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A Review Note on the Applications of Linear Operators in Hilbert Space

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Abstract

Hilbert Spaces are the closest generalization to infinite dimensional spaces of the Euclidean Spaces. We Consider Linear transformations defined in a normed space and we see that all of them are Continuous if the Space is finite Dimensional Hilbert Space Provide a user-friendly framework for the study of a wide range of subjects from Fourier Analysis to Quantum Mechanics. The adjoint of an Operator is defined and the basic properties of the adjoint operation are established. This allows the introduction of self Adjoint Operators are the subject of the section.

Keywords: linear space, norm of a vector, inner product, orthogonal vector, adjoint of an operator

1. Introduction

The Concept of Hilbert Space was put forwarded by David Hilbert in his work on Quadratic forms in infinitely many Variables. We take a Closer look at Linear Continuous map between Hilbert Spaces [1]. These are called bounded operators and branch of Functional Analysis Called “Operator Theory” [2]. Next we derive an important inequality which has many interesting applications in the theory of inner product spaces and as a consequence we obtain that each inner product space is a normed Vector spaces with the norm [3], i.e. the inner product generates this form. Moreover there are several essential algebraic identities, variously and ambiguously called Polarization Identities. These and other closely related identities are of constant use. Now we are in position to state and prove the above mentioned important inequality known as Cauchy-Schwartz Buniakowski inequality (briefly we say CSB inequality) and we shall also use this to define the concept of angle by means of a formula [1]. The theory of Hilbert Space that Hilbert and Others developed has not only greatly enriched the world of Mathematics but has Proven Extremely useful in the development of Scientific Theories Particularly Quantum Mechanics.

2. Definition 1

A Hilbert space is a complex Banach space whose norm induced from an inner produce [4] i.e., in which there is defined a complex function (x, y) of vectors x & y and α, β are scalars with the following properties

- i. $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$
- ii. $(\overline{x}, y) = (y, x)$
- iii. $(x, x) = \|x\|^2$

2.1 Remark 1.1

Every polynomial equation of the n^{th} degree with complex co-efficient has exactly n complex roots [5].

In accordance with the above remarks the scalars in this example are understood to be the complex number.

Consider the space l_2^n with the inner product of two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ defined by

$$(x, y) = \sum_{i=1}^n x_i \overline{y_i}$$

Now we are going to prove that l_2^n is a Hilbert space.

2.1.1 Proof

By using Hilbert Space definition $[(x, y)]$ of complex function, α, β are scalars]

- i. $(\alpha x + \beta y, z) = \sum_{i=1}^n (\alpha x_i + \beta y_i) \overline{z_i}$
 $= \alpha \sum_{i=1}^n x_i \overline{z_i} + \beta \sum_{i=1}^n y_i \overline{z_i}$
 $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z) \forall x, y, z \in l_2^n$
- ii. $(\overline{x}, y) = \sum_{i=1}^n \overline{x_i} \overline{y_i}$
 $= \sum_{i=1}^n \overline{x_i} \overline{y_i}$
 $= \sum_{i=1}^n y_i \overline{x_i}$
 $= (y, x)$
- iii. $(x, x) = \sum_{i=1}^n x_i \overline{x_i}$
 $= \sum_{i=1}^n |x_i|^2$
 $(x, x) = \|x\|^2$

Therefore (x, y) is an inner product on l_2^n .

Therefore l_2^n is a Hilbert space.

2.2 Theorem 1.1 (Schwartz inequality)

If x and y are any two vectors in Hilbert space then $|(x, y)| \leq \|x\| \|y\|$ [6–8].

