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Relaxation Dynamics of Point Vortices

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Abstract

We study a model describing relaxation dynamics of point vortices, from quasi-stationary state to the stationary state. It takes the form of a mean field equation of Brownian point vortices derived from Chavanis, and is formulated by our previous work as a limit equation of the patch model studied by Robert-Someria. This model is subject to the micro-canonical statistic laws; conservation of energy, that of mass, and increasing of the entropy. We study the existence and nonexistence of the global-in-time solution. It is known that this profile is controlled by a bound of the negative inverse temperature. Here we prove a rigorous result for radially symmetric case. Hence E/M^2 large and small imply the global-in-time and blowup in finite time of the solution, respectively. Where E and M denote the total energy and the total mass, respectively.

Keywords: point vortex, quasi-equilibrium, relaxation dynamics

1. Introduction

Our purpose is to study the system

$$\begin{aligned} \omega_t + \nabla \cdot \omega \nabla^\perp \psi &= \nabla \cdot (\nabla \omega + \beta \omega \nabla \psi) \quad \text{in } \Omega \times (0, T), \\ \frac{\partial \omega}{\partial \nu} + \beta \omega \frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} &= 0, \quad \omega|_{t=0} = \omega_0(x) \end{aligned} \quad (1)$$

with

$$-\Delta \psi = \omega \quad \text{in } \Omega, \quad \psi|_{\partial \Omega} = 0, \quad \beta = -\frac{\int_{\Omega} \nabla \omega \cdot \nabla \psi}{\int_{\Omega} \omega |\nabla \psi|^2}, \quad (2)$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, ν is the outer unit normal vector on $\partial \Omega$, and

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}, \quad \nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}, \quad x = (x_1, x_2). \quad (3)$$

The unknown $\omega = \omega(x, t) \in \mathbf{R}$ stands for a mean field limit of many point vortices,

$$\omega(x, t)dx = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}(dx). \quad (4)$$

It was derived, first, for Brownian point vortices by [1, 2], with $\beta = \beta(t)$ standing for the inverse temperature. Then, [3, 4] reached it by the Lynden-Bell theory [5] of relaxation dynamics, that is, as a model describing the movement of the mean field of many point vortices, from quasi-stationary state to the stationary state. This model is consistent to the Onsager theory [6–12] on stationary states and also the patch model proposed by [13, 14], that is,

$$\omega(x, t) = \sum_{i=1}^{N_p} \sigma_i 1_{\Omega_i(t)}(x), \quad (5)$$

where N_p , σ_i , and $\Omega_i(t)$ denote the number of patches, the vorticity of the i -th patch, and the domain of the i -th patch, respectively [15–17].

This chapter is concerned on the one-sided case of

$$\omega_0 = \omega_0(x) > 0. \quad (6)$$

If this initial value is smooth, there is a unique classical solution to (1)–(4) local in time, denoted by $\omega = \omega(x, t)$, with the maximal existence time $T = T_{\max} \in (0, +\infty]$. More precisely, the strong maximum principle to (1) guaranttes

$$\omega = \omega(x, t) > 0 \quad \text{on } \bar{\Omega} \times [0, T). \quad (7)$$

Then, the Hopf lemma to the Poisson equation in (2) ensures

$$\frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} < 0, \quad (8)$$

and hence the well-definedness of

$$-\beta = \frac{\int_{\Omega} \nabla \omega \cdot \nabla \psi}{\int_{\Omega} \omega |\nabla \psi|^2}. \quad (9)$$

We confirm that system (1)–(3) satisfies the requirements of isolated system of thermodynamics. First, the mass conservation is derived from (1) as

$$\frac{d}{dt} \int_{\Omega} \omega = 0, \quad (10)$$

because

$$\nu \cdot \nabla^{\perp} \psi \Big|_{\partial \Omega} = 0 \quad (11)$$

holds by (2). Second, the energy conservation follows as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_2^2 &= (\nabla \psi, \nabla \psi_t) = (\omega_t, \psi) \\ &= (\omega \nabla^{\perp} \psi, \nabla \psi) - (\nabla \omega + \beta \omega \nabla \psi, \nabla \psi) \\ &= -(\nabla \omega, \nabla \psi) - \beta \int_{\Omega} \omega |\nabla \psi|^2 = 0 \end{aligned} \quad (12)$$

by (1) and (2), because

$$\nabla^\perp \psi \cdot \nabla \psi = 0, \quad (13)$$

where $(\cdot, \cdot)$ denotes the L^2 inner product. Third, the entropy increasing is achieved, writing (1) as

$$\omega_t = \nabla \cdot \omega (-\nabla^\perp \psi + \nabla (\log \omega + \beta \psi)), \quad \frac{\partial}{\partial \nu} (\log \omega + \beta \psi) \Big|_{\partial \Omega} = 0. \quad (14)$$

In fact, it then follows that

$$\int_{\Omega} \omega_t (\log \omega + \beta \psi) = \int_{\Omega} \omega \nabla^\perp \psi \cdot \nabla (\log \psi + \beta \psi) - \omega |\nabla (\log \omega + \beta \psi)|^2 dx \quad (15)$$

with

$$\begin{aligned} \int_{\Omega} \omega \nabla^\perp \psi \cdot \nabla (\log \omega + \beta \psi) &= \int_{\Omega} \nabla \omega \cdot \nabla^\perp \psi \\ &= \int_{\partial \Omega} \omega \nu \cdot \nabla^\perp \psi - \int_{\Omega} \omega \nabla \cdot (\nabla^\perp \psi) = 0 \end{aligned} \quad (16)$$

from (11) and

$$\nabla^\perp \cdot \nabla = \nabla \cdot \nabla^\perp = 0. \quad (17)$$

Since

$$\int_{\Omega} \omega_t \log \omega = \frac{d}{dt} \int_{\Omega} \omega (\log \omega - 1), \quad \int_{\Omega} \omega_t \psi = \frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_2^2 = 0, \quad (18)$$

We thus end up with the mass conservation

$$M = \int_{\Omega} \omega, \quad (19)$$

the energy conservation

$$E = \|\nabla \psi\|_2^2 = (\psi, \omega), \quad (20)$$

and the entropy increasing

$$\frac{d}{dt} \int_{\Omega} \omega (\log \omega - 1) = - \int_{\Omega} \omega |\nabla (\log \omega + \beta \psi)|^2 \leq 0. \quad (21)$$

Henceforth, $C > 0$ stands for a generic constant. In the previous work [4] we studied radially symmetric solutions and obtained a criterion for the existence of the solution global in time. Here, we refine the result as follows, where $B(0, 1)$ denotes the unit ball.

