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β_I -Compactness, β_I^* -Hyperconnectedness and β_I -Separatedness in Ideal Topological Spaces

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Abstract

Let (X, τ, I) be an ideal topological space. A subset A of X is said to be β -open if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$, and it is said to be β_I -open if there is a set $O (\in \tau)$ with the property (1) $O - A \in I$ and (2) $A - \text{cl}(\text{int}(\text{cl}(O))) \in I$. The set A is called β_I -compact if every cover of A by β_I -open sets has a finite sub-cover. The set A is said to be $c\beta_I$ -compact, if every cover $\{O_\lambda : \lambda \in \Lambda\}$ of A by β -open sets, Λ has a finite subset Λ_0 such that $A - \cup \{O_\lambda : \lambda \in \Lambda_0\} \in I$. The set A is said to be countably β_I -compact if every countable cover of A by β_I -open sets has a finite sub-cover. An ideal topological space (X, τ, I) is said to be β_I^* -hyperconnected if $X - \text{cl}^*(A) \in I$ for every non-empty β_I -open subset A of X . Two subsets A and B of X is said to be β_I -separated if $\text{cl}_{\beta_I}(A) \cap B = \emptyset = A \cap \text{cl}_{\beta_I}(B)$. Moreover, A is called a β_I -connected set if it can't be written as a union of two β_I -separated subsets. An ideal topological space (X, τ, I) is called β_I -connected space if X is β_I -connected. In this article, we give some important properties of β_I -open sets, β_I -compact spaces, $c\beta_I$ -compact spaces, β_I^* -hyperconnected spaces, and β_I -connected spaces.

Keywords: β -open set, β_I -open set, β_I -compact, $c\beta_I$ -compact, β_I^* -hyperconnected

1. Introduction

Let (X, τ) be a topological space. A subset A of X is said to be a β -open set [1] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. For example, consider the topology $(X, \tau) = (\{a, b, c\}, \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\})$. Then $\emptyset, X, \{a, b\}, \{a, c\}$ are the β -open sets of (X, τ) . A subset A of X is said to be *semi-open* set [1] if $A \subseteq \text{cl}(\text{int}(A))$. A subset A of X is said to be α -open set [2] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. A subset A of X is said to be *pre-open* set [3] if $A \subseteq \text{int}(\text{cl}(A))$. A subset A of X is said to be *regular-open* set [4] if $A = \text{int}(\text{cl}(A))$. A subset A of X is said to be β^* -open set [5] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A))) \cup \text{int}(\text{cl}_\delta(A))$ (please see [5] for the notation $\text{cl}_\delta(A)$). A subset A of X is said to be $\hat{\beta}$ -generalized-closed set [6] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq O$ whenever $A \subseteq O$ and O is open in X .

An ideal I on a set X is a nonempty collection of subsets of X which satisfies the conditions: (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Let (X, τ) be a topological space and I be an ideal in X . Then we call (X, τ, I) an ideal topological space. For example, let $X = \{a, b, c\}$. Then $I = \{\emptyset, \{a\}\}$ is an ideal on X . To see this, we note that the subsets of $\emptyset$ is itself, and the subsets of $\{a\}$ are $\{a\}$ and $\emptyset$. Note that all of these subsets are in I . Next, we observe that $\emptyset \cup \emptyset = \emptyset \in I$, $\emptyset \cup \{a\} = \{a\} \in I$ and $\{a\} \cup \{a\} = \{a\} \in I$. Thus, $I = \{\emptyset, \{a\}\}$ is an ideal on X .

For the concepts that were not discussed here please refer to [5, 7–13].

Topology is a new subject of mathematics, being born in the nineteenth century. However, the involvement of topology is clear in the other branches of math [12].

Topology is also seen in some fields of science. In particular, it is applied in biochemistry [14] and information systems [15].

Topology as a mathematical system is fundamentally comprised of open sets, among others. Open sets were generalized in a couple of different ways over the past. To mention a few, Stone [4] presented regular open set. Levine [1] presented semi-open sets. Najasted [2] presented α -open sets. Mashhour et al. [3] presented pre-open sets. Abd El-Monsef et al. [7] presented β -open set. Among these generalization, this study focused on one—the β -open sets.

Abd El-Monsef et al. [7] also presented the concepts β -continuous and β -open mappings. They gave some of their properties. Recently, β -open sets were investigated by many math enthusiast. For example, Abid [16] utilized β -open sets to gain some properties of non-semi-predense set. Tahiliani [13] presented an operation involving β -open sets which paved way to the creation of β - γ -open sets. Kannan and Nagaveni [6] generalized β -open set, and named it $\hat{\beta}$ -generalized closed set. Mubarki et al. [5] also generalized β -open set, and named it β^* -open set. El-Mabhouth and Mizyed [17] also generalized β -open set, and named it βc -open set. Akdag and Ozkan [8] made an investigation of β -open sets in soft topological spaces. Arockiarani and Arokia Lancy [9] introduced $g\beta$ -closed set and $gs\beta$ -closed set (these were defined using β -open sets).

The notion of ideal topological spaces was introduced by Kuratowski [18]. Later, Vaidyanathaswamy [19] studied the concept in point set topology. Tripathy and Shravan [20, 21], Tripathy and Acharjee [22], Tripathy and Ray [23], among others, were also some of those who studied ideal topological spaces.

This study have important applications in some areas of mathematics. In particular, β_I -compactness, β_I^* -hyperconnectedness and β_I -separatedness can be investigated in the areas of measure theory, continuum theory and dimension theory just as the parallel notions (compactness, hypercompactness, and separatedness, respectively) were studied in those areas. The purpose of this paper is to introduce and study a notion of connectedness, hypercompactness, and separatedness relative to the family of all β -open sets in some ideal topological spaces.

2. β_I -compactness in ideal spaces

In this section, we gave some important properties of β_I -open sets in τ_ω -spaces.

Recall, a topological space (X, τ) is said to be a τ_ω -space if for every subset A of X , it is always true that $\text{int}(\text{cl}(A)) = \text{int}(\text{cl}(\text{int}A))$. For example, let $X = \{w, x, y, z\}$. Then $\tau_1 = \{\emptyset, \{w\}, \{w, x\}, \{w, y\}, \{w, x, y\}, X\}$ is a τ_ω -space, while $\tau_2 = \{\emptyset, \{w, x, y\}, X\}$ is not. Also, a discrete space is a τ_ω -space, while an indiscrete space is not.

Lemma 1.1 characterizes β -open sets in a τ_ω -space.

Lemma 1.1. Let (X, τ) be a τ_ω -space and I be an ideal in X . A set $A (\subseteq X)$ is a β -open set precisely if there is a set $O (\in \tau)$ with the property that $O \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(O)))$.