2.2.1 Proof

When $y = 0$, the result is clear for both sides vanish

$$\text{i.e., } |(x, y)| = |(x, 0)| = |0| = 0$$

$$\|x\| \|y\| = \|x\| \|0\| = \|x\| 0 = 0$$

$$|(x, y)| = \|x\| \|y\| = 0$$

When $y \neq 0$

Take any scalar $\alpha \in C$ [Complex banach space] always $\|x - \alpha y\|^2 \geq 0$

$$\begin{aligned} (x - \alpha y, x - \alpha y) &\geq 0 \\ (x, x) - (x, \alpha y) - (\alpha y, x) + (\alpha y, \alpha y) &\geq 0 \\ (x, x) - \bar{\alpha}(x, y) - \alpha(y, x) + \alpha \bar{\alpha}(y, y) &\geq 0 \\ \|x\|^2 - \bar{\alpha}(x, y) - \overline{\alpha(x, y)} + \alpha \bar{\alpha} \|y\|^2 &\geq 0 \end{aligned} \quad (1)$$

Put $\alpha = \frac{(x, y)}{\|y\|^2}$ where $(x, y) \in C$.

$\because y \neq 0$ and $\|y\| \neq 0$.

So choose $\alpha = \frac{(x, y)}{\|y\|^2}$, $\bar{\alpha} = \frac{\overline{(x, y)}}{\|y\|^2}$.

From Eq. (1) becomes

$$\begin{aligned} \|x\|^2 - \frac{\overline{(x, y)}}{\|y\|^2} (x, y) - \frac{(x, y)}{\|y\|^2} \overline{(x, y)} + \frac{(x, y)}{\|y\|^2} \frac{\overline{(x, y)}}{\|y\|^2} \|y\|^2 &\geq 0 \\ \|x\|^2 - \frac{|(x, y)|^2}{\|y\|^2} - \frac{|(x, y)|^2}{\|y\|^2} + \frac{|(x, y)|^2}{\|y\|^2} &\geq 0 \\ \|x\|^2 - \frac{|(x, y)|^2}{\|y\|^2} &\geq 0 \\ \frac{|(x, y)|^2}{\|y\|^2} &\leq \|x\|^2 \\ |(x, y)|^2 &\leq \|x\|^2 \|y\|^2 \\ \therefore |(x, y)| &\leq \|x\| \|y\| \end{aligned}$$

2.3 Result 1

An inner product space is a normal linear space [9].

2.3.1 Proof

To prove.

$\|x\| \geq 0$ and $\|x\| = 0$ if $x = 0$

$$\|x\| = \sqrt{(x, x)} \Rightarrow \|x\|^2 = (x, x) \geq 0$$

So that $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

Now we have to show that $\|x + y\| \leq \|x\| + \|y\|$

$$\begin{aligned} \|x + y\|^2 &= [(x + y), (x + y)] \\ &= (x, x) + (x, y) + (y, x) + (y, y) \end{aligned}$$

$$\begin{aligned}
 &= \|x\|^2 + (x, y) + \overline{(x, y)} + \|y\|^2 \\
 &= \|x\|^2 + 2 \operatorname{Re}(x, y) + \|y\|^2 \\
 &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
 \|x + y\|^2 &\leq (\|x\| + \|y\|)^2 \\
 \|x + y\| &\leq \|x\| + \|y\|
 \end{aligned}$$

Now we can prove that $\|\alpha x\| = |\alpha|\|x\|$.
Consider

$$\begin{aligned}
 \|\alpha x\|^2 &= (\alpha x, \alpha x) \\
 &= \alpha \bar{\alpha} (x, x) \\
 \|\alpha x\|^2 &= |\alpha|^2 \|x\|^2 \\
 \|\alpha x\| &= |\alpha| \|x\|
 \end{aligned}$$

An inner product is a normed linear space.

2.4 Result 1.1

An inner product in Hilbert space is jointly continuous [10].

2.4.1 Proof

Since $x_n \rightarrow x$ and $y_n \rightarrow y \Rightarrow (x_n, y_n) \rightarrow (x, y)$

We have.