Theorem 1 *Let*

$$\Omega = B(0, 1), \quad \omega_0 = \omega_0(r), \quad \omega_{0r} < 0, \quad 0 < r = |x| \leq 1. \quad (22)$$

Then there is $C_0 > 0$ such that

$$C_0 \|\omega_0\|_2^3 \leq E \underline{\omega} \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_\infty \leq C, \quad t \geq 0, \quad (23)$$

where

$$\underline{\omega} = \min_{\Omega} \omega_0 > 0. \quad (24)$$

Theorem 2 Under the assumption of (22) there is $\delta_0 > 0$ such that

$$\frac{E}{M^2} < \delta_0 \Rightarrow T < +\infty. \quad (25)$$

Remark 1 Since

$$\begin{aligned} \|\omega_0\|_2^3 &= \left(\int_{\Omega} \omega_0^2 \right)^{3/2} \geq \left(\underline{\omega}^{2/3} \int_{\Omega} \omega_0^{4/3} \right)^{3/2} \\ &= \underline{\omega} \left(\int_{\Omega} \omega_0^{4/3} \right)^{3/2} \geq \underline{\omega} |\Omega|^{-1/2} \left(\int_{\Omega} \omega_0 \right)^2 = \underline{\omega} |\Omega|^{-1/2} M^2 \end{aligned} \quad (26)$$

the assumption (23) implies

$$\frac{E}{M^2} \geq C_0 |\Omega|^{-1/2}. \quad (27)$$

Therefore, roughly, the conditions $E/M^2 \gg 1$ and $E/M^2 \ll 1$ imply $T = +\infty$ and $T < +\infty$, respectively.

Remark 2 The assumption (22) implies

$$\beta = \beta(t) < 0, \quad 0 \leq t < T, \quad (28)$$

and then we obtain Theorem 1. In other words, the conclusion of this theorem arises from (28), without (22).

Remark 3 Since

$$\frac{E}{M^2} = \frac{\int_{\Omega} |\nabla \psi|^2}{\left(\int_{\Omega} \omega \right)^2} \quad (29)$$

it holds that

$$\frac{E}{M^2} = \|\nabla c\|_2^2, \quad c = \frac{(-\Delta)^{-1} \omega_0}{\int_{\Omega} \omega_0} = \frac{\psi_0}{\int_{\partial\Omega} -\frac{\partial \psi_0}{\partial \nu}}, \quad (30)$$

where $\psi_0 = (-\Delta)^{-1} \omega_0$.

The system (1)–(4) thus obeys a profile of the micro-canonical ensemble. In a system associated with the canonical ensemble, the inverse temperature β is a constant in (1) independent of t , with the third equality in (2) eliminated:

$$\begin{aligned} \omega_t + \nabla \cdot \omega \nabla^\perp \psi &= \nabla \cdot (\nabla \omega + \beta \omega \nabla \psi), \quad \frac{\partial \omega}{\partial \nu} + \beta \omega \frac{\partial \psi}{\partial \nu} \Big|_{\partial\Omega} = 0, \quad \omega|_{t=0} = \omega_0(x) > 0 \\ -\Delta \psi &= \omega, \quad \psi|_{\partial\Omega} = 0. \end{aligned} \quad (31)$$

Then there arise the mass conservation

$$\frac{d}{dt} \int_{\Omega} \omega = 0, \quad (32)$$

and the free energy decreasing

$$\frac{d}{dt} \int_{\Omega} \omega (\log \omega - 1) + \frac{\beta}{2} |\nabla \psi|^2 dx = - \int_{\Omega} \omega |\nabla (\log \omega + \beta \psi)|^2 \leq 0. \quad (33)$$

The system (31) without vortex term,

$$\begin{aligned} \omega_t &= \nabla \cdot (\nabla \omega + \beta \omega \nabla \psi), \quad \frac{\partial \omega}{\partial \nu} + \beta \omega \frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad \omega|_{t=0} = \omega_0(x) > 0 \\ -\Delta \psi &= \omega, \quad \psi|_{\partial \Omega} = 0. \end{aligned} \quad (34)$$

is called the Smoluchowski-Poisson equation. This model is concerned on the thermodynamics of self-gravitating Brownian particles [18] and has been studied in the context of chemotaxis [19–23]. We have a blowup threshold to (34) as a consequence of the quantized blowup mechanism [19, 23]. The results on the existence of the bounded global-in-time solution [24–26] and blowup of the solution in finite time [27] are valid even to the case that β is a function of t as in $\beta = \beta(t)$. provided with the vortex term $\nabla \cdot \omega \nabla^\perp \psi$ on the right-hand side. We thus obtain the following theorems.

Theorem 3 *It holds that*

$$-\beta(t) \leq \delta, \quad \|\omega_0\|_1 < 8\pi\delta^{-1} \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_\infty \leq C \quad (35)$$

in (31), where $\delta > 0$ is arbitrary.

Theorem 4 *It holds that*

$$-\beta(t) \geq \delta, \quad \|\omega_0\|_1 > 8\pi\delta^{-1} \Rightarrow \exists \omega_0 > 0, \quad \|\omega_0\|_1 > 8\pi\delta^{-1} \quad \text{such that } T < +\infty \quad (36)$$

in (31), where $\delta > 0$ is arbitrary.

Remark 4 *In the context of chemotaxis in biology, the boundary condition of ψ is required to be the form of Neumann zero. The Poisson equation in (34) is thus replaced by*

$$-\Delta \psi = \omega - \frac{1}{|\Omega|} \int_{\Omega} \omega, \quad \frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} = 0 \quad (37)$$

or

$$-\Delta \psi + \psi = \omega, \quad \frac{\partial \psi}{\partial \nu} \Big|_{\partial \Omega} = 0 \quad (38)$$

by [28] and [29], respectively. In this case there arises the boundary blowup, which reduces the value 8π in Theorems 3–4 to 4π . The value 8π in Theorems 3–4, therefore, is a consequence of the exclusion of the boundary blowup [30]. This property is valid even for (37) or (38) of the Poisson part, if (22) is assumed.

Remark 5 *The requirement to ω_0 in Theorem 4 is the concentration at an interior point, which is not necessary in the case of (22). Hence Theorems 3 and 4 are refined as*

$$-\beta(t) \leq \delta, \quad \|\omega_0\|_1 < 8\pi\delta^{-1} \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_\infty \leq C \quad (39)$$

and

$$-\beta(t) \geq \delta, \quad \|\omega_0\|_1 > 8\pi\delta^{-1} \Rightarrow T < +\infty, \quad (40)$$

if (22) holds in (35). The main task for the proof of Theorems 1 and 2, therefore, is a control of $\beta = \beta(t)$ in (1).

This paper is composed of four sections and an appendix. Section 2 is devoted to the study on the stationary solutions, and Theorems 1 and 2 are proven in Sections 3 and 4, respectively. Then Theorem 4 is confirmed in Appendix.

2. Stationary states

First, we take the canonical system (31) with β independent of t . By (32) and (33), its stationary state is defined by

$$\log \omega + \beta\psi = \text{constant}, \quad \omega = \omega(x) > 0, \quad \int_{\Omega} \omega = M. \quad (41)$$

Then it holds that

$$\omega = \frac{Me^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}} \quad (42)$$

and hence

$$-\Delta\psi = \frac{Me^{-\beta\psi}}{\int_{\Omega} e^{-\beta\psi}}, \quad \psi|_{\partial\Omega} = 0. \quad (43)$$

There arises an ordered structure in $\beta < 0$, as observed by [11], as a consequence of a quantized blowup mechanism [19, 20, 31]. In the micro-canonical system (1) and (2), the value β in (43) has to be determined by E besides M .

