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Variance Balanced Design

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Abstract

In this chapter binary, ternary and n-ary variance balanced design is constructed using balanced incomplete block, resolvable balanced incomplete block, semi regular group divisible, factorial, fractional factorial designs. Constructed variance balanced designs are with v , $(v + 1)$, $(v + 2)$ and $(v + r)$ treatments. Method of construction of variance balanced designs are supported by suitable examples. It is found that all most all variance balanced designs are with high efficiency factors.

Keywords: incidence matrix, C – Matrix, resolvable balanced incomplete block designs, eigen values, balanced and group divisible designs

1. Introduction

In literature balanced incomplete block designs are either variance balanced (VB), efficiency balanced (EB) or pairwise balanced. Raghvarao ([1], Theorem 4.5.2) discussed that among the class of connected designs the balanced designs are the most efficient designs. A design is said to be variance balanced, if the variance of the estimate of each of the possible elementary treatment contrast is the same, i.e., if t_i denotes the estimate of i^{th} treatment effects, then $\text{Var} (t_i - t_j)$ is constant for all $i \neq j$.

Chakrabarti [2] gave useful concept of C – matrix of design. It is known that balanced incomplete block designs are the most efficient but do not exist for all parametric specifications, and they are equi replicated and have equal block sizes. In some situations, balanced block designs with equal replicates or unequal block size or both are needed. The variance balanced designs can have both equal and unequal number of replications and block sizes. The importance of variance balanced designs in the context of experimental material is well known, as it yields optimal designs apart from ensuring simplicity in the analysis. Many practical situations demand designs with varying block sizes (Pearce, [3], or resolvable VB designs with unequal replications Mukerjee and Kageyama [4]). Rao [5], Hedayat and Federer [6], Raghavarao [7] and Puri and Nigam [8] defined that a design is said to be variance balanced, if every normalized estimable linear function of treatment effect can be estimated with the same precision. They also discussed the necessary and sufficient conditions for the existence of such designs. John [9], Jones et al. [10], Kageyama [11, 12], Kageyama et al. [13], Pal and Pal [14], Roy [15], Sinha [16, 17], Sinha and Jones [18] and Tyagi [19] gave some more methods for constructing block designs with unequal treatment replications and unequal block sizes. Khatri [20], along with a method of construction of VB designs, gave a formula to measure over-all A-efficiency of variance balanced designs. Das and Ghosh [21] gave the methods of construction of variance balanced designs with augmented blocks and treatments. Mukerjee and Kageyama [22] introduced resolvable variance balanced designs. A technique for constructing variance balanced designs, which is based on

the unionizing block principle of Headayat and Federer [6], was described in Calvin [23]. Calvin and Sinha [24] extended his technique to produce designs with more than two distinct block sizes that permit fewer replications. Agarwal and Kumar [25] gave a method of construction of variance balanced designs which is associated with group divisible (GD) designs. Rao [5] observed that, if the information matrix C of a block design satisfied

$$C = \theta \left[I_v - \frac{1}{v} E_{vv} \right]$$

where, θ is non zero eigen value of C matrix, I_v is an identity matrix of order v , E_{vv} is the matrix with v rows and v columns where, all the elements are unity, then such design is called Variance balanced designs. Since balanced incomplete block design (BIBD) satisfies this property and hence, balanced incomplete block design is a particular case of Variance balanced designs.

Das and Ghosh [21] defined generalized efficiency balanced (GEB) design, which include both VB as well as EB designs. Ghosh [26], Ghosh and Karmaker [27], Ghosh and Devecha [28], Ghosh, Divecha, and Kageyama [29], Ghosh et al. [30], Ghosh, et al. [31, 32] obtained several methods for construction of VB designs. Ghosh and Joshi [30] constructed VB design through GD design. Again, Ghosh and Joshi [33] Constructed VB Design through Triangular design. Kageyama [10] recommended the use of non-binary VB design, when binary VB designs are not available for given values of parameters. Ghosh and Ahuja [34] carried out VB design using fractional factorial designs. Agarwal and Kumar [35, 36] developed some methods of constructing ternary VB designs with $(v + s)$ treatments ($s \geq 1$), having blocks of unequal sizes, through block designs with v treatment. Ghosh, Kageyama and Joshi [37] developed Ternary VB designs using BIB and GD design. Ghosh et al. [37] further obtained more VB designs using Latin square type PBIB design. Ghosh [38] studied the robustness of variance balanced design against the loss of k treatments and one block. Ghosh et al. [39] discuss construction of VB design using factorial designs. Hedayat and Stufken [40] established a relation between pair wise balanced and variance balanced designs. Jones [41] discussed the property of incomplete block designs. Gupta and Jones [42] constructed equal replicated VB designs.

2. Method of construction

Method of construction of Variance balanced design with equal/unequal replication sizes and equal/unequal block sizes is carried out in this chapter. Section 3 discusses the construction of variance balanced design using Hadamard matrix. While construction of variance balanced design using semi regular group divisible design is discussed in Section 4. Variance balanced design is constructed by augmenting n more blocks which is discussed in Section 5. Construction of variance balanced design with $(v + 1)$ treatments using unreduced balanced incomplete block design is shown in Section 6. Section 7 discusses the construction of variance balanced design using 2^n symmetrical factorial experiments. Variance balanced design is constructed using incidence matrix also, and is shown in section – 8.

3. Variance balanced design using Hadamard matrix

Theorem – 3.1: Equi-replicated Variance balanced design with parameters $v = n - 1$, $b = n$, $r = n/2$, $k = \{n - 1, n/2 - 1\}$ and $C_{im} = \frac{(3n-4)}{(n-1)(n-2)}$ can always be constructed

from a Hadamard matrix of size n by deleting its first row and then considering rows as treatments and columns as blocks.

Proof: Consider a Hadamard matrix of size n . Delete its first row. The size of this matrix become $(n - 1) \times n$. We replace -1 by 0 , and call this matrix by N . This matrix contains $(n-1)$ rows and n columns where, element “1” occurs $n/2$ times in each row, $(n - 1)$ times in first column, and $(n/2-1)$ times in the remaining columns. Consider matrix N as an incidence matrix of a variance balanced design, where rows are treatments and columns are blocks, so, $v = n-1$, $b = n$, $r = n/2$ and $k = \{n - 1, n/2-1\}$.

For variance balanced design, $C_{im} = \sum_j^b \frac{n_{ij}n_{mj}}{n_j}$, where, $i \neq m = 1$ to v .

C_{im} is computed as $C_{1m} = \frac{1}{n-1} + \frac{1}{\frac{n}{2}-1} = \frac{(3n-4)}{(n-1)(n-2)}$.

We can verify that, C_{im} gives same constant value for each pair of treatments. Now a block design is said to variance balanced design, if C matrix satisfies, $C = \theta (I_v - E_{vv}/v)$, where, θ is non zero eigen value of C matrix with multiplicity $(v - 1)$, where,

$$C = \text{diag}(r_1, r_2, \dots, r_v) - N K^{-1} N'.$$

$$C = \begin{bmatrix} n/2 & 0 & \dots & 0 \\ 0 & \frac{n}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n/2 \end{bmatrix} - \begin{bmatrix} n(n-2) & 3n-4 & \dots & 3n-4 \\ 3n-4 & n(n-2) & \dots & 3n-4 \\ \vdots & \vdots & \ddots & \vdots \\ 3n-4 & 3n-4 & \dots & n(n-2) \end{bmatrix} / (n-1)(n-2)$$

After simplification C reduces to

$$C = \frac{n(n-2)(n-3) + 6n - 8}{2(n-1)(n-2)} \left[I_v - \frac{E_{vv}}{v} \right] \quad (1)$$

Where, $\theta = \frac{n(n-2)(n-3) + 6n - 8}{2(n-1)(n-2)}$ denotes the non-zero eigen value of C matrix with multiplicity $(n - 2)$.

Eq. (1) satisfy the condition of variance balanced design. Hence, this is an equi-replicated and two unequal block sizes variance balanced design.

3.1 Efficiency factor of a variance balanced design

The efficiency factor of a variance balanced design is defined as

$$E = \frac{\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD}}{\text{Var}(\hat{t}_i - \hat{t}_m)_{VB}}$$

Where, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r) \sigma^2 = \frac{2}{n/2} \sigma^2$ and

$$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta) \sigma^2 = \frac{2}{\frac{n(n-2)(n-3) + 6n - 8}{2(n-1)(n-2)}} \sigma^2$$

$$E = \frac{n(n-2)(n-3) + 6n - 8}{n(n-1)(n-2)}$$

Example-3.1 Construct a variance balanced design from a Hadamard matrix of size 8.

Using Theorem – 3.1, we construct a variance balanced design from a Hadamard matrix of size 8 as following:

Hadamard Matrix of size 8

1 1 1 1 1 1 1 1.

1 -1 1 -1 1 -1 1 -1

1 1 -1 -1 1 1 -1 -1

1 -1 -1 1 1 -1 -1 1 N = 1

1 1 1 1 -1 -1 -1 -1

1 -1 1 -1 -1 1 -1 1

1 1 -1 -1 -1 -1 1 1

1 -1 -1 1 -1 1 1 -1

Incidence matrix of a Variance balanced design

1 0 1 0 1 0 1 0

1 1 0 0 1 1 0 0

0 0 1 1 0 0 1

1 1 1 1 0 0 0 0

1 0 1 0 0 1 0 1

1 1 0 0 0 0 1 1

1 0 0 1 0 1 1 0

N gives the incidence matrix of an equi-replicated and un equal block sizes variance balanced design with parameters $v = 7$, $b = 8$, $r = 4$, $k = \{7, 3\}$, $C_{im} = 10/21$ and information matrix,

$$C = \begin{bmatrix} 4 & 0 & \dots & 0 \\ 0 & 4 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 4 \end{bmatrix} - \begin{bmatrix} 48 & 20 & \dots & 20 \\ 20 & 48 & \dots & 20 \\ \vdots & \vdots & \ddots & \vdots \\ 20 & 20 & \dots & 48 \end{bmatrix} / 42$$

After simplification, C reduces to

$$C = \frac{280}{84} \left[I_7 - \frac{E_{77}}{7} \right] = \theta \left[I_7 - \frac{E_{77}}{7} \right] \quad (2)$$

Where, $\theta = \frac{10}{3}$, is the non zero eigen value of C matrix with multiplicity 6. Hence, it is a variance balanced design, with $\hat{t}_i = (1/\theta)Q_i = (3/10)Q_i$,

$$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta)\sigma^2 = (6/10)\sigma^2. \text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r)\sigma^2 = \frac{2}{4}\sigma^2$$

and Efficiency factor, $E = 5/6$. This shows that efficiency factor is very high.

4. Variance balanced design through semi regular group divisible designs

In this section, we discuss the construction of variance balanced design by adding the blocks of semi-regular group divisible design with its groups, provided the following conditions (i) block sizes, $k = \lambda_2$, (ii) $\lambda_1 = 0$ and (iii) number of groups are considered as number of blocks, are satisfied.

Theorem – 4.1 Let the parameters of a semi regular group divisible design are v , b , r , k , $\lambda_1 = 0$, λ_2 , m and n , where $k = \lambda_2$. By adding the b blocks of this semi regular group divisible design with number of groups as blocks, an equi-replicated and unequal block sizes variance balanced design is constructed with parameters $v_1 = v$, $b_1 = b + mn$, $r_1 = r + n$, $k_1 = \{k, n\}$ and $C_{im} = \lambda_2 / k$ or $C_{im} = \lambda_1 / k + n/n$.

Proof: Consider a semi regular group divisible design with parameters v , b , r , k , $\lambda_1 = 0$, $\lambda_2 = k$, m and n , where, m denotes number of groups and n number of treatments per group. Denote N as the incidence matrix of the resulting design. Consider one group as one block. Here, there are m groups and hence, we have m more blocks. Add b blocks of the semi-regular group divisible design with its m more blocks, provided m blocks are repeated n times. Hence, $v_1 = v$, $b_1 = b + mn$,

$r_1 = r + n$ and $k_1 = \{k, n\}$. We can check, $C_{im} = \sum_j^b \frac{n_{ij}n_{mj}}{n_j}$, where $i \neq m = 1$ to v , for each pair of treatment as following. $C_{1m} = \frac{\lambda_2}{k}$, for those pair of treatments, which occur λ_2 times. Since, $\lambda_2 = k$ and hence, $C_{im} = 1$. Again, for those pair of treatments for which $\lambda_1 = 0$, $C_{1m} = \frac{\lambda_1}{k} + \frac{n}{n} = 1$. For variance balanced design, C_{im} should be the same for each pair of treatments. Hence, $\frac{\lambda_1}{k} + \frac{n}{n} = \frac{\lambda_2}{k}$. This implies that, using this method, we can construct a variance balanced design from those semi - regular group divisible designs in which $(\lambda_2 - \lambda_1) = k$ holds true.

Again, a block design is said to be a variance balanced design, if C matrix satisfies, $C = \theta (I_v - E_{vv}/v)$, where, θ is non zero eigen value of C matrix with multiplicity $(v - 1)$, and, $C = \text{diag} (r_1, r_2, \dots, r_v) - N K^{-1} N'$.

$$C = \begin{bmatrix} r+n & 0 & \dots & 0 \\ 0 & r+n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r+n \end{bmatrix} - \begin{bmatrix} \frac{r+k}{k} & 1 & \dots & 1 \\ 1 & \frac{r+k}{k} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \frac{r+k}{k} \end{bmatrix}$$

Diagonal elements = $[k(r+n) - (r+k)]/k$, and off diagonal elements = $-k/k = -1$. After simplification, C reduces to

$$C = \frac{k(r+n) - r}{k} \left[I_v - \frac{E_{vv}}{v} \right] \quad (3)$$

Where, $\theta = \frac{k(r+n)-r}{k}$ denotes the non-zero eigen value of C matrix with multiplicity $(v - 1)$.

Eq. (3) satisfy the condition of variance balanced design. Hence, this is equi replicated and two unequal block sizes variance balanced design with parameters $v_1 = v$, $b_1 = b + mn$, $r_1 = r + n$, $k_1 = \{k, n\}$.

4.1 Efficiency factor

The efficiency factor of a variance balanced design is defined as

$$E = \frac{\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD}}{\text{Var}(\hat{t}_i - \hat{t}_m)_{VB}}, \text{ where, } \text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r) \sigma^2 = \frac{2}{r+n} \sigma^2,$$

$$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta) \sigma^2 = \frac{2}{\frac{k(r+n)-r}{k}} \sigma^2 \text{ and } E = \frac{k(r+n) - r}{k(r+n)}$$

Example – 4.1 Construct a variance balanced design with parameters $v_1 = 6$, $b_1 = 18$, $r_1 = 8$, $k_1 = \{3, 2\}$ from a semi regular group divisible design SR – 20, having parameters $v = 6$, $b = 12$, $r = 6$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $m = 3$ and $n = 2$. Where, group is (3,2).

Three groups each with 2 treatments are (1 4), (2 5), (3 6).

Blocks of the semi-regular group divisible design, SR – 20 are.

(1 2 3), (2 4 6), (3 4 5), (1 5 6), (1 2 3), (2 4 6), (3 4 5), (1 5 6), (1 2 6), (1 3 5), (2 3 4), (4 5 6),

Using Theorem – 4.1, incidence matrix of the variance balanced design is given as.

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

N_1 gives the incidence matrix of an equi-replicated and un-equal block sizes variance balanced design with parameters $v_1 = 6$, $b_1 = 18$, $r_1 = 8$, $k_1 = \{3, 2\}$ $C_{im} = 1$ and information matrix,

$$C = \begin{bmatrix} 8 & 0 & \dots & 0 \\ 0 & 8 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 8 \end{bmatrix} - \begin{bmatrix} 18 & 6 & \dots & 6 \\ 6 & 18 & \dots & 6 \\ \vdots & \vdots & \ddots & \vdots \\ 6 & 6 & \dots & 18 \end{bmatrix} / 6$$

After simplification, C reduces to

$$C = 6 \left[I_7 - \frac{E_{77}}{7} \right] = \theta \left[I_6 - \frac{E_6}{6} \right]$$

Where, $\theta = 6$ is the non zero eigen value of C matrix with multiplicity 5. Hence, it is a variance balance design with $\hat{t}_i = (1/\theta) Q_i = (1/6) Q_i$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta)\sigma^2 = (2/6)\sigma^2$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r)\sigma^2 = (2/8)\sigma^2$, and Efficiency factor, $E = 3/4$. This shows that efficiency factor is very high.

5. Variance balanced design through augmenting $n (\geq 1)$ blocks

In this section, variance balanced designs are obtained through balanced incomplete block design by augmenting one and more than one blocks, such that each augmented block contains each of the v treatments. The resulting design is an un-equal replicated and un-equal blocks sizes variance balanced design.

Theorem – 5.1 Let N be the incidence matrix of a balanced incomplete block design with parameters v , b , r , k and λ . Let n blocks are added with the blocks of the given balanced incomplete block design. The incidence matrix N_1 defined as

$$N_1 = [(N)_{v \times b} \mathbf{1}_{v \times 1}]$$

gives variance balanced design with parameters $v_1 = v$, $b_1 = b + n$, $r_1 = \{(r + n), b\}$, $k_1 = \{k, v\}$, where, N_1 is the incidence matrix of Variance balanced design.

Proof: Consider a balanced incomplete block design with parameters v , b , r , k and λ , whose incidence matrix is denoted by N . Next n more blocks are augmented, hence, for resulting design, $v_1 = v$, $b_1 = b + n$, $r_1 = (r + n)$, $k_1 = \{k, v\}$.

$$C_{im} = \frac{\lambda}{k} + \frac{n}{v} = \frac{\lambda v + nk}{vk}.$$

Again, a block design is said to variance balanced design, if C matrix satisfies,

$C = \theta (I_v - E_{vv}/v)$, where, θ is non zero eigen value of C matrix with multiplicity $(v - 1)$ and $C = \text{diag}(r_1, r_2, \dots, r_v) - N K^{-1} N'$.

$$C = \begin{bmatrix} r+n & 0 & \dots & 0 \\ 0 & r+n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r+n \end{bmatrix} - \begin{bmatrix} k(b+n) & \lambda v + nk & \dots & \lambda v + nk \\ \lambda v + nk & k(b+n) & \dots & \lambda v + nk \\ \vdots & \vdots & \ddots & \vdots \\ \lambda v + nk & \lambda v + nk & \dots & k(b+n) \end{bmatrix} / vk$$

Diagonal elements are $[vk(r+n) - k(b+n)]/vk$, and off diagonal elements are $-(\lambda v + nk)/vk$. After simplification, C reduces to C =

$$\begin{bmatrix} vk(r+n) - k(b+n) & -(\lambda v + nk) & \dots & -(\lambda v + nk) \\ -(\lambda v + nk) & vk(r+n) - k(b+n) & \dots & -(\lambda v + nk) \\ \vdots & \vdots & \ddots & \vdots \\ -(\lambda v + nk) & -(\lambda v + nk) & \dots & vk(r+n) - k(b+n) \end{bmatrix} / vk$$

$$\text{Finally, } C = \frac{(\lambda v + nk)}{k} \left[I_v - \frac{E_{vv}}{v} \right] \quad (4)$$

where, $\theta = \frac{(\lambda v + nk)}{k}$ is the non zero eigen value of C matrix with multiplicity $(v-1)$. Eq. (4) satisfy the condition of variance balanced design. Hence, this is equi replicated and two unequal block sizes variance balanced design with parameters $v_1 = v$, $b_1 = b + n$, $r_1 = (r + n)$, $k_1 = \{k, v\}$.

5.1 Efficiency factor

The efficiency factor of a variance balanced design is defined as

$$E = \frac{\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD}}{\text{Var}(\hat{t}_i - \hat{t}_m)_{VB}}, \text{ where, } \text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r) \sigma^2 = \frac{2}{r+n} \sigma^2,$$

$$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta) \sigma^2 = \frac{2}{\frac{(\lambda v + nk)}{k}} \sigma^2 = \frac{2k}{(\lambda v + nk)} \text{ and } E = \frac{(\lambda v + nk)}{k(r+n)}.$$

Example – 5.1 Construct a variance balanced design with parameters $v_1 = 9$, $b_1 = 15$, $r_1 = 7$, $k_1 = \{3, 9\}$ from a balanced incomplete block design having parameters $v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$.

Blocks of the balanced incomplete block design are (1 2 3), (4 5 6), (7 8 9), (1 4 7), (2 5 8), (3 6 9), (1 6 8), (2 4 9), (3 5 7), (1 5 9), (2 6 7), (3 4 8). Let $n = 3$.

Using Theorem – 5.1, incidence matrix of the variance balanced design is given as.

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

N_1 gives the incidence matrix of an equi replicated and un - equal block sizes variance balanced design with parameters $v_1 = 9$, $b_1 = 15$, $r_1 = 7$, $k_1 = \{3, 9\}$ $C_{im} = 2/3$ and information matrix,

$$C = \begin{bmatrix} 7 & 0 & .. & 0 \\ 0 & 7 & .. & 0 \\ : & : & : & : \\ 0 & 0 & .. & 7 \end{bmatrix} - \begin{bmatrix} 45 & 18 & .. & 18 \\ 18 & 45 & .. & 18 \\ : & : & .. & : \\ 18 & 18 & .. & 45 \end{bmatrix} / 27$$

After simplification, C reduces to $C = \begin{bmatrix} 144 & -18 & .. & -18 \\ -18 & 144 & .. & -18 \\ : & : & : & : \\ -18 & -18 & .. & 144 \end{bmatrix} / 27$

Finally,

$$C = \frac{162}{27} \left[I_9 - \frac{E_{99}}{9} \right] = \frac{18}{3} \left[I_9 - \frac{E_{99}}{9} \right] = \theta \left[I_9 - \frac{E_{99}}{9} \right] \quad (5)$$

Where, $\theta = 6$ is the non zero eigen value of C matrix with multiplicity 8. Hence, it is a variance balance design with $\hat{t}_i = (1/\theta) Q_i = (1/6) Q_i$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta)\sigma^2 = (2/6)\sigma^2$. $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r)\sigma^2 = (2/7)\sigma^2$ and Efficiency factor, $E = 6/7$. This shows that efficiency factor is very high.

6. Variance balanced design with $(v + 1)$ treatments

Variance balanced design with $(v + 1)$ treatments is constructed by reinforcing one treatment in each block of a balanced incomplete block design.

6.1 Variance balanced designs with $(v + 1)$ treatments from a series of balanced incomplete block design with parameters v , $b = v_{C_2}$, $r = v - 1_{C_{2-1}}$, $k = 2$ and $\lambda = 1$

In this section, method of the construction of variance balanced design with $(v + 1)$ treatments is discussed. Variance balanced design with $(v + 1)$ treatments can always be constructed through a balanced incomplete block design by reinforcing one treatment and augmenting n blocks. Let the parameters of a balanced incomplete block design are v , $b = v_{C_2}$, $r = v - 1_{C_{2-1}}$, $k = 2$ and $\lambda = 1$, provided $v(r - 1)$ must be divisible by $(k + 1) = 3$. This is shown in Theorem – 6.1.

Theorem – 6.1: Let the parameters of an unreduced balanced incomplete block design are v , $b = v_{C_2}$, $r = v - 1_{C_{2-1}}$, $k = 2$ and $\lambda = 1$, whose incidence matrix is denoted by N . Let balanced incomplete block design is reinforced by one treatment up to b blocks and augmented with n blocks, such that each block contains each of the v treatments, where, $n = 1, 2, \dots$. The incidence matrix N_1 defined by

$$N_1 = \begin{bmatrix} N_{v \times b} & E_{v \times n} \\ \mathbf{1}_{1 \times b} & \mathbf{0}_{1 \times n} \end{bmatrix}$$

gives the incidence matrix of a Variance balanced design with parameters $v_1 = (v + 1)$, $b_1 = b + n$, $r_1 = \{r + n, b\}$, $k = \{3, v\}$, where, $E_{v \times n}$ is a matrix of v rows and n columns with elements as 1, $\mathbf{1}$ is a vector of one row and b columns, $\mathbf{0}$ is a vector of one row and n columns, provided $n = \frac{v(r-1)}{3}$, n being integer.

Proof: Let us consider a unreduced balanced incomplete block designs with parameters $b = v_{C_2}$, $r = v - 1_{C_{2-1}}$, $k = 2$ and $\lambda = 1$, provided v is divisible by k . This series

of balanced incomplete block design is reinforced by one more treatment and augmented by n blocks such that $(v + 1)^{\text{th}}$ treatment appears in each of the b blocks and each of the n more blocks contains each of the v treatment once and only once, hence, $v_1 = v + 1$, $b_1 = b + n$, $r_1 = \{r + n, b\}$, $k = \{3, v\}$ become the parameters of the resulting variance balanced design. Let us check the C_{im} ($i \neq m = 1$ to v) value for each pair of treatments, C_{im} value for any pair of treatments among v treatments is computed as

$$C_{im} = \frac{1}{3} + \frac{n}{v} \quad (6)$$

Again,

$$C_{im} = \frac{r}{3}, i = 1, \dots, v, \text{ and } m = v + 1 \quad (7)$$

For variance balanced design, C_{im} for each pair of treatment must be same and hence, from (6) and (7), $\frac{1}{3} + \frac{n}{v} = \frac{r}{3}$, or, $\frac{n}{v} = \frac{(r-1)}{3}$, hence, $n = \frac{v(r-1)}{3}$.

Again, a block design is said to variance balanced design, if C matrix satisfies,

$C = \theta (I_v - E_{vv}/v)$, where, θ is non zero eigen value of C matrix with multiplicity $(v - 1)$ and $C = \text{diag} (r_1, r_2, \dots, r_v) - N K^{-1} N'$.

$$C = \begin{bmatrix} r+n & 0 & \dots & 0 \\ 0 & r+n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b \end{bmatrix} - \begin{bmatrix} vr+3n & v+3n & \dots & v+3n \\ v+3n & vr+3n & \dots & v+3n \\ \vdots & \vdots & \ddots & \vdots \\ v+3n & v+3n & \dots & bv \end{bmatrix} / 3v$$

Diagonal elements are (i) $[3v(r + n) - (vr + 3n)]/3v$ and (ii) $2bv$, and off diagonal elements are $-(v+3n)/3v$. After simplification, C reduces to

$$C = \begin{bmatrix} 3v(r+n)-(vr+3n) & -(v+3n) & \dots & -(v+3n) \\ -(v+3n) & 2v(r+n)-(vr+3n) & \dots & -(v+3n) \\ \vdots & \vdots & \ddots & \vdots \\ -(v+3n) & -(v+3n) & \dots & 2bv \end{bmatrix} / 3v$$

For variance balanced design, all the diagonal elements must be same and hence, $[3v(r + n) - (vr + 3n)] = 2bv$. This shows that one can use either of diagonal element. In this section, we use $[3v(r + n) - (vr + 3n)]$ as a diagonal element.

Finally,

$$C = \frac{(3n + 2r + 1)}{3} \left[I_v - \frac{E_{vv}}{v} \right] \quad (8)$$

Where, $\theta = \frac{(3n+2r+1)}{3}$ is the non zero eigen value of C matrix with multiplicity v . Eq. (8) satisfy the condition of variance balanced design. Hence, this is an unequal replicated and unequal block sizes variance balanced design with parameters $v_1 = v$, $b_1 = b + n$, $r_1 = \{(r + n), b\}$, $k_1 = \{3, v\}$.

6.2 Efficiency factor

Since the resulting variance balanced design has two unequal replications and hence, there are two efficiency factors. The efficiency factor of a variance balanced design is defined as.

$E_1 = \frac{\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1}}{\text{Var}(\hat{t}_i - \hat{t}_m)_{VB}}, \text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1} = (2/r) \sigma^2 = \frac{2}{r+n} \sigma^2$, where, t_i and t_m are any two treatments among v treatments, that is, $i \neq m = 1$ to v . $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD2} = (\frac{1}{r+n} + \frac{1}{b}) \sigma^2$, where, $i \neq m = 1$ to v and $m = (v + 1)$.

$$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta) \sigma^2 = \frac{2}{\frac{(3n+2r+1)}{3}} \sigma^2 = \frac{6}{(3n + 2r + 1)}.$$

$$E_1 = \frac{(3n + 2r + 1)}{3(r + n)}. \text{ Again, } E_2 = \frac{\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD2}}{\text{Var}(\hat{t}_i - \hat{t}_m)_{VB}} = \frac{(b + r + n)(3n + 2r + 1)}{6b(r + n)}$$

Example – 6.1: Construct a variance balanced design with parameters $v_1 = 6$, $b_1 = 15$, $r_1 = \{9, 10\}$, $k_1 = \{3, 5\}$ from a balanced incomplete block design having parameters $v = 5$, $b = 10$, $r = 4$, $k = 2$, $\lambda = 1$.

Blocks of the balanced incomplete block design are.

(1 2), (1 3), (1 4), (1 5), (2 3), (2 4), (2 5), (3 4), (3 5), (4 5). Let $n = \frac{v(r-1)}{3} = 5$. Hence, five blocks are augmented.

Using Theorem –6.1, incidence matrix of the variance balanced design is given as.

$$N_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

N_1 gives the incidence matrix of an unequal replicated and unequal block sizes variance balanced design with parameters $v_1 = 6$, $b_1 = 15$, $r_1 = \{9, 10\}$, $k_1 = \{3, 5\}$, $C_{im} = 20/15$ and information matrix,

$$C = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix} - \begin{bmatrix} 35 & 20 & 20 & 20 & 20 & 20 \\ 20 & 35 & 20 & 20 & 20 & 20 \\ 20 & 20 & 35 & 20 & 20 & 20 \\ 20 & 20 & 20 & 35 & 20 & 20 \\ 20 & 20 & 20 & 20 & 35 & 20 \\ 20 & 20 & 20 & 20 & 20 & 50 \end{bmatrix} / 15$$

After simplification, C reduces to

$$C = \begin{bmatrix} 100 & -20 & -20 & -20 & -20 & -20 \\ -20 & 100 & -20 & -20 & -20 & -20 \\ -20 & -20 & 100 & -20 & -20 & -20 \\ -20 & -20 & -20 & 100 & -20 & -20 \\ -20 & -20 & -20 & -20 & 100 & -20 \\ -20 & -20 & -20 & -20 & -20 & 100 \end{bmatrix} / 15$$

$$\text{Finally, } C = \frac{120}{15} \left[I_6 - \frac{66}{6} \right] = 8 \left[I_9 - \frac{E_{99}}{9} \right] = \theta \left[I_6 - \frac{E_{66}}{6} \right] \quad (9)$$

Where, $\theta = 8$ is the non zero eigen value of C matrix with multiplicity 5. Hence, it is a variance balanced design with $\hat{t}_i = (1/\theta) Q_i = (1/8) Q_i$,

$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta)\sigma^2 = (2/8)\sigma^2$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1} = (2/r)\sigma^2 = (2/9)\sigma^2$ and $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD2} = (\frac{1}{r+n} + \frac{1}{b})\sigma^2 = (\frac{1}{9} + \frac{1}{10})\sigma^2 = (19/90)\sigma^2$ with.

Efficiency factor, $E_1 = 8/9$ and $E_2 = 38/45$. This shows that efficiency factor is very high.

6.3 Variance balanced designs with $(v + 1)$ treatments from a series of balanced incomplete block design with parameters $v, b = v_{C_k}, r = v - 1_{C_{k-1}}, k$ and $\lambda = v - 2_{C_{k-2}}$

In Section 6.1, method of the construction of variance balanced design with $(v + 1)$ treatments is discussed with block sizes $k = 2$. In this section, we have extended the method of construction of variance balanced designs with $(v + 1)$ treatments through a balanced incomplete block design for any value of k by reinforcing one treatment and augmenting n blocks. Let the parameters of a balanced incomplete block design are $v, b = v_{C_k}, r = v - 1_{C_{k-1}}, k$ and $\lambda = v - 2_{C_{k-2}}$, provided, $v(r - \lambda)$ must be divisible by $(k + 1)$. This is shown in Theorem – 6.2.

Theorem – 6.2 Let the parameters of a balanced incomplete block design are $v, b = v_{C_k}, r = v - 1_{C_{k-1}}, k$ and $\lambda = v - 2_{C_{k-2}}$. whose incidence matrix is denoted by N . Let balanced incomplete block design is reinforced by one treatment up to b blocks and augmented with n blocks, such that each block contains each of the v treatments, where, $n = 1, 2, \dots$. The incidence matrix N_1 defined by

$$N_1 = \begin{bmatrix} N_{v \times b} & E_{v \times n} \\ \mathbf{1}_{1 \times b} & \mathbf{0}_{1 \times n} \end{bmatrix}$$

gives the incidence matrix of a Variance balanced design with parameters $v_1 = (v + 1), b_1 = b + n, r_1 = \{r + n, b\}, k_1 = \{(k + 1), v\}$, where, $E_{v \times n}$ is a matrix of v rows and n columns with elements as 1, $\mathbf{1}$ is a vector of one row and b columns, $\mathbf{0}$ is a vector of one row and n columns, provided $n = \frac{v(r-\lambda)}{(k+1)}$, n being integers.

Proof: Let us consider a series of balanced incomplete block designs with parameters $v, b = v_{C_k}, r = v - 1_{C_{k-1}}, k$ and $\lambda = v - 2_{C_{k-2}}$, provided n is divisible by $\frac{v(r-\lambda)}{(k+1)}$. This series of balanced incomplete block design is reinforced by one more treatment and augmented by n blocks such that $(v + 1)^{\text{th}}$ treatment appears in each of the b blocks and each of the n more blocks contains each of the v treatment once and only once, hence, $v_1 = v + 1, b_1 = b + n, r_1 = \{r + n, b\}, k_1 = \{(k + 1), v\}$ are the parameters of the resulting variance balanced design. Let us check the C_{im} ($i \neq m = 1$ to v) value for each pair of treatments. C_{im} value for any pair of treatments among v treatments is computed as

$$C_{im} = \frac{1}{(k + 1)} + \frac{n}{v} \quad (10)$$

$$\text{Again, } C_{im} = \frac{r}{(k + 1)}, i = 1, \dots, v \text{ and } m = v + 1 \quad (11)$$

For variance balanced design, C_{im} for each pair of treatment, must be same and hence, from (10) and (11), $\frac{1}{(k+1)} + \frac{n}{v} = \frac{r}{(k+1)}$, or, $\frac{n}{v} = \frac{(r-\lambda)}{(k+1)}$, hence, $n = \frac{v(r-\lambda)}{(k+1)}$.

Again, a block design is said to variance balanced design, if C matrix satisfies,

$C = \theta (I_v - E_{vv}/v)$, where, θ is non zero eigen value of C matrix with multiplicity $(v - 1)$ and $C = \text{diag}(r_1, r_2, \dots, r_v) - N K^{-1} N'$.

$$C = \begin{bmatrix} r+n & 0 & .. & 0 \\ 0 & r+n & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .. & b \end{bmatrix} - \begin{bmatrix} vr+n(k+1) & \lambda v+n(k+1) & .. & \lambda v+n(k+1) \\ \lambda v+n(k+1) & vr+n(k+1) & .. & \lambda v+n(k+1) \\ \vdots & \vdots & .. & \vdots \\ \lambda v+n(k+1) & \lambda v+n(k+1) & .. & bv \end{bmatrix} / v(k+1)$$

Diagonal elements are (i) $[v(k+1)(r+n) - (vr+n(k+1))]/v(k+1)$ and (ii) kbv and off diagonal elements are $-(\lambda v+n(k+1))/v(k+1)$. After simplification, C reduces to

$$C = \begin{bmatrix} vk(n+r) + n(v-k-1) & -(\lambda v+n(k+1)) & .. & -(\lambda v+n(k+1)) \\ -(\lambda v+n(k+1)) & vk(n+r) + n(v-k-1) & .. & -(\lambda v+n(k+1)) \\ \vdots & \vdots & .. & \vdots \\ -(\lambda v+n(k+1)) & -(\lambda v+n(k+1)) & .. & kbv \end{bmatrix} / v(k+1)$$

For variance balance design, all the diagonal elements must be same and hence, $vk(n+r) + n(v-k-1) = kbv$. This shows that we can use either of diagonal element. In this section, we used $vk(n+r) + n(v-k-1)$ as a diagonal element.

$$\text{Finally, } C = \frac{(n(k+1) + kr + \lambda)}{(k+1)} \left[I_v - \frac{E_{vv}}{v} \right] \quad (12)$$

Where, $\theta = \frac{n(k+1) + kr + \lambda}{(k+1)}$ is the non zero eigen value of C matrix with multiplicity v . Eq. (12) satisfy the condition of variance balanced design. Hence, this is an unequal replicated and unequal block sizes variance balanced design with parameters $v_1 = v$, $b_1 = b + n$, $r_1 = \{(r+n), b\}$, $k_1 = \{3(k+1) v\}$.

6.3.1 Efficiency factor of this variance balanced design

Since the resulting variance balanced design is a two unequal replicated design and hence, there are two efficiency factors. The efficiency factor of a variance balanced design is defined as.

$E_1 = \frac{\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1}}{\text{Var}(\hat{t}_i - \hat{t}_m)_{VB}}$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1} = (2/r) \sigma^2 = \frac{2}{r+n} \sigma^2$, where t_i and t_m are any two treatments among v treatments, $i \neq m = 1$ to v . $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD2} = (\frac{1}{(r+n)} + \frac{1}{b}) \sigma^2$, where, $i \neq m = 1$ to v and $m = (v+1)$.

$$\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta) \sigma^2 = \frac{2}{\frac{n(k+1) + kr + \lambda}{(k+1)}} \sigma^2 = \frac{2(k+1)}{[n(k+1) + kr + \lambda]} \sigma^2$$

$$E_1 = \frac{[n(k+1) + kr + \lambda]}{(r+n)(k+1)} \text{ and } E_2 = \frac{[n(k+1) + kr + \lambda](b+r+n)}{2b(r+n)(k+1)}$$

Example – 6.2: Construct a variance balanced design with parameters $v_1 = 6$, $b_1 = 15$, $r_1 = \{9, 10\}$, $k_1 = \{3, 5\}$ from a balanced incomplete block design having parameters $v = 6$, $b = 20$, $r = 10$, $k = 3$, $\lambda = 4$.

Blocks of the balanced incomplete block design are.

(1 2 3), (1 2 4), (1 2 5), (1 2 6), (1 3 4), (1 3 5), (1 3 6), (1 4 5), (1 4 6), (1 5 6), (2 3 4), (2 3 5), (2 3 6), (2 4 5), (2 4 6), (2 5 6), (3 4 5), (3 4 6), (3 5 6), (4 5 6). Let $n = \frac{v(r-\lambda)}{(k+1)} = 9$. Hence, nine blocks are augmented. Using Theorem – 6.2, incidence matrix of the variance balanced design is given as.

$$N_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

N_1 gives the incidence matrix of an unequal replicated and un - equal block sizes variance balanced design with parameters $v_1 = 7$, $b_1 = 29$, $r_1 = \{19, 20\}$, $k_1 = \{4, 6\}$; $C_{im} = 10/4$ and information matrix,

$$C = \begin{bmatrix} 19 & 0 & .. & 0 \\ 0 & 19 & .. & 0 \\ : & : & : & : \\ 0 & 0 & .. & 20 \end{bmatrix} - \begin{bmatrix} 48 & 30 & .. & 30 \\ 30 & 48 & .. & 30 \\ : & : & .. & : \\ 30 & 30 & .. & 60 \end{bmatrix} / 12$$

$$\text{After simplification, } C \text{ reduces to } C = \begin{bmatrix} 180 & -30 & -30 \\ -30 & 180 & .. & -30 \\ : & : & : & : \\ -30 & -30 & .. & 180 \end{bmatrix} / 12$$

$$\text{Finally, } C = \frac{210}{12} \left[I_7 - \frac{E_{77}}{7} \right] = \frac{35}{2} \left[I_7 - \frac{E_{77}}{7} \right] = \theta \left[I_7 - \frac{E_{77}}{7} \right] \quad (13)$$

Where, $\theta = 35/2$ is the non zero eigen value of C matrix with multiplicity 6. Hence, it is a variance balanced design with $\hat{t}_i = (1/\theta) Q_i = (2/35) Q_i$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta)\sigma^2 = (4/35)\sigma^2$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1} = (2/r)\sigma^2 = (2/19)\sigma^2$ and $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD2} = (\frac{1}{r+n} + \frac{1}{b})\sigma^2 = (\frac{1}{9} + \frac{1}{10})\sigma^2 = (39/380)\sigma^2$ with efficiency factor, $E_1 = 35/38$ and $E_2 = 273/304$. This shows that efficiency factor is very high.

Nonexistence of variance balanced design by reinforcing $(v + t)$ treatments.

Variance balanced design cannot be constructed from a balanced incomplete block design by reinforcing $(v + t)$ treatments, $t = 1, 2, \dots$

Because for variance balanced design C_{im} must be same for each pair of treatments. In this case $C_{im} = \frac{\lambda}{k+1}$, $i \neq m = 1$ to v and $C_{i(v+1)} = \frac{r}{k+1}$, where, $t = 1$. As per the condition of variance balanced design, $\frac{r}{k+1} = \frac{\lambda}{k+1}$, which is not possible as $(r - \lambda) > 0$. If two treatments are reinforced, then $\frac{r}{k+1} = \frac{\lambda}{k+1} = \frac{b}{k+1}$ must holds true, but $b \neq r \neq \lambda$.

7. Variance balanced design using 2^n symmetrical factorial experiments

This section discusses the construction of variance balanced design using 2^n symmetrical factorial experiment.

Theorem: 7.1 Let us consider a 2^n factorial experiment. By deleting the control treatment and all the main effects, equi-replicated and unequal block sizes variance balanced design is obtained with parameters $v = n$, $b = 2^n - n - 1$, $r = 2^{n-1}$, $k = \{2, 3, 4 \dots, n\}$.

Proof: consider a 2^n treatment combination of a 2^n factorial experiment. Delete its control treatment and all main effects. Consider each treatment combination as one block. So, we have $(2^n - 1 - n)$ blocks with $v (= n)$ treatments, as factors are considered as treatments. Consider this matrix as an incidence matrix of a block design, whose all elements are either zero or 1 only. So, the design is binary.

Since each treatment is repeated $(2^{n-1} - 1)$ times, design is equi replicated and unequal block sizes with $k = \{2, 3, \dots, n\}$. Let the incidence matrix of the block design is given by

$$N = \begin{bmatrix} 0 & 1 & 1 & 1 & \vdots & 1 \\ 1 & 0 & 0 & 1 & \vdots & 1 \\ 1 & 1 & 1 & 0 & \vdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 1 & 1 & \vdots & 1 \end{bmatrix}$$

The incidence matrix of the block design is a variance balanced, if the C matrix of the block design satisfy $C = \theta[I_v - \frac{1}{v}E_{vv}]$, where, θ is non-zero eigen value of C matrix.

$$C = \begin{bmatrix} 2^{n-1} - 1 & 0 & \dots & 0 \\ 0 & 2^{n-1} - 1 & \dots & 0 \\ 0 & 0 & \dots & 2^{n-1} - 1 \end{bmatrix} - \begin{bmatrix} Y & X & \dots & X \\ X & Y & \dots & X \\ \vdots & \vdots & \vdots & \vdots \\ X & X & \dots & Y \end{bmatrix}$$

where, $Y = \frac{\binom{n-1}{1}}{2} + \frac{\binom{n-1}{2}}{3} + \frac{\binom{n-1}{3}}{4} + \dots + \frac{1}{n}$, and

$X = \frac{1}{2} + \frac{n-2}{3} + \frac{n-3}{4} + \dots + \frac{n-(n-1)}{n}$ and $r = 2^{n-1} - 1$

After simplification, C reduces to

$$C = \begin{bmatrix} 2^{n-1} - 1 - Y & -X & \dots & -X \\ -X & 2^{n-1} - 1 - Y & \dots & -X \\ \vdots & \vdots & \vdots & \vdots \\ -X & -X & \dots & 2^{n-1} - 1 - Y \end{bmatrix}$$

Finally,

$$C = (2^{n-1} - 1 - Y + X) \left[I_v - \frac{E_{vv}}{v} \right] \quad (14)$$

Where, $\theta = (2^{n-1} - 1 - Y + X)$ is the non-zero eigen value of C matrix with multiplicity $(v - 1)$. Eq. (14) satisfy the condition of variance balanced design. Hence, this is an equal replicated and unequal block sizes variance balanced design

with parameters $v = n$, $b = 2^n - n - 1$, $r = 2^{n-1} - 1$, $k = \{2, 3, 4 \dots, n\}$. Efficiency factor, $E = \frac{(2^{n-1}-1-Y+X)}{(2^{n-1}-1)}$.

Example: 7.1. Construct a variance balanced design with parameters $v = 4$, $b = 11$, $r = 7$, $k = \{2, 3, \dots, n\}$

Using Theorem 7.1, incidence matrix of the variance balanced design is given by

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 33 & 17 & 17 & 17 \\ 17 & 33 & 17 & 17 \\ 17 & 17 & 33 & 17 \\ 17 & 17 & 17 & 33 \end{bmatrix} / 12$$

After simplification, C reduces to

$$C = \begin{bmatrix} 51 & -17 & -17 & -17 \\ -17 & 51 & -17 & -17 \\ -17 & -17 & 51 & -17 \\ -17 & -17 & -17 & 51 \end{bmatrix} / 12$$

Finally,

$$C = \frac{68}{12} \left[I_4 - \frac{E_{44}}{4} \right] = \frac{17}{3} \left[I_4 - \frac{E_{44}}{4} \right] = \theta \left[I_4 - \frac{44}{4} \right] \quad (15)$$

Where, $\theta = 17/3$ is the non zero eigen value of C matrix with multiplicity 3. Hence, it is a variance balanced design with $\hat{t}_i = (1/\theta) Q_i = (3/17) Q_i$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta) \sigma^2 = (6/17) \sigma^2$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD} = (2/r) \sigma^2 = (2/7) \sigma^2$ and efficiency factor, $E = 17/21$.

This result is due to Ghosh, Sinogia and Ghosh (2018).

8. Construction of variance balanced designs using some incidence matrix

Theorem 8.1: Let I_n denotes the identity matrix of order n , j_n is a column Vector of one, 0_n is the row vectors having all elements zero. An incidence matrix N

defined as $N = \begin{bmatrix} j'_n & 0_{1 \times \frac{n}{2}} \\ I_n & E_{n \times n/2} \end{bmatrix}$

gives the incidence matrix of a Variance balanced designs, with parameters $v = n + 1$, $b = n + \frac{n}{2}$, $r = \{n, 1 + \frac{n}{2}\}$ and $k = \{2, n\}$, where, n is even.

Proof: Proof is obvious.

Example 8.1. Let $n = 6$. So, the incidence matrix N using Theorem 8.1 is given by

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix} / 2$$

After simplification, C reduces to

$$C = \begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 & -1 & -16 \\ -1 & -1 & -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 \end{bmatrix} / 2$$

Finally,

$$C = \frac{7}{2} \left[I_7 - \frac{E_{77}}{7} \right] = \frac{7}{2} \left[I_7 - \frac{E_{77}}{7} \right] = \theta \left[I_7 - \frac{E_{77}}{7} \right] \quad (16)$$

where, $\theta = 7/2$ is the non zero eigen value of C matrix with multiplicity 6. Hence, it is a variance balanced design with parameters $v = 7, b = 9, r = \{6, 4\}, k = \{2, 6\}$. $\hat{t}_i = (1/\theta) Q_i = (2/7) Q_i$; $\text{Var}(\hat{t}_i - \hat{t}_m)_{VB} = (2/\theta)\sigma^2 = (4/7)\sigma^2$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD1} = (2/r)\sigma^2 = (2/4)\sigma^2$, $\text{Var}(\hat{t}_i - \hat{t}_m)_{RBD2} = (\frac{1}{6} + \frac{1}{4})\sigma^2 = (5/12)\sigma^2$ with efficiency factor, $E_1 = 7/8$ and efficiency factor, $E_2 = 35/48$.

This result is due to Ghosh, Sinojia and Ghosh (2018).

9. Conclusions

In this chapter, we have constructed Variance balanced designs using balanced incomplete block, group divisible, resolvable semi - regular group divisible, symmetrical factorial and fractional factorial designs. It is observed that efficiency

factor of all most all variance balanced design is high. Variance balanced designs constructed in sections 3 to 6 are new and extended methods, while Section 7 and 8, discuss the review work of Ghosh, Sinojia and Ghosh (2018).

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References

- [1] Raghavarao D. Constructions and Combinatorial Problems in Design of Experiments. New York: Wiley; 1971
- [2] Chakrabarti MC. On the C-matrix in design of experiments. Journal of the Indian Statistical Association. 1963;1: 8-23. 8C23
- [3] Pearce SC. Experimenting with blocks of natural size. Biometrics. 1964; 20(4):699-706. 699C706. DOI: 10.2307/2528123
- [4] Kageyama S. Reduction of associate classes for block designs and related combinatorial arrangements. Hiroshima Mathematical Journal. 1974;4:527C618
- [5] Rao VR. A note on balanced designs. The Annals of Mathematical Statistics. 1958;29(1):290-294. 290C294. DOI: 10.1214/aoms/1177706729
- [6] Hedayat A, Federer WT. Pairwise and variance balanced incomplete block design. Annals of the Institute of Statistical Mathematics. 1974;26(1): 331-338. DOI: 10.1007/BF02479828
- [7] Raghavarao D. On balanced unequal block designs. Biometrika. 1962;49(3/4):561-562. 561C562. DOI: 10.1093/biomet/49.3-4.561
- [8] Puri PD, Nigam AK. Balanced block designs. Communications in Statistics – Theory and Methods. 1977;6(12): 1171-1179. 1171C1179. DOI: 10.1080/03610927708827560
- [9] John PWM. Balanced designs with unequal number of replications. Annals of Mathematical Statistics. 1964;35: 897C899. DOI: 10.1214/aoms/1177703597
- [10] Jones B, sinha, K., & Kageyama, S. Further equireplicated variance balanced designs with unequal block sizes. Utilitas Mathematica. 1987;32: 5C10
- [11] Kageyama S. Construction of balanced block designs. Utilitas Mathematica. 1976;9:209C229
- [12] Kageyama S. Existence of variance-balanced binary designs with fewer experimental units. Statistics & Probability Letters. 1988;7(1):27-28. 27C28. DOI: 10.1016/0167-7152(88)90083-1
- [13] Kulshreshtha AC, Dey A, Saha GM. Balanced designs with unequal replications and unequal block sizes. Annals of Mathematical Statistics. 1972; 43:1342-1345
- [14] Pal S, Pal S. Nonproper variance balanced designs and optimality. Communications in Statistics - Theory and Methods. 1988;17:1685C1695. DOI: 10.1080/03610928808829706
- [15] Roy J. On the efficiency factor of block designs. Sankhya. 1958;19: 181C188
- [16] Sinha K. A note on equireplicated balanced block designs from BIB designs. Journal of the Indian Society of Agricultural Statistics. 1988;42:150C153
- [17] Sinha K. Some new equireplicated balanced block designs. Statistics & Probability Letters. 1989;8:89. DOI: 10.1016/0167-7152(89)90089-8
- [18] Sinha K, Jones B. Further equireplicated balanced block designs with unequal block sizes. Statistics & Probability Letters. 1988;6:229C330
- [19] Tyagi BN. On a class of variance balanced block designs. Journal of Statistical Planning and Inference. 1979; 3:333-336. DOI: 10.1016/0378-3758(79)90029-6
- [20] Khatri CG. A note on variance balanced designs. Journal of Statistical Planning and Inference. 1982;6(2):

173-177. 173C177. DOI: 10.1016/0378-3758(82)90086-6

[21] Das MN, Ghosh DK. Balancing incomplete block designs. *Sankhyā: The Indian Journal of Statistics, Series B.* 1985;**47**(1):67-77. 67C77

[22] Mukerjee R, Kageyama S. On resolvable and affine resolvable variance-balanced designs. *Biometrika.* 1985;**72**(1):165-172. 165C172. DOI: 10.1093/biomet/72.1.165

[23] Calvin JA. A new class of variance balanced designs. *Journal of Statistical Planning and Inference.* 1986;**14**(2-3): 251-254. DOI: 10.1016/0378-3758(86)90162-X

[24] Calvin JA, Sinha K. A method of constructing variance balanced designs. *Journal of Statistical Planning and Inference.* 1989;**23**(1):127-131. DOI: 10.1016/0378-3758(89)90045-1

[25] Agarwal GG, Kumar S. On a class of variance balanced designs associated with GD designs. *Calcutta Statistics Association Bulletin.* 1984;**33**(3-4): 187-190. 187C190

[26] Ghosh DK. A series of generalized efficiency balanced designs. *Gujarat Statistical Review.* 1988;**15**(1):33-38. 33C38

[27] Ghosh DK, Karmokar PK. Some series of efficiency balanced designs. *Australian & New Zealand Journal of Statistics.* 1988;**30**(1):47-51. 47C51

[28] Ghosh DK, Divecha J. Construction of variance balanced designs. *TEC.R.* No.2, Dept. of Maths. and Stat., Saurashtra Uni. 1989:1-13. 1C13

[29] Ghosh DK, Divecha J, Kageyama S. Equi - replicate variance balanced designs from group divisible designs. *Journal of the Japan Statistical Society.* 1991;**21**(1):205-209. 205C209. DOI: 10.1111/j.1467-842X.1988.tb00610.x

[30] Ghosh DK, Joshi K. Note on construction of variance balanced designs through group divisible designs. *Utilitas Mathematica.* 1991;**39**:249-253. 249C253

[31] Ghosh DK, Karmokar PK, Divecha J. On Comparison of ternary efficiency balanced and ternary variance balanced designs. *Utilitas Mathematica.* 1991;**40**: 25-27

[32] Ghosh DK, Anita S, Kageyama S. Construction of variance-balanced and efficiency balanced ternary block designs. *Journal of Japan Statistical Association Soc.* 1994;**24**(2):201-208

[33] Ghosh DK, Joshi K. Construction of variance balanced designs through triangular PBIB designs. *Calcutta Statistical Association Bulletin.* 1995;**45** (1-2):111-118

[34] Ghosh DK, Sangeeta A. On variance balanced designs. *Journal of Modern Applied Statistical Methods.* 2017;**16**(2): 124-137

[35] Agarwal GG, Kumar S. A note on construction of variance balanced designs. *Journal of the Indian Society of Agricultural Statistics.* 1985;**37**(2): 181-183. 181C183

[36] Agarwal GG, Kumar S. Construction of balanced ternary designs. *Australian and New Zealand Journal of Statistics.* 1986;**28**(2):251-255. 251C255. DOI: 10.1111/j.1467-842X.1986.tb00605.x

[37] Ghosh DK, Joshi K, Kageyama S. Ternary variance balanced designs through BIB and GD designs. *Journal of the Japan Statistical Society.* 1993;**23**(1): 75-81. 75C81

[38] Ghosh DK. Robustness of Variance balanced designs against the loss of one block. *Indian Journal of Statistics and Applications.* 2012;**1**:17-30. 17C30

[39] Ghosh DK, Sinojia CN, Ghosh S. On a class of variance balanced designs.

International Journal of Statistics and Probability. 2018;7(3):112-120

[40] Hedayat A, Stufken JA. Relation between pair wise balanced and variance balanced block designs. Journal of the American Statistical Association. 1989;**84**(407):753-755. DOI: 10.1080/01621459.1989.10478830

[41] Jones RM. On a property of incomplete blocks. Journal of the Royal Statistical Society: Series B. 1959;**21**: 172-179. Available from: <http://www.jstor.org/stable/2983939>

[42] Gupta SC, Jones B. Equi-replicate balanced block designs with unequal block sizes. Biometrika. 1983;**70**(2): 443-440. 443C440. DOI: 10.1093/biomet/70.2.433