$\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$ and

$\|y_n - y\| \rightarrow 0$ as $n \rightarrow \infty$ and

Now consider

$$\begin{aligned}
 |(x_n, y_n), (x, y)| &= |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)| \\
 &\leq |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x, y)| \\
 &\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \quad (\text{by Schwartz Inequality}) \\
 &\rightarrow 0 \text{ as } y_n \rightarrow y \text{ and } x_n \rightarrow x \\
 |(x_n, y_n), (x, y)| &\rightarrow 0 \text{ as } n \rightarrow \infty \\
 (x_n, y_n) &\rightarrow (x, y) \text{ as } n \rightarrow \infty
 \end{aligned}$$

2.5 Theorem 1.1 (parallelogram law in Hilbert space)

If x and y are any two vectors in Hilbert space then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (2)$$

2.5.1 Proof

$$\begin{aligned}
 \|x + y\|^2 &= (x + y, x + y) \\
 &= (x, x) + (x, y) + (y, x) + (y, y)
 \end{aligned}$$

$$= \|x\|^2 + (x, y) + \overline{(x, y)} + \|y\|^2 \quad (3)$$

$$\begin{aligned} \|x - y\|^2 &= (x - y, x - y) \\ &= (x, x) - (x, y) - (y, x) + (y, y) \\ &= \|x\|^2 - (x, y) - \overline{(x, y)} + \|y\|^2 \end{aligned} \quad (4)$$

Adding (3) and (4) we get,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

2.6 Theorem 1.2 (polarization identity)

If x and y are any two vectors in Hilbert space then

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \quad (5)$$

2.6.1 Proof

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) \\ &= (x, x) + (x, y) + (y, x) + (y, y) \\ &= \|x\|^2 + (x, y) + \overline{(x, y)} + \|y\|^2 \end{aligned} \quad (6)$$

$$\begin{aligned} \|x - y\|^2 &= (x - y, x - y) \\ &= (x, x) - (x, y) - (y, x) + (y, y) \\ &= \|x\|^2 - (x, y) - \overline{(x, y)} + \|y\|^2 \end{aligned} \quad (7)$$

Subtracting (6) and (7) we get

$$\|x + y\|^2 - \|x - y\|^2 = 2(x, y) + 2(y, x) \quad (8)$$

Replace y by iy in Eq. (8)

$$\begin{aligned} \|x + iy\|^2 - \|x - iy\|^2 &= 2(x, iy) + 2(iy, x) \\ &= 2i(x, y) + 2i(y, x) \\ &= -2i(x, y) + 2i(y, x) \end{aligned}$$

Multiply both sides by i

$$i\|x + iy\|^2 - i\|x - iy\|^2 = 2(x, y) - 2(y, x) \quad (9)$$

Adding (8) and (9) we get

$$\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 = 4(x, y). \quad (10)$$

3. Definition 2

Let B be an arbitrary Banach space. A convex set in B is a non-empty subset S with the property that if x and y are in S then

$$z = x + t(y - x) = (1 - t)x + ty$$

it also in S for all real number t such that $0 \leq t \leq 1$.

A convex set is a non-empty set which contains the segment joining any pairs of its points.

Since C is convex it is non-empty and contains $\frac{x+y}{2}$ whenever it contains x and y [11].

3.1 Theorem 2.1

3.1.1 Application of parallelogram law

A closed convex subset C of a Hilbert space H contains a unit vector of smallest norm [12].

3.1.2 Proof

Step 1:

Since C is a convex set, it is non empty and contains $\frac{x+y}{2} \in C$ whenever $x, y \in C$.

Let $d = \inf \{\|x\|/x \in C\}$ then

$$d \leq \|x\| \forall x \in C \quad (11)$$

Then there exist a sequence $\{x_n\}$ of vectors in C such that $\{x_n\} \rightarrow d$ as

$$n \rightarrow \infty \quad (12)$$

Let $x_m, x_n \in C$.

$\because C$ is convex, $\frac{x_m+x_n}{2} \in C$

By (11), $d \leq \left\| \frac{x_m+x_n}{2} \right\|$

$$\begin{aligned} d &\leq \frac{\|x_m + x_n\|}{2} \\ 2d &\leq \|x_m + x_n\| \\ 4d^2 &\leq \|x_m + x_n\|^2 \\ -4d^2 &\geq -\|x_m + x_n\|^2 \end{aligned} \quad (13)$$

By parallelogram law

$$\begin{aligned} \|x_m + x_n\|^2 + \|x_m - x_n\|^2 &= 2\|x_m\|^2 + 2\|x_n\|^2 \\ \|x_m - x_n\|^2 &= 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \\ \|x_m - x_n\|^2 &\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \text{ by Eq. (13)} \\ &\leq 2d^2 + 2d^2 - 4d^2 \text{ by Eq. (12) as } n \rightarrow \infty. \\ \|x_m - x_n\| &\rightarrow 0 \text{ as } n \rightarrow \infty, m \rightarrow \infty. \end{aligned}$$

$\therefore \{x_n\}$ is a Cauchy sequence in C .

