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Vertex Decomposability of Path Complexes and Stanley's Conjectures

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Abstract

Monomials are the link between commutative algebra and combinatorics. With a simplicial complex Δ , one can associate two square-free monomial ideals: the Stanley-Reisner ideal I_Δ whose generators correspond to the non-face of Δ , or the facet ideal $I(\Delta)$ that is a generalization of edge ideals of graphs and whose generators correspond to the facets of Δ . The facet ideal of a simplicial complex was first introduced by Faridi in 2002. Let G be a simple graph. The edge ideal $I(G)$ of a graph G was first considered by R. Villarreal in 1990. He studied algebraic properties of $I(G)$ using a combinatorial language of G . In combinatorial commutative algebra, one can attach a monomial ideal to a combinatorial object. Then, algebraic properties of this ideal are studied using combinatorial properties of combinatorial object. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$ -graded ring and M a $\mathbb{Z}^n$ -graded R -module. Then, Stanley conjectured that $\text{depth}(M) \leq \text{sdepth}(M)$. He also conjectured that each Cohen-Macaulay simplicial complex is partition-able. In this chapter, we study the relation between vertex decomposability of some simplicial complexes and Stanley's conjectures.

Keywords: vertex decomposable, simplicial complex, Matroid, path

1. Introduction

Let $R = K[x_1, \dots, x_n]$, where K is a field. Fix an integer $n \geq t \geq 2$ and let G be a directed graph. A sequence $x_{i_1}, \dots, x_{i_t}$ of distinct vertices is called a path of length t if there are $t - 1$ distinct directed edges $e_1, \dots, e_{t-1}$ where e_j is a directed edge from x_{i_j} to $x_{i_{j+1}}$. Then, the path ideal of G of length t is the monomial ideal $I_t(G) = (x_{i_1} \dots x_{i_t} : x_{i_1}, \dots, x_{i_t} \text{ is a path of length } t \text{ in } G)$ in the polynomial ring $R = K[x_1, \dots, x_n]$. The distance $d(x, y)$ of two vertices x and y of a graph G is the length of the shortest path from x to y . The path complex $\Delta_t(G)$ is defined by

$$\Delta_t(G) = \langle \{x_{i_1}, \dots, x_{i_t}\} : x_{i_1}, \dots, x_{i_t} \text{ is a path of length } t \text{ in } G \rangle.$$

Path ideals of graphs were first introduced by Conca and De Negri [1, 2] in the context of monomial ideals of linear type. Recently, the path ideal of cycles has been extensively studied by several mathematicians. In [3], it is shown that $I_2(C_n)$ is sequentially Cohen-Macaulay, if and only if, $n = 3$ or $n = 5$. Generalizing this result, in [4], it is proved that $I_t(C_n)$, ($t > 2$), is sequentially Cohen-Macaulay, if and only if $n = t$ or $n = t + 1$ or $n = 2t + 1$. Also, the Betti numbers of the ideal $I_t(C_n)$ and $I_t(L_n)$ is computed explicitly in [5]. In particular, it has been shown that [6, 7]:

Theorem 1.1 ([1, Corollary 5.15]). *Let n, t, p and d be integers such that $n \geq t \geq 2$, $n = (t + 1)p + d$, where $p \geq 0$ and $0 \leq d < (t + 1)$. Then,*

- i. *The projective dimension of the path ideal of a graph cycle C_n or line L_n is given by,*

$$\text{pd}(I_t(C_n)) = \begin{cases} 2p, & d \neq 0 \\ 2p - 1, & d = 0 \end{cases} \quad \text{pd}(I_t(L_n)) = \begin{cases} 2p - 1, & d \neq t \\ 2p, & d = t \end{cases} \quad (1)$$

- ii. *The regularity of the path ideal of a graph cycle C_n or line L_n is given by,*

$$\begin{aligned} \text{reg}(I_t(C_n)) &= (t - 1)p + d + 1 \\ \text{reg}(I_t(L_n)) &= \begin{cases} p(t - 1) + 1, & d < t \\ p(t - 1) + t, & d = t \end{cases} \end{aligned} \quad (2)$$

In [8] it has been shown that $\Delta_t(G)$ is a simplicial tree if G is a rooted tree and $t \geq 2$. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$ -graded ring and M a $\mathbb{Z}^n$ -graded R -module. Then, Stanley [9] conjectured that

$$\text{depth}(M) \leq \text{sdepth}(M) \quad (3)$$

He also conjectured in [10] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan, and Yassemi in [11–14] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. In this chapter, we first study algebraic properties of $\Delta_t(C_n)$. In Section 1, we recall some definitions and results, which will be needed later. In Section 2, for all $t > 2$ we show that the following conditions are equivalent:

- i. $\Delta_t(C_n)$ is matroid;
- ii. $\Delta_t(C_n)$ is vertex decomposable;
- iii. $\Delta_t(C_n)$ is shellable;
- iv. $\Delta_t(C_n)$ is Cohen-Macaulay;
- v. $n = t$ or $t + 1$.