Equality (21), however, still ensures (41) and hence (42) in the stationary state even for (1)–(3). Writing

$$v = -\beta\psi, \quad \mu = \frac{-\beta M}{\int_{\Omega} e^{-\beta\psi}}, \quad (44)$$

we obtain

$$-\Delta v = \mu e^v \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0, \quad \frac{E}{M^2} = \frac{\|\nabla v\|_2^2}{\left(\int_{\Omega} e^{\frac{v}{2}}\right)^2} \quad (45)$$

by (30) and (43).

This system is the stationary state of (1) and (2) introduced by [4]. The first two equalities

$$-\Delta v = \mu e^v, \quad v|_{\partial\Omega} = 0 \quad (46)$$

comprise a nonlinear elliptic eigenvalue problem and the unknown eigenvalue μ is determined by the third equality,

$$\frac{E}{M^2} = \frac{\|\nabla v\|_2^2}{\left(\int_{\Omega} -\frac{\partial v}{\partial \nu}\right)^2}. \quad (47)$$

The elliptic theory ensures rather deailed features of the set of solutions to (46). Here we note the following facts [31].

1. There is $\bar{\mu} = \mu(\Omega) > 0$ such that the problem (46) does not admit a solution for $\mu > \bar{\mu}$.
2. Each $\mu \leq 0$ admits a unique solution.
3. Each $0 < \delta < \bar{\mu}$ admits a constant $C = C(\delta) > 0$ such that $\|v\|_{\infty} \leq C$ for any solution $v = v(x)$.
4. There is a family of solutions $\{(\mu, v)\}$ such that $\mu \downarrow 0$ and $\|v\|_{\infty} \rightarrow +\infty$.

We show the following theorem, consistent to Theorem 2.

Theorem 5 *If $\Omega = B(0, 1) \subset \mathbf{R}^2$, there is $\delta > 0$ such that any solution (v, μ) to (45) admits*

$$\frac{E}{M^2} \geq \delta. \quad (48)$$

Proof: If $\mu = 0$, it holds that $v = 0$. We have $\pm v > 0$ exclusively in Ω , provided that $\pm \mu > 0$, respectively. By the elliptic theory [32], therefore, any solution v to (46) is radially symmetric as in $v = v(r)$, $r = |x|$. We have, furthermore, $\pm v_r < 0$ in $0 < r \leq 1$, if $\pm \mu > 0$, respectively.

Then it holds that $\psi = \psi(r)$, and hence

$$-\frac{1}{r}(r\psi_r)_r = \omega \quad \text{in } 0 < r \leq 1, \quad \psi|_{r=1} = 0 \quad (49)$$

by (42) and (43), which implies

$$-r\psi_r(r) = \int_0^r s\omega(s)ds > 0, \quad 0 < r \leq 1. \quad (50)$$

We thus obtain $\mu \neq 0$, in particular.

If $\mu < 0$ we have $\beta > 0$ by (44), and therefore, $\psi_r > 0$ in $0 < r \leq 1$ by $v_r > 0$ there. It is a contradiction, and hence $\mu > 0$. In this case, the solution $v = v(r)$ to (46) is explicit [31]. The numbers of the solution is 0, 1, and 2, according to $\mu > 2$, $\mu = 2$, and $0 < \mu < 2$, respectively, and if $0 < \mu \leq 2$ the solutions $v = v_{\pm}$ are given as

$$v_{\pm}(r) = \log \frac{8\gamma_{\pm}r}{(1 + \gamma_{\pm}r^2)^2}, \quad \gamma_{\pm} = \frac{4}{\mu} \left\{ 1 - \frac{\mu}{4} \pm \left(1 - \frac{\mu}{2} \right)^{1/2} \right\}. \quad (51)$$

In fact, we have $\gamma_+ = \gamma_-$ for $\mu = 2$.

This solution is parametrized by

$$\sigma = \int_{\Omega} \mu e^v \in (0, 8\pi). \quad (52)$$

Hence each $0 < \sigma < 8\pi$ admits a unique solution (v, μ) to (46), and $v = v_+$ and $v = v_-$ according as $\sigma \geq 4\pi$ and $\sigma \leq 4\pi$, respectively. It holds also that $\mu \downarrow 0$ if either

$\sigma \uparrow 8\pi$ or $\sigma \downarrow 0$. Thus we have only to confirm that E/M^2 is bounded, both as $\sigma \uparrow 8\pi$ and $\sigma \downarrow 0$.

As $\sigma \uparrow 8\pi$, we have

$$v = v_+(x) \rightarrow 4 \log \frac{1}{|x|} \quad \text{locally uniformly on } \overline{\Omega} \setminus \{0\} \quad (53)$$

and hence

$$\|\nabla v\|_2^2 \rightarrow +\infty, \quad \int_{\partial\Omega} -\frac{\partial v}{\partial \nu} \rightarrow 8\pi, \quad (54)$$

which implies

$$\lim_{\sigma \uparrow 8\pi} \frac{E}{M^2} = +\infty. \quad (55)$$

As $\sigma \downarrow 0$, on the other hand, we have

$$v = v_-(x) \rightarrow 0 \quad \text{uniformly in } \overline{\Omega}. \quad (56)$$

Since $\mu \downarrow 0$, furthermore, there arises that

$$\gamma = \gamma_- = \frac{4}{\mu} \left\{ 1 - \frac{\mu}{4} - \left(1 - \frac{\mu}{2} \right)^{1/2} \right\} = \mu(1 + o(1)). \quad (57)$$

It holds also that

$$v(r) = \log \frac{8\gamma}{\mu} - 2 \log(1 + \mu r^2) \quad (58)$$

and hence

$$v_r(r) = -\frac{4\mu r}{(1 + \mu r^2)^2} = -4\mu r(1 + o(1)) \quad \text{uniformly on } \overline{\Omega}. \quad (59)$$

Then, (59) implies

$$\begin{aligned} \|\nabla v\|_2^2 &= 2\pi \int_0^1 v_r^2 r \, dr = 2\pi \cdot 16\mu^2 \cdot \int_0^1 r^3 \, dr \cdot (1 + o(1)) \\ &= 8\pi\mu^2(1 + o(1)) \end{aligned} \quad (60)$$

as well as

$$\left(\int_{\partial\Omega} -\frac{\partial v}{\partial \nu} \right)^2 = 16\mu^2 \cdot 2\pi(1 + o(1)). \quad (61)$$

It thus follows that

$$\lim_{\sigma \downarrow 0} \frac{E}{M^2} = \frac{1}{4} \quad (62)$$

and hence the conclusion. $\square$

3. Proof of Theorem 1

The first observation is the following lemma.

Lemma 1 Under the assumption of (22), it holds that

$$\beta = \beta(t) < 0, \quad \omega_r(r, t) < 0, \quad 0 < r \leq 1, \quad 0 \leq t < T. \quad (63)$$

Proof: We have (7) and hence

$$\psi_r(r, t) < 0, \quad 0 < r \leq 1, \quad 0 \leq t < T \quad (64)$$

by (49), which implies, in particular,

$$\beta = -\frac{(\nabla\omega, \nabla\psi)}{\int_{\Omega} \omega |\nabla\psi|^2} < 0 \quad (65)$$

at $t = 0$ by (22).

