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Topology in Photonic Discrete-Time Quantum Walks: A Comprehensive Review

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Abstract

We present a comprehensive review of photonic implementations of discrete-time quantum walks (DTQW) in the spatial and temporal domains. Moreover, we introduce a novel scheme for DTQWs using transverse spatial modes of single photons and programmable spatial light modulators (SLM) to manipulate them. We discuss current applications of such photonic DTQW architectures in quantum simulation of topological effects in photonic systems.

Keywords: quantum walks, spatial-multiplexing, time-multiplexing, spatial light modulators, geometric phase, Zak phase, topology

1. Introduction

Quantum computation is an interdisciplinary field that encompasses several interconnected branches such as quantum algorithms, quantum information, and quantum communication. There are several advantages associated with quantum information processing that have positioned quantum computation as a key resource in advanced modern science and technologies. Among the promising conjectures predicted by quantum information and communication, we find the development of more powerful algorithms that may allow to significantly increase the processing capacity and may enable the quantum simulation of complex physical systems and mathematical problems for which we know no classical digital computer algorithm that could efficiently simulate them at present.

Quantum algorithms are the main building blocks of quantum information and quantum communication strategies. Nevertheless, building superior quantum algorithms is a challenging task due to the complexities of quantum mechanics itself, and because quantum algorithms are required to demonstrate that they can outperform their classical counterparts, in order to be considered an evolutionary advantage. Therefore quantum algorithms must be more efficient than any existing classical protocol. In this context, quantum walks, i.e., the quantum mechanical counterpart of classical random walks, can be regarded as a sophisticated tool for building quantum algorithms for quantum information and quantum communication that has been shown to constitute a universal model for quantum computation [1–13].

The quantum walk is one of the most striking manifestations of how quantum interference leads to a strong departure between quantum and classical phenomena [2, 3, 14]. In the discrete version of the quantum walk, namely the discrete-time

quantum walk (DTQW) [15], the time evolution is described in terms of a series of discrete time-steps. DTQWs provide for a flexible architecture for the investigation of a large number of complex topological and holonomical effects, in the experimental [16–18] and theoretical domains [19–31]. Moreover, DTQWs are robust algorithm for modeling a large number of time-varying processes, ranging from energy transfer in chains of spins [32, 33] to energy transport in biological systems [34]. Furthermore, DTQWs allow to study multi-dimensional quantum interference effects [35–38] and can outline a route for authentication of quantum complexity [39, 40] and universal quantum computation [41]. In addition, quantum walks involving multiple particles guarantee a relentless tool for encoding quantum information in an exponentially large Hilbert space [42], as well as for simulations in quantum chemical, biological and physical systems [43], in 1D and 2D geometries [44–46].

In this Chapter, we present a comprehensive review of photonic realizations of DTQW in both, the spatial [47] and the temporal [48] realms, based on spatial-multiplexing and time-multiplexing techniques, respectively. Moreover, we present a novel scheme for photonic DTQW exploiting transverse spatial modes of photons and programmable spatial light modulators (SLM) to manipulate the modes [3]. In contrast to all previous multiplexed implementations, this novel approach warrants quantum simulation of arbitrary discrete time-steps, only limited by the spatial resolution of the SLM itself. We also deliberate about possible applications of such photonic DTQW platforms in quantum simulation of topological phenomena in photonic systems, and the implementation of non-local quantum coin operations, based on two-photon hybrid entanglement. Part of this review is based on the work by the Author, selected as the cover story of a Special Issue on Quantum Topology, for the journal Crystals (MDPI) in 2017 [2].

2. Theoretical framework

As a starter, we describe the theoretical framework for the mathematical description of DTQWs, and applications in the generation and detection of non-trivial geometric-phase structures, in 1D DTQW platforms. The basic discrete step in the DTQW is mathematically described by a unitary quantum evolution operator $U(\theta) = TR_{\vec{n}}(\theta)$, with $R_{\vec{n}}(\theta)$ a rotation operation along an arbitrary direction, represented by the 3D vector $\vec{n} = (n_x, n_y, n_z)$, represented by the following expressions:

$$R_{\vec{n}}(\theta) = \begin{pmatrix} \cos(\theta) - in_z \sin(\theta) & (in_x - n_y) \sin(\theta) \\ (in_x + n_y) \sin(\theta) & \cos(\theta) + in_z \sin(\theta) \end{pmatrix}$$

written in the well-known 2x2 Pauli basis [49]. We note that the rotation operation acts on polarization in the case of photons, or on spin in the case of atoms or ions. In the Pauli basis the y-rotation operation is expressed as:

$$R_y(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

This unitary operation is followed by a spin- or polarization-dependent translation T , which can be mathematically expressed by:

$$T = \sum_x |x+1\rangle\langle x| \otimes |H\rangle\langle H| + |x-1\rangle\langle x| \otimes |V\rangle\langle V|,$$

with $H = (1, 0)^T$ and $V = (0, 1)^T$.

The quantum evolution operator for a discrete time-step is generated by a Hamiltonian $H(\theta)$, such that $U(\theta) = e^{-iH(\theta)}$ ($\hbar = 1$), where:

$$H(\theta) = \int_{-\pi}^{\pi} dk \left[E_{\theta}(k) \vec{n}(k) \cdot \vec{\sigma} \right] \otimes |k\rangle\langle k|$$

and $\vec{\sigma}$ are the Pauli matrices, which readily reveals the spin-orbit coupling workings in the system. The discrete-time quantum walk described by the unitary operator $U(\theta)$ has readily been experimentally implemented in a number of devices such as photonic, cold-atom and trapped-ion devices [47, 48, 50–52]. It has been shown to display chiral symmetry and exhibit a Dirac-like dispersion relation, expressed as $\cos(E_{\theta}(k)) = \cos(k) \cos(\theta)$. In general, the spectrum of the system will depend on the selected branch cut. Here, we select the branch cut to be at the quasi-energy gap [53, 54].