(see Theorem 2.6).

In Section 3, for all $t \geq 2$ we show that $\Delta_t(G)$ is vertex decomposable if and only if $G = H(p, n, q)$ or $G = H(p, n)$. In Section 4, vertex decomposability path complexes of Dynkin graphs are shown. In Section 5 as an application of our results, we show

that if $n = t$ or $t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$ and $K[\Delta_t(G)]$, where $G = H(p, n, q)$ or $G = H(p, n)$.

2. Preliminaries

In this section, we recall some definitions and results which will be needed later.

Definition 2.1. A simplicial complex Δ over a set of vertices $V = \{x_1, \dots, x_n\}$, is a collection of subsets of V , with the property that:

- a. $\{x_i\} \in \Delta$, for all i ;
- b. if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set).

An element of Δ is called a *face* of Δ and complement of a face F is $V \setminus F$ and it is denoted by F^c . Also, the complement of the simplicial complex $\Delta = \langle F_1, \dots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \dots, F_r^c \rangle$. The *dimension* of a face F of Δ , $\dim F$, is $|F| - 1$ where $|F|$ is the number of elements of F and $\dim \emptyset = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of Δ is a subset F of V with $F \notin \Delta$. We denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ . The maximal faces of Δ under inclusion are called *facets* of Δ . The *dimension* of the simplicial complex Δ , $\dim \Delta$, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called *pure*.

Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ be the facet set of Δ . It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta = \langle F_1, \dots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex Γ is called a *subcomplex* of Δ , if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

For $v \in V$, the subcomplex of Δ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \{F \in \Delta : v \notin F\}. \quad (4)$$

The *link* of a face $F \in \Delta$, denoted by $\text{link}_\Delta(F)$, is a simplicial complex on V with the faces, $G \in \Delta$ such that, $G \cap F = \emptyset$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\text{link}_\Delta(v)$.

$$\text{link}_\Delta(v) = \{F \in \Delta : v \notin F, F \cup \{v\} \in \Delta\}. \quad (5)$$

Let Δ be a simplicial complex over n vertices $\{x_1, \dots, x_n\}$. For $F \subset \{x_1, \dots, x_n\}$, we set:

$$x_F = \prod_{x_i \in F} x_i. \quad (6)$$

We define the *facet ideal* of Δ , denoted by $I(\Delta)$, to be the ideal of S generated by $\{x_F : F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ , denoted by I_Δ , is the ideal of S generated by square-free monomials $\{x_F : F \in \mathcal{N}(\Delta)\}$. Also, we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ .

Definition 2.2. A simplicial complex Δ on $\{x_1, \dots, x_n\}$ is said to be a *matroid* if, for any two facets F and G of Δ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_i\}) \cup \{x_j\}$ is a facet of Δ .

Definition 2.3. A simplicial complex Δ is recursively defined to be *vertex decomposable*, if it is either a simplex, or else has some vertex v so that,

- a. Both $\Delta \setminus v$ and $\text{link}_\Delta(v)$ are vertex decomposable, and
- b. No face of $\text{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.

A vertex v which satisfies in condition (b) is called a *shedding vertex*.

Definition 2.4. A simplicial complex Δ is *shellable*, if the facets of Δ can be ordered $F_1, \dots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \dots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2 . Otherwise, Δ is connected. It is well known that

$$\text{Matroid} \Rightarrow \text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay}$$

Definition 2.5. Given a simplicial complex Δ on V , we define $\Delta^\vee$, the *Alexander dual* of Δ , by

$$\Delta^\vee = \{V \setminus F : F \notin \Delta\}. \quad (7)$$

It is known that for the complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$. Let $I \neq 0$ be a homogeneous ideal of S and $\mathbb{N}$ be the set of non-negative integers. For every $i \in \mathbb{N} \cup \{0\}$, one defines:

$$t_i^S(I) = \max \left\{ j : \beta_{i,j}^S(I) \neq 0 \right\} \quad (8)$$

where $\beta_{i,j}^S(I)$ is the i, j -th graded Betti number of I as an S -module. The *Castelnuovo-Mumford regularity* of I is given by

$$\text{reg}(I) = \sup \{ t_i^S(I) - i : i \in \mathbb{Z} \}. \quad (9)$$

We say that the ideal I has a *d-linear resolution*, if I is generated by homogeneous polynomials of degree d and $\beta_{i,j}^S(I) = 0$, for all $j \neq i + d$ and $i \geq 0$. For an ideal that has a d -linear resolution, the Castelnuovo-Mumford regularity would be d . If I is a graded ideal of S , then we write (I_d) for the ideal generated by all homogeneous polynomials of degree d belonging to I .