Since $\omega = \omega(r, t)$ and $\psi = \psi(r, t)$, we obtain $\nabla^\perp \psi = 0$, and hence

$$\omega_t = \omega_{rr} + \frac{1}{r} \omega_r + \beta \psi_r \omega_r - \beta \omega^2 \quad (66)$$

by (1). Then $z = \omega_r$ satisfies

$$\begin{aligned} z_t &= z_{rr} - \frac{1}{r^2} z + \frac{1}{r} z_r + \beta \psi_{rr} z + \beta \psi_r z_r - 2\beta \omega z, \quad 0 < r \leq 1, \quad 0 \leq t < T \\ z|_{r=0} &= 0, \quad z|_{t=0} = \omega_{0r}(r) < 0, \quad 0 < r \leq 1 \end{aligned} \quad (67)$$

and

$$z = -\beta \omega \psi_r, \quad r = 1, \quad 0 \leq t < T. \quad (68)$$

Putting

$$m(t) = \min_{\partial\Omega} z(\cdot, t) = \omega_r(\cdot, t)|_{r=1}, \quad (69)$$

we obtain $m(0) < 0$ from the assumption. If there is $0 < t_0 < T$ such that

$$m(t) < 0, \quad 0 \leq t < t_0 < T, \quad m(t_0) = 0, \quad (70)$$

we obtain $z(r, t) > 0$ for $0 \leq t < t_0$, $0 < r \leq 1$, and $t = t_0$, $0 < r < 1$ by the strong maximum principle. By (64), we have (65) for $0 \leq t \leq t_0$, that is,

$$\beta = -\frac{\int_0^1 \psi_r z r \, dr}{\int_0^1 \omega \psi_r^2 r \, dr} < 0, \quad 0 \leq t \leq t_0, \quad (71)$$

and hence

$$z = -\beta \omega \psi_r < 0 \quad r = 1, \quad t = t_0, \quad (72)$$

a contradiction. It holds that $z = \omega_r < 0$ for $0 \leq t < T$, $r = 1$, and hence

$$\beta = -\frac{\int_0^1 \psi_r \omega_r r \, dr}{\int_0^1 \omega \psi_r^2 r \, dr} < 0, \quad 0 \leq t < T. \quad \square \quad (73)$$

The proof of Theorem 3 relies on the fact

$$\beta \geq -C, \quad \int_{\Omega} \omega(\log \omega - 1) \leq C \Rightarrow T = +\infty, \quad \|\omega(\cdot, t)\|_{\infty} \leq C. \quad (74)$$

This property is known for the Smoluchoski-Poisson equation (34), but the proof is valid even to (31) with vortex term. Having (21), therefore, we have to provide the inequality $\beta \geq -C$.

The inequality $\beta < 0$, on the other hand, is sufficient for the following arguments.

Lemma 2 *If $\beta \leq 0$, $0 \leq t < T$, it holds that*

$$\omega \geq \underline{\omega} \equiv \min_{\bar{\Omega}} \omega_0 > 0 \quad \text{on } \bar{\Omega} \times [0, T). \quad (75)$$

Proof: Since (17) we obtain

$$\begin{aligned} \omega_t + \nabla^{\perp} \psi \cdot \nabla \omega &= \Delta \omega + \beta \nabla \psi \cdot \nabla \omega + \beta \Delta \psi \\ &= \Delta \omega + \beta \nabla \psi \cdot \nabla \omega - \beta \omega^2 \\ &\geq \Delta \omega + \beta \nabla \psi \cdot \nabla \omega \quad \text{in } \Omega \times (0, T) \end{aligned} \quad (76)$$

with

$$-\frac{\partial \omega}{\partial \nu} = \beta \omega \frac{\partial \psi}{\partial \nu} > 0 \quad \text{on } \partial \Omega \times [0, T) \quad (77)$$

by (8). Then the result follows from the comparison theorem. $\square$

Lemma 3 *Under the assumption of the previous lemma, there is $C_0 = C_0(\Omega) > 0$ such that*

$$C_0 \|\omega_0\|_2^3 \leq E \underline{\omega} \Rightarrow \|\omega(\cdot, t)\|_2 \leq \|\omega_0\|_2, \quad -\beta(t) \leq \alpha \equiv \frac{\|\omega_0\|_2^2}{E \underline{\omega}}, \quad 0 \leq t < T. \quad (78)$$

Proof: Using (11) and (17), we obtain

$$\begin{aligned} \int_{\Omega} [\nabla \cdot (\omega \nabla^{\perp} \psi)] \omega &= \int_{\Omega} \omega \nabla \omega \cdot \nabla^{\perp} \psi = \frac{1}{2} \int_{\Omega} \nabla \omega^2 \cdot \nabla^{\perp} \psi \\ &= -\frac{1}{2} \int_{\Omega} \omega^2 \nabla \cdot \nabla^{\perp} \psi = 0. \end{aligned} \quad (79)$$

Hence (1) with (2) implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 &= -\beta \int_{\Omega} \omega \nabla \psi \cdot \nabla \omega = -\frac{\beta}{2} (\nabla \psi, \nabla \omega^2) \\ &= -\frac{\beta}{2} \int_{\partial \Omega} \omega^2 \frac{\partial \psi}{\partial \nu} + \frac{\beta}{2} (\Delta \psi, \omega^2) \leq -\frac{\beta}{2} \|\omega\|_3^3 \end{aligned} \quad (80)$$

by $\beta \leq 0$ and (88). Since

$$\int_{\Omega} \nabla \omega \cdot \nabla \psi = \int_{\partial \Omega} \omega \frac{\partial \psi}{\partial \nu} + \int_{\Omega} \omega (-\Delta \psi) \leq \int_{\Omega} \omega^2 \quad (81)$$

follows from (8), furthermore, it holds that

$$-\beta = \frac{\int_{\Omega} \nabla \omega \cdot \nabla \psi}{\int_{\Omega} \omega |\nabla \psi|^2} \leq \underline{\omega}^{-1} \cdot \frac{\|\omega\|_2^2}{\|\nabla \psi\|_2^2} = \frac{1}{E\underline{\omega}} \|\omega\|_2^2. \quad (82)$$

Then inequality (80) induces

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 \leq \frac{1}{2E\underline{\omega}} \|\omega\|_2^2 \cdot \|\omega\|_3^3. \quad (83)$$

Here we use the Gagliardo-Nirenberg inequality (see (4.16) of [19]) in the form of

$$\|\omega\|_3^3 \leq C \|\omega\|_{H^1} \cdot \|\omega\|_2^2 = C \|\omega\|_2^2 (\|\nabla \omega\|_2 + \|\omega\|_2), \quad (84)$$

to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 &\leq \frac{C}{E\underline{\omega}} \|\omega\|_2^4 (\|\nabla \omega\|_2 + \|\omega\|_2) \\ &\leq \frac{1}{2} \|\nabla \omega\|_2^2 + \frac{C^2}{8(E\underline{\omega})^2} \|\omega\|_2^8 + \frac{C}{2E\underline{\omega}} \|\omega\|_2^5 \end{aligned} \quad (85)$$

and hence

$$\frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 \leq \frac{C}{E\underline{\omega}} \|\omega\|_2^5 \left(\frac{C}{E\underline{\omega}} \|\omega\|_2^3 + 1 \right). \quad (86)$$

Then, Poincaré-Wirtinger's inequality ensures

$$\frac{d}{dt} \|\omega\|_2^2 + \mu \|\omega\|_2^2 \leq \frac{C}{E\underline{\omega}} \left(\frac{C}{E\underline{\omega}} \|\omega\|_2^6 + \|\omega\|_2^3 \right) \|\omega\|_2^2, \quad (87)$$

where $\mu = \mu(\Omega) > 0$ is a constant.