$\because C$ is a closed set in complete Space H

$\Rightarrow C$ is complete

There exist a vector x in C such that $x_n \rightarrow x$

i.e.) $x = \lim x_n$

$$\begin{aligned}\|x\| &= \|\lim x_n\| \\ &= \lim \|x_n\| = d \\ &= \inf \left\{ \frac{\|x\|}{x} \in C \right\}\end{aligned}$$

$\Rightarrow \|x\|$ is smallest.

$\therefore x$ is a vector in C with smallest norm.

Step 2:

To prove uniqueness of x .

Suppose there exist a vector x in C with $\|x'\| = d$ and $x \neq x'$ in C

$\because C$ is convex, $\frac{x+x'}{2} \in C$

By Eq. (11)

$$d \leq \left\| \frac{x+x'}{2} \right\| \quad (14)$$

By parallelogram law,

$$\begin{aligned}\left\| \frac{x+x'}{2} \right\|^2 + \left\| \frac{x-x'}{2} \right\|^2 &= 2\left\| \frac{x}{2} \right\|^2 + 2\left\| \frac{x'}{2} \right\|^2 \\ \left\| \frac{x+x'}{2} \right\|^2 &= 2\left\| \frac{x}{2} \right\|^2 + 2\left\| \frac{x'}{2} \right\|^2 - \left\| \frac{x+x'}{2} \right\|^2 \\ &\leq 2\left\| \frac{x}{2} \right\|^2 + 2\left\| \frac{x'}{2} \right\|^2 \\ &\leq \frac{2\|x\|^2}{2} + \frac{2\|x'\|^2}{2} \\ &\leq \frac{2d^2}{4} + \frac{2d^2}{4} \\ &\leq \frac{4d^2}{4} \\ \left\| \frac{x+x'}{2} \right\|^2 &\leq d^2 \\ \left\| \frac{x+x'}{2} \right\| &\leq d\end{aligned}$$

Which is a contradiction to Eq. (14)

Therefore our assumption on $x \neq x'$ is wrong

Hence $x = x'$

3.2 Theorem 2.2 (orthogonal complements)

Two Vectors x and y in a Hilbert space H are said to be orthogonal (written as $x \perp y$) if $(x, y) = 0$ [9]

$$\begin{aligned}\text{i. } x \perp y &\Leftrightarrow (x, y) = 0 \\ &\Leftrightarrow (\overline{x}, \overline{y}) = \overline{0}\end{aligned}$$

$$\Leftrightarrow (y, x) = 0$$

$$\Leftrightarrow y \perp x$$

$$\text{ii. } (0, x) = 0 \forall x \in H$$

$$\Rightarrow 0 \perp x \forall x \in H$$

Therefore 0 is orthogonal to every vector x in H

$$\text{iii. } x \perp x \Leftrightarrow (x, x) = 0$$

$$\Leftrightarrow \|x\|^2 = 0$$

$$\Leftrightarrow \|x\| = 0$$

$$\Leftrightarrow x = 0$$

This means that 0 is the only vector orthogonal to itself

3.3 Theorem 2.3 (Pythagorean theorem)

Geometric fact about orthogonal vectors in the Pythagorean theorem such that $x \perp y$ implies [9]

$$\|x + y\|^2 = \|x - y\|^2 = \|x\|^2 + \|y\|^2$$

3.3.1 Proof

Since

$$x \perp y \Leftrightarrow (x, y) = 0$$

$$\Leftrightarrow (\overline{x}, \overline{y}) = \overline{0}$$

$$\Leftrightarrow (y, x) = 0$$

$$\Leftrightarrow y \perp x$$

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) \\ &= (x, x) + (x, y) + (y, x) + (y, y) \end{aligned}$$