Definition 2.6. A graded ideal I is componentwise linear if (I_d) has a linear resolution for all d .

Also, we write $I_{[d]}$ for the ideal generated by the squarefree monomials of degree d belonging to I .

Definition 2.7. A graded S -module M is called *sequentially Cohen-Macaulay* (over K), if there exists a finite filtration of graded S -modules,

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = M \quad (10)$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \dots < \dim(M_r/M_{r-1}). \quad (11)$$

The Alexander dual allows us to make a bridge between (sequentially) Cohen-Macaulay ideals and (componentwise) linear ideals.

Definition 2.8 (Alexander duality). For a square-free monomial ideal $I = (M_1, \dots, M_q) \subset S = K[x_1, \dots, x_n]$, the Alexander dual of I , denoted by $I^\vee$, is defined to be:

$$I^\vee = P_{M_1} \cap \dots \cap P_{M_q} \quad (12)$$

where, P_{M_i} is prime ideal generated by $\{x_j : x_j | M_i\}$.

Theorem 2.1 ([7, Proposition 8.2.20], [5, Theorem 3]). Let I be a square-free monomial ideal in $S = K[x_1, \dots, x_n]$.

- i. The ideal I is componentwise linear ideal if and only if $S/I^\vee$ is sequentially Cohen-Macaulay.
- ii. The ideal I has a q -linear resolution if and only if $S/I^\vee$ is Cohen-Macaulay of dimension $n - q$.

Remark 2.1. Two special cases, we will be considering in this paper, are when G is a cycle C_n , or a line graph L_n on vertices $\{x_1, \dots, x_n\}$ with edges

$$\begin{aligned} E(C_n) &= \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}; \\ E(L_n) &= \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}\} \end{aligned}$$

Remark 2.2. All Cohen-Macaulay simplicial complexes of positive dimension are connected.

3. Vertex decomposability path complexes of cycles

As the main result of this section, it is shown that $\Delta_t(C_n)$ is matroid, vertex decomposable, shellable, and Cohen-Macaulay if and only if $n = t$ or $n = t + 1$. For the proof, we shall need the following lemmas and propositions.

Lemma 3.1. Let $\Delta_t(P_n)$ be a simplicial complex on the path $P_n = \{x_1, \dots, x_n\}$ and $2 \leq t \leq n$. Then $\Delta_t(P_n)$ is vertex decomposable.

Proof. If $t = n$, then $\Delta_n(P_n)$ is a simplex which is vertex decomposable. Let $2 \leq t < n$ then one has

$$\Delta_t(P_n) = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{n-t+1}, \dots, x_n\} \rangle. \quad (13)$$

So $\Delta_t(P_n) \setminus x_n = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{n-t}, \dots, x_{n-1}\} \rangle$. Now, we use induction on the number of vertices of P_n and by induction hypothesis $\Delta_t(P_n) \setminus x_n$ is vertex decomposable. On the other hand, it is clear that $\text{link}_{\Delta_t(P_n)} \{x_n\} = \langle \{x_{n-t+1}, \dots, x_{n-1}\} \rangle$. Thus, $\text{link}_{\Delta_t(P_n)} \{x_n\}$ is a simplex which is not a facet of $\Delta_t(P_n) \setminus x_n$. Therefore, $\Delta_t(P_n)$ is vertex decomposable.

Lemma 3.2. Let $\Delta_2(C_n)$ be a simplicial complex on $\{x_1, \dots, x_n\}$. Then $\Delta_2(C_n)$ is vertex decomposable.

