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4-Dimensional Canards with Brownian Motion

Shuya Kanagawa and Kiyoyuki Tchizawa

Abstract

Generally speaking, it is impossible to analyze slow-fast system with Brownian motion. If it becomes possible to do using a new approach, we can evaluate the rigidity of the original system. What kind of the behavior of such a system we have? Using non-standard analysis, on a “hyper finite time line” by Anderson, the Brownian motions are described by step functions. Then, the original differential equations are described by the difference equations due to using non-standard analysis. When constructing the difference equations, the corresponding measure is extended topologically. Because the interval of the difference is according to the hyper finite time line, the topological space is well defined. In this paper, we propose a two-region economic model with Brownian motions. This concrete example gives us new results.

Keywords: canard solution, slow-fast system, nonstandard analysis, Brownian motion, stochastic differential equation

1. Introduction

Consider a slow-fast system in $\mathbf{R}^4$ with a 2-dimensional slow manifold:

$$\begin{cases} \varepsilon \frac{dx}{dt} = h(x, y, \varepsilon) \\ \frac{dy}{dt} = g(x, y, \varepsilon) \end{cases}, \quad (1)$$

where ε is infinitesimal and

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : \mathbf{R}^4 \rightarrow \mathbf{R}^2, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : \mathbf{R}^4 \rightarrow \mathbf{R}^2, \\ x = x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2, \quad y = y(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbf{R}^2.$$

The above expression form is based on Nelson's [1]. The slow-fast system (1) is applied to many fields, e.g. electronic circuits, neuron systems, etc. In these applications, effectiveness of random noises always exists. We take up this point of view as being one of the main problems.

Now, let us consider a stochastic differential equation for a slow-fast system with a Brownian motion $B(t)$ as the random noises modifying the slow fast system (1): For $t \in [0, T]$, $T > 0$

$$\begin{cases} \varepsilon dx = h(x, y, \varepsilon) dt \\ dy = g(x, y, \varepsilon) dt + \sigma dB, \end{cases} \quad (2)$$

where $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in \mathbb{R}^2$ is a 2-dimensional standard Brownian motion and $\sigma > 0$ is a positive constant which gives a standard deviation for the Brownian motion $B(t)$.

Since the Brownian motion $B(t)$ is almost surely non-differentiable everywhere, it is difficult to analyze slow-fast system (2).

On the other hand, Anderson [2] showed that the Brownian motion is described by step functions using non-standard analysis on a hyper finite time line by the following definition. (See also [3, 4]).

Definition 1. Let $N_t = \frac{t}{\Delta t}$, $0 \leq t \leq T$ and $N = N_T$. Assume that a sequence of i.i.d. random variables $\{\Delta B_k, k = 1, \dots, N\}$ has the distribution

$$P\{\Delta B_k = \sqrt{\Delta t}\} = P\{\Delta B_k = -\sqrt{\Delta t}\} = \frac{1}{2} \quad (3)$$

for each $k = 1, \dots, N$. An extended Wiener process $\{B(t), t \geq 0\}$ is defined by

$$B(t) = \sum_{k=1}^{Nt} \Delta B_k, \quad 0 \leq t \leq T. \quad (4)$$

Rewriting the system (2) via step functions on the hyper finite time line, the following system (5) is obtained.

$$\begin{cases} \varepsilon(x_n - x_{n-1}) = h(x_{n-1}, y_{n-1}, \varepsilon) \Delta t \\ y_n - y_{n-1} = g(x_{n-1}, y_{n-1}, \varepsilon) \Delta t + \sigma \Delta B_n, \end{cases} \quad (5)$$

for $n = 1, 2, \dots, N$, where $\Delta B_n = B(n\Delta t) - B((n-1)\Delta t)$, $\Delta t = \frac{T}{N}$ and N is a hyper finite natural number.

Since the system (5) is equivalent to the system (2), taking $B(t)$ in Definition 1, we prove the existence of the solution for the system (2) in Section 3.

2. Slow-fast system in $\mathbb{R}^4$ with co-dimension 2

We assume that the system (1) satisfies the following conditions (A1) $\sim$ (A5):

(A1) h is of class $\mathbf{C}^1$ and g is of class $\mathbf{C}^2$.

(A2) The slow manifold $S = \{(x, y) \in \mathbb{R}^4 | h(x, y, 0) = 0\}$ is a two-dimensional differential manifold and intersects the set $V = \{(x, y) \in \mathbb{R}^4 | \det(\frac{\partial h}{\partial x}(x, y, 0)) = 0\}$ transversely.

