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XEFGM Fracture Analysis of Functionally Graded Materials under Mixed Mode and Asymmetric Loading

Nathera A. Saleh and Haider Khazal

Abstract

This research describes crack analysis in the functionally graded materials (FGMs) by adopting the extended element free Galerkin method (XEFGM) under mixed mode and asymmetric loading. These loads are somewhat similar to fatigue loads because, numerically, they are load values from zero and then directly to the critical load. The meshfree method can be easily simulated the fracture problems against the traditional numerical method because it is not dependent on mesh. Triangles technique in the process of numerical integration at regions of discontinuity, functions of enrichment, and as well as the appropriate support field to contain numerical points and nodes to from the shape functions are used in this study. In addition, incompatible interaction integration technique has used to determine the stress intensity factors (SIFs). Two study cases with different crack positions were studied and compared with the experimental works of the relevant reference literature, where accurate and identical results were obtained.

Keywords: Crack propagation, incompatible interaction integration technique, meshfree method, functionally graded materials

1. Introduction

Functionally graded materials (FGMs) arose from the realization that it was necessary to meet ultra-high temperature and cryogenic requirements. The goals of strength, flexibility, and fatigue resistance inspired the early research. This aim in achieving a smooth and perfect spatial variation is most effectively met by gathering different materials with those favorable characteristics, thereby avoiding the detriment effect such as stress concentration and residual stress found in discrete interface. The gradual change in material properties from the original composite has been shown to increase efficiency by mitigating failure and maintaining the intended benefits of merging two or more different materials. Functionally graded materials are found in nature, for example in bones, teeth, wood, and bamboo [1]. As a result of the microstructure and mechanical properties of functionally graded materials (FGMs) that different from their corresponding from the conventional composites, they have to be used in many engineering, military, medical and space sciences applications. Numerical methods have been used and applied in the

analysis of fracture problems in these materials because they have the ability to give realistic results [2–11]. Popovich et al. [12], Zhao et al. [13] studied fatigue load that exerted on certain types of functionally graded materials. Most recently, digital image correlation technique was used with numerical verification on a stepwise functionally graded material made of glass and epoxy to find the path growth of the crack and the stress concentration values [14, 15]. On the other hand, the optical method was used to analyze the crack path in a material made continuously from the graded materials and the stress concentration factor and T-stress were also calculated [16]. One of these methods that have been adopted in this research is the extended weak Galerkin formulation-element free method (XEFGM). The element-free Galerkin method with the sub-triangle technique to enhance the accuracy of the Gauss squared near the crack to determine stress intensity factors in isotropic and anisotropic materials can give accurate results in analyzing fracture structures. The incompatible interaction integral technique is applied to extract stress intensity factors (SIFs). The method of sub-triangulation is used to hence the discontinuity terms. With low degrees of freedom, results can be obtained that are highly consistent with those of the relevant reference literature.

2. Field Equations

In elastic materials, Hooke's law can be employed in the following equation that clears the relationship between the stresses and strains in specific material.

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \quad (1)$$

$\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain respectively; while $\mathbf{D}$ represents the material matrix that varied with the displacement x in the functionally graded materials can be find as:

$$\mathbf{D} = \frac{E(x)}{1-\nu^2} \begin{bmatrix} 1 & \nu(x) & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu(x))/2 \end{bmatrix} \quad (\text{Plane stress}) \quad (2)$$

and

$$\mathbf{D} = \frac{E(x)(1-\nu(x))}{(1+\nu(x))(1-2\nu(x))} \begin{bmatrix} 1 & \frac{\nu(x)}{1-\nu(x)} & 0 \\ \frac{\nu(x)}{1-\nu(x)} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu(x)}{2(1-\nu(x))} \end{bmatrix} \quad (\text{Plane strain}) \quad (3)$$

Where E and ν represent Young's Modulus and Poisson's ratio that varied with the displacement x respectively, Eqs. (2) and (3) are useful because they give a change in the properties of the material at each point of the material in relation to the displacement.