Proof: Suppose that A is a β -open set. Then $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. Consider $O = \text{int}(A)$ (note that O is open). Since (X, τ) is a τ_ω -space, $\text{int}(\text{cl}(A)) = \text{int}(\text{cl}(\text{int}(A)))$. Hence, $O \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(A))) = \text{cl}(\text{int}(\text{cl}(\text{int}(A)))) = \text{cl}(\text{int}(\text{cl}(O)))$.

Conversely, suppose that there is a set $O (\in \tau)$ with the property that $O \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(O)))$. Since $O \subseteq A$, we have $\text{cl}(O) \subseteq \text{cl}(A)$. And so, $\text{int}(\text{cl}(O)) \subseteq \text{int}(\text{cl}(A))$. Therefore, $\text{cl}(\text{int}(\text{cl}(O))) \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. Thus, $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. □

Next, we define β_I -open set.

Definition 1.1. Let (X, τ) be a topological space and I be an ideal in X . A subset A of X is called β -open with respect to the ideal I , or a β_I -open set, if there exists an open set O such that (1) $O - A \in I$, and (2) $A - \text{cl}(\text{int}(\text{cl}(O))) \in I$.

For example, let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, and $I = \{\emptyset, \{b\}\}$ (note that τ is a topology on X , and I is an ideal on X). Then $A = \{b, c\}$ is a β_I -open set. To see this, consider $O = \{b, c\}$. Then O is an open set. Observe that $O - A = \{b, c\} - \{b, c\} = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(O))) = \{b, c\} - \text{cl}(\text{int}(\text{cl}(\{b, c\}))) = \{b, c\} - \text{cl}(\text{int}(\{b, c\})) = \{b, c\} - \text{cl}(\{b, c\}) = \{b, c\} - \{b, c\} = \emptyset \in I$. Thus, $A = \{b, c\}$ is β -open with respect to the ideal I .

Lemma 1.2 says that an open set is a β_I -open set, and an element of the ideal is a β_I -open set. One may see [24] to gain more insights relative to these ideas. While, Lemma 1.3 says that in a τ_ω -space a β -open set is also a β_I -open set.

Lemma 1.2. Let (X, τ) be a topological space and I be an ideal in X . Then the following statements are true.

- i. Every open set is a β_I -open set.
- ii. Every element of I is a β_I -open set.

Proof: (i) Let A be an open set. Note that $A - A = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(A))) \subseteq A - \text{cl}(A) = \emptyset \in I$. Thus, A is β_I -open. (ii) Let $A \in I$. Consider $O = \emptyset$. Note that $O - A = \emptyset - A = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(O))) = A - \emptyset = A \in I$. Thus, A is β_I -open. □

Lemma 1.3. Let (X, τ) be a τ_ω -space and I be an ideal in X . Then every β -open set is a β_I -open set.

Proof: Let A be a β -open set. By Lemma 1.1 there exists an open set O such that $O \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(O)))$. Hence $O - A = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(O))) = \emptyset \in I$. Thus, A is β_I -open. □

Let (X, τ) be a topology and I be an ideal in X . We say that I is countably additive if $\cup \{A_i : i \in \mathbb{N}\} \in I$ whenever $\{A_i : i \in \mathbb{N}\}$ is a (countable) family of elements of I .

Lemma 1.4 says that in a τ_ω -space, if I is the minimal ideal, then the β_I -open sets are precisely the β -open sets.

Lemma 1.4. Let (X, τ) be a τ_ω -space and I be an ideal in X . If I is not countably additive, then the following statements are equivalent.

- i. $I = \{\emptyset\}$.
- ii. A is a β -open set precisely when A is a β_I -open set.

Proof: (i) $\Rightarrow$ (ii) Suppose that $I = \{\emptyset\}$. Let A be a β -open set. By Lemma 1.3, A is a β_I -open set. For the converse, let A be a β_I -open set and O be an open set with $O - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(O))) \in I$. Because $I = \{\emptyset\}$, we have $O - A = \emptyset$ and $A - \text{cl}(\text{int}(\text{cl}(O))) = \emptyset$. Hence, $O \subseteq A$ and $A \subseteq \text{cl}(\text{int}(\text{cl}(O)))$, that is $O \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(O)))$. Therefore, by Lemma 1.1 A is β -open.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds, and that $I \neq \{\emptyset\}$. Let D be a non-empty element of I . By Lemma 1.2, D is β_I -open. Thus, by assumption D is β -open. Now, by Lemma 1.1, there exists $O_1 \in \tau$ with $O_1 \subseteq D \subseteq \text{cl}(\text{int}(\text{cl}(O_1)))$. Since D is an element of I and $O_1 \subseteq D$, we have $O_1 \in I$. Hence, $O_1 \cup D \in I$. By Lemma 1.1, $O_1 \cup D$ is a β_I -open set. Hence, by assumption $O_1 \cup D$ is a β -open set. And so, again there exists $O_2 \in \tau$ with $O_2 \subseteq (O_1 \cup D) \subseteq \text{cl}(\text{int}(\text{cl}(O_2)))$. Since $O_1 \cup D \in I$ and $O_2 \subseteq O_1 \cup D$, we have $O_2 \in I$. Hence, $O_1 \cup O_2 \cup D \in I$. Thus, by Lemma 1.1, $O_1 \cup O_2 \cup D$ is a β_I -open set. By assumption $O_1 \cup O_2 \cup D$ is a β -open set. And so, again there exists $O_3 \in \tau$ with $O_3 \subseteq (O_1 \cup O_2 \cup D) \subseteq \text{cl}(\text{int}(\text{cl}(O_3)))$. Since $O_1 \cup O_2 \cup D \in I$ and $O_3 \subseteq O_1 \cup O_2 \cup D$, we have $O_3 \in I$. Hence, $O_1 \cup O_2 \cup O_3 \cup D \in I$. Continuing in this fashion we obtain a countably infinite subset $\{O_1, O_2, O_3, \dots\}$ of I with $O_1 \cup O_2 \cup O_3 \cup \dots \in I$. This is a contradiction since I is not countably additive. Thus, $I = \{\emptyset\}$. $\square$

Next, we define β_I -compact set, β_I -compact space, compatible β_I -compact set, and compatible β_I -compact space.

Definition 1.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be β_I -compact if for every cover $\{O_\lambda : \lambda \in \Lambda\}$ of A by β_I -open sets, Λ has a finite subset Λ_0 , such that $\{O_\lambda : \lambda \in \Lambda_0\}$ still covers A . A space X is said to be a β_I -compact space if it is β_I -compact as a subset.