$$= \|x\|^2 + \|y\|^2 \text{ by}$$

$$(x, y) = 0, (y, x) = 0 \tag{15}$$

$$\begin{aligned} \|x - y\|^2 &= (x - y, x - y) \\ &= (x, x) - (x, y) - (y, x) + (y, y) \end{aligned}$$

$$= \|x\|^2 + \|y\|^2 \text{ by}$$

$$(x, y) = 0, (y, x) = 0 \tag{16}$$

From Eq. (15) and (16)

$$\|x + y\|^2 = \|x - y\|^2 = \|x\|^2 + \|y\|^2$$

4. Definition 3

A Vector x is said to be orthogonal to a non-empty set S (written as $x \perp S$) if $x \perp y \forall y \in S$ [7].

4.1 Definition 3.1

The orthogonal complement of S denoted by $S^\perp$ is the set of all vectors orthogonal to s i.e., $S^\perp = \{x/x \in H \text{ and } x \perp s\}$.

i.e., $x \in S^\perp \Leftrightarrow x \perp s$ [10].

The following statements are the easy consequence of the definition

- i. $\{0\}^\perp = H$
- ii. $H^\perp = \{0\}$
- iii. $S \cap S^\perp \subseteq \{0\}$
- iv. $S_1 \subseteq S_2 \Rightarrow S_2^\perp \subseteq S_1^\perp$
- v. $S^\perp$ is a closed subspace of H .

4.1.1 Theorem 3.1.1

If M is a proper closed linear subspace of a Hilbert Space H . Then there exists a non-zero vectors z_0 in H such that orthogonal to M . i.e., $z_0 \perp M$ [10].

4.1.1.1 Proof

Let x be a vector not in M and let d be the distance from r to M . Then by theorem "Let M be a closed linear subspace of a Hilbert Space H , let x be a vector not in M and let d be the distance from x to M . Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$ "

We define $z_0 = x - y_0, x \in H, y_0 \in M$

$$\Rightarrow y_0 \in H$$

$$\Rightarrow x - y_0 \in H$$

$$\Rightarrow z_0 \in H$$

$$d = d(x, M) = \inf \{\|x - m\| : m \in M\}$$

$$\Rightarrow d \leq \|x - m\| \forall m \in M \text{ and} \quad (17)$$

$$\|x - m\| \geq 0 \forall m \in M$$

$$\inf \{\|x - m\| : m \in M\} \geq 0$$

$$\Rightarrow d \geq 0$$

Claim:- $d > 0$,

If $d = 0$ then $\inf \{\|x - m\| : m \in M\} = 0$.

Then there exists a sequence $\{m_n\}$ in M such that $\|x - m_n\| = 0$ as $n \rightarrow \infty$

$\Rightarrow \{m_n\} \rightarrow x$ as $n \rightarrow \infty$.

$\Rightarrow x \in M$, since M is closed.

This is a contradiction to $x \notin M$. $d \neq 0$.

Hence $d > 0$.

$$\|z_0\| = \|x - y_0\| = d$$

$$\Rightarrow \|z_0\| = d > 0$$

$$\Rightarrow \|z_0\| > 0$$

$$\Rightarrow z_0 \neq 0$$

$$x \in M \Rightarrow x \in H \text{ since } M \subseteq H$$

$$y_0 \in M \Rightarrow y_0 \in H$$

$$\Rightarrow x - y_0 \in H$$

$$\Rightarrow z_0 \in H$$

This proves the existence of non-zero vector z_0 in H .

We conclude the proof by showing that if y is an arbitrary vector in M then $z_0 \perp M$.