Proof. Since $\Delta_2(C_n) = \langle \{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\} \rangle$ then we have

$$\Delta_2(C_n) \setminus x_n = \langle \{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-2}, x_{n-1}\} \rangle. \quad (14)$$

By Lemma 2.1 $\Delta_2(C_n) \setminus x_n$ is vertex decomposable. Also, it is trivial that $\text{link}_{\Delta_2(C_n)} \{x_n\} = \langle \{x_{n-1}\}, \{x_1\} \rangle$ is vertex decomposable and no face

of $\text{link}_{\Delta_2(C_n)}\{x_n\}$ is a facet of $\Delta_2(C_n)\setminus x_n$. Therefore, $\Delta_2(C_n)$ is vertex decomposable. $\square$

Lemma 3.3. *Let $\Delta_t(C_n)$ be a simplicial complex on $\{x_1, \dots, x_n\}$ and $3 \leq t \leq n - 2$. Then $\Delta_t(C_n)$ is not Cohen-Macaulay.*

Proof. It suffices to show that $I_{\Delta_t(C_n)^\vee}$ has not a linear resolution. Since $I_{\Delta_t(C_n)^\vee} = I(\Delta_t(C_n)^c)$, then one can easily check that $I_{\Delta_t(C_n)^\vee} = I_{n-t}(C_n)$. By Theorem 1.1 we have

$$\text{reg}\left(I_{\Delta_t(C_n)^\vee}\right) = (n - t - 1)p + d + 1. \quad (15)$$

Since $3 \leq t \leq n - 2$ then one has $\text{reg}\left(I_{\Delta_t(C_n)^\vee}\right) \neq n - t$ and by Theorem 2.1 $\Delta_t(C_n)$ is not Cohen-Macaulay.

Proposition 3.1. *Let $\Delta_t(C_n)$ be a simplicial complex on $\{x_1, \dots, x_n\}$ and $t \geq 3$. Then $\Delta_t(C_n)$ is vertex decomposable if and only if $n = t$ or $t + 1$.*

Proof. By Lemma 3.3, it suffices to show that if $n = t$ or $t + 1$, then $\Delta_t(C_n)$ is vertex decomposable. If $n = t$, then $\Delta_n(C_n)$ is a simplex which is vertex decomposable.

If $t = n - 1$, then we have

$$\Delta_{n-1}(C_n) = \langle \{x_1, \dots, x_{n-1}\}, \{x_2, \dots, x_n\}, \{x_3, \dots, x_n, x_1\}, \dots, \{x_n, x_1, \dots, x_{n-2}\} \rangle.$$

Now, we use induction on the number of vertices of C_n and show that $\Delta_{n-1}(C_n)$ is vertex decomposable. It is clear that $\Delta_{n-1}(C_n)\setminus x_n = \langle \{x_1, \dots, x_{n-1}\} \rangle$ is a simplex, which is vertex decomposable.

On the other hand,

$$\text{link}_{\Delta_{n-1}(C_n)}\{x_n\} = \langle \{x_1, \dots, x_{n-2}\}, \dots, \{x_{n-1}, x_1, \dots, x_{n-3}\} \rangle = \Delta_{n-2}(C_{n-1}). \quad (16)$$

By induction hypothesis $\text{link}_{\Delta_{n-1}(C_n)}\{x_n\}$ is vertex decomposable. It is easy to see that no face of $\text{link}_{\Delta_{n-1}(C_n)}\{x_n\}$ is a facet of $\Delta_{n-1}(C_n)\setminus x_n$. Therefore, $\Delta_{n-1}(C_n)$ is vertex decomposable.

Proposition 3.2. *$\Delta_2(C_n)$ is a matroid if and only if $n = 3$ or 4 .*

Proof. If $n = 3$ or 4 , then it is easy to see that $\Delta_2(C_n)$ is a matroid. Now, we prove the converse. It suffices to show that $\Delta_2(C_n)$ is not a matroid for all $n \geq 5$. We consider two facets $\{x_1, x_2\}$ and $\{x_{n-1}, x_n\}$. Then, we have $(\{x_1, x_2\} \setminus \{x_1\}) \cup \{x_{n-1}\} = \{x_2, x_{n-1}\}$ and $(\{x_1, x_2\} \setminus \{x_1\}) \cup \{x_n\} = \{x_2, x_n\}$. Since $\{x_2, x_{n-1}\}$ and $\{x_2, x_n\}$ are not the facets of $\Delta_2(C_n)$. So $\Delta_2(C_n)$ is not matroid for all $n \geq 5$.

For the simplicial complexes, one has the following implication:

$$\text{Matroid} \Rightarrow \text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen - Macaulay}$$

Note that these implications are strict, but by the following theorem, for path complexes, the reverse implications are also valid.

Theorem 3.1. *Let $t \geq 3$. Then the following conditions are equivalent:*

- i. $\Delta_t(C_n)$ is matroid;
- ii. $\Delta_t(C_n)$ is vertex decomposable;
- iii. $\Delta_t(C_n)$ is shellable;
- iv. $\Delta_t(C_n)$ is Cohen-Macaulay;
- v. $n = t$ or $t + 1$.