Definition 1.3. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be countably β_I -compact if for every countable cover $\{O_n : n \in \mathbb{N}\}$ of A by β_I -open sets, $\mathbb{N}$ has a finite subset $\{i_j : j = 1, 2, \dots, k\}$ with the property that $\{O_{i_j} : j = 1, 2, \dots, k\}$ still covers A . A space X is said to be a countably β_I -compact space if it is countably β_I -compact as a subset.

Definition 1.4. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be compatible β_I -compact, or simply $c\beta_I$ -compact, if for every cover $\{O_\lambda : \lambda \in \Lambda\}$ of A by β -open sets, Λ has a finite subset Λ_0 , such that $A - \cup \{O_\lambda : \lambda \in \Lambda_0\} \in I$. An ideal topological space (X, τ, I) is said to be $c\beta_I$ -compact space if it is $c\beta_I$ -compact as a subset.

Theorem 1.1 says that in an ideal τ_ω -space in which I is the minimal ideal, the notions β -compact, β_I -compact and $c\beta_I$ -compact coincides.

Theorem 1.1. Let (X, τ) be a τ_ω -space and $I = \{\emptyset\}$. Then the following statements are equivalent.

- i. (X, τ, I) is a β -compact space.
- ii. (X, τ, I) is a β_I -compact space.
- iii. (X, τ, I) is a $c\beta_I$ -compact space.

Proof: (i) $\Rightarrow$ (ii) Suppose that (i) holds. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β -open sets that covers X . By assumption, Λ has a finite subset, say Λ_0 , with the property that $\{O_\lambda : \lambda \in \Lambda_0\}$ still covers X . By Lemma 1.3 (iii), $\{O_\lambda : \lambda \in \Lambda_0\}$ is also a family of β_I -open sets. Hence, $\{O_\lambda : \lambda \in \Lambda_0\}$ is a finite covering of X by β_I -open sets. Therefore, X is β_I compact.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β -open sets that covers X . Since $I = \{\emptyset\}$, by Lemma 1.4 $\{O_\lambda : \lambda \in \Lambda\}$ is also a family of β_I -open sets that covers X . By assumption, Λ has a finite subset, say Λ_0 , with the property that $\{O_\lambda : \lambda \in \Lambda_0\}$ still covers X . Note that $\{U_\lambda : \lambda \in \Lambda_0\}$ is also a family of β -open sets, and $X - \cup_{\lambda \in \Lambda_0} O_\lambda = \emptyset \in I$. Therefore, X is $c\beta_I$ compact.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β -open sets that covers X . By assumption, Λ has a finite subset, say Λ_0 , with the property that

$X - \cup_{\lambda \in \Lambda_0} O_\lambda \in I$. Since $I = \{\emptyset\}$, $X - \cup_{\lambda \in \Lambda_0} O_\lambda = \emptyset$, that is $X \subseteq \cup_{\lambda \in \Lambda_0} O_\lambda$. Hence, $\{O_\lambda : \lambda \in \Lambda_0\}$ is covering of X . Therefore, X is β compact. $\square$

Theorem 1.2 presents a characterization of β_I -compact spaces.

Theorem 1.2. *Let (X, τ, I) be an ideal topological space. Then the following are equivalent.*

i. (X, τ, I) is β_I -compact.

ii. If $\{F_\lambda : \lambda \in \Lambda\}$ is a family of β_I -closed sets with $\cap\{F_\lambda : \lambda \in \Lambda\} = \emptyset$, then Λ has a finite subset, say Λ_0 , with the property $\cap\{F_\lambda : \lambda \in \Lambda_0\} = \emptyset$.

Proof: (i) $\Rightarrow$ (ii) Suppose that (i) holds. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of β_I -closed sets with the property $\cap\{F_\lambda : \lambda \in \Lambda\} = \emptyset$. Then $\cup\{F_\lambda^C : \lambda \in \Lambda\} = (\cap\{F_\lambda : \lambda \in \Lambda\})^C = X$. Hence, $\{F_\lambda^C : \lambda \in \Lambda\}$ is a family of β_I -open sets which covers X . By assumption, Λ has a finite subset, say Λ_0 , with the property $\cup\{F_\lambda^C : \lambda \in \Lambda_0\} = X$, i.e. $\cap\{F_\lambda : \lambda \in \Lambda_0\} = \emptyset$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β_I -open sets that covers X , i.e. $\cup\{O_\lambda : \lambda \in \Lambda\} = X$. Then $\cap\{O_\lambda^C : \lambda \in \Lambda\} = (\cup\{O_\lambda : \lambda \in \Lambda\})^C = \emptyset$. Note that O^C is β_I -closed since O is β_I -open. By assumption, Λ has a finite subset, say Λ_0 , with the property that $\cap\{O_\lambda^C : \lambda \in \Lambda_0\} = \emptyset$. Thus, $\cup\{O_\lambda : \lambda \in \Lambda_0\} = (\cap\{O_\lambda^C : \lambda \in \Lambda_0\})^C = X$, that is $\{O_\lambda : \lambda \in \Lambda_0\}$ is a family of β_I -open sets that covers X . $\square$

Theorem 1.3 presents a characterization of β_I -compact spaces.

Theorem 1.3. *Let (X, τ) be a topological space and I be an ideal in X . Then the following are equivalent.*

i. (X, τ, I) is $c\beta_I$ -compact.

ii. If $\{F_\lambda : \lambda \in \Lambda\}$ is a family of β -closed sets with $\cap\{F_\lambda : \lambda \in \Lambda\} = \emptyset$, then Λ has a finite subset, say Λ_0 , with the property that $\cap\{F_\lambda : \lambda \in \Lambda_0\} \in I$.

Proof: (i) $\Rightarrow$ (ii) Suppose that (i) holds. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of β -closed sets such that $\cap\{F_\lambda : \lambda \in \Lambda\} = \emptyset$. Note that $\cup\{F_\lambda^C : \lambda \in \Lambda\} = (\cap\{F_\lambda : \lambda \in \Lambda\})^C = X$. Hence, $\{F_\lambda^C : \lambda \in \Lambda\}$ is a family of β -open sets that covers X . By assumption, Λ has a finite subset, say Λ_0 , with the property $X - \cup\{F_\lambda^C : \lambda \in \Lambda_0\} \in I$, i.e. $\cap\{F_\lambda : \lambda \in \Lambda_0\} \in I$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β -open sets that covers X , i.e. $\cup\{O_\lambda : \lambda \in \Lambda\} = X$. Note that $\cap\{O_\lambda^C : \lambda \in \Lambda\} = (\cup\{O_\lambda : \lambda \in \Lambda\})^C = \emptyset$. By assumption, Λ has a finite subset, say Λ_0 , with the property $\cap\{O_\lambda^C : \lambda \in \Lambda_0\} \in I$, i.e. $X - \cup\{O_\lambda : \lambda \in \Lambda_0\} \in I$. $\square$

Remark 1.1. [25] *Let (X, τ, I) and (Y, σ, J) be ideal spaces, and $f : X \rightarrow Y$ be a fuction. Then:*

i. $f(I) = \{f(A) : A \in I\}$ is an ideal in Y , where $f(A) = \{f(a) : a \in A\}$; And,

ii. if f is bijective, then $f^{-1}(J) = \{f^{-1}(B) : B \in J\}$ is an ideal in X , where $f^{-1}(B) = \{f^{-1}(b) : b \in B\}$.