Then, the pli set $PL = \{(x, y) \in S \cap V\}$ is a one-dimensional differentiable manifold.

(A3) Either the value of g_1 or that of g_2 is nonzero at any point of PL .

Note that the pli set PL divides the slow manifolds $S \setminus PL$ into three parts depending on the signs of the two eigenvalues of $\frac{\partial h}{\partial x}(x, y, 0)$.

First consider the following reduced system which is obtained from (1) with $\varepsilon = 0$:

$$\begin{cases} 0 = h(x, y, 0) \\ \frac{dy}{dt} = g(x, y, 0). \end{cases} \quad (6)$$

By differentiating $h(x, y, 0)$ with respect to t , we have

$$\frac{\partial h}{\partial x}(x, y, 0) \frac{dx}{dt} + \frac{\partial h}{\partial y}(x, y, 0) g(x, y, 0) = 0. \quad (7)$$

Then (6) becomes the following:

$$\begin{cases} \frac{dx}{dt} = - \left[\frac{\partial h}{\partial x}(x, y, 0) \right]^{-1} \frac{\partial h}{\partial y}(x, y, 0) g(x, y, 0) \\ \frac{dy}{dt} = g(x, y, 0) \end{cases}, \quad (8)$$

where $(x, y) \in S \setminus PL$. To avoid degeneracy in (8), we consider the following system:

$$\begin{cases} \frac{dx}{dt} = \left\{ -\det \left[\frac{\partial h}{\partial x}(x, y, 0) \right]^{-1} \right\} \left[\frac{\partial h}{\partial x}(x, y, 0) \right]^{-1} \frac{\partial h}{\partial y}(x, y, 0) g(x, y, 0) \\ \frac{dy}{dt} = \left\{ \det \left[\frac{\partial h}{\partial x}(x, y, 0) \right]^{-1} \right\} g(x, y, 0) \end{cases}. \quad (9)$$

The phase portrait of the system (9) is the same as that of (8) except the region where $\det \left[\frac{\partial h}{\partial x}(x, y, 0) \right] = 0$, but only the orientation of the orbit is different.

Definition 2. A singular point of (9), which is on PL , is called a pseudo singular point of (1).

(A4) $\text{rank} \left[\frac{\partial h}{\partial x}(x, y, 0) \right] = 2$ for any $(x, y) \in S \setminus PL$.

From (A4), the implicit function theorem guarantees the existence of a unique function $y = \xi(x)$ such that $h(x, \xi(x), 0) = 0$. By using $y = \xi(x)$, we obtain the following system:

$$\frac{dx}{dt} = \left\{ -\det \left[\frac{\partial h}{\partial x}(x, \xi(x), 0) \right]^{-1} \right\} \left[\frac{\partial h}{\partial x}(x, \xi(x), 0) \right]^{-1} \frac{\partial h}{\partial y}(x, \xi(x), 0) g(x, \xi(x), 0). \quad (10)$$

(A5) All singular points of (10) are non-degenerate, that is, the linearization of (10) at a singular point has two nonzero eigenvalues.

Definition 3. Let λ_1, λ_2 be two eigenvalues of the linearization of (10) at a pseudo singular point. The pseudo singular point with real eigenvalues is called a pseudo singular saddle point if $\lambda_1 < 0 < \lambda_2$ and a pseudo singular node point if $\lambda_1 < \lambda_2 < 0$ or $\lambda_1 > \lambda_2 > 0$.

The following theorem is established (see, e.g. [5]).

Theorem 1. Let (x_0, y_0) be a pseudo singular point. If $\text{trace} \left[\frac{\partial h}{\partial x}(x_0, y_0, 0) \right] < 0$, then there exists a solution which first follows the attractive part and the repulsive part after crossing PL near the pseudo singular point.

Remark 1. The condition $\text{trace} \left[\frac{\partial h}{\partial x}(x_0, y_0, 0) \right] < 0$ implies that one of eigenvalues of $\left[\frac{\partial h}{\partial x}(x_0, y_0, 0) \right]$ is equal to zero and the other one is negative. Notice that the system has two kinds of vector fields: one is 2-dimensional slow

and the other is 2-dimensional fast one. The condition provides the state of the fast vector field.

Remark 2. The singular solution in Theorem 1 is called a canard in $\mathbf{R}^4$ with 2-dimensional slow manifold. As a result, it causes a delayed jumping. The study of canards requires still more precise topological analysis on the slow vector field.

In the next section, we show that a canard exists for the system (2) in which the orbit of the canard of the system (1) is moved to another one by a Brownian motion $B(t)$.