Proof. (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) is well known.

(iv) $\Rightarrow$ (v): Follows from Lemma 3.3 and Proposition 2.1.

(v) $\Rightarrow$ (i): If $n = t$, then $\Delta_t(C_n)$ is a simplex which is a matroid.

If $n = t + 1$, then

$$\Delta_t(C_n) = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \{x_3, \dots, x_{t+1}, x_1\}, \dots, \{x_{t+1}, x_1, \dots, x_{t-1}\} \rangle.$$

For any two facets F and G of $\Delta_t(C_n)$ one has $|F \cap G| = t - 1$. We claim that for any two facets F and G of $\Delta_t(C_n)$ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_i\}) \cup \{x_j\}$ is a facet of $\Delta_t(C_n)$. We have to consider two cases. If $x_i \in F$ and $x_i \notin G$, then we choose $x_j \in G$ such that $x_j \notin F$. Thus, $(F \setminus \{x_i\}) \cup \{x_j\} = G$ which is a facet of $\Delta_t(C_n)$.

For other case, if $x_i \in F$ and $x_i \in G$, then we choose $x_j \in G$ such that x_j is the same x_i . Therefore, $(F \setminus \{x_i\}) \cup \{x_i\} = F$ is a facet of $\Delta_t(C_n)$, which completes the proof.

4. Vertex decomposability path complexes of trees

As the main result of this section, for all $t \geq 2$, we characterize all such trees whose $\Delta_t(G)$ is vertex decomposable. Let $H(p, n, q)$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the path P_n and q pendant vertices to the other pendant vertex of P_n . Also, let $H(p, n)$ be graph obtained by attaching p pendant vertices to one pendant vertex of the path P_n .

Remark 4.1. Let $P_n = \{x_1, \dots, x_n\}$ be a path on vertices $\{x_1, \dots, x_n\}$ and $H(2, n)$ be a graph obtained by attaching two pendant vertices to pendant vertex x_n . Then, $\Delta_t(H(2, n))$ is vertex decomposable for all $t \geq 2$.

Proof. By Lemma 3.1 proof is trivial.

Proposition 4.1. Let $P_n = \{x_1, \dots, x_n\}$ be a path on vertices $\{x_1, \dots, x_n\}$ and $H(p, n)$ be a graph obtained by attaching p pendant vertices to pendant vertex x_n . Then $\Delta_t(H(p, n))$ is vertex decomposable for all $t \geq 2$.

Proof. We prove the proposition by induction on p the number of pendant vertices to pendant vertex x_n of P_n . If $p = 0$ or 1 , then $H(p, n)$ is a path and by Lemma 2.1 $\Delta_t(H(p, n))$ is vertex decomposable. If $p = 2$, then by remark 4.1 $\Delta_t(H(p, n))$ is vertex decomposable. Now, let $p > 2$ and $\{y_1, \dots, y_p\}$ be p pendant vertices to pendant vertex x_n of P_n , then one has

$$H(p, n) \setminus \{y_1\} = H(p - 1, n) \quad (17)$$

and

$$\Delta_t(H(p, n)) \setminus \{y_1\} = \Delta_t(H(p - 1, n)). \quad (18)$$

Therefore by induction hypothesis $\Delta_t(H(p - 1, n))$ is vertex decomposable. So $\Delta_t(H(p, n)) \setminus \{y_1\}$ is vertex decomposable. If $t = 3$, then we have

$$\text{link}_{\Delta_3(H(p, n))} \{y_1\} = \langle \{x_{n-1}, x_n\}, \{y_2, x_n\}, \dots, \{y_p, x_n\} \rangle \quad (19)$$

It is easy to see that $\text{link}_{\Delta_3(H(p, n))} \{y_1\}$ is vertex decomposable and y_1 is a shedding vertex. If $t = 2$ or $t > 3$, one has

$$\text{link}_{\Delta_t(H(p, n))} \{y_1\} = \langle \{x_{n-t+2}, \dots, x_n\} \rangle. \quad (20)$$

thus $\text{link}_{\Delta_t(H(p,n))} \{y_1\}$ is a simplex, which is not a facet of $\Delta_t(H(p,n)) \setminus \{y_1\}$, therefore $\Delta_t(H(p,n))$ is vertex decomposable.