Next, we define β_I -open, β_I -irresolute, and β_I -continuous functions.

Definition 1.5. Let (X, τ, I) and (Y, σ, J) be ideal topological spaces. A function $f : X \rightarrow Y$ is said to be

- i. β -open if $f(A)$ is β -open for every β -open set A ,
- ii. β -irresolute if $f^{-1}(B)$ is β -open for every β -open set B , and
- iii. β -continuous if $f^{-1}(B)$ is β -open for every open set B .
- iv. β_I -open if $f(A)$ is β_J -open for every β_I -open set A ,
- v. β_I -irresolute if $f^{-1}(B)$ is β_I -open for every β_J -open set B , and
- vi. β_I -continuous if $f^{-1}(B)$ is β_I -open for every open set B .

Theorem 1.4 says that given a β -irresolute function, if the domain is compatibly compact, then so is the image of f . On the other hand, Theorem 1.5 say that given an open surjection, if the co-domain is compatibly compact, then so is the domain.

Theorem 1.4. Let (X, τ) and (Y, σ) be topological spaces, I be an ideal in X , and $f : X \rightarrow Y$ be a β -irresolute function. If X is a $c\beta_I$ -compact space, then $(f(X), \{f(X) \cap B : B \in \sigma\}, f(I))$ is a $c\beta_{f(I)}$ -compact space.

Proof: Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β -open sets that covers $f(X)$. Because f is a β -irresolute, $\{f^{-1}(O_\lambda) : \lambda \in \Lambda\}$ is a family of β -open sets that covers X . Because X is $c\beta_I$ -compact, Λ has a finite subset, say Λ_0 , with the property $X - \bigcup \{f^{-1}(O_\lambda) : \lambda \in \Lambda_0\} \in I$. Hence, by Remark 1.1 $f(X) - \bigcup \{O_\lambda : \lambda \in \Lambda_0\} = f(X - \bigcup \{f^{-1}(O_\lambda) : \lambda \in \Lambda_0\}) \in f(I)$. $\square$

Theorem 1.5. Let (X, τ) and (Y, σ) be topological spaces, J be an ideal in Y , and $f : X \rightarrow Y$ be a β -open surjection (surjective function). If Y is $c\beta_J$ -compact, then X is $c\beta_{f^{-1}(J)}$ -compact.

Proof: Let $\{O_\lambda : \lambda \in \Lambda\}$ be a family of β -open sets that covers X . Because f is a β -open surjection, $\{f(O_\lambda) : \lambda \in \Lambda\}$ is a family β -open sets that covers Y . Because Y is $c\beta_J$ -compact, Λ has a finite subset, say Λ_0 , with the property $Y - \bigcup \{f(O_\lambda) : \lambda \in \Lambda_0\} \in J$. Hence, $X - \bigcup \{O_\lambda : \lambda \in \Lambda_0\} = f^{-1}(Y - \bigcup \{f(O_\lambda) : \lambda \in \Lambda_0\}) \in f^{-1}(J)$. $\square$

The next theorem says that in a τ_ω -space and when $I = \{\emptyset\}$, the family of all countably β_I -compact space contains all $c\beta_I$ -compact space.

Theorem 1.6. Let (X, τ, I) be an ideal τ_ω -space and $I = \{\emptyset\}$. If X is $c\beta_I$ -compact, then it is also countably β_I -compact.

Proof: Let $\{O_n : n \in \mathbb{N}\}$ be a countable family β_I -open sets that covers X . Because X is $c\beta_I$ -compact, $\mathbb{N}$ has a finite subset $\{i_j : j = 1, 2, \dots, k\}$ with the property that $X - \bigcup \{O_{i_j} : j = 1, 2, \dots, k\} \in I$. Because $I = \{\emptyset\}$, $X = \bigcup \{O_{i_j} : j = 1, 2, \dots, k\} \in I$.

By Lemma 1.4 $\{O_{i_j} : j = 1, 2, \dots, k\}$ is also a family of β -open sets. Hence,

$\{O_{i_j} : j = 1, 2, \dots, k\}$ is a finite subcover of X by β -open sets. $\square$

3. β_I^* -hyperconnectedness in ideal spaces

The concept $*$ -hyperconnectedness was introduced by Ekici et al. [26], and the concept $I*$ -hyperconnectedness was introduced by Abd El-Monsef et al. [27].

These insights motivated us to create the concept called β_I^* -hyperconnectedness. One may see [28] to gain more insights on these ideas.

Definition 1.6. Let (X, τ) be a topological space and I be an ideal on X . A function $(\)^*(I, \tau) : P(X) \rightarrow P(X)$ given by $A^*(I, \tau) = \{x \in X : A \cup U \notin I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ is called a local of A with respect to τ and I .

Example 1.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ (note that τ is a topology on X and I is an ideal on X). Then, $\emptyset^* = \emptyset$, $\{a\}^* = \{c\}$, $\{b\}^* = \{c\}$, $\{c\}^* = X$, $\{a, b\}^* = \{c\}$, $\{a, c\}^* = X$, $\{b, c\}^* = X$ and $X^* = X$.

Definition 1.7. Let (X, τ) be a topological space and I be an ideal on X . The Kuratowski closure operator $Cl(\)^*(I, \tau) : P(X) \rightarrow P(X)$ for the topology $\tau^*(I, \tau)$ is given by $Cl(A)^*(I, \tau) = A \cup A^*$.

Example 1.2. Consider the ideal space of Example 3. Then we have,
 $Cl(\emptyset)^* = \emptyset \cup \emptyset^* = \emptyset \cup \emptyset = \emptyset$, $Cl(\{a\})^* = \{a\} \cup \{a\}^* = \{a\} \cup \{c\} = \{a, c\}$,
 $Cl(\{b\})^* = \{b\} \cup \{b\}^* = \{b\} \cup \{c\} = \{b, c\}$, $Cl(\{c\})^* = \{c\} \cup \{c\}^* = \{c\} \cup X = X$,
 $Cl(\{a, b\})^* = \{a, b\} \cup \{a, b\}^* = \{a, b\} \cup \{c\} = X$, $Cl(\{a, c\})^* = \{a, c\} \cup \{a, c\}^* = \{a, c\} \cup X = X$,
 $Cl(\{b, c\})^* = \{b, c\} \cup \{b, c\}^* = \{b, c\} \cup X = X$, and $Cl(X)^* = X \cup X^* = X \cup X = X$.