3. Photonic DTQWs

3.1 Multiplexed DTQWs in the spatial domain

The original strategy for implementation of photonic DTQW via spatial-mode multiplexing was first introduced by Broome *et al.* [47]. The dimension of the Hilbert space for the spatial DTQW is determined by $2n + 1$ multiplexed longitudinal spatial modes of single photons coupled to a coin operation encoded in the two-dimensional spin or polarization subspace $\{|H\rangle, |V\rangle\}$. The discrete spatial modes of single photons $\{|j\rangle\}$ are labeled as $j = \pm(n - 2k)$ with $k = 0, 1, \dots, \lfloor n/2 \rfloor$, where n denotes the walker's discrete-time step. Single-photons created via SPDC (Spontaneous Parametric Down-Conversion) in a non-linear PPKTP crystals are injected into a free-space reference spatial mode $|j\rangle = |0\rangle$. This reference mode is sequentially spatially multiplexed by a concatenation of calcite polarizing beam-displacers (CBD). Arbitrary coin states are prepared by a polarizing beam-splitter in combination with a half- (HWP) and quarter wave-plates (QWP). In due course, a combination of a HWP and a CBD implements a single discrete-step evolution. By concatenating n of such unitary arrangements one can implement n steps of a DTQW (see **Figure 1(a)** for reference). Coincident detection of photons at Avalanche Photo Detectors (APDs) (4.4 ns time window) herald a successful run of the walk. The typical number steps implemented with spatial-multiplexed schemes is of order $n \approx 10$ [47].

3.2 Multiplexed DTQW in the temporal domain

The strategy for implementation of photonic DTQW via temporal-mode multiplexing was first introduced in Ref. [48]. The dimension of the Hilbert space for the DTQW is determined by a unique spatial mode $|j\rangle = |0\rangle$ and 2^n multiplexed temporal modes $|k\rangle$ (for $k = 1, 2, \dots, 2^n$), with n the discrete time-step number. The spatial mode is coupled to a coin operator in a two dimensional polarization subspace ($|H\rangle, |V\rangle$) (see **Figure 1(b)**). Analogue single-photon states (on average) are generated via an attenuated pulsed diode laser. The initial states of the photons injected in the DTQW are controlled by means of half-wave plates (HWPs) and quarter-wave plates (QWPs), in order to produce eigenstates of the chirality operator $|\psi_0^{\pm}\rangle = |0\rangle \otimes (|H\rangle \pm i|V\rangle)/\sqrt{2}$. Inside the loop, the unitary rotation ($R_n(\theta)$) is implemented by a HWP with its optical axis oriented in the direction $\theta/2$. The

Figure 1. Depicted architecture for experimental realization of DTQW (a) via spatial-mode multiplexing, (b) via temporal-mode multiplexing (see text for details).

polarization-dependent translation operator T is realized in the temporal domain via a polarizing beam splitter (PBS) in addition to a calibrated temporal delay-line using polarization preserving optical fibers, in which horizontally polarized light follows a longer path. The resulting calibrated temporal delay between both polarization components corresponds to a single step in the DTQW ($x \pm 1$). Polarization controllers (PCs) are introduced to compensate for arbitrary polarization rotations in the fibers. After implementing the polarization-dependent temporal difference, the so-called “time-bins” are recombined in a single spatial mode by means of a second PBS and they are directed into the fiber loops. Detection is accomplished by coupling a portion of the photons out of the loop, via a beam sampler (BS) with a probability of 5 % per step. Compensation HWPs (CHWPs) are introduced to correct for unintended dichroism introduced by the beam sampler (BS). Single-photon detectors (SPD) and avalanche photo-diodes (APDs) are employed to detect the photon arrival-time and to determine its polarization component. The probability that a photon undergoes a full round-trip is given by the overall coupling efficiency ($> 70\%$) and the total losses in the system resulting in $\eta = 0.50$. The average number of photons per input pulse is determined by neutral density (ND) filters, and is typically below $\langle n \rangle < 0.003$ to reduce contribution from multi-photon events. Such a scheme enables for implementation of a large number of discrete-time steps (typically $n \approx 20$) in a compact scheme, thus reducing the footprint characterizing spatially multiplexed architectures.

3.3 DTQW using spatial light modulators (SLM) and transverse spatial modes

We will identify the lattice points of a DTQW in a 1D geometry by the transverse spatial modes of a single photon. More specific, for photonic propagation in z -direction, the lattice sites in 1D will correspond to the position x (or y) of the

transverse propagation plane. The Hilbert space of the quantum walker will be given by the discrete basis $\{|j\rangle : j \in \mathbb{Z}\}$, where $|0\rangle$ corresponds to the spatial mode aligned with the optical axis and $\{|j > 0\rangle\}$ ($\{|j < 0\rangle\}$) correspond to the upper (lower) modes, as depicted in **Figure 2(a)**. The use of transverse modes of photons for DTQW has been demonstrated for a single step by Francisco *et al.* in an intricate setup [55], encoding the subspace of the quantum coin in the upper and lower regions of the x -axis. Here, we propose a physically more intuitive approach: the quantum coin is encoded in the 2-Dimensional polarization degrees of the photon that is in the horizontal/vertical basis, i.e., $\{|H\rangle, |V\rangle\}$. In this manner, the polarization dependent translation operator T can be expressed as:

$$T = \sum_j |j+1\rangle\langle j| \otimes |H\rangle\langle H| + |j-1\rangle\langle j| \otimes |V\rangle\langle V|. \quad (1)$$