Writing

$$y(t) = \frac{C}{E\underline{\omega}} \|\omega\|_2^3, \quad (88)$$

we obtain

$$\frac{d}{dt} \|\omega\|_2^2 + \mu \|\omega\|_2^2 \leq (y^2 + y) \|\omega\|_2^2, \quad (89)$$

and therefore, if

$$y^2 + y < \mu/2 \quad (90)$$

holds at $t = 0$, it keeps to hold that

$$\frac{d}{dt} \|\omega\|_2^2 \leq 0 \quad (91)$$

and (90) for $0 \leq t < T$. Then, we obtain

$$\|\omega(\cdot, t)\|_2 \leq \|\omega_0\|_2, \quad 0 \leq t < T, \quad (92)$$

and hence

$$-\beta(t) \leq \frac{\|\omega_0\|_2^2}{E\omega} = \alpha, \quad 0 \leq t < T \quad (93)$$

by (82).

The condition $y(0) < \frac{\mu}{2}$ means

$$C_0 \|\omega_0\|_2 \leq E\omega \quad (94)$$

for $C_0 > 0$ sufficiently large, and hence we obtain the conclusion. $\square$

Proof of Theorem 1: By the parabolic regularity, it suffices to show that

$$\|\omega(\cdot, t)\|_\infty \leq C, \quad 0 \leq t < T \quad (95)$$

under the assumption. We have readily shown

$$\|\omega(\cdot, t)\|_2 \leq C, \quad 0 \leq -\beta(t) \leq C, \quad 0 \leq t < T \quad (96)$$

by Lemma 3. Then, the conclusion (95) is obtained similarly to (34). See [26] for more details.

In fact, we have

$$\begin{aligned} \int_{\Omega} [\nabla \cdot (\omega \nabla^\perp \psi)] \omega^p &= - \int_{\Omega} \omega \nabla^\perp \psi \cdot \nabla \omega^p = -p \int_{\Omega} \omega^p \nabla^\perp \psi \cdot \nabla \omega \\ &= -\frac{p}{p+1} \int_{\Omega} \nabla^\perp \psi \cdot \nabla \omega^{p+1} = \frac{p}{p+1} \int_{\Omega} \omega^{p+1} \nabla \cdot (\nabla^\perp \psi) = 0 \end{aligned} \quad (97)$$

for $p > 0$ by (11) and (34). Then it follows that

$$\begin{aligned} \frac{1}{p+1} \frac{d}{dt} \int_{\Omega} \omega^{p+1} + \frac{4p}{(p+1)^2} \|\nabla \omega^{\frac{p+1}{2}}\|_2^2 &= -\beta \int_{\Omega} \omega \nabla \psi \cdot \nabla \omega^p \\ &= -\beta \cdot \frac{p}{p+1} \int_{\Omega} \nabla \psi \cdot \nabla \omega^{p+1} \leq -\beta \frac{p}{p+1} \int_{\Omega} \omega^{p+1} (-\Delta \psi) \\ &= -\beta \frac{p}{p+1} \int_{\Omega} \omega^{p+1} \leq C \int_{\Omega} \omega^{p+2} \end{aligned} \quad (98)$$

by $\beta < 0$ and (8). Then, Moser's iteration scheme ensures (95) as in [33].

4. Proof of Theorem 2

We begin with the following lemma.

Lemma 4 *Under the assumption of (22), it holds that*

$$-\beta(t) \geq \delta, \quad 0 \leq t < T, \quad M = \|\omega_0\|_1 > \frac{8\pi}{\delta} \Rightarrow T < +\infty \quad (99)$$

in (31), where $\delta > 0$ is a constant.

Proof: We have $\omega = \omega(r, t)$ and $\psi = \psi(r, t)$ for $r = |x|$ under the assumption, which implies $\nabla^\perp \psi = 0$. Then we obtain

$$\nabla \cdot \omega \nabla^\perp \psi = \nabla \omega \cdot \nabla^\perp \psi = 0 \quad (100)$$

by (17). It holds also that

$$\begin{aligned} \nabla \cdot (\omega \nabla \psi) &= \nabla \cdot \left(\omega \psi_r \frac{x}{r} \right) = \left(\nabla \cdot \frac{x}{r} \right) \omega \psi_r + \frac{x}{r} \cdot \nabla (\omega \psi_r) \\ &= \frac{1}{r} \omega \psi_r + (\omega \psi_r)_r = \frac{1}{r} (r \omega \psi_r)_r, \end{aligned} \quad (101)$$

and therefore, there arises that

$$\omega_t = \frac{1}{r} (r \omega_r + \beta r \omega \psi_r)_r, \quad \omega_r + \beta \omega \psi_r|_{r=1} = 0. \quad (102)$$

from (31).

Then (102) implies

$$\begin{aligned} \frac{d}{dt} \int_0^1 \omega r^3 \, dr &= \int_0^1 \omega_t r^3 \, dr = \int_0^1 (r \omega_r + \beta r \omega \psi_r)_r r^2 \, dr \\ &= - \int_0^1 2r^2 (\omega_r + \beta \omega \psi_r) \, dr \\ &= -2r^2 \omega|_{r=0}^{r=1} + \int_0^1 4r \omega - 2\beta \omega \psi_r r^2 \, dr. \end{aligned} \quad (103)$$

Here we use (50) derived from the Poisson part of (31), that is,

$$-r \psi_r(r, t) = A(r, t) \equiv \int_0^r s \omega(s, t) ds. \quad (104)$$

Putting

$$\lambda = \int_0^1 \omega r \, dr = \frac{M}{2\pi}, \quad (105)$$

we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^1 \omega r^3 \, dr &= -2\omega|_{r=1} + 4\lambda + 2\beta \int_0^1 A A_r \, dr \\ &= -2\omega|_{r=1} + 4\lambda + \beta A^2|_{r=0}^{r=1} \\ &= -2\omega|_{r=1} + 4\lambda + \beta \lambda^2 \\ &< 4\lambda \left(\beta + \frac{M}{8\pi} \right) \leq 4\lambda \left(-\delta + \frac{M}{8\pi} \right). \end{aligned} \quad (106)$$

Since $-\delta + \frac{M}{8\pi} < 0$, therefore, $T = +\infty$ is impossible, and we obtain $T < +\infty$. $\square$

Lemma 5 Under the assumption (22), there is $\delta > 0$ such that

$$\frac{E}{M^2} < \delta, \quad \beta(t) \leq 0, \quad 0 \leq t < T \Rightarrow \beta(t) \leq -\frac{1}{CE^{1/2}} \quad 0 \leq t < T. \quad (107)$$