Definition 1.8. Let (X, τ) be a topological space and I be an ideal on X . The Kuratowski interior operator $Int(\)^*(I, \tau) : P(X) \rightarrow P(X)$ for the topology $\tau^*(I, \tau)$ is given by $Int(A)^*(I, \tau) = X - Cl(X - A)^*$.

Definition 1.9 is taken from [26], while Definition 1.10 is taken from [29].

Definition 1.9. [26] An ideal space (X, τ, I) is called $*$ -hyperconnected if $cl^*(A) = X$ for all non-empty open set $A \subseteq X$.

Definition 1.10. [29] An ideal space (X, τ, I) is called I^* -hyperconnected if $X - cl^*(A) \in I$ for all non-empty open set $A \subseteq X$.

A notion similar to Definition 1.9 and Definition 1.10 is presented next.

Definition 1.11. An ideal topological space (X, τ, I) is said to be β_I^* -hyperconnected space if $X - cl^*(A) \in I$ for every non-empty β_I -open subset A of X .

The next theorem says that the family of all β_I^* -hyperconnected space contains all I^* -hyperconnected space.

Theorem 1.7. Let (X, τ) be a topological space, and I be an ideal in X . If X is I^* -hyperconnected, then it is β_I^* -hyperconnected also.

Proof: Let X be I^* -hyperconnected, and A be a non-empty open set. Because X is I^* -hyperconnected, we have $X - cl(A)^* \in I$ for all non-empty open set $A \subseteq X$. And, because an open set is also a β_I -open set, we have $X - cl(A)^* \in I$ for all non-empty β_I -open set $A \subseteq X$. Hence, X is β_I^* -hyperconnected. $\square$

The next lemma is clear.

Lemma 1.5. Let (X, τ) be a topological space. Then the intersection of any family of ideals on X is an ideal on X .

Theorem 1.8 is taken from [29]. It says that when I is the minimal ideal, then the notions $*$ -hyperconnected and I^* -hyperconnected are equivalent.

Theorem 1.8. [29] Let (X, τ) be a topological space, and $I = \{\emptyset\}$. Then, X is $*$ -hyperconnected if and only if it is I^* -hyperconnected.

The next remark is clear.

Remark 1.2. If (X, τ) is a clopen topological space (a space in which every open set is also closed), then A is open if and only if A is β -open.

To see this, let A be an open set. Since τ is clopen, A is closed also. Hence, $cl(int(cl(A))) = A$. Thus, A is a β -open set. Conversely, if A is a β -open set, then $A \subseteq cl(int(cl(A)))$. This implies that A must be open.

Theorem 1.9 says that in a clopen τ_ω -space, with respect to the minimal ideal I , the notions β_I^* -hyperconnected and I^* -hyperconnected are equivalent.

Theorem 1.9. Let (X, τ) be a clopen τ_ω -space, and $I = \{\emptyset\}$. Then, X is I^* -hyperconnected if and only if it is β_I^* -hyperconnected.

Proof: Suppose that X is I^* -hyperconnected. Let A is a non-empty element of τ . Then $X - cl^*(A) \in I$. By Remark 1.2 and Lemma 1.5, every open set is absolutely β_I -open. Thus, $X - cl^*(A) \in I$ for all β_I -open set A ($\neq \emptyset$). Therefore, X is β_I^* -hyperconnected also. Conversely, suppose that X is β_I^* -hyperconnected. Let A be a non-empty β_I -open set. Then $X - cl^*(A) \in I$. By Remark 1.2 and Lemma 1.5, every β_I -open set is absolutely open. Thus, $X - cl(A)^* \in I$ for all open set A ($\neq \emptyset$). Therefore, X is I^* -hyperconnected also. $\square$

Corollary 1.1 says that in a clopen τ_ω -space, relative to the minimal ideal I , the notions β_I^* -hyperconnected, I^* -hyperconnected, and $*$ -hyperconnected are equivalent.

Corollary 1.1. Let (X, τ) be a clopen τ_ω -space and $I = \{\emptyset\}$. Then the following statements are equivalent.

- i. X is I^* -hyperconnected.
- ii. X is β_I^* -hyperconnected.
- iii. X is β_I^* -hyperconnected.

Theorem 1.3 may be an important property.

Remark 1.3. If an ideal τ_ω space $(X, \tau, \{\emptyset\})$ is a β_I^* -hyperconnected space, then $X - cl^*(A) \in I$ for every non-empty β -open subset A of X .

To see this, let A is a non-empty β -open set. By Lemma 1.4 A is β_I -open. Since X is β_I^* -hyperconnected, $X - cl^*(A) \in I$.

Theorem 1.10 is a characterization of β_I^* -hyperconnected space.

Theorem 1.10. Let (X, τ) be an topological space and I be an ideal in X . Then the following statements are equivalent.

- i. X is a β_I^* -hyperconnected space.
- ii. $\text{Int}(A)^* \in I$ for all proper β_I -closed subset A of X .

Proof: (i) $\Rightarrow$ (ii) Suppose that (i) holds. Let B be β_I -closed. Then $X - B$ is β_I -open. Since $B \neq X$, $X - B \neq \emptyset$. Hence, by assumption we have $\text{Int}(B)^* = X - \text{Cl}(X - B)^* \in I$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Let A ($\neq X$) be a non-empty β_I -open set. Then $X - A$ is a non-empty β_I -open set. Hence, by assumption we have $X - \text{cl}(A)^* = X - \text{cl}(X - (X - A))^* = \text{int}(X - A)^* \in I$. Thus, X is β_I^* -hyperconnected. $\square$

4. β_I -separatedness in ideal spaces

In this section, we present the concepts β_I -separated sets and β_I -connected sets. We also present some of their important properties.

Let (X, τ, I) be an ideal topological space and A be a subset of X . The β -closure of A , denoted by $cl_\beta(A)$, is the smallest β -closed set that contains A . The β_I -closure of A , denoted by $cl_{\beta_I}(A)$, is the smallest β_I -closed set that contains A .

Next, we define β_I -separated set, β_I -connected set, and β_I -connected space.