For an unbiased coin operator, we have:

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (2)$$

Considering the following initial state for the quantum walker $|\psi_0\rangle = |0\rangle|H\rangle$, the temporal evolution of the initial quantum state after n steps will be given by:

$$|\psi_n\rangle = (TR)^n |\psi_0\rangle = \frac{1}{\sqrt{n+1}} \sum_{j=0}^n e^{i\phi_{n-2j}} |n-2j\rangle |\theta_{n-2j}\rangle, \quad (3)$$

with $\phi_{n-2j} = 0$ or π , and $|\theta_{n-2j}\rangle = \cos(\theta_{n-2j})|H\rangle + \sin(\theta_{n-2j})|V\rangle$ the polarization state of the coin in the $(n-2j)$ -th spatial mode. As an example, for $n = 4$:

Figure 2. (a) Discretization of a single-photon spatial amplitude profile in transverse modes along the x -direction. (b) Sketch of the proposed optical setup for preparing the n -th step walker-coin state (3) encoded in the transverse modes and polarization of a single-photon field: SMF, single mode fiber for spatial filtering; SLM, programmable spatial light modulator (see text for details). (c) Phase masks addressed at the phase-only SLM for preparing the state given by Eq. (6) with $n = 4$ and 5 . The dashed rectangles indicate empty transverse modes. (d) Optical module that implements one step ($|\psi_n\rangle \rightarrow |\psi_{n+1}\rangle$) of the 1D DTQW proposed here (see text for details). (e) Numbering convention of the spatial modes exiting the beam-displacer [47].

$$\begin{aligned}
|\psi_4\rangle &= \alpha|+4\rangle|H\rangle + | +2\rangle \left(\frac{3|H\rangle + |V\rangle}{\sqrt{10}} \right) + |0\rangle \left(\frac{|V\rangle - |H\rangle}{\sqrt{2}} \right) - | -2\rangle \left(\frac{|V\rangle - |H\rangle}{\sqrt{2}} \right) \\
&\quad - | -4\rangle|V\rangle.
\end{aligned} \tag{4}$$

Thus, the corresponding probability distribution characterizing the quantum walker, after n steps, will be given by:

$$P_n(j) = |(\langle H|j\rangle|\psi_n\rangle)|^2 + |(\langle V|j\rangle|\psi_n\rangle)|^2. \tag{5}$$

In order to analyze DTQW in 1D, within the framework described above, we present a realistic optical setup which can be divided into two modules. The first module is destined to prepare the initial state of Eq. (3) for an arbitrary value of n , only limited, in principle, by the resolution of the SLM used to manipulate the transverse spatial modes of photons. The second module, is destined to implement a single step in the protocol, namely, the unitary operation $U = TR$, with T and R given by Eqs. (1) and (2), respectively. With such preparation module, the probability distributions given by Eq. (5) can be measured directly. In addition, by concatenating it with the one-step propagation module, it will be possible to implement an arbitrary step in the quantum walk from n to $n + 1$. Therefore, in principle it is possible to simulate 1D DTQW for arbitrary steps n , (with n a large number) surpassing the number of steps that can be implemented with time- or spatial-multiplexing approaches. In what follows, we describe the proposed preparation and propagation modules, in addition to the measurement module required to estimate the probability $P_n(j)$.

3.3.1 DTQW preparation optical module

In **Figure 2(b)** a sketch of the optical module proposed in order to prepare the input state of the quantum walker, corresponding to the n -th step of a DTQW (Eq. (3)) employing polarization degrees of freedom and transverse spatial modes of single photons. The preparation module is divided into two submodules: the first submodule, is employed to prepare the spatial degrees of freedom of the input state, and the second submodule, is employed to spatial modes with the polarization degree of freedom. A key element for the appropriate implementation of such preparation module are state-of-the-art programmable spatial light modulators (SLMs). Such SLM devices, typically based on liquid crystal display (LCD) technologies, consist of a two-dimensional array of pixels, which when properly programmed, can control the phase, amplitude and polarization of the incident light field [56]. Recently, they have been deployed in a vast number of quantum information and communication protocols [57–60].

Let us consider $\int d\vec{r} \psi(\vec{r})|1\vec{r}\rangle \otimes |H\rangle$ as the quantum state of a monochromatic single-photon multi-mode field horizontally polarized in the paraxial approximation, here $\vec{r} = (x, y)$ is the position coordinate in the transverse plane, and $\psi(\vec{r})$ is the normalized transverse probability amplitude. Such single-photon states can be generated, for example, from a spontaneous parametric down-conversion (SPDC) single photon source. The transverse amplitude $\psi(\vec{r})$ can be manipulated using the technique developed by Prosser *et al.* [61]. Within this approach, it is possible to prepare arbitrary states of the form $\sum_j \beta_j |j\rangle$ with $\sum_j |\beta_j|^2 = 1$, where $\{|j\rangle\}$ represent

the orthogonal transverse spatial modes, in the x -direction. In brief, this technique utilizes an SLM which modulates the phase information of the incident profile $\psi(\vec{r})$ while leaving unaffected its amplitude or polarization. For simplicity, such phase information is assumed to be uniform across the entire surface of modulation. Next, a phase mask based on an array of d rectangular regions, each region corresponding to a blazed diffraction grating, is displayed on the liquid crystal screen (an example of phase mask for $d = 7$ is depicted in the inset of **Figure 2(b)**). The single photon phase profile is modulated by this mask and, in the far field, light beam it is diffracted into different orders ($0, \pm 1, \dots$) as it reaches regions with blazed gratings; otherwise, the beam propagates straight to the zeroth order. By choosing the first $+1$ order to prepare the states, the modulus of its complex coefficients will be evaluated according to the phase-modulation depth of each grating, which determines the intensity diffracted to the selected order. In addition, the phase of the coefficients will be a constant value added to the gratings. Finally, the $+1$ diffraction order is filtered by a slit diaphragm, such that the emerging photon is in a coherent superposition of d transverse “slit” modes $\{|j\rangle\}$. More specific, the states can be prepared as:

$$|\chi_n\rangle = \left(\frac{1}{\sqrt{n+1}} \sum_{j=0}^n e^{i\phi_{n-2j}} |n-2j\rangle \right) \otimes |H\rangle, \quad (6)$$

where $\phi_{n-2j} = 0$ or π , and n a positive integer. For a given n , one configures a phase mask for the SLM with $d = n + 1$ slit/diffraction gratings, symmetrically distributed starting from the highest modes $j = \pm n$. **Figure 2(c)** shows typical examples of masks for $n = 4$ and $n = 5$. As a technical remark, since the states we intend to prepare are uniform in phase (see Eqs. (3) and (6)), the phase-modulation depth of the gratings displayed a liquid crystal SLM will be a constant. Therefore, we can set it to be equal to 2π , ideally achieving 100% of diffraction efficiency in $+1$ order.

In order to prepare the state given by Eq. (3) starting by the input state given by Eq. (6), it is required to implement polarization rotations conditioned on the transverse-mode positions, as described by the unitary operator

$$\sum_{j=0}^n |n-2j\rangle \langle n-2j| \otimes \mathcal{R}(\vartheta_{n-2j}), \quad (7)$$

where

$$\mathcal{R}(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \quad (8)$$

transforms $|H\rangle$ into an arbitrary state of linear polarization. By applying this rotation on $|\chi_n\rangle$ with the appropriate $\mathcal{R}(\vartheta_{n-2j})$ ‘s, one generates the desired state $|\psi_n\rangle$ using the preparation module.

Spatially-dependent polarization rotations can be implemented by means of an SLM programmed for such task [62]. There are several different techniques for the various types of existing SLMs, which enable each pixel of the SLM device to work effectively as programmable polarization rotator [63, 64]. The details of these techniques are beyond the scope of the present work. With such programmable SLM, the transformation (7) onto the state (6) can be implemented by manipulating the transverse spatial modes of $|\chi_n\rangle$ on the liquid crystal display screen and

applying proper modulation on the input polarization $|H\rangle$. As depicted in **Figure 2(b)**, this procedure is achieved using a $4f$ lens system which will image the filtered output field of the phase-only SLM onto a polarization-rotator SLM.

This concludes the description of the proposed preparation optical module for arbitrary walker-coin state in the n -th step of a 1D DTQW, encoded in polarization and transverse spatial modes, respectively, of single photons. As previously states, the largest number of steps to be implemented n will be limited by the resolution of SLM. To illustrate this, consider a phase-only SLM with $2N$ pixels in the direction where the transverse modes are encoded (say x-direction). If each spatial mode is encoded in a row, and separated by another row of pixels, both with one-pixel width, it would be possible to define N distinguishable modes. In turn, this would enable us, in principle, to prepare the walker-coin state (3) up to $n = \lfloor N/2 \rfloor$. For a standard SLM with $2N = 1920$ [56], then $n = 480$, which represents a much larger figure than the maximum number of steps that can be implemented with multiplexed schemes. After the preparation module, one can determine the probability distribution (5) by recording the photon count rates at each of the $2n + 1$ output transverse modes of the second SLM (see **Figure 2(b)**), appropriately normalizing to the total number of counts. This can be achieved with an array of $2n + 1$ avalanche photo-diodes (APDs) or with a single-photon detector scanning along the transverse modes. The detection apparatus has to be located right after the second SLM, in order to prevent the diffracted modes to interfere and alter the probability distribution. Alternatively, as will be described below, a transverse-to-longitudinal mode conversion can be implemented which would enable to locate the detector at greater distances from the preparation module.

3.3.2 DTQW one-step propagation module

The quantum coin operator is the quantum analogue of a walker throwing a coin, and deciding whether to proceed to the left or to right, depending on whether the coin falls heads or tail. By encoding the left and right information in the 2 dimensional photon polarization basis $|H\rangle$ and $|V\rangle$, the quantum operator corresponding to flipping a coin R , as presented in Eq. (2), can be readily implemented by using a polarization half wave plate (HWP) oriented at $\pi/4^\circ$. Moreover, in order to implement the polarization-dependent translation operation T , as described in Eq. (1), it is straightforward to employ a birefringent element. However, this element, should also prevent the transverse modes from propagating in free space, in order to limit unwanted diffraction and interference, which would seriously hamper the characterization of the walker's translation. To maintain the discrete lattice structure of the protocol, while working with transverse modes which are properly discretized in the plane of state preparation but which are not properly discretized after the single-step due to free space propagation, one must apply a discretization procedure along all propagation planes. This can eventually be achieved by introducing a cylindrical lens with focal distance f , located at a distance f to the second SLM utilized in the preparation module, this is schemed in **Figure 2(d)**. In this way, the transverse modes at the output preparation plane are transformed into longitudinal modes, along all the remaining propagation planes. Once this transverse to longitudinal model conversion is enforced, one can simply use a polarizing calcite beam-displacer or a polarizing beam splitter in order to implement the polarization translation T . As illustrated in the inset of **Figure 2(d)**, such optical element may be oriented transmit vertically polarized light (V) and introduce a lateral beam displacement into the neighboring mode on horizontally polarized light (H). In summary, for an input state $|\psi_n\rangle$ given by (3), the one-step propagation module TR consists of a HWP oriented at $\pi/4^\circ$ and a calcite beam-displacer in order to

implement R and T , respectively, in addition to a cylindrical lens which enables transverse-to-longitudinal mode conversion. After this single-step propagation module, it is possible to detect the probability distribution $P_{n+1}(j)$ (Eq. (5)), as described above. The entire procedure is depicted in **Figure 2(d)**. Furthermore, **Figure 2(e)**, describes our convention for labeling spatial modes after propagation through the calcite beam-displacer.