Eq. (1) can be put as:

$$\boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad (4)$$

where **C** represents the compliance matrix.
In FGM

$$E = E(x_1, x_2) = E(x) \tag{5}$$

$$\nu = \nu(x_1, x_2) = \nu(x) \tag{6}$$

In the above two equations, it is clear that the Young's Modulus and Poisson's ratio change with the displacement and this gives the impression for the behave of properties of the functionally graded materials as shown in **Figure 1**.

The stresses and displacements in functionally graded materials near the crack tip can be obtained by depending on angular stress and displacement functions as [2]:

$$\sigma_{11} = \frac{1}{\sqrt{2\pi r}} \left[K_I f_{11}^I(\theta) + K_{II} f_{11}^{II}(\theta) \right] \tag{7}$$

$$\sigma_{22} = \frac{1}{\sqrt{2\pi r}} \left[K_I f_{22}^I(\theta) + K_{II} f_{22}^{II}(\theta) \right] \tag{8}$$

$$\sigma_{12} = \frac{1}{\sqrt{2\pi r}} \left[K_I f_{12}^I(\theta) + K_{II} f_{12}^{II}(\theta) \right] \tag{9}$$

$$u_1 = \frac{1}{G_{tip}} \sqrt{\frac{r}{2\pi}} \left[K_I g_1^I(\theta) + K_{II} g_1^{II}(\theta) \right] \tag{10}$$

$$u_2 = \frac{1}{G_{tip}} \sqrt{\frac{r}{2\pi}} \left[K_I g_2^I(\theta) + K_{II} g_2^{II}(\theta) \right] \tag{11}$$

$$G_{tip} = \frac{E_{tip}}{[2(1 + \nu_{tip})]} \tag{12}$$

Figure 1.
FGM body with crack,

The mechanical properties in Eq. (11) to (13) will be extracted at the tip of crack. $f_{ij}^I(\theta)$, $f_{ij}^{II}(\theta)$, $g_{ij}^I(\theta)$ and $g_{ij}^{II}(\theta)$ ($i, j = 1, 2$) represent the standard angular functions [17, 18]. Eqs. (7) to (13) can calculate the values of stresses and displacements at each point of the material supported by the enrichment functions.

3. XEFGM Structure

In the two-dimensional solid elasticodynamics as depicted in **Figure 2**, the equilibrium equation that gives the relationship between the stresses and traction load $\mathbf{t}$ can be depicted in matrix form as:

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in problem domain } \Omega \quad (13)$$

Where the following Eqs. (15) to (17) represent boundary equations.

$$\boldsymbol{\sigma} \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t \quad (14)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad (15)$$

$$\boldsymbol{\sigma} \mathbf{n} = 0 \quad \text{on } \Gamma_e \quad (16)$$

And the differential operator matrix $\mathbf{L}$ is

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (17)$$

EFGM adopted the moving least squares (MLS) approximation [19] to find shape function of the numerical method. The governing equation of the relevant problem:

Figure 2.
2D cracked body.

$$\int_{\Omega} (\mathbf{L}\delta\mathbf{u})^T (\mathbf{D}\mathbf{L}\mathbf{u}) d\Omega - \int_{\Omega} \delta\mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T \bar{\mathbf{t}} d\Gamma - \int_{\Gamma_u} \delta\lambda^T (u - \bar{u}) d\Gamma - \int_{\Gamma_u} \delta\mathbf{u}^T \lambda d\Gamma = 0 \quad (18)$$

where λ represents the Lagrange multiplier variable. The MLS shape function ϕ_i at node i a point $\mathbf{x}$ can be defined as [20]:

$$\phi_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) [\mathbf{A}(\mathbf{x})]^{-1} w(\mathbf{x} - \mathbf{x}_i) \mathbf{p}(\mathbf{x}_i) \quad (19)$$

$\mathbf{p}(\mathbf{x})$ is the basis function

$$\mathbf{p}^T(\mathbf{x}) = [1 \quad xy] \quad (20)$$

And $\mathbf{A}$ can be extracted as

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^n w(\mathbf{x} - \mathbf{x}_i) \mathbf{p}(\mathbf{x}_i) \mathbf{p}^T(\mathbf{x}_i) \quad (21)$$