Definition 1.12. Let (X, τ, I) be an ideal topological space. A pair of subsets, say A and B , of X is said to be β_I -separated if $cl_{\beta_I}(A) \cap B = \emptyset = A \cap cl_{\beta_I}(B)$.

Definition 1.13. Let (X, τ, I) be an ideal topological space and A be a subset of X . Then A is said to be β_I -connected if it cannot be expressed as a union of two β_I -separated sets. A topological space X is said to be β_I -connected if it is β_I -connected as a subset.

Recall, a topological space (X, τ) is said to be a τ_ζ -space if for every pair of subsets A and B of X , it is always true that $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$ and $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. For example, a discrete space is a τ_ζ -space, while an indiscrete space is not. Also, if $X = \{a, b, c\}$, then $\tau = \{\emptyset, \{a\}, X\}$ is not a τ_ζ -space. Let (X, τ) be a τ_ζ -space and I be an ideal in X . Then we call (X, τ, I) an ideal τ_ζ -space.

Lemma 1.6 present sufficient conditions for two sets to be β_I -separated.

Lemma 1.6. Let (X, τ) be a topological space and I be an ideal in X . If $A (\neq \emptyset)$ is β -open and $B (\neq \emptyset)$ is β_I -open with $A \cap B = \emptyset$, then they are β_I -separated.

Proof: Suppose that A and B is not β_I -separated, that is $\text{cl}_{\beta_I}(A) \cap B \neq \emptyset$ or $A \cap \text{cl}_{\beta_I}(B) \neq \emptyset$. Because $A \cap B = \emptyset$, we have $A \subseteq B^C$ and $B \subseteq A^C$. If A is β -open, then A^C is β -closed. Similarly, if B is β_I -open, then B^C is β_I -closed. Thus, $B^C \cap B \supseteq \text{cl}_{\beta_I}(A) \cap B \neq \emptyset$, or $A \cap A^C \supseteq A \cap \text{cl}_{\beta_I}(B) \neq \emptyset$. A contradiction. $\square$

Lemma 1.7 says that in a τ_ω -space every β_I -connected space is connected. Recall, a space is connected if it cannot be written as a union of two non-empty open sets.

Lemma 1.7. Let (X, τ) be a topology and I be an ideal in X . If X is β_I -connected, then it is connected.

Proof: Suppose that to the contrary X is not connected. Let A and B be non-empty disjoint elements of τ with $X = A \cup B$. Note that A and B are β -open and β_I -open also. Because $A = B^C$ and $B = A^C$, A and B are also β -closed and β_I -closed. And so, $A = \text{cl}_{\beta_I}(A)$ and $B = \text{cl}_{\beta_I}(B)$. Thus, $\text{cl}_{\beta_I}(A) \cap B = A \cap B = \emptyset$ and $A \cap \text{cl}_{\beta_I}(B) = A \cap B = \emptyset$. This implies that X is β_I -separated, that is X is not β_I -connected. $\square$

Remark 1.4. Let (X, τ) be a topology and I be an ideal in X . If $Y \subseteq X$, then $I_Y = \{Y \cap A : A \in I\}$ is an ideal in the relative topology (Y, τ_Y) .

To see this, for the first property, let $B \in I_Y$ and $A \subseteq B$. Then $A \subseteq B \subseteq Y$. Now, if $A \in I_Y$, then there exist $C \in I$ such that $Y \cap C = A$. Note that $A \subseteq B \subseteq C$. Hence, $A, B \in I$. Thus, $A = Y \cap A \in I_Y$. Next, for the second, let $D, E \in I_Y$. Then $D \subseteq Y$ and $E \subseteq Y$. if $D \in I_Y$, then there exist $F \in I$ such that $Y \cap F = D$. Similarly, if $E \in I_Y$, then there exist $G \in I$ such that $Y \cap G = E$. Since I is an ideal, $F \cup G \in I$. Now, because $D \cup E \subseteq F \cup G$, $D \cup E \in I$. Thus, $D \cup E = (D \cup E) \cap Y \in I_Y$.

The next statement, Theorem 1.11, presents a way to construct β_I -open sets in a subspace.

Theorem 1.11. Let (X, τ, I) be an ideal τ_ζ -space and Y be a clopen (a set that is open and closed at the same time) set. If A is a β_I -open subset of X (τ, I), then $A \cap Y$ is a β_{I_Y} -open set in Y (τ_Y, I_Y).

Proof: Let A be a β_I -open set in X (τ, I). Then there exists an open set U' such that $U' - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(U')))) \in I$. Let $U = U' \cap Y$. Then

$$\begin{aligned} U - (A \cap Y) &= U \cap (A \cap Y)^C \\ &= (U' \cap Y) \cap (A^C \cup Y^C) \\ &= (U' \cap Y \cap A^C) \cup (U' \cap Y \cap Y^C) \\ &= U' \cap Y \cap A^C \\ &= (U' - A) \cap Y \in I_Y. \end{aligned} \quad (1)$$

Moreover, since X is a τ_ζ -space and Y is clopen

$$\begin{aligned} (A \cap Y) - \text{cl}(\text{int}(\text{cl}(U))) &= (A \cap Y) - \text{cl}(\text{int}(\text{cl}(U' \cap Y))) \\ &= (A \cap Y) - \text{cl}(\text{int}(\text{cl}(U'))) \cap Y \\ &= [A - \text{cl}(\text{int}(\text{cl}(U')))] \cap Y \in I_Y. \end{aligned} \quad (2)$$

This shows that $A \cap Y$ is β_{I_Y} -open in $Y(\tau_Y, I_Y)$. $\square$

Corollary 1.2. Let (X, τ, I) be an ideal τ_ζ -space and Y be a clopen set. If A is a β_I -closed subset of $X(\tau, I)$, then $A \cap Y$ is a β_{I_Y} -closed set in $Y(\tau_Y, I_Y)$.

Proof: If A is β_I -closed, then A^C is β_I -open. By Theorem 1.11, $A^C \cap Y$ is β_{I_Y} -open. Hence, $A \cap Y = (A^C \cap Y)^C$ is β_{I_Y} -closed in Y . $\square$

The next remark is clear. We shall be using it in showing some of the succeeding theorems.

Remark 1.5. Let (X, τ, I) be an ideal topological space and $Y \subseteq X$. Then $I_Y = \{A \cap Y : A \in I\}$ is a subset of I .

Proof: If A is a β_{I_Y} -open set in Y , then there exists an open set $O \in \tau_Y$ with $O - A \in I_Y$ and $A - \text{cl}(\text{int}(\text{cl}(O))) \in I_Y$. Because $\tau_Y \subseteq \tau$ and by Remark 1.5, there exists an open set $O \in \tau$ with $O - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(O))) \in I$. Thus, A is β_I -open in $X(\tau, I)$.