4. DTQW: applications in topology and geometry

Geometric phases acquired during quantum evolution of a particle can have different origins. The Berry phase [65] is a type of geometric phase that can be assigned to quantum particles which return their initial state adiabatically, while recording the path information on a geometric phase (Φ), defined as [65, 66]:

$$e^{i\Phi} = \langle \psi_{\text{ini}} | \psi_{\text{final}} \rangle. \quad (9)$$

A number of physical consequences can be attached to geometric phases, such as the modification of material properties in solids, for example the conductivity in Graphene [67, 68], the emergence of surface edge-states in topological insulators, whose surface electrons experience a geometric phase [69], the modification of molecular chemical reactions [70], and more recently geometric phases have been predicted to have implications for quantum technology, via the elusive Majorana particle [71].

In this review, we report on the progress in the characterization of geometry and topology of DTQW architectures consisting of a unitary step U given by a sequence of two non-commuting rotations in parameter space, followed by a spin-dependent translation. The topological parameter space of the DTQW architecture we analyze does not present continuous 1D topological boundaries. Unlike the “split-step” quantum walk [16, 19], or other analogous systems recently studied in the literature, the platform we report only presents a discrete number of Dirac points, where the quasi-energy gap closes. At these discrete Dirac points, the Zak Phase difference is not defined; therefore, these discrete points represent topological boundaries of zero dimension. Here we ascribe a topological boundary at the set of points where the topological invariant is not defined, namely at the discrete points where the quasi-energy gap closes. Such gapless points can be considered topological defects in parameter space. Since the system has topological defects, we argue the system is topologically non-trivial. We demonstrate the non-trivial topological landscape of the system by calculating different holonomic and geometric quantities, such as the Zak phase, which corresponds to the Berry phase in the Brillouin zone.

5. Topology and the geometric Zak phase

The physical concept of geometric phase, such as Berry or Zak phase, is intimately linked to the concept of holonomy of a manifold. Holonomy from a geometrical standpoint: within the framework of differential geometry, an holonomy group H_x at a given point in space x for an oriented n -dimensional manifold M endowed with a given metric g_{ij} can be assigned via the (parallel) transport of a vector field $V \in TM_x$ along all possible closed curves C , starting and ending at the same point x . The condition for parallel transport is mathematically represented by the following expression:

$$t^\mu \nabla_\mu V = 0, \quad (10)$$

here t^μ is the tangent vector to the curve C and ∇_μ the Levi-Civita connection of (M, g_{ij}) , representing a unique torsion free connection which satisfies $\nabla g_{\mu\nu} = 0$. By the Levi-Civita conditions together with (10), one can derive that:

$$t^\mu \nabla_\mu (g_{ij} V^i V^j) = 0,$$

stating that the vector field norm $\|V\| = g_{ij} V^i V^j$ is conserved upon travel along the closed curve C . Nevertheless, the resulting vector V_C after travel will not necessarily coincide with V , in general it will be rotated in the form:

$$V_C = R_x(C)V,$$

where $R(C)$ is an element of $SO(n)$. Therefore, a rotation matrix $R_x(C)$ corresponding to any pair (x, C) with C an arbitrary curve in the manifold can be assigned. The set of rotations $R_x(C)$ at a fixed point x can be obtained by considering all possible curves C forming a group, which turns out to be equal or smaller than $SO(n)$. This set is known as the holonomy group H_x at the fixed point x . For simply connected manifolds M , the holonomy groups at points x and y are isomorphic. In such cases, we refer to the holonomy H of M . Otherwise, the definition of holonomy group becomes point dependent.

The geometric concept of holonomy can be depicted for a manifold M which is embedded in space R^n . An illustrative example is the sphere S^2 with its canonical metric given by:

$$g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2. \quad (11)$$

This sphere represents a surface $x_1^2 + x_2^2 + x_3^2 = 1$ embedded in a space R^3 . The canonical metric g_{S^2} is the distance element within this surface. For a given curve C , the holonomy element $H(C)$ is a rotation $R(\alpha)$ where $\alpha(C)$ is the solid angle subtended by the curve at the center of the sphere. It can be instructive to verify this explicitly. Consider a unit vector r in R^3 , which parameterizes the points of the sphere in the form

$$r = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta).$$

By taking a vector $\nu \in TM_x$, with TM_x the tangent plane to the manifold, to be transported along a curve C in S^2 . By describing the travel by a parameter t , which can be interpreted as the traveling “time” along the path, the parallel transport condition can be expressed in a compact form. The vector will always be orthogonal to r , with $V \times r = 0$; if not, it would have a component orthogonal to the surface. Nevertheless, this condition is not enough since ν could reside in the tangent plane $T_x S^2$ of any point x along the curve. In this case, a velocity Ω with non-zero component in r -direction could make the vector rotate. In order to avoid such rotations, it is customary to insure that Ω has no components in the r -direction, a condition which can be expressed as $\Omega \times r = 0$. The resulting angular velocity Ω should be different from zero; however, since ν has a conserved component in $\dot{r}$ and $\dot{r}$ produces a rotation in R^3 as well as V . These two conditions are mathematically expressed as:

$$\dot{\nu} = \Omega \times \nu, \quad \Omega = r \times \dot{r}. \quad (12)$$

For the applications in DTQW that we intend to consider, we can express the condition above in terms of a complex unit vector ψ , defined by:

$$\psi = \frac{1}{2}(v + iv'), \quad v' = r \times v.$$

In order to find the solid angle $\alpha(C)$, it possible to define a local orthogonal basis, with vector elements u and v . These elements are explicitly given as:

$$u(r) = (-\sin \phi, \cos \phi, 0), \quad (13)$$

$$v(r) = (-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta). \quad (14)$$