Proof: First, Lemma 1 implies

$$\omega \geq \omega_* \equiv \omega|_{r=1}. \quad (108)$$

Second, we have

$$\begin{aligned} \int_{\Omega} \nabla \psi \cdot \nabla \omega &= \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} \omega + \int_{\Omega} (-\Delta \psi) \omega = \omega_* \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} + \|\omega\|_2^2 \\ &= \omega_* \int_{\Omega} \Delta \psi + \|\omega\|_2^2 = \|\omega\|_2^2 - \omega_* M, \end{aligned} \quad (109)$$

and hence

$$-\beta = \frac{\int_{\Omega} \nabla \psi \cdot \nabla \omega}{\int_{\Omega} \omega |\nabla \psi|^2} = \frac{\|\omega\|_2^2 - \omega_* M}{\int_{\Omega} \omega |\nabla \psi|^2}. \quad (110)$$

Here, we use the Gagliardo-Nirenberg inequality in the form of

$$\|w\|_4^2 \leq C \|w\|_2 \|w\|_{H^1}, \quad (111)$$

which implies

$$\begin{aligned} \int_{\Omega} \omega |\nabla \psi|^2 &\leq \|\omega\|_2 \|\nabla \psi\|_4^2 \leq C \|\omega\|_2 \|\nabla \psi\|_2 \|\nabla \psi\|_{H^1} \\ &\leq CE^{1/2} \|\omega\|_2^2 \end{aligned} \quad (112)$$

by the elliptic estimate of the Poisson equation in (2),

$$\|\psi\|_{H^2} \leq C \|\omega\|_2. \quad (113)$$

We have, on the other hand,

$$\omega_* M \leq \frac{M}{E} \int_{\Omega} \omega |\nabla \psi|^2 \quad (114)$$

by (110), and therefore,

$$-\beta \geq \frac{1}{CE^{1/2}} - \frac{E}{M} \geq \frac{1}{2CE^{1/2}}, \quad (115)$$

provided that

$$\frac{E}{M^2} < \left(\frac{1}{2C} \right)^2. \quad (116)$$

Then the conclusion follows. □

Proof of Theorem 2: By Lemma 5, there is $\delta_0 > 0$ such that

$$\frac{E}{M^2} < \delta \Rightarrow -\beta \geq \frac{1}{CE^{1/2}} \equiv \delta_1, \quad (117)$$

and then, Lemma 4 ensures

$$M > \frac{8\pi}{\delta_1} \Rightarrow T < +\infty. \quad (118)$$

The assumption in (118) means

$$\frac{E}{M^2} < \left(\frac{1}{8\pi c}\right)^2, \quad (119)$$

and hence we obtain the conclusion. $\square$

Appendix Proof of Theorem 4

This theorem is valid to the general case of Ω and ω_0 without (22). We assume $\delta = 1$ without loss of generation, so that

$$\beta \leq -1. \quad (120)$$

We follow the argument [27] concerning (34) with the Poisson part replaced by (42) or (43). Thus we have to take case of the vortex term $\nabla \cdot \omega \nabla^\perp \psi$, time varying $\beta = \beta(t)$, and the Dirichlet boundary condition in (31).

We recall the cut-off function used in [34] (see also Chapter 5 of [19]). Hence each $x_0 \in \overline{\Omega}$ and $0 < R \leq 1$ admit $\varphi = \varphi_{x_0, R} \in C^2(\overline{\Omega})$ with

$$\left. \frac{\partial \varphi}{\partial \nu} \right|_{\partial \Omega} = 0, \quad 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ in } \Omega \cap B(x_0, R/2), \quad \varphi = 0 \text{ in } \Omega \setminus B(x_0, R), \quad (121)$$

and

$$|\nabla \varphi| \leq CR^{-1} \varphi^{1/2}, \quad |\nabla^2 \varphi| \leq CR^{-2} \varphi^{1/2}. \quad (122)$$

In more details, we take a cut-off function, denoted by ψ , satisfying (121), using a local conformal mapping, and then put $\varphi = \psi^4$.

Let

$$\varphi \in C^2(\overline{\Omega}), \quad \left. \frac{\partial \varphi}{\partial \nu} \right|_{\partial \Omega} = 0. \quad (123)$$

be given. First, we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \omega \varphi &= \int_{\Omega} \omega \nabla^\perp \psi \cdot \nabla \varphi - (\nabla \omega + \beta \omega \nabla \psi) \cdot \nabla \varphi \, dx \\ &= \int_{\Omega} \omega \nabla^\perp \psi \cdot \nabla \varphi + \omega \Delta \varphi - \beta \omega \nabla \psi \cdot \nabla \varphi \, dx \end{aligned} \quad (124)$$

by (11). It holds that

$$\begin{aligned}
 \int_{\Omega} \omega \nabla \psi \cdot \nabla \varphi &= \iint_{\Omega \times \Omega} \omega(x, t) [\nabla_x G(x, x') \cdot \nabla \varphi(x)] \omega(x', t) \, dx dx' \\
 &= \iint_{\Omega \times \Omega} \omega(x, t) \varphi_{x_0, 2R}(x') [\nabla_x G(x, x') \cdot \nabla \varphi(x)] \omega(x', t) \, dx dx' \\
 &\quad + \iint_{\Omega \times \Omega} \omega(x, t) (1 - \varphi_{x_0, 2R}(x')) [\nabla_x G(x, x') \cdot \nabla \varphi(x)] \omega(x', t) \, dx dx' \\
 &= I + II.
 \end{aligned} \tag{125}$$

Let, furthermore, $x_0 \in \Omega$ and $0 < R \ll 1$ in the above equality. Then,

$$\varphi = |x - x_0|^2 \varphi_{x_0, R} \tag{126}$$

satisfies the requirement (123).

It holds that

$$\nabla \varphi = 2(x - x_0) \varphi_{x_0, R} + |x - x_0|^2 \nabla \varphi_{x_0, R} \tag{127}$$

and hence

$$|\nabla \varphi| \leq C|x - x_0| \left(\varphi_{x_0, R} + |x - x_0| R^{-1} \varphi_{x_0, R}^{1/2} \right) \leq C|x - x_0| \varphi_{x_0, R}^{1/2}. \tag{128}$$

We obtain, furthermore,

$$|x' - x_0| \geq 2R, \quad |x - x_0| \leq R \Rightarrow |x - x'| \geq R, \tag{129}$$

and hence

$$|\nabla_x G(x, x')| \leq CR^{-1} \tag{130}$$

in this case. Then it follows that

$$|II| \leq CR^{-1} M \int_{\Omega} |x - x_0| \varphi_{x_0, R}^{1/2} \omega(x, t) \, dx \leq CR^{-1} M^{3/2} A^{1/2}, \tag{131}$$

where

$$A = \int_{\Omega} |x - x_0|^2 \varphi_{x_0, R} \omega. \tag{132}$$

We have, on the other hand,

$$\begin{aligned}
 I &= \iint_{\Omega \times \Omega} \omega(x, t) \varphi_{x_0, 2R}(x') [\nabla_x G(x, x') \cdot \nabla \varphi(x)] \omega(x', t) \, dx dx' \\
 &= \frac{1}{2} \iint_{\Omega \times \Omega} [\varphi_{x_0, 2R}(x') \nabla \varphi(x) \cdot \nabla_x G(x, x') + \varphi_{x_0, 2R}(x) \nabla \varphi(x') \cdot \nabla_{x'} G(x, x')] \omega \otimes \omega,
 \end{aligned} \tag{133}$$

where $G = G(x, x')$ is the Green's function to

$$-\Delta \psi = \omega, \quad \omega|_{\partial \Omega} = 0 \tag{134}$$

and

$$\omega \otimes \omega = \omega(x, t) \omega(x', t) \, dx dx'. \quad (135)$$

Here we use the local property of the Green's function

$$G(x, x') = \Gamma(x - x') + K(x, x'), \quad K \in C^2(\overline{\Omega} \times \Omega) \cap C^2(\Omega \times \overline{\Omega}), \quad (136)$$

where

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|} \quad (137)$$

stands for the fundamental solution to $-\Delta$.