The converse follows from Theorem 1.11. $\square$

The next statement, Lemma 1.8, characterizes β_I -open sets in subspaces.

Lemma 1.8. Let (X, τ, I) be an ideal τ_ζ -space, $Y (\subseteq X)$ be clopen, and $\tau_Y \subseteq \tau$. If $A \subseteq Y$, then A is β_{I_Y} -open in $Y(\tau_Y, I_Y)$ if and only if it is β_I -open in $X(\tau, I)$.

Proof: If A is a β_{I_Y} -open set in Y , then there exists an open set $O \in \tau_Y$ with $O - A \in I_Y$ and $A - \text{cl}(\text{int}(\text{cl}(O))) \in I_Y$. Because $\tau_Y \subseteq \tau$ and by Remark 1.5, there exists an open set $O \in \tau$ with $O - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(O))) \in I$. Thus, A is β_I -open in $X(\tau, I)$.

The converse follows from Theorem 1.11. $\square$

The next statement, Theorem 1.12, provides a way of determining the closure of a set in the subspace.

Theorem 1.12. Let (X, τ, I) be an ideal τ_ζ -space, Y be clopen, and $\tau_Y \subseteq \tau$. If $A \subseteq X$, then $\text{cl}_{\beta_{I_Y}}(A \cap Y) = \text{cl}_{\beta_I}(A) \cap Y$.

Proof: Since $\text{cl}_{\beta_I}(A)$ is a β_I -closed set in X , by Lemma 1.8 $\text{cl}_{\beta_I}(A) \cap Y$ is a β_{I_Y} -closed set in Y . Hence, $\text{cl}_{\beta_{I_Y}}(A \cap Y) \supseteq \text{cl}_{\beta_{I_Y}}(\text{cl}_{\beta_I}(A) \cap Y) = \text{cl}_{\beta_I}(A) \cap Y$. But, $\text{cl}_{\beta_{I_Y}}(A \cap Y) = \text{cl}_{\beta_I}(A \cap Y) \subseteq \text{cl}_{\beta_I}(A) \cap \text{cl}_{\beta_I}(Y) = \text{cl}_{\beta_I}(A) \cap Y$. Therefore, $\text{cl}_{\beta_{I_Y}}(A \cap Y) = \text{cl}_{\beta_I}(A) \cap Y$. $\square$

Definition 1.14. Let (X, τ, I) be an ideal topological space, and (Y, τ_Y, I_Y) be a subspace. A pair of subsets, say A and B , of X is said to be β_{I_Y} -separated in Y if $\text{cl}_{\beta_{I_Y}}(A) \cap B = \emptyset = A \cap \text{cl}_{\beta_Y}(B)$, where $\text{cl}_{\beta_Y}(B) = \text{cl}_\beta(B) \cap Y$.

Definition 1.15. Let (X, τ, I) be an ideal topological space, and (Y, τ_Y, I_Y) be a subspace. A subset A of Y is said to be β_{I_Y} -connected if it cannot be expressed as a union of two β_{I_Y} -separated sets. The subspace Y is said to be β_{I_Y} -connected if it is β_{I_Y} -connected as a subset.

The next statement, Theorem 1.13, says that if two sets are separated in the mother space, then they are also separated in the subspace.

Theorem 1.13. Let (X, τ) be a τ_ζ -space, I be an ideal, $Y (\subseteq X)$ be clopen, and $\tau_Y \subseteq \tau$. If A and B are β_I -separated in X , then they are β_{I_Y} -separated in Y .

Proof: If A and B are β_I -separated in X , then by Theorem 1.12 $\emptyset = \text{cl}_{\beta_I}(A) \cap B = (\text{cl}_{\beta_I}(A) \cap Y) \cap B = \text{cl}_{\beta_{I_Y}}(A) \cap B$ and $\emptyset = A \cap \text{cl}_\beta(B) = (A \cap \text{cl}_\beta(B)) \cap Y = A \cap (\text{cl}_\beta(B) \cap Y) = A \cap \text{cl}_{\beta_Y}(B)$. Thus, A and B are β_{I_Y} -separated. $\square$

The next statement, Remark 1.6, says that if two non-empty sets, which expresses X as a disjoint union, is β_I -separated, then one must be β -open and the other must be β_I -open.

Remark 1.6. Let (X, τ) be a topological space and I be an ideal. If X is β_I -separated (say, $X = A \cup B$ with $A \neq \emptyset$, $B \neq \emptyset$, and $\text{cl}_{\beta_I}(A) \cap B = \emptyset = A \cap \text{cl}_\beta(B)$), then A is β -open while B is β_I -open.

To see this, if A and B is β_I -separated, then $\text{cl}_{\beta_I}(A) \cap B = \emptyset$ and $A \cap \text{cl}_\beta(B) = \emptyset$. Hence, $A^C = \text{cl}_\beta(B)$ and $B^C = \text{cl}_{\beta_I}(A)$. Thus, A^C is β -closed and B^C is β_I -closed. Accordingly, A is β -open and B is β_I -open.

The next statement, Theorem 1.14, characterizes β_I -connected spaces.

Theorem 1.14. *Let (X, τ) be a topological space and I be an ideal X . Then, X is β_I -connected if and only if it cannot be expressed as a union of two a non-empty disjoint sets in which one is a β -open set and the other is a β_I -open set.*

Proof: Suppose that X is β_I -connected, and we can express X as a union of two non-empty disjoint β -open set and β_I -open set, say $A \cup B = X$ (with A , a β -open set, and B , a β_I -open set) and $A \cap B = \emptyset$. If $A \cup B = X$ and $A \cap B = \emptyset$, then $A^c = B$ and $B^c = A$. Since A is β -open, B is β -closed. Also, since B is β_I -open, A is β_I -closed. Hence, $\text{cl}_{\beta_I}(A) \cap B = A \cap B = \emptyset$ and $A \cap \text{cl}_{\beta}(B) = A \cap B = \emptyset$. Thus, the pair A and B is β_I -separated. This is a contradiction.

The converse follows from Remark 1.6. □

The next statement, Theorem 1.15, says that two separated set cannot contain portions of a connected set.

