/H_p)/(N_p/H_p)$.

Proof: Let define $f : G_p/H_p \rightarrow G_p/N_p, f(aH_p) = aN_p$. First let show f is well-defined: If $aH_p = bH_p$ then we need to prove whether or not $f(aH_p) = f(bH_p)$. Since $aH_p = bH_p$ then $ab^{-1} \in H_p$ and $H_p \leq N_p, ab^{-1} \in N_p$. Since $ab^{-1} \in N_p$ then $aN_p = bN_p$ and so that $f(aH_p) = f(bH_p)$, i.e., f is well defined. f is homomorphism. And using the First Isomorphism Theorem we can conclude the result.

4. Conclusion

There are some papers such as solvable partial groups, topological structures of partial group, Transitivity Theorem-Thompson Theorem of partial groups, k-partial groups, primitive pairs of partial groups so on related with the partial group in the literature. It is known that every group is partial but the converse is not true. That is why some structures are different from each other for the usual group and partial group.

In this chapter, some structures of partial groups (Clifford Semigroup) are sought to demonstrate algebraically. At the beginning of the chapter (preliminaries section), several fundamental results of partial groups with some numerical examples are given from the literature. For example, if A and B be two usual groups such that the intersection of them is equal to $\{1 = e\}$, then the union of subgroups of A and subgroups of B is a partial group. Partial normal groups and partial quotient groups have introduced an analog of the group theory. By using them, a number of isomorphism theorems are proved for partial groups with several other ideas. All results are obtained using closely group theory as algebraic approximations. Readers also may consider/investigate other structures/properties of the partial groups different from the group as algebraically.

Conflict of interest

The author declare no conflict of interest.

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