Let

$$\rho_{x_0, R}^2(x, x') = \varphi_{x_0, 2R}(x') \nabla \varphi(x) \cdot \nabla_x K(x, x') + \varphi_{x_0, 2R} \nabla \varphi(x') \cdot \nabla_{x'} K(x, x'). \quad (138)$$

Since (128) implies

$$\begin{aligned} |\varphi_{x_0, 2R}(x') \nabla \varphi(x)| &\leq C \varphi_{x_0, 2R}(x') |x - x_0| \varphi_{x_0, R}^{1/2}(x) \\ &\leq C |x - x_0| \varphi_{x_0, R}^{1/2}(x), \end{aligned} \quad (139)$$

it holds that

$$|\rho_{x_0, R}^1(x, x')| \leq C \left(|x - x_0| \varphi_{x_0, R}^{1/2}(x) + |x' - x_0| \varphi_{x_0, R}^{1/2}(x') \right). \quad (140)$$

Then, we obtain

$$I = \frac{1}{2} \iint_{\Omega \times \Omega} \rho_{x_0, R}^0(x, x') \omega \otimes \omega + III \quad (141)$$

with

$$|III| \leq CM^{3/2} A^{1/2} \leq CR^{-1} M^{3/2} A^{1/2}, \quad (142)$$

where

$$\rho_{x_0, R}^0(x, x') = \nabla \Gamma(x - x') \cdot (\varphi_{x_0, 2R}(x') \nabla \varphi(x) - \varphi_{x_0, 2R}(x) \nabla \varphi(x')). \quad (143)$$

Here, we have

$$\nabla \Gamma(x) = -\frac{x}{2\pi |x|^2}, \quad (144)$$

and therefore,

$$\rho_{x_0, R}^0(x, x') = \rho_{x_0, R}^2(x, x') + \rho_{x_0, R}^3(x, x') \quad (145)$$

fo

$$\rho_{x_0, R}^2(x, x') = -\frac{1}{2\pi} \frac{x - x'}{|x - x'|^2} \varphi_{x_0, 2R}(x') \cdot (\nabla \varphi(x) - \nabla \varphi(x')) \quad (146)$$

$$\rho_{x_0, R}^3(x, x') = -\frac{1}{2\pi} \frac{x - x'}{|x - x'|^2} (\varphi_{x_0, 2R}(x') - \varphi_{x_0, 2R}(x)) \cdot \nabla \varphi(x). \quad (147)$$

Since (128) implies

$$|\rho_{x_0,R}^3(x, x')| \leq CR^{-1} |\nabla \varphi(x)| \leq CR^{-1} |x - x_0| \varphi_{x_0,R}^{1/2}(x), \quad (148)$$

there arises that

$$I = \frac{1}{2} \iint_{\Omega \times \Omega} \rho_{x_0,R}^2(x, x') \omega \otimes \omega + IV, \quad (149)$$

with

$$|IV| \leq CR^{-1} M^{3/2} A^{1/2}, \quad (150)$$

similarly.

We have, furthermore,

$$\begin{aligned} \nabla \varphi(x) - \nabla \varphi(x') &= 2(x - x') \varphi_{x_0,R}(x) + 2(x' - x_0) (\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')) \\ &\quad + |x' - x_0|^2 (\nabla \varphi_{x_0,R}(x) - \nabla \varphi_{x_0,R}(x')) + (|x - x_0|^2 - |x' - x_0|^2) \nabla \varphi_{x_0,R}(x), \end{aligned} \quad (151)$$

and hence

$$\rho_{x_0,R}^2(x, x') = -\frac{1}{\pi} \varphi_{x_0,2R}(x') \varphi_{x_0,R}(x) + \rho_{x_0,R}^4(x, x') + \rho_{x_0,R}^5(x, x') + \rho_{x_0,R}^6(x, x') \quad (152)$$

with

$$\begin{aligned} |\rho_{x_0,R}^4(x, x')| &\leq C|x - x'|^{-1} \varphi_{x_0,2R}(x') |x' - x_0| |\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')| \\ &\leq CR^{-1} |x' - x_0| \varphi_{x_0,2R}(x'), \end{aligned} \quad (153)$$

$$\begin{aligned} |\rho_{x_0,R}^5(x, x')| &\leq C|x - x'|^{-1} \varphi_{x_0,2R}(x') |x' - x_0|^2 |\nabla \varphi_{x_0,R}(x) - \nabla \varphi_{x_0,R}(x')| \\ &\leq CR^{-2} |x' - x_0|^2 \varphi_{x_0,2R}(x') \\ &\leq CR^{-1} |x' - x_0| \varphi_{x_0,2R}(x'), \end{aligned} \quad (154)$$

and

$$\begin{aligned} |\rho_{x_0,R}^6(x, x')| &\leq C|x - x'| \varphi_{x_0,2R}(x') ||x - x_0|^2 - |x' - x_0|^2| \cdot |\nabla \varphi_{x_0,R}(x)| \\ &\leq CR^{-1} (|x - x_0| + |x' - x_0|) \varphi_{x_0,R}(x) \varphi_{x_0,2R}(x') \\ &\leq C(R^{-1} |x - x_0| \varphi_{x_0,R}(x) + R^{-1} |x' - x_0| \varphi_{x_0,2R}(x')) \end{aligned} \quad (155)$$

by

$$|x - x_0|^2 - |x' - x_0|^2 = |(x - x', x + x' - 2x_0)| \leq |x - x'| (|x - x_0| + |x' - x_0|). \quad (156)$$

The residual terms are thus treated similarly, and it follows that

$$\left| I + \frac{1}{2\pi} \int_{\Omega} \omega \varphi_{x_0,R} \cdot \int_{\Omega} \omega \varphi_{x_0,2R} \right| \leq CR^{-1} M^{3/2} A^{1/2}, \quad (157)$$

which results in

$$\left| \int_{\Omega} \omega \nabla \psi \cdot \nabla \varphi + \frac{1}{2\pi} \int_{\Omega} \omega \varphi_{x_0,R} \cdot \int_{\Omega} \omega \varphi_{x_0,2R} \right| \leq CR^{-1} M^{3/2} A^{1/2}. \quad (158)$$

We can argue similarly to the vortex term in (124). This time, from

$$\nabla^{\perp} \Gamma(x) \cdot x = 0 \quad (159)$$

it follows that

$$\left| \int_{\Omega} \omega \nabla^{\perp} \psi \cdot \nabla \varphi \right| \leq CR^{-1} M^{3/2} A^{1/2}. \quad (160)$$