Let α be any scalar then

$$\|z_0 - \alpha y\| = \|x - y_0 - \alpha y\|$$

$$= \|x - (y_0 + \alpha y)\|$$

$$\geq \alpha \|z_0\|$$

$$\|z_0 - \alpha y\| \geq \|z_0\|$$

$$\Rightarrow (z_0 - \alpha y)(z_0 - \alpha y) \geq (z_0, z_0)$$

$$(z_0, z_0) - (z_0, \alpha y) - (\alpha y, z_0) + (\alpha y, \alpha y) \geq (z_0, z_0)$$

$$-\bar{\alpha}(z_0, y) - \alpha(y, z_0) + \alpha\bar{\alpha}(y, y) \geq 0 \quad (18)$$

Put $\alpha = \beta(z_0, y)$ for an arbitrary real β then $\bar{\alpha} = \beta(\overline{z_0, y})$ i.e. (18) becomes

$$\Rightarrow -\beta(\overline{z_0, y})(z_0, y) - \beta(z_0, y)(y, z_0) + \beta(z_0, y)\beta(\overline{z_0, y})\|y\|^2 = 0$$

$$\Rightarrow -\beta|z_0, y|^2 - \beta|(z_0, y)|^2 + \beta^2|(z_0, y)|^2\|y\|^2 \geq 0$$

$$\Rightarrow -2\beta|(z_0, y)|^2 + \beta^2|z_0, y|^2\|y\|^2 \geq 0$$

$$\Rightarrow \beta|(z_0, y)|^2(-2 + \beta\|y\|^2) \geq 0$$

$$\Rightarrow \beta|(z_0, y)|^2(\beta\|y\|^2 - 2) \geq 0 \quad (19)$$

Clearly

$$|(z_0, y)| = 0$$

Suppose

$$|(z_0, y)| > 0$$

Choose β arbitrary smallest +ve such that $\beta\|y\|^2 < 2$

$$\begin{aligned} & (\beta \|y\|^2 - 2) < 0 \\ \Rightarrow & \beta |(z_0, y)|^2 (\beta \|y\|^2 - 2) < 0 \end{aligned}$$

This is a contradiction to the Eq. (19)

$$\begin{aligned} & |(z_0, y)| = 0 \\ \Rightarrow & (z_0, y) = 0 \end{aligned}$$

Therefore

$$\begin{aligned} & z_0 \perp y, y \in M \\ \Rightarrow & z_0 \perp M. \end{aligned}$$

Hence it is proved.

5. Definition 4

5.1 Adjoint of an operator

Let H be a Hilbert Space and T be an operator on H then the mapping $T^* : H \rightarrow H$ defined by $(Tx, y) = (x, T^*y) \forall x, y \in H$ is called the adjoint of T . We verify that T^* is actually an operator on H [13]

i. To prove that T^* is linear

i.e., To prove

$$T^*(y + z) = T^*(y) + T^*(z)$$

$$T^*(\alpha y) = \alpha T^*(y)$$

$$(x, T^*(y + z)) = (Tx, y + z)$$

$$= (x, T^*y) + (x, T^*z)$$

$$= (x, T^*y + T^*z)$$

$$(x, T^*(y + z)) = (x, T^*(y) + T^*(z))$$

$$\Rightarrow T^*(y + z) = T^*(y) + T^*(z)$$

$$(x, T^*(\alpha y)) = (Tx, \alpha y)$$

$$= \overline{\alpha}(Tx, y)$$

$$= \overline{\alpha}(x, T^*y)$$

$$(x, T^*(y + z)) = (x, \alpha T^*(y))$$

$$\Rightarrow T^*(\alpha y) = \alpha T^*(y)$$

Therefore T is linear

ii. To prove that T^* is continuous

$$0 \leq \|T^*(x)\|^2$$

$$= (T^*x, T^*x)$$

$$\begin{aligned}
 &= (TT^*x, x) \\
 &= |(TT^*x, x)| \\
 &\leq \|TT^*x\| \|x\| \\
 &\leq \|T\| \|T^*x\| \|x\| \text{ (by Schwartz Inequality)} \\
 &\|T^*x\| \leq \|T\| \|x\| \\
 &\sup\{\|T^*x\|/\|x\| \leq 1\} \leq \|T\| \text{ if } \|x\| \leq 1 \\
 &\|T^*\| \leq \|T\|
 \end{aligned}$$

Since T is bounded, T^* is also bounded.

Hence T^* is an operator on H .

These facts tell us that $T \rightarrow T^*$ is an mapping of $B(H)$ into itself

This mapping is called the adjoint operation on $B(H)$. [$B(H)$ is a Banach Space].