And

$$w(r) = \begin{cases} \frac{2}{3} - 4r_s^2 + 4r_s^3 & r_s \leq \frac{1}{2} \\ \frac{4}{3} - 4r_s + 4r_s^2 - \frac{4}{3}r_s^3 & \frac{1}{2} < r_s \leq 1 \\ 0 & r_s > 1 \end{cases} \quad (22)$$

Where r_s is the size of the influence domain for node i .

The externally enriched displacement approximation $\mathbf{u}^h$ of a model point $\mathbf{x}$ [21–23]:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^n \phi_i(\mathbf{x}) \mathbf{u}_i + \sum_{k=1}^{m_t} \phi_k \sum_{\alpha=1}^4 Q_{\alpha}(\mathbf{x}) \mathbf{b}_k \quad (23)$$

For isotropic FGMs, enrichment functions are applied to enrich the MLS formulation [24, 25]:

$$Q(r, \theta) = \left(\sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right) \quad (24)$$

Therefore the final discretized system equations,

$$\begin{bmatrix} \mathbf{K} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{q} \end{bmatrix} \quad (25)$$

The vectors in Eq. (26)

$$\mathbf{Q} = - \int_{\Gamma_u} \mathbf{N}^T \phi d\Gamma \quad (26)$$

$$\mathbf{q} = - \int_{\Gamma_u} \mathbf{N}^T \bar{\mathbf{u}} \quad (27)$$

$$\lambda(\mathbf{x}) = \sum_{i=1}^{n_\lambda} N_i(\mathbf{x}) \lambda_i \quad (28)$$

$\mathbf{U}$ is the global displacement vector can be defined as:

$$\mathbf{U} = \{\mathbf{u} \ \mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \ \mathbf{b}_4\}^T \quad (29)$$

Where $\mathbf{b}_1$ to $\mathbf{b}_4$ represent the enrichment, function terms.
And

$$\mathbf{K}_{ij}^n = \begin{bmatrix} K_{ij}^{uu} & K_{ij}^{ub} \\ K_{ij}^{bu} & K_{ij}^{bb} \end{bmatrix} \quad (30)$$

$$\mathbf{F}_i^m = \left\{ \mathbf{F}_i^u \quad \mathbf{F}_i^{b_1} \quad \mathbf{F}_i^{b_2} \quad \mathbf{F}_i^{b_3} \quad \mathbf{F}_i^{b_4} \right\}^T \quad (31)$$

where

$$\mathbf{K}_{ij}^{rs} = \int_{\Omega} (\mathbf{B}_i^r)^T \mathbf{D} \mathbf{B}_j^s d\Omega \quad (r, s = u, b) \quad (32)$$

$$\mathbf{F}_i^u = \int_{\Omega} \phi_i^t \mathbf{b} d\Omega + \int_{\Gamma_t} \phi_i^T \bar{\mathbf{t}} d\Gamma \quad (33)$$

$$\mathbf{F}_i^{b_a} = \int_{\Omega} \phi_i^T Q_a \mathbf{b} d\Omega + \int_{\Gamma_t} \phi_i^T Q_a \bar{\mathbf{t}} d\Gamma \quad (a = 1, 2, 3, 4) \quad (34)$$

$\mathbf{B}_i^u$ and $\mathbf{B}_i^b$ are matrices of shape function derivatives:

$$\mathbf{B}_i^u = \begin{bmatrix} \phi_{i,x} & 0 \\ 0 & \phi_{i,y} \\ \phi_{i,y} & \phi_{i,x} \end{bmatrix} \quad (35)$$

$$\mathbf{B}_i^b = \begin{bmatrix} \mathbf{B}_i^{b_1} & \mathbf{B}_i^{b_2} & \mathbf{B}_i^{b_3} & \mathbf{B}_i^{b_4} \end{bmatrix} \quad (36)$$