Lemma 4.1. *Let $p = 2$ and $q \geq 2$, Then $\Delta_t(H(2,n,q))$ is vertex decomposable for all $2 \leq t \leq n + 2$.*

Proof. Let $H(2,n,q)$ denote the double starlike tree obtained by attaching two pendant vertices $\{y_1, y_2\}$ to pendant vertex x_1 of path P_n and $\{y'_1, \dots, y'_q\}$ be pendant vertices to pendant vertex x_n of P_n . So by proposition 3.2 $\Delta_t(H(2,n,q)) \setminus \{y_1\}$ is vertex decomposable. Now, we prove that $\text{link}_{\Delta_t(H(2,n,q))} \{y_1\}$ is vertex decomposable. If $t = 3$, then $\text{link}_{\Delta_3(H(2,n,q))} \{y_1\} = \langle \{x_1, x_2\}, \{x_1, y_2\} \rangle$ which is vertex decomposable. If $t = n + 2$, then

$$\text{link}_{\Delta_{n+2}(H(2,n,q))} \{y_1\} = \langle \{x_1, \dots, x_n, y'_1\}, \{x_1, \dots, x_n, y'_2\}, \dots, \{x_1, \dots, x_n, y'_q\} \rangle.$$

It is easy to see that $\text{link}_{\Delta_{n+2}(H(2,n,q))} \{y_1\}$ is vertex decomposable. If $t = 2$ or $4 \leq t \leq n + 1$, then we have $\text{link}_{\Delta_t(H(2,n,q))} \{y_1\} = \langle \{x_1, \dots, x_{t-1}\} \rangle$. Thus, $\text{link}_{\Delta_t(H(2,n,q))} \{y_1\}$ is a simplex, which is vertex decomposable. It is clear that y_1 is a shedding vertex. $\square$

Proposition 4.2. *Let Q_1, Q_2 be two paths of maximum length k in tree G and y be a leaf of G such that $y \in Q_1 \cap Q_2$, $|Q_1 \cap Q_2| = L$. Then $\Delta_t(G)$ is not vertex decomposable.*

Proof. Suppose $Q_1 = \{y_1, y_2, \dots, y_{k-L}, x_1, x_2, \dots, x_{L-1}, y\}$ and

$Q_2 = \{y'_1, y'_2, \dots, y'_{k-L}, x_1, x_2, \dots, x_{L-1}, y\}$ be two paths of length k in G such that $Q_1 \cap Q_2 = \{x_1, x_2, \dots, x_{L-1}, y\}$ and $\deg(y) = 1$. Since $\text{link}_{\Delta_k(G)} \{x_1, \dots, x_{L-1}, y\}$ is disconnected and pure of positive dimension. By remark 1.11 $\Delta_k(G)$ is not Cohen-Macaulay and hence, $\Delta_k(G)$ is not vertex decomposable. $\square$

Proposition 4.3. *Let G be a double starlike tree such that is not a path. Then $\Delta_t(G)$ is vertex decomposable for all $2 \leq t \leq n + 2$.*

Proof. Let $G = H(p,n,q)$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the path P_n and q pendant vertices to the other pendant vertex of P_n . We prove the theorem by induction on p the number of pendant vertices to pendant vertex x_1 of P_n . If $p = 0$ or $p = 1$, then by proposition 3.2 $\Delta_t(G)$ is vertex decomposable. If $p = 2$, then by Lemma 4.3 $\Delta_t(G)$ is vertex decomposable. Now, let $p > 2$ and $\{y_1, \dots, y_p\}$ be p pendant vertices to pendant vertex x_1 of P_n . Since $G \setminus \{y_1\}$ is again double starlike tree on $p - 1$ pendant vertices. Therefore, by induction hypothesis, $\Delta_t(G \setminus \{y_1\})$ is vertex decomposable. So $\Delta_t(G \setminus \{y_1\}) = \Delta_t(G) \setminus \{y_1\}$ is vertex decomposable. Let $t = 2$, then $\text{link}_{\Delta_2(G)} \{y_1\} = \langle \{x_1\} \rangle$ is simplex and vertex decomposable. Let $t = 3$, then $\text{link}_{\Delta_3(G)} \{y_1\} = \langle \{x_2, x_1\}, \{y_2, x_1\}, \dots, \{y_p, x_1\} \rangle$ is vertex decomposable. Let $3 < t \leq n + 1$, then $\text{link}_{\Delta_t(G)} \{y_1\} = \langle \{x_1, x_2, \dots, x_{t-1}\} \rangle$ is simplex and vertex decomposable. Let $t = n + 2$, then $\text{link}_{\Delta_{n+2}(G)} \{y_1\} = \langle \{x_1, \dots, x_n, y_1\}, \{x_1, \dots, x_n, y_2\}, \dots, \{x_1, \dots, x_n, y_p\} \rangle$ is a path complex of a starlike tree which is vertex decomposable. It is easy to see that no face of $\text{link}_{\Delta_t(G)} \{y_1\}$ is a facet of $\Delta_t(G) \setminus \{y_1\}$. So $\Delta_t(G)$ is vertex decomposable.