3. Canards with Brownian motion

Let us prove the following theorem.

Theorem 2. *In the system (3), if there exists $\{k_n\}$ such that*

$$|x_n - x_{n-1}| \leq \varepsilon k_n, \quad n = 1, 2, \dots, N \quad (11)$$

and

$$\sup_{1 \leq n \leq N} k_n \leq K \quad (12)$$

for some K hyper finite, then there exists a solution of (5) which is called canard in the sense of Remark 2.

Proof. From the condition (11), we have

$$|x_n - x_{n-1}| = \left| \frac{1}{\varepsilon} h(x_{n-1}, y_{n-1}, \varepsilon) \Delta t \right| \leq \varepsilon k_n. \quad (13)$$

ε is an arbitrary constant, therefore putting $\varepsilon = \frac{1}{N}$ we have from (13)

$$|h(x_{n-1}, y_{n-1}, \varepsilon)| \leq \frac{\varepsilon^2 k_n}{\Delta t} \leq \frac{k_n}{N}, \quad (14)$$

for each $n \geq 1$.

From Definition 2, the following is satisfied for the pseudo-singular point (x_0, y_0) of (1);

$$\begin{cases} \left\{ -\det \left[\frac{\partial h}{\partial x}(x_0, y_0, 0) \right]^{-1} \right\} \left[\frac{\partial h}{\partial x}(x_0, y_0, 0) \right]^{-1} \frac{\partial h}{\partial y}(x_0, y_0, 0) g(x_0, y_0, 0) = 0 \\ h(x_0, y_0, 0) = 0 \end{cases} \quad (15)$$

Assume that σ^2 in the Brownian motion $B(t)$ is sufficiently small. Let (x_ξ, y_ξ) be a pseudo-singular point of (2) or (5). Note that (2) is equivalent to (5) in the sense of nonstandard analysis. Then there exists a positive number ξ such that

$$\left\{ -\det \left[\frac{\partial h}{\partial x}(x_\xi, y_\xi, 0) \right]^{-1} \right\} \left[\frac{\partial h}{\partial x}(x_\xi, y_\xi, 0) \right]^{-1} \frac{\partial h}{\partial y}(x_\xi, y_\xi, 0) (g(x_\xi, y_\xi, 0) \Delta t + \sigma \Delta B_\xi) \approx 0, \quad (16)$$

where $\Delta t = \frac{T}{N}$.

In this situation, as $\sigma \approx 0$

$$(x_\xi, y_\xi) \approx (x_0, y_0). \quad (17)$$

Therefore, the eigenvalues of the linearized system (2) at the point (x_ξ, y_ξ) keeps the almost same as the eigenvalues of the system (1) at the point (x_0, y_0) .

On the other hand, there exists a canard of (1) from Theorem 1. Since $\frac{k_n}{N}$ is small enough, the solution of (5) also first follows the attractive part and the repulsive part follows after crossing PL near the pseudo singular point like as the canard of (1).

4. Concrete models

4.1 Two-region business cycle model

As a concrete model, we consider a two-region business cycle model between two nations A and B including a Brownian motion $B(t) = (B_1(t), B_2(t))$ as followings. See [6] for more details of the two-region business cycle model.

$$\begin{cases} \varepsilon \frac{dx_1}{dt} = -\frac{1-\alpha+m_1}{\theta}y_1 + \frac{m_2}{\theta}y_2 - \left(\frac{\varepsilon}{\theta} + 1 - \alpha\right)(x_1 - a) + \frac{1-n_1}{\theta}\varphi_1(x_1 - a) + \frac{n_2}{\theta}\varphi_2(x_2 - a) \\ \varepsilon \frac{dx_2}{dt} = \frac{m_1}{\theta}y_1 - \frac{1-\alpha+m_2}{\theta}y_2 + \frac{1-n_1}{\theta}\varphi_1(x_1 - a) - \left(\frac{\varepsilon}{\theta} + 1 - \alpha\right)(x_2 - a) + \frac{1-n_2}{\theta}\varphi_2(x_2 - a) \\ dy_1 = (x_1 - a)dt + \sigma dB_1(t) \\ dy_2 = (x_2 - a)dt + \sigma dB_2(t) \end{cases} \quad (18)$$

for $0 \leq t \leq T$, where $x_1(t)$ and $x_2(t)$ are exports of A and B, $m_1(t)$ and $m_2(t)$ are imports of A and B, $y_1(t) - \frac{q}{1-\alpha}$ and $y_2(t) - \frac{q}{1-\alpha}$ are national income identities of A and B for some constants q and α , respectively. (See [6] for more details.)