$$\mathbf{B}_i^u = \begin{bmatrix} (\phi_i Q_\alpha)_{,x} & 0 \\ 0 & (\phi_i Q_\alpha)_{,y} \\ (\phi_i Q_\alpha)_{,y} & (\phi_i Q_\alpha)_{,x} \end{bmatrix} \quad (37)$$

$$\alpha = 1, 2, 3, 4 \quad (38)$$

$$\boldsymbol{\varepsilon} = \mathbf{L} \mathbf{u}^h \quad (39)$$

$$\boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon} \quad (40)$$

Eq. (40) and (41) represent stress and strain for whole body of FG material calculated by numerical method.

The interaction integral and J integral can be defined [28–30]:

$$M = \int_A \left\{ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \frac{1}{2} (\sigma_{ik} u_{ik}^{aux} + \sigma_{ik}^{aux} u_i) \delta_{1j} \right\} q_{,j} dA + \int_A \left\{ \sigma_{ij} (c_{ijkl}^{tip} - c_{ijkl}(x)) \sigma_{kl,1}^{aux} \right\} q dA \quad (46)$$

$$J_{local} = (K_I^2 + K_{II}^2) / E_{tip} \quad (47)$$

$$J_{local}^s = \frac{(K_I + K_I^{aux})^2 + (K_{II} + K_{II}^{aux})^2}{E_{tip}} = J_{local} + J_{local}^{aux} + M_{local} \quad (48)$$

$$J_{local}^{aux} = \left[(K_I^{aux})^2 + (K_{II}^{aux})^2 \right] / E_{tip} \quad (49)$$

and M_{local} is given by

$$M_{local} = 2(K_I K_I^{aux} + K_{II} K_{II}^{aux}) / E_{tip} \quad (50)$$

$$\begin{aligned} K_I &= M_{local}^{(1)} E_{tip} / 2, & (K_I^{aux} = 1.0, K_{II}^{aux} = 0.0), K_{II} \\ &= M_{local}^{(2)} E_{tip} / 2, & (K_I^{aux} = 0.0, K_{II}^{aux} = 1.0). \end{aligned} \quad (51)$$

The Eq. (51) is used to calculate the stress intensity factors during the fracture analysis in functionally graded materials. The crack propagation criterion (Maximum hoop stress) was applied by the depend on procedure that was adopted by Erdogan and Sih [31].

5. Numerical examples

5.1 Example 1

The first example is that there is a crack in the FGM beam that undergoes a three-point bend as shown in **Figure 4**, and in this problem, the beams are homogeneous and gradient (along with the X_2 direction). **Figure 4a** and **b** show the geometry of the sample and BCs for two different boundary conditions: the states (a) and (b), respectively. Also, this figure shows the complete nodal distributions, and the adaptive background cells visualize the distribution of Gauss points and fertilization nodes around the crack for Case (a). Note that the nodal and background cell distributions are valid for both conditions. The material properties (**Table 1**) of the monolithic beam used are as follows:

$$E = 2890 \text{ MPa}, \nu = 0.4, K_{Ic} = 1.09 \text{ MPa} \sqrt{m}.$$

A 64 x 28 back grid and 1856 non-uniform distribution nodes are adopted in this case (**Figure 5**). 2875 nodes in finite element method was previously used by Kim and Paulino [3]. **Figure 6** depicts the comparison of the crack path of a homogeneous case (b) beam obtained by current work with the experimental results reported by Galvez et al. [32] With numerical simulation by [3]. The reasonably well output between the numerical and experimental results are obtained. Note that in this case, the gradient of the material does not affect the path of the crack. **Figure 7** shows the effect of increasing the slit length by current numerical simulations of (b) condition on the slit path compared to the experiment available for a

Figure 4. Three-point bending cracked beam. (a) Case (a). (b) Case (b). (c) The nodal allocation. (d) back grid structure. (e) Gauss points allocation.

gradient beam. **Figures 8 and 9** depict results of KI and KII for the different relative size of the J-integral domain (r_J) respectively. The results of the proposed method remain accurate for a wide range of r_J values and the integral field size $J(r_J)$ does not significantly influence the values of SIFs.