5.2 Theorem 4.1

The adjoint operation $T \rightarrow T^*$ on $B(H)$ has the following properties [9]

- i. $(T_1 + T_2)^* = T_1^* + T_2^*$
- ii. $(\alpha T)^* = \bar{\alpha} T^*$
- iii. $(T_1 T_2)^* = T_2^* T_1^*$
- iv. $T^{**} = T$
- v. $\|T^*\| = \|T\|$
- vi. $\|T^* T\| = \|T\|^2$

5.2.1 Proof

- i. $(T_1 + T_2)^* = T_1^* + T_2^*$

$$\begin{aligned}
 (x, (T_1 + T_2)^* y) &= ((T_1 + T_2)x, y) \\
 &= (T_1 x + T_2 x, y) \\
 &= (T_1 x, y) + (T_2 x, y) \\
 &= (x, T_1^* y) + (x, T_2^* y) \\
 (x, (T_1 + T_2)^* y) &= (x, T_1^* y) + (x, T_2^* y) \\
 (T_1 + T_2)^* y &= T_1^* y + T_2^* y \\
 (T_1 + T_2)^* y &= (T_1^* + T_2^*) y \forall y \in H \\
 (T_1 + T_2)^* &= T_1^* + T_2^*
 \end{aligned}$$
- ii. To prove $(\alpha T)^* = \bar{\alpha} T^*$

$$(x, (\alpha T)^* y) = ((\alpha T)x, y)$$

$$\begin{aligned}
 &= (\alpha Tx, y) \\
 &= \alpha(Tx, y) \\
 &= \alpha(x, T^*y) \\
 (x, (\alpha T)^*y) &= \alpha(x, T^*y) \forall x \in H \\
 (\alpha T)^* &= \bar{\alpha} T^*
 \end{aligned}$$

iii. To prove $(T_1 T_2)^* = T_2^* T_1^*$

$$\begin{aligned}
 (x, (T_1 T_2)^* y) &= ((T_1 T_2)x, y) \\
 &= (T_1(T_2 x), y) \\
 &= (T_2 x, T_1^* y) \\
 (x, (T_1 T_2)^* y) &= (x, T_2^* T_1^* y) \\
 (T_1 T_2)^* y &= T_2^* T_1^* y \forall y \in H \\
 (T_1 T_2)^* &= T_2^* T_1^*
 \end{aligned}$$

iv. To prove that $T^{**} = T$

$$\begin{aligned}
 (x, T^{**} y) &= (T^* x, y) \forall x \in H \\
 &= \overline{(y, T^* x)} \\
 &= \overline{(Ty, x)} \forall x \in H \\
 &= (x, Ty) \\
 T^{**} y &= Ty \\
 T^{**} &= T
 \end{aligned}$$

v. To prove that $\|T^*\| = \|T\|$

$$\begin{aligned}
 0 &\leq \|T^*(x)\|^2 \\
 &= (T^* x, T^* x) \\
 &= (TT^* x, x) \\
 &= |(TT^* x, x)| \\
 &\leq \|TT^* x\| \|x\| \\
 &\leq \|T\| \|T^* x\| \|x\| \text{ [by Schwartz Inequality].} \\
 \|T^* x\| &\leq \|T\| \|x\| \text{ if } \|x\| = 1. \\
 \sup\{\|T^* x\|/\|x\| = 1\} &\leq \|T\| \\
 \|T^* x\| &\leq \|T\|
 \end{aligned} \tag{20}$$

We know that $T = T^{**}$.

$$\begin{aligned}
 \Rightarrow \|T\| &= \|T^{**}\| \\
 &\leq \|T^*\| \\
 \|T\| &\leq \|T^*\|
 \end{aligned} \tag{21}$$

From Eq. (20) and (21) we get

$$\|T^*\| = \|T\|$$

vi. To prove that $\|T^*T\| = \|T\|^2$.