On the other hand, the phase α of ψ can be expressed as:

$$\psi = n \exp(i\alpha), \quad n = \frac{1}{2}(u + iv).$$

Note that α depends on the choice of u and v , but the phase change due to the transport along C does not. Such phase change is expressed as:

$$\alpha(C) = \oint_C d\alpha = \Im(\oint_C n^* \cdot dn) = \Im \int \int_{Int(C)} dn \wedge dn^*,$$

the previous step makes use of the Stokes. It can be noted that the integrand is invariant under the Gauge transformations, meaning:

$$n' = n \exp(i\mu(r)).$$

This integral can be written explicitly in terms of the coordinate system, obtaining:

$$\alpha(C) = \Im \int \int_{Int(C)} d\theta d\phi (\partial_\theta n^* \cdot \partial_\phi n - \partial_\theta n \cdot \partial_\phi n^*), \quad (15)$$

$$\alpha(C) = \int \int_{IntC} \sin \theta d\theta d\phi, \quad (16)$$

which is the solid angle subtended by C .

Within the framework of quantum mechanics, one can replace the mathematical complex vector $\psi(\theta, \phi)$ by a quantum state vector $|\psi(X)\rangle$, where X s are the coordinates describing the parameter space.

A complex basis $|n(X)\rangle$ for any X can be introduced, and the relative phase of $|\psi(X)\rangle$ can be defined:

$$|\psi\rangle = |n(X)\rangle \exp(i\gamma).$$

This phase is of course is base dependent, but the holonomy is independent from that choice of basis. Holonomy can be defined by an adiabatic travel around a curve C , in parameter space. Upon the travel, the resulting wave function accumulates an additional phase due to the non-trivial holonomy of such space. In other words,

$$\langle \psi_{\text{ini}} | \psi_{\text{final}} \rangle = \exp(i\alpha(C)).$$

The phase $\alpha(C)$ is known as the Berry phase [65]. The condition of parallel transport becomes in this context

$$\Im\langle\psi|d\psi\rangle = 0.$$

By simple generalization of the arguments given above in the differential geometrical context, it follows that this phase is simply

$$\alpha(C) = \iint_{Int(C)} \Im \langle dn | \wedge | dn \rangle. \quad (17)$$

Note that the phase is dependent on the choice of path C .

The natural language for an holonomy in this context is in terms of principal bundles. There exists a natural metric g_{ij} in the parameter space of the problem. This issue was studied in [72], where the authors considered the following tensor

$$T_{ij} = \langle \partial_i n | (1 - |n\rangle \langle n|) | \partial_j n \rangle.$$

This tensor is Gauge invariant

$$|n(X)\rangle \rightarrow |n(X)\rangle \exp(-i\mu(r)).$$

One may define a “distance” between two states by

$$\Delta_{12} = 1 - |\langle \psi_1 | \psi_2 \rangle|^2.$$

The interpretation of distance is as follows. For two states $|\psi_1\rangle$ and $|\psi_2\rangle$ which differ only by a global phase are defined, we have $\Delta_{12} = 0$. Taking the limit $1 \rightarrow 2$ and using the fact that the states are normalized we obtain

$$ds^2 = \langle dn | (1 - |n\rangle \langle n|) | dn \rangle = T_{ij} dX^i dX^j = g_{ij} dX^i dX^j, \quad (18)$$

this follows from the fact that the product of a symmetric tensor by an antisymmetric one is zero. Note that, for a 2-dimensional spin system:

$$|+\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\frac{\phi}{2}} \\ \sin \frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} \sin \frac{\theta}{2} e^{i\frac{\phi}{2}} \\ -\cos \frac{\theta}{2} e^{-i\frac{\phi}{2}} \end{pmatrix}$$

gives the canonical metric on S^2 (18).

A subtle remark is in order, in relation to the colloquial use of the words geometry, holonomy, and topology. The holonomy of a manifold M in a geometry context is geometrical, in the sense that the notion of parallel transport described above is related to the Levi-Civita connection ∇_i , which is constructed for a particular metric tensor g_{ij} defined on the manifold M . Nevertheless, such holonomy in general is not a topological invariant for M . More specific, two different complete metrics g_{ij} and g'_{ij} may exist, defined on the same manifold M , but possessing different holonomy groups [2]. In the context of quantum physics, the Berry phase or Zak phase may nevertheless describe topological phenomena. In fact, the description of the Berry phase above has a formal analogy with the concept of holonomy.

We can define such a geometric phase as the holonomy for an abstract connection in a principal bundle $P(U(1), X)$, where X the parameter space X . The curvature of this connection is defined in (17), also known as the Berry curvature which is Gauge invariant with flux given by Berry phase $\alpha(C)$. For closed manifolds, such

fluxes describe a Chern class of the bundle. These classes take integer values and are invariant under Gauge transformations. Such classes describe different bundles in parameter space X and are topological invariant, basically meaning that they do not depend on the choice of the metric in the underlying manifold X [65].

In the following section, we present applications of these mathematical concepts within the context of quantum mechanical problems, in particular of DTQWs.