X_2	E (MPa)	ν	$K_{cr}(Mpa \sqrt{m})$
0	1780	0.41	0.99
60	4000	0.39	1.19

Table 1.
Material characteristics of the graduated beam.

Figure 5.
Nodes of enrichment around the crack prior to the final step of crack propagation.

Figure 6.
Comparison of crack paths for a homogeneous beam (Case b).

5.2 Example 2

Figure 10 depicts the configuration and mechanical properties (**Table 2**) of the case study (2) that is bending four points with vertical cracks that perpendicular on the gradient of material.

Figure 7.
The effect of increasing the crack length by current numerical simulations of case (b) on the crack path in a graded beam.

Figure 8.
 K_I values for case (b) with different relative rJ .

Rousseau and Tippur [33] applied ξ that is zero on the left side of the stepping part, and one on the right side (**Figure 10**). In current work, A 64 x 28 back grid and 2070 non-uniform distribution nodes are adopted in this case (**Figures 11 and 12**), while more than 10,000 element and 30,000 nodes were adopted by [33] to study this case.

The results of the current research work give high accuracy with related references as depicted in **Tables 3 and 4**. The consistent of present research can be depicted in **Tables 3–5**. **Figure 13** gives a comparison of the effect of increasing the slit length on the slit path of the current work with experimental work [33] at

ξ	$K_I(\text{Mpa } \sqrt{m})$			$K_{II}(\text{Mpa } \sqrt{m})$		
	Present work	Ref. [8]	Ref. [4]	Present work	Ref. [8]	Ref. [4]
0.17	2.087	2.087	2.088	-0.116	-0.117	-0.127
0.58	2.689	2.694	2.695	-0.087	-0.085	-0.094
1	2.588	—	2.598	-0.010	—	-0.013

Table 3.
Stress intensity factors rapprochement at $d_{\max} = 2$ and $r_J = 1.2$.

ξ	$P_{cr} \text{ (N)}$			Crack initiation angle θ_0 (deg.)		
	Present work	Ref. [8]	Ref. [4]	Present work	Ref. [8]	Ref. [4]
0.17	255	250	249.3	6.901	7.009	7
0.58	298	300	298	4.020	3.998	4
1	295	—	289.9	0.601	—	0.5

Table 4.
Critical load of and crack initiation angle rapprochement at $d_{\max} = 2$ and $r_J = 1.2$.

$\xi = 0.17; r_J = 1.2$	$d_{\max}$		
	1.7	2	2.3
$K_I(\text{Mpa } \sqrt{m})$	2.081	2.087	2.091
$K_{II}(\text{Mpa } \sqrt{m})$	-0.112	-0.116	-0.111

Table 5.
SIFs for different support domain size.

Figure 13.
Comparison of effect of increasing the crack length on the propagation path for current research and experimental work [33]. ($\Delta a = 1$ mm for red line, and $\Delta a = 3$ mm for green line).

Figure 14.
KI values for case (2) vs. relative rJ .

Figure 15.
 K_{II} for case (2) vs. relative rJ .

6. Conclusion

The work mentioned in this work submitted the development of the XEFGM method to simulate crack propagation and compute stress intensity factors in mixed-mode fracture analysis of FGM beams under mixed mode and asymmetric loading. It has been demonstrated that XEFGM needs much less DOF than

traditional FEM and XFEM to give the same accuracy levels. The adopt of sub-triangle technique for numerical integration, proper support field, and enrichment functions at the crack site has been shown to significantly increase the resolution of the solution. In addition, numerical simulation showed little effect of increasing incision length on the propagation path compared to the available experimental and numerical results. It is clear in this example that the growth of the crack is moving towards the soft side. There is no effect for increasing the crack length on the propagation path for current research with experimental work.

Additionally, the gradation of materials has no effect on the incision path of bending cases. Use of incompatible interaction integration method provides very accurate results for SIFs values.

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