$$\begin{aligned} 0 &\leq \|Tx\|^2 \\ &= (Tx, Tx) \\ &= (Tx, T^{**}x) \\ &= (T^*Tx, x) \\ &= |(T^*Tx, x)| \\ &\leq \|T^*Tx\| \|x\| \text{ by Schwartz Inequality} \\ &\leq \|T^*T\| \|x\| \|x\| \\ \|Tx\|^2 &\leq \|TT^*\| \|x\|^2 \text{ if } \|x\| = 1 \\ \|Tx\|^2 &\leq \|TT^*\| \text{ if } \|x\| = 1 \\ \sup\{\|Tx\|/\|x\| = 1\} &\leq \|TT^*\| \\ \|T\|^2 &\leq \|TT^*\| \end{aligned} \tag{22}$$

$$\begin{aligned} \Rightarrow \|TT^*\| &= \|T\| \|T^*\| \\ &\leq \|T\| \|T\| \\ &\leq \|T\|^2 \end{aligned} \tag{23}$$

From Eq. (22) and (23) we get.

$$\|T^2\| = \|TT^*\|$$

6. Definition 5

6.1 Self Adjoint operator

An operator A on a Hilbert Space H is said to be self Adjoint if $A = A^*$. Since $0^* = 0$ and $I^* = I$, 0 and I are self-Adjoint operator [14].

6.2 Theorem 5.1

The Self-Adjoint operator in $B(H)$ from the closed real linear subspace of $B(H)$ and a real banch space which contains the identity transformation [14].

6.2.1 Proof

We will notice here about the product of two self-adjoint operators.

Let S denote the set of all Self-Adjoint operator in $B(H)$.

To prove $\underline{S}$ is a real linear subspace of $\underline{B(H)}$.

Let

$$A_1, A_2 \in S$$

$\Rightarrow A_1$ and A_2 are Self Adjoint.

$\Rightarrow A_1^* = A_1$ and $A_2^* = A_2$.

Let α, β are real,

$$\begin{aligned} (\alpha A_1 + \beta A_2)^* &= (\alpha A_1)^* + (\beta A_2)^* \\ &= \bar{\alpha} A_1^* + \bar{\beta} A_2^* \\ &= \alpha A_1^* + \beta A_2^* \\ &= \alpha A_1 + \beta A_2 \\ (\alpha A_1 + \beta A_2)^* &= \alpha A_1 + \beta A_2 \\ \Rightarrow \alpha A_1 + \beta A_2 &\in S \end{aligned}$$

Therefore S is real linear subspace of $B(H)$.

Further if $\{A_n\}$ is a sequence of self Adjoint operators which converges to an operator A . Then it is easy to see that A is also self Adjoint.

i.e.) Let $\{A_n\}$ be a sequence in S such that $A_n \rightarrow A$.

$$\begin{aligned} \|A - A^*\| &= \|A - A_n + A_n - A_n^* + A_n^* - A^*\| \\ &\leq \|A - A_n\| + \|A_n - A_n^*\| + \|A_n^* - A^*\| \\ &\leq \|A - A_n\| + \|A_n - A\| \text{ by } \|T^*\| = \|T\|. \\ &\leq 2\|A_n - A\| \rightarrow 0 \text{ as } A_n \rightarrow A \end{aligned}$$

Therefore $\|A_n - A\| \leq 0$

$\Rightarrow \|A_n - A\| = 0$ (since norm cannot be -ve)

$\Rightarrow A - A^* = 0$

$\Rightarrow A = A^*$

$\Rightarrow A$ is Self Adjoint

Therefore $A \in S$.

Hence S is closed in $B(H)$.

Therefore $B(H)$ is complete, S is complete.

Hence S is a real Banach Space.

$\therefore I^* = I$, I is self-Adjoint.

$\Rightarrow I \in S[(H) \text{ Contains identity transformation}]$.

Hence it is proved.

7. Conclusion

Even though every Hilbert Space is a Banach space, but there exist plenty of Banach space which are not Hilbert Spaces. However the converse is not true [13]. The Parallelogram Identity gives a criterion for normed space to become an inner product space [15]. It is important to emphasize that every finite dimensional normed Linear Space is a Hilbert Space [2]. Since every finite dimensional normed space is complete. The Theorems of this section allows us to define the adjoint Operator of a bounded Operator [3]. Finally we studied the Self adjoint Operator and its Properties. The results included here are Classical and can be found in the following Reference books in Functional Analysis.

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