5.1 Applications via spatial multiplexing: split-step DTQW

In this Section we analyze in detail two cases of topologically non-trivial Zak phase landscape, where the Zak phase is the equivalent to the Berry phase across the Brillouin zone. The first, the so-called split-step DTQW is implemented by applying two consecutive conditional translations T and rotations R characterized by rotation parameters $\theta_{1,2}$, such that the unitary step becomes $U(\theta_1, \theta_2) = TR(\theta_1)TR(\theta_2)$, as described in detail in [19]. The so-called “split-step” DTQW has been demonstrated to exhibit non-trivial topology characterized by distinct topological sectors, which are in turn delimited by continuous linear 1D topological boundaries. Such topological sectors are typically characterized by topological invariants, for DTQWs this is typically the winding number W , which can take binary integer values $W = 0, 1$.

The dispersion relation, which expresses the quasi-energy E as a function of the quasi-momentum k and the DTQW parameters $\theta_{1,2}$ for the split-step DTQW, results in [19]:

$$\cos(E_{\theta,\phi}(k)) = \cos(k) \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2).$$

In order to decompose the DTQW Hamiltonian of the system in terms of Pauli matrices $H_{QW} = E(k)\vec{n} \cdot \vec{\sigma}$ becomes [16], we require to know (x,y,z) components of the 3D-norm [19]:

$$\begin{aligned} n_{\theta_1,\theta_2}^x(k) &= \frac{\sin(k) \sin(\theta_1) \cos(\theta_2)}{\sin(E_{\theta_1,\theta_2}(k))}, \\ n_{\theta_1,\theta_2}^y(k) &= \frac{\cos(k) \sin(\theta_1) \cos(\theta_2) + \sin(\theta_2) \cos(\theta_1)}{\sin(E_{\theta_1,\theta_2}(k))}, \\ n_{\theta_1,\theta_2}^z(k) &= \frac{-\sin(k) \cos(\theta_2) \cos(\theta_1)}{\sin(E_{\theta_1,\theta_2}(k))}. \end{aligned} \quad (19)$$

We now turn to our second example of topologically non-trivial DTQW.

5.2 Applications via temporal multiplexed: DTQW with non-commuting rotations

As a second non-trivial example, we introduce a DTQW consisting of two sequential non-commuting rotations R_1 and R_2 , which constitute the main building block of the unitary step U in the DTQW [2]. While the first rotation R_1 is performed along the y -direction by an angle θ , the second rotation R_2 is performed along the x -direction, by an angle ϕ . In this manner, the unitary step becomes $U(\theta, \phi) = TR_x(\phi)R_y(\theta)$, where $R_x(\phi)$ is also given in the Pauli basis [49] by:

$$R_x(\phi) = \begin{pmatrix} \cos(\phi) & i \sin(\phi) \\ i \sin(\phi) & \cos(\phi) \end{pmatrix}.$$

The 3D-norm required for expressing the Hamiltonian in the Pauli basis, results in:

$$\begin{aligned} n_{\theta,\phi}^x(k) &= \frac{-\cos(k) \sin(\phi) \cos(\theta) + \sin(k) \sin(\theta) \cos(\phi)}{\sin(E_{\theta,\phi}(k))}, \\ n_{\theta,\phi}^y(k) &= \frac{\cos(k) \sin(\theta) \cos(\phi) + \sin(k) \sin(\phi) \cos(\theta)}{\sin(E_{\theta,\phi}(k))}, \\ n_{\theta,\phi}^z(k) &= \frac{-\sin(k) \cos(\theta) \cos(\phi) + \cos(k) \sin(\theta) \sin(\phi)}{\sin(E_{\theta,\phi}(k))}. \end{aligned} \quad (20)$$

The dispersion relation for the DTQW with non-commuting rotations results in:

$$\cos(E_{\theta,\phi}(k)) = \cos(k) \cos(\theta) \cos(\phi) + \sin(k) \sin(\theta) \sin(\phi), \quad (21)$$

it can be easily verified that we recover a Dirac-like dispersion relation for $\phi = 0$, as expected.

As readily mentioned, the described system exhibits a non-trivial phase diagram consisting of a large number of discrete gapless points for different quasi-momenta. Such singular points can be regarded as topological defects in parameter space. Each gapless points represent topological boundaries of dimension zero, where topological invariant, such as the winding number W , are not defined. As anticipated, in contrast to the “split-step” DTQW described in previous sections, this system does not contain continuous topological boundaries. We calculated analytically the gapless Dirac points and zero-dimension topological boundaries for the system by using basic trigonometric considerations. It can be readily demonstrated that there are 13 discrete points for different values of quasi-momentum k where the gap closes. This is depicted in **Figure 3**. Different symbols correspond to different values

Figure 3. Phase diagram for DTQW with non-commuting rotations. The symbols indicate gapless Dirac points where quasi-energy gap closes for different values of quasi-momentum: Squares ($k = 0$), pentagons ($|k| = \pi$), rombooids ($k = +\pi/2$), and circles ($k = -\pi/2$). The discrete Dirac points represent topological boundaries of dimension zero, and endow the system with a non-trivial topology [2].

of quasi-momenta. Namely, pentagons correspond to Dirac points for $|k| = \pi$, rombooids correspond to Dirac points for $k = +\pi/2$, squares correspond to Dirac points for $k = 0$, and circles correspond to Dirac points for $k = -\pi/2$. Such holonomic structure in itself is topologically non-trivial, and was studied in [2] for the first time.

6. Geometric phase calculation

We now provide expressions for the geometric phase, the so-called Zak phase acquired due to quantum evolution across the Brillouin Zone, in the two aforementioned scenarios. These two scenarios are characterized by a generic Hamiltonian of the form:

$$H \sim n_x \sigma_x + n_y \sigma_y + n_z \sigma_z. \quad (22)$$

The specific Hamiltonians for each scenario differ by a constant factor, and by the specific expressions of the normal vector n_i (with $i = x, y, z$). Since the eigenvectors of the Hamiltonian are the only quantities of interest for the present problem, overall constants can be safely ignored.