Now, let us introduce a difference equations for the system (18). Then, the relations $\Delta t = \frac{T}{N}$ and $t_k = k \frac{T}{N}$, $k = 0, 1, \dots, N$ are satisfied, where N is a hyper finite. Put

$$\begin{cases} \frac{\varepsilon}{\Delta t} \{x_1(t_k) - x_1(t_{k-1})\} = -\frac{1-\alpha+m_1}{\theta}y_1(t_{k-1}) + \frac{m_2}{\theta}y_2(t_{k-1}) \\ \quad - \left(\frac{\varepsilon}{\theta} + 1 - \alpha\right)(x_1(t_{k-1}) - a) + \frac{1-n_1}{\theta}\varphi_1(x_1(t_{k-1}) - a) + \frac{n_2}{\theta}\varphi_2(x_2(t_{k-1}) - a) \\ \frac{\varepsilon}{\Delta t} \{x_2(t_k) - x_2(t_{k-1})\} = \frac{m_1}{\theta}y_1(t_{k-1}) - \frac{1-\alpha+m_2}{\theta}y_2(t_{k-1}) \\ \quad + \frac{1-n_1}{\theta}\varphi_1(x_1(t_{k-1}) - a) - \left(\frac{\varepsilon}{\theta} + 1 - \alpha\right)(x_2(t_{k-1}) - a) + \frac{1-n_2}{\theta}\varphi_2(x_2(t_{k-1}) - a) \\ y_1(t_k) - y_1(t_{k-1}) = (x_1(t_{k-1}) - a)\Delta t + \sigma\{B_1(t_k) - B_1(t_{k-1})\} \\ y_2(t_k) - y_2(t_{k-1}) = (x_2(t_{k-1}) - a)\Delta t + \sigma\{B_2(t_k) - B_2(t_{k-1})\} \end{cases} \quad (19)$$

Furthermore put

$$\varphi_1(x - a) = \varphi_2(x - a) = (1 - \alpha) \left(\theta x + x^2 - \frac{x^3}{3} \right). \quad (20)$$

4.2 Simulation results

In this section, let us provide computer simulations for the two-region business cycle model using the above Eqs. (19) and (20). In (19), we assume that two Brownian motions $B_1(t)$ and $B_2(t)$ are mutually independent and note that

$$B_1(t_k) - B_1(t_{k-1}) \sim N(0, \Delta t \sigma_1^2), \quad B_2(t_k) - B_2(t_{k-1}) \sim N(0, \Delta t \sigma_2^2), \quad (21)$$

for each $1 \leq k \leq \frac{T}{\Delta t}$.

Figure 1, except for the axes, shows the pli set $PL = \{(x, y) \in S \cap V\}$ with the pseudo singular point $(1, -1)$ of (1) defined by (9).

Putting some parameters in (19), we have the following results for some orbits of $\{(x_1(t), x_2(t)), 0 \leq t \leq T\}$ satisfying the Eq. (1) or (2).

Figure 2 shows an orbit of $\{(x_1(t), x_2(t)), 0 \leq t \leq T = 0.8\}$ satisfying the Eq. (5) with $\sigma_1 = \sigma_2 = 0$ and starting from $(0.8, -0.8)$ near the pseudo singular point $(1, -1)$. In **Figures 2–4**, $\varepsilon = 0.01$. From **Figure 2** the speed of the orbit $(x_1(t), x_2(t))$ for $0 \leq t \leq 0.2$ is not only very fast, but also the orbit jumps near the pseudo singular point $(1, -1)$. The orbit turns at the point $(-2, 2)$ and returns on the line. $\Delta t = 0.001$ in **Figures 2–6**.

Figure 3 shows an orbit of $\{(x_1(t), x_2(t)), 0 \leq t \leq T = 0.8\}$ satisfying the Eq. (5) with $\sigma_1 = \sigma_2 = 0.4$ and starting from $(0.8, -0.8)$ near the pseudo singular point $(1, -1)$. From **Figure 3** we observe that the orbit moves on the line from $(0.8, -0.8)$ and separates from the line at $t = 0.2$ by the Brownian motion $B(t)$.

Figure 4 shows an orbit of $\{(x_1(t), x_2(t)), 0 \leq t \leq T = 4.64\}$ satisfying the Eq. (5) with $\sigma_1 = \sigma_2 = 0$ and starting from $(0.8, -0.8)$ near the pseudo singular point $(1, -1)$. The orbit separates from the line at $t = 4.61$.