Concerning the principal term of (124), we use

$$\Delta \varphi = 4\varphi_{x_0,R} + 4(x - x_0) \cdot \nabla \varphi_{x_0,R} + |x - x_0|^2 \Delta \varphi_{x_0,R}. \quad (161)$$

From

$$|(x - x_0) \cdot \nabla \varphi_{x_0,R}| \leq CR^{-1} |x - x_0| \varphi_{x_0,R}^{1/2} \quad (162)$$

and

$$\begin{aligned} ||-x_0|^2 \Delta \varphi_{x_0,R}| &\leq CR^{-2} |x - x_0|^2 \varphi_{x_0,R}^{1/2} \\ &\leq CR^{-1} |x - x_0| \varphi_{x_0,R}^{1/2}, \end{aligned} \quad (163)$$

it follows that

$$\begin{aligned} \left| \int_{\Omega} \omega \Delta \varphi - 4 \int_{\Omega} \omega \varphi_{x_0,R} \right| &\leq C \int_{\Omega} R^{-1} |x - x_0| \varphi_{x_0,R} \omega \\ &\leq CR^{-1} M^{1/2} A^{1/2}. \end{aligned} \quad (164)$$

Let $M_1 = M_{x_0,R}$ and $M_2 = M_{x_0,2R}$ for

$$M_{x_0,R} = \int_{\Omega} \omega \varphi_{x_0,R}. \quad (165)$$

Then, using (120), we end up with

$$\frac{dA}{dt} \leq 4M_1 - \frac{M_1^2}{2\pi} + CR^{-1} (M^{3/2} + M^{1/2}) A^{1/2} + C(M_2 - M_1). \quad (166)$$

Inequality (166) implies $T < +\infty$ if $A(0) \ll 1$, as is observed by [27] (see also Chapter 5 of [19]). Here we describe the proof for completeness.

The first observation is the monotonicity formula

$$\left| \frac{d}{dt} \int_{\Omega} \omega \varphi \right| \leq C(M + M^2) \|\nabla \varphi\|_{C^1}, \quad (167)$$

derived from (124) and the symmetry of the Green's function: $G(x, x') = G(x', x)$. The proof is the same as in (34) and is omitted.

Second, we put $I_1 = I_{x_0,R}$ and $I_2 = I_{x_0,2R}$ for

$$I_{x_0,R} = \int_{\Omega} |x - x_0|^2 \omega \varphi_{x_0,R}. \quad (168)$$

Then it holds that

$$\begin{aligned} M_2 - M_1 &\leq \int_{R < |x-x_0| < 2R} \varphi_{x_0,2R} \omega \\ &\leq 2R^{-1} \int_{\Omega} |x - x_0| \varphi_{x_0,2R} \omega \leq 2M^{1/2} R^{-1} I_2^{1/2} \end{aligned} \quad (169)$$

and

$$\begin{aligned} A_2 &= A_1 + \int_{\Omega} |x - x_0|^2 (\varphi_{x_0,2R} - \varphi_{x_0,R}) \omega \\ &\leq A_1 + 4R^2 \int_{\Omega} (\varphi_{x_0,2R} - \varphi_{x_0,R}) \omega, \end{aligned} \quad (170)$$

which implies

$$\begin{aligned} \frac{dA_1}{dt} &\leq 4M_1 - \frac{M_1^2}{2\pi} + CR^{-1} (M^{3/2} + M^{1/2}) A_1^{1/2} \\ &\quad + C (M^{3/2} + M^{1/2}) \left\{ \int_{\Omega} (\varphi_{x_0,2R} - \varphi_{x_0,R}) \omega \right\}^{1/2}. \end{aligned} \quad (171)$$

Here, we use (167) to ensure

$$\left| \frac{d}{dt} \left(4M_1 - \frac{M_1^2}{2\pi} \right) \right| \leq C(M + M^2) R^{-2} \quad (172)$$

and

$$\left| \frac{d}{dt} \int_{\Omega} (\varphi_{x_0,2R} - \varphi_{x_0,R}) \omega \right| \leq C(M + M^2) R^{-2}. \quad (173)$$

Then, it follows that

$$4M_1 - \frac{M_1^2}{2\pi} \leq 4M_1(0) - \frac{M_1(0)^2}{2\pi} + CBa(R^{-1}t^{1/2}) \quad (174)$$

and

$$\begin{aligned} \int_{\Omega} (\varphi_{x_0,2R} - \varphi_{x_0,R}) \omega &\leq \int_{\Omega} (\varphi_{x_0,2R} - \varphi_{x_0,R}) \omega_0 + CBa(R^{-1}t^{1/2}) \\ &\leq 2R^{-2} A_2(0) + CBa(R^{-1}t^{1/2}) \end{aligned} \quad (175)$$

for

$$B = M^{3/2} + M^{1/2}, \quad a(s) = s^2 + s. \quad (176)$$

Thus we obtain

$$\begin{aligned} \frac{dA_1}{dt} &\leq 4M_1(0) - \frac{M_1(0)^2}{2\pi} + CR^{-1}BA_1^{1/2} + CBA_2(0)^{1/2} + CBa\left(R^{-1}t^{1/2}\right) \\ &= J(0) + CBa\left(R^{-1}t^{1/2}\right) + CBR^{-1}A_1^{1/2} \end{aligned} \quad (177)$$

for

$$J = 4M_1 - \frac{M_1^2}{4\pi} + CBR^{-1}A_2^{1/2}. \quad (178)$$

Assume $M_1(0) > 8\pi$, and put

$$-4\delta = 4M_1(0) - \frac{M_1(0)^2}{2\pi} < 0. \quad (179)$$

Let, furthermore,

$$\frac{1}{R^2} \int_{\Omega} |x - x_0|^2 \varphi_{x_0, 2R} \omega_0 \leq \eta. \quad (180)$$

Now we define s_0 by

$$CBa(s_0) = \delta \quad (181)$$

in (177), and take $0 < \eta \ll 1$ such that

$$\eta \leq \delta s_0^2. \quad (182)$$

Then, if R and T_0 satisfy $R^{-2}T_0 = \eta\delta^{-1}$, it holds that

$$A_1(0) \leq R^2\eta < 2\delta T_0. \quad (183)$$

Making $0 < \eta \ll 1$, furthermore, we may assume

$$\begin{aligned} J(0) + CBR^{-1}A_1(0)^{1/2} &\leq -4\delta + CBR^{-1}A_2(0)^{1/2} \\ &\leq -4\delta + CB\eta^{1/2} \leq -3\delta, \end{aligned} \quad (184)$$

which results in

$$\begin{aligned} \frac{dA_1}{dt} &\leq J(0) + CBa\left(R^{-1}T_0^{1/2}\right) + BR^{-1}A_1(t)^{1/2} \\ &= J(0) + \delta + CBR^{-1}A_1^{1/2}, \quad 0 \leq t < T_0, \end{aligned} \quad (185)$$

provided that $T \geq T_0$.

A continuation argument to (184)–(185) guarantees

$$\frac{dA_1}{dt} \leq -2\delta, \quad 0 \leq t < T_0, \quad (186)$$

and then we obtain

$$A_1(T_0) \leq A_1(0) - 2\delta T_0 < 0 \quad (187)$$

by (183), a contradiction. □

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