Theorem 1.15. *Let (X, τ) be a topological space, I be an ideal X , and A be a β_I -connected set. If $A \subseteq H \cup G$ where H and G is a pair of β_I -separated sets, then either $A \subseteq H$ or $A \subseteq G$.*

Proof: Suppose that to the contrary, $A = (A \cap H) \cup (A \cap G)$ with $A \cap H \neq \emptyset$ and $A \cap G \neq \emptyset$. Since H and G is a pair of β_I -separated sets, $\text{cl}_{\beta_I}(A \cap H) \cap (A \cap G) \subseteq \text{cl}_{\beta_I}(H) \cap (G) = \emptyset$ and $(A \cap H) \cap \text{cl}_{\beta}(A \cap G) \subseteq H \cap \text{cl}_{\beta}(G) = \emptyset$. Thus, $\text{cl}_{\beta_I}(A \cap H) \cap (A \cap G) = \emptyset$ and $(A \cap H) \cap \text{cl}_{\beta}(A \cap G) = \emptyset$. Therefore, A can be expressed as a union of two β_I -separated sets $A \cap H$ and $A \cap G$. A contradiction. □

The next statement, Theorem 1.16, says that subsets of each of two separated sets are also separated.

Theorem 1.16. *Let (X, τ) be a topological space, I be an ideal in X , and, A and B be β_I -separated sets. If $C \subseteq A$ ($C \neq \emptyset$) and $D \subseteq B$ ($D \neq \emptyset$), then C and D are also β_I -separated.*

Proof: Suppose that A and B are β_I -separated. Then $\text{cl}_{\beta_I}(A) \cap B = \emptyset$ and $A \cap \text{cl}_{\beta}(B) = \emptyset$. Thus, $\text{cl}_{\beta_I}(C) \cap D \subseteq \text{cl}_{\beta_I}(A) \cap B = \emptyset$ and $C \cap \text{cl}_{\beta}(D) = A \cap \text{cl}_{\beta}(B) = \emptyset$. Hence, $\text{cl}_{\beta_I}(C) \cap D = \emptyset = C \cap \text{cl}_{\beta}(D)$. Therefore, C and D is β_I -separated. □

5. Conclusion

With the important concepts and results which intertwined with those introduced by other authors, this chapter is very interesting. The construction of the different theorems were realized using the definitions or properties of β -open sets, β_I -compact spaces, β_I^* -hyperconnected spaces, β_I -separated spaces. Also, some properties focusing on generalizing ideals in ideal topological space theory were realized.

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References

- [1] Levine N. Semi-open sets and semi-continuity in topological spaces. The American Mathematical Monthly. 1963; **70**(1):36-41
- [2] Njåstad O. On some classes of nearly open sets. Pacific Journal of Mathematics. 1965; **15**(3):961-970
- [3] Mashhour AS, Abd El-Monsef ME, El-Deeh SN. On pre-continuous and weak pre-continuous mappings. Proceedings of the Mathematical and Physical Society of Egypt. 1982; **53**:47-53
- [4] Stone MH. Applications of the theory of boolean rings to general topology. Transactions of the American Mathematical Society. 1937; **41**(3):375-481
- [5] Mubarki AM, Al-Rshudi MM, Al-Juhani MA. β^* -open sets and β^* -continuity in topological spaces. Journal of Taibah University for Science. 2014; **8**(2):142-148
- [6] Kannan K, Nagaveni N. On $\hat{\beta}$ -generalized closed sets in topological spaces. International Journal of Mathematical Analysis. 2012; **6**(57): 2819-2828
- [7] Abd El-Monsef ME. β -open sets and β -continuous mappings. Bulletin of the Faculty of Science. Assiut University. 1983; **12**:77-90
- [8] Akdag M, Ozkan A. On soft β -open sets and soft β -continuous functions. The Scientific World Journal. Hindawi. 2014; **2014**:1-6. DOI: 10.1155/2014/843456
- [9] Arockiarani I, Arokia LA. Generalized soft $g\beta$ -closed sets and soft $gs\beta$ -closed sets in soft topological spaces. International Journal of Mathematical Archive. 2013; **4**(2):1-7
- [10] Dugundji J. Topology. Boston: Ally and Bacon; 1966
- [11] Kuratowski K. Topology. Poland: Scientific Publishers. 1996
- [12] Morris SA. Topology without Tears. Australia: University of New England; 1989
- [13] Tahiliani S. Operation approach to β -open sets and applications. Mathematical Communications. 2011; **16**(2):577-591
- [14] Bhattacharyya P. Semi-generalized closed sets in topology. Indian Journal of Mathematics. 1987; **29**(3):375-382
- [15] Skowron A. On topology information systems. Bulletin of the Polish Academy of Sciences. 1989; **3**: 87-90
- [16] Abid MY. Non semi-pre-denseness in topological spaces. Journal of Kerbala University. 2007; **5**(2):159-163
- [17] El-Mabhouh A, Mizyed A. On the topology generated by βc -open sets. International Journal of Mathematical Sciences and Engineering Applications (IJMSEA). 2015; **9**(1):223-232
- [18] Kuratowski K. Topologie. Bulletin of the American Mathematical Society. 1934; **40**:787-788
- [19] Vaidyanathaswamy R. Set Topology, Chelsea, New York. Lubbock, Texas: University of New Mexico, Albuquerque, New Mexico Texas Technological College; 1960
- [20] Shravan K, Tripathy BC. Generalised closed sets in multiset topological space. Proyecciones (Antofagasta). 2018; **37**(2):223-237
- [21] Shravan K, Tripathy BC. Multiset ideal topological spaces and local functions. Proyecciones (Antofagasta). 2018; **37**(4):699-711

[22] Chandra Tripathy B, Acharjee S. On (γ, δ) -bitopological semi-closed set via topological ideal. *Proyecciones (Antofagasta)*. 2014;**33**(3):245-257

[23] Tripathy BC, Ray GC. Mixed fuzzy ideal topological spaces. *Applied Mathematics and Computation*. 2013; **220**:602-607

[24] Michael FI. On semi-open sets with respect to an ideal. *European Journal of Pure and Applied Mathematics*. 2013; **6**(1):53-58

[25] Newcomb RL. Topologies which are compact modulo an ideal [Ph.d. dissertation]. Santa Barbara, CA: University of California; 1967

[26] Erdal E, Takashi N. *-hyperconnected ideal topological spaces. *Analele științifice Ale Universității "Al.I. Cuza" Din Iași (S.N.) Matematică*. 2012;**LVIII**:121-129

[27] Maheshwari SN, Thakur SS. On α -compact spaces. *Bulletin of the Institute of Mathematics*. 1985;**13**(4):341-347

[28] Nasef AA, Radwan AE, Esmaeel RB. Some properties of α -open sets with respect to an ideal. *International Journal of Pure and Applied Mathematics*. 2015; **102**(4):613-630

[29] Abd El-Monsef ME, Nasef AA, Radwan AE, Ibrahim FA, Esmaeel RB. Some properties of semi-open sets with respect to an ideal. *International Electronic Journal of Pure and Applied Mathematics*. 2015;**9**(3):167-179. DOI: 10.12732/iej pam.v9i3.6