In general the Hamiltonian in the Pauli basis is given by the following matrix:

$$H = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} \quad (23)$$

and is characterized by the eigenvalues, which represent the eigenenergies of the system:

$$\lambda = \pm \sqrt{n_x^2 + n_y^2 + n_z^2}. \quad (24)$$

By diagonalizing this generic Hamiltonian, we find thee normalized eigenvectors for the generic Hamiltonian are given by:

$$|V_{\pm}\rangle = \begin{pmatrix} \frac{n_x + in_y}{\sqrt{2n_x^2 + 2n_y^2 + 2n_z^2 \mp 2n_z \sqrt{n_x^2 + n_y^2 + n_z^2}}} \\ \frac{n_z \mp \sqrt{n_x^2 + n_y^2 + n_z^2}}{\sqrt{2n_x^2 + 2n_y^2 + 2n_z^2 \mp 2n_z \sqrt{n_x^2 + n_y^2 + n_z^2}}} \end{pmatrix}. \quad (25)$$

It is to be noted that the scaling factor $n_i \rightarrow \lambda n_i$ does not affect the result. As mentioned, this results from the fact that two Hamiltonians differing by a constant have the same eigenvectors.

The geometric Zak phase ($\Phi_{Zak} = Z$) for the positive and negative bands ($\pm$), is expressed as:

$$Z_{\pm} = i \int_{-\pi/2}^{\pi/2} dk \langle V_{\pm} | \partial_k V_{\pm} \rangle. \quad (26)$$

We will now apply these concepts to the specific examples reviewed in the previous sections.

6.1 Split-step DTQW

We will calculate the Zak phase for two types of DTQW, the first one is the so-called split-step DTQW [19, 73]. It consists of a DTQW with unitary step U given by the following expression $U(\theta_1, \theta_2) = TR(\theta_1)TR(\theta_2)$. Such unitary step can be readily implemented via spatial multiplexing, as described in [19, 73]. For the unitary step characterizing the split-step DTQW, the components of the normal vector n_i for decomposing the Hamiltonian in terms of Pauli operators can be written in the following manner:

$$\begin{aligned} n_{\theta_1, \theta_2}^x(k) &= \frac{\sin(k) \sin(\theta_1) \cos(\theta_2)}{\sin(E_{\theta_1, \theta_2}(k))}, \\ n_{\theta_1, \theta_2}^y(k) &= \frac{\cos(k) \sin(\theta_1) \cos(\theta_2) + \sin(\theta_2) \cos(\theta_1)}{\sin(E_{\theta_1, \theta_2}(k))}, \\ n_{\theta_1, \theta_2}^z(k) &= \frac{-\sin(k) \cos(\theta_2) \cos(\theta_1)}{\sin(E_{\theta_1, \theta_2}(k))}. \end{aligned} \quad (27)$$

In particular, we consider the case in which the normal vector $n = (n_x, n_y, n_z)$ is fully transversal, meaning that $n_z = 0$. By setting the angle parameters such that $n_z = 0$, it can be easily demonstrated that the normalized Hamiltonian eigenvectors are of the form:

$$|V_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi(k)} \\ \mp 1 \end{pmatrix}, \quad \tan \phi(k) = \frac{n_y}{n_x}. \quad (28)$$

There are two possible angle choices that lead to $n_z = 0$, these are $\theta_1 = 0$ or $\theta_2 = 0$. For either of these angle choices, the Zak phase for the positive and negative band take equivalent values, of the form [2]:

$$Z = Z_{\pm} = i \int_{-\pi/2}^{\pi/2} dk \langle V_{\pm} | \partial_k V_{\pm} \rangle, \quad (29)$$

$$Z = i \int_{-\pi/2}^{\pi/2} dk \langle V_{\pm} | \partial_k V_{\pm} \rangle = \phi(-\pi/2) - \phi(\pi/2), \quad (30)$$

from where it follows that

$$Z = \frac{\tan(\theta_2)}{\tan(\theta_1)}. \quad (31)$$

A numerical simulation of the Zak phase for the split-step DTQW is depicted in **Figure 4a**.

6.2 DTQW with non-commuting rotations

The particular DTQW with non-commuting rotations presented in previous Sections can be readily implemented via temporal multiplexing approaches. To this end, we recall that the unitary step results in $U(\theta, \phi) = TR_x(\phi)R_y(\theta)$. The Cartesian components of the 3D-norm n_i ($i = x, y, z$) are as follows:

$$n_x = -\cos(k)a + \sin(k)b, \quad (32)$$

Figure 4. (a) Non-trivial geometric Zak phase landscape for DTQW with non-commuting rotations obtained by numeric integration, (b) Geometric Zak phase landscape for “split-step” DTQW obtained analytically [2].

$$n_y = \cos(k)b + \sin(k)a, \quad (33)$$

$$n_z = \cos(k)c - \sin(k)d, \quad (34)$$

where

$$a = \sin(\phi) \cos(\theta), \quad (35)$$

$$b = \cos(\phi) \sin(\theta), \quad (36)$$

$$c = \sin(\phi) \sin(\theta), \quad (37)$$

$$d = \cos(\phi) \cos(\theta), \quad (38)$$

angular functions as defined above. N_1 is can be expressed as:

$$N_1 = n_x + in_y = -\exp(-ik)(a - ib). \quad (39)$$

In this scenario, calculation of Zak phase in terms of the Hamiltonian eigenvectors