Figure 5 shows an orbit of $\{(x_1(t), x_2(t)), 0 \leq t \leq T = 2.75\}$ satisfying the Eq. (5) with $\sigma_1 = \sigma_2 = 0$ and starting from $(0.8, -0.8)$ near the pseudo singular point $(1, -1)$, where $\varepsilon = 0.004$. The orbit with $\varepsilon = 0.004$ separates from the line at

Figure 1.
Pli set PL .

Figure 2.
 $\Delta t = 0.001$, $0 \leq t \leq 0.8$, $a = 0.6$, $\varepsilon = 0.01$, $\theta = 0.5$, $m_1 m_1 = 0.1$, $m_2 = 0.2$, $n_1 = 0.2$, $n_2 = 0.2$, $\sigma_1 = \sigma_2 = 0$.

Figure 3.
 $\Delta t = 0.001$, $0 \leq t \leq 0.8$, $a = 0.6$, $\varepsilon = 0.01$, $\theta = 0.5$, $m_1 = 0.1$, $m_2 = 0.2$, $n_1 = 0.4$, $n_2 = 0.4$, $\sigma_1 = \sigma_2 = 0.4$.

$t = 2.61$. On the other hand the orbit with $\varepsilon = 0.01$ separates from the line at $t = 4.61$ in **Figure 4**. Therefore we see that the orbit changes according to ε . **Figure 6** shows an orbit of $\{(x_1(t), x_2(t)), 0 \leq t \leq T = 0.8\}$ satisfying the Eq. (5) with $\sigma_1 = \sigma_2 = 0.4$ and starting from $(0.8, -0.8)$ near the pseudo singular point $(1, -1)$, where $\varepsilon = 0.004$. The orbit with $\varepsilon = 0.004$ separates from

Figure 4.
 $\Delta t = 0.001, 0 \leq t \leq 4.64, a = 0.6, \varepsilon = 0.01, \theta = 0.5, m_1 = 0.1, m_2 = 0.2, n_1 = 0.2, n_2 = 0.2, \sigma_1 = \sigma_2 = 0.$

Figure 5.
 $\Delta t = 0.001, 0 \leq t \leq 2.75, a = 0.6, \varepsilon = 0.004, \theta = 0.5, m_1 = 0.1, m_2 = 0.2, n_1 = 0.2, n_2 = 0.2, \sigma_1 = \sigma_2 = 0.$

the line at $t = 0.1$. On the other hand, in **Figure 3**, the orbit with $\varepsilon = 0.01$ separates from the line at $t = 0.2$. Then, the orbit changes according to ε also in the non-random case.

Figure 6.
 $\Delta t = 0.001$, $0 \leq t \leq 0.8$, $a = 0.6$, $\varepsilon = 0.004$, $\theta = 0.5$, $m_1 = 0.1$, $m_2 = 0.2$, $n_1 = 0.2$, $n_2 = 0.2$,
 $\sigma_1 = \sigma_2 = 0.4$.

5. Conclusion

Brownian motions are described by non-differentiable functions almost surely. In order to overcome the difficulty in the system (2) we consider the system (5) using nonstandard analysis. The system (5) makes us possible to analyze the canard with Brownian motions. As the difference equations is determined by according to the hyper finite time line, the measure is extended effectively to do this analysis. In **Figures 1–6** obtained by the simulations, we observe the effects of Brownian motions which change the orbit of $(x_1(t), x_2(t))$.

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References

- [1] Nelson E. Internal set theory: A new approach to nonstandard analysis. *Bulletin of the American Mathematical Society*. 1977;**83**:1165-1198
- [2] Anderson RM. A non-standard representation for brownian motion and ito integration. *Israel Journal of Mathematics*. 1976;**25**(2):1546
- [3] Kanagawa S, Tchizawa K. Proof of Ito's Formula for Ito's Process in Nonstandard Analysis. *Applied Mathematics*. 2019;**10**(2):561-567
- [4] Kanagawa S, Tchizawa K. Extended wiener process in nonstandard analysis. *Applied Mathematics*. 2020;**11**(2): 247254
- [5] Tchizawa K. Four-dimensional canards and their center manifold. *Extended Abstracts Spring 2018. Trends in Mathematics*. 2019;**11**:193-199. Springer Nature Switzerland AG
- [6] Miki H, Nishino H, Tchizawa K. On the possible occurrence of duck solutions in domestic and two-region business cycle models. *Nonlinear Studies*. 2012;**18**:39-55