Now, we are ready that prove the main result of this section.

Theorem 4.1. *Let G be a tree such that is not a path. Then $\Delta_t(G)$ is vertex decomposable for all $t \geq 2$ if and only if $G = H(p,n,q)$ or $H(p,n)$.*

Proof. ($\Rightarrow$) We prove by contradiction. Suppose $G \neq H(p,n,q)$ and $G \neq H(p,n)$. So there exists two paths of maximum length k in G which contain L common

vertices such that one of these vertices is a leaf. Therefore, by proposition 4.2 $\Delta_k(G)$ is not vertex decomposable, which is a contradiction. ($\Leftarrow$) By proposition 4.1 and Proposition 4.3, the proof is completed.

5. Vertex decomposability path complexes of dynkin graphs

Dynkin diagrams first appeared [15] in the connection with classification of simple Lie groups. Among Dynkin diagrams a special role is played by the simply laced Dynkin diagrams A_n, D_n, E_6, E_7 , and E_8 . Dynkin diagrams are closely related to coxeter graphs that appeared in geometry (see [16]).

After that, Dynkin diagrams appeared in many branches of mathematics and beyond, in particular in representation theory. In [17], Gabriel introduced a notion of a quiver (directed graph) and its representations. He proved the famous Gabriel's theorem on representation of quivers over algebraic closed field. Let Q be a finite quiver and $\overline{Q}$ the undirected graph obtained from Q by deleting the orientation of all arrows.

Theorem 5.1. (Gabriel theorem). *A connected quiver Q is of Finite type if and only if the graph $\overline{Q}$ is one of the following simply laced Dynkin diagrams: A_n, D_n, E_6, E_7 or E_8 .*

Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_j, y_j\}$ be whisker of L_n at x_j with $3 \leq j \leq n-1$.

We obtain some condition that $\Delta_t(L_n \cup W(x_j))$ is vertex decomposable, as an application of our results, vertex decomposability path complexes of Dynkin graphs are shown. Throughout this section, we assume $L_n \cup W(x_j)$ be an undirected graph. By Lemma 3.1, we have the following corollary:

Corollary 5.1. *Let A_n be a Dynkin graph and $2 \leq t \leq n$. Then $\Delta_t(A_n)$ is vertex decomposable.*

Proposition 5.1. *Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_{n-1}, y_{n-1}\}$ be whisker of L_n at x_{n-1} .*

Then, $\Delta_t(L_n \cup W(x_{n-1}))$ is vertex decomposable for all $2 \leq t \leq n$.

Proof. Then by proposition 4.1 proof is trivial.

Corollary 5.2. *Let D_n be a Dynkin graph and $n \geq 4$. Then $\Delta_t(D_n)$ is vertex decomposable for all $2 \leq t \leq n$.*

Proof. We know that $D_n = L_n \cup W(x_{n-1})$. So by proposition 4.3 $\Delta_t(D_n)$ is vertex decomposable. $\square$

Theorem 5.2. *Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_j, y_j\}$ be whisker of L_n at x_j with $3 \leq j \leq n-2$.*

Then, $\Delta_t(L_n \cup W(x_j))$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $n \geq t > \alpha$, where $\alpha = \min \{d(y_j, x_1), d(y_j, x_n)\}$.

Proof. We first show that $\Delta_t(L_n \cup W(x_j))$ is not vertex decomposable for all $4 \leq t \leq \alpha$. It is well known that if Δ is a Cohen-Macaulay simplicial complex, then $\text{link}_\Delta\{F\}$ is Cohen-Macaulay for each face F of Δ .

Also, we know that all Cohen-Macaulay complexes of positive dimension are connected. It is easy to see that $\text{link}_{\Delta_t(L_n \cup W(x_j))}\{x_j, y_j\}$ is disconnected and pure of positive dimension.

This implies that $\Delta_t(L_n \cup W(x_j))$ is not Cohen-Macaulay and hence $\Delta_t(L_n \cup W(x_j))$ is not vertex decomposable without loss of generality we can assume that $\alpha = d(y_j, x_1)$ if $t = 2$ or $n \geq t > \alpha$, one has:

$$\Delta_t(L_n \cup W(x_j)) = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{j-1}, x_j, \dots, x_{j+t-2}\}, \{y_j, x_j, x_{j+1}, \dots, x_{j+t-2}\}, \dots, \{x_{n-t+1}, \dots, x_n\} \rangle$$

So $\Delta_t(L_n \cup W(x_j)) \setminus y_j = \Delta_t(L_n)$ and by Lemma [4, 6, 7, 9, 10, 18–20] $\Delta_t(L_n \cup W(x_j)) \setminus y_j$ is vertex decomposable. On the other hand, we have $\text{link}_{\Delta_t(L_n \cup W(x_j))} \{y_j\} = \langle \{x_j, x_{j+1}, \dots, x_{j+t-2}\} \rangle$ that is a simplex and vertex decomposable.

If $t = 3$, then

$$\Delta_3(L_n \cup W(x_j)) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \dots, \{x_{j-1}, x_j, y_j\}, \{y_j, x_j, x_{j+1}\}, \dots, \{x_{n-2}, x_{n-1}, x_n\} \rangle$$

and $\Delta_3(L_n \cup W(x_j)) \setminus y_j = \Delta_3(L_n)$ which is vertex decomposable.

It is easy to see that $\text{link}_{\Delta_t(L_n \cup W(x_j))} \{y_j\} = \langle \{x_{j-1}, x_j\}, \{x_j, x_{j+1}\} \rangle$ is vertex decomposable and y_j is a shedding vertex.

Corollary 5.3. *Let E_6 be a Dynkin graph. Then $\Delta_t(E_6)$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $t = 5$.*

Proof. Since $E_6 = L_5 \cup W(x_3)$, so by Theorem 5.2 the proof is completed.

Corollary 5.4. *Let E_7 be a Dynkin graph. Then $\Delta_t(E_7)$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $5 \leq t \leq 6$.*

Proof. We know that $E_7 = L_6 \cup W(x_3)$ and the proof follow from Theorem 5.2.

Corollary 5.5. *Let E_8 be a Dynkin graph. Then $\Delta_t(E_8)$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $5 \leq t \leq 7$.*

6. Stanley decompositions

Let R be any standard graded K -algebra over an infinite field K , i.e., R is a finitely generated graded algebra $R = \bigoplus_{i \geq 0} R_i$ such that $R_0 = K$ and R is generated by R_1 . There are several characterizations of the depth of such an algebra. We use the one that $\text{depth}(R)$ is the maximal length of a regular R -sequence consisting of linear forms. Let $x_F = \prod_{i \in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \dots, x_n\}$. The K -subspace $x_F K[Z]$ of $S = K[x_1, \dots, x_n]$ is the subspace generated by monomials $x_F u$, where u is a monomial in the polynomial ring $K[Z]$. It is called a squarefree Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let Δ be a simplicial complex on $\{x_1, \dots, x_n\}$. A squarefree Stanley decomposition $\mathcal{D}$ of $K[\Delta]$ is a finite direct sum $\bigoplus_i u_i K[Z_i]$ of squarefree Stanley spaces, which is isomorphic as a $\mathbb{Z}^n$ -graded K -vector space to $K[\Delta]$, i.e.

$$K[\Delta] \cong \bigoplus_i u_i K[Z_i] \quad (21)$$

We denote by $\text{sdepth}(\mathcal{D})$ the minimal dimension of a Stanley space in $\mathcal{D}$ and we define $\text{sdepth}(K[\Delta]) = \max \{\text{sdepth}(\mathcal{D})\}$, where $\mathcal{D}$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in [9] the upper bound for the depth of $K[\Delta]$ as the following:

$$\text{depth}(K[\Delta]) \leq \text{sdepth}(K[\Delta]) \quad (22)$$

Also, we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1, \dots, x_n\}$ with facets $G_1, \dots, G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \cup_{i=1}^t [F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ . Here, the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [10, 18] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable [16–20]. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan, and Yassemi [14] proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1, \dots, x_n\}$ we have that $\text{depth}(K[\Delta]) \leq \text{sdepth}(K[\Delta])$ if and only if Δ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable [4, 6, 7, 19, 20]. Then as a consequence of our results, we obtain:

Corollary 6.1. *if $n = t$ or $t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$.*

Corollary 6.2. *Let G be a tree such that is not a path. if $G = H(p, n, q)$ or $G = H(p, n)$ then $\Delta_t(G)$ is vertex decomposable for all $t \geq 2$ and Stanley's conjecture holds for $K[\Delta_t(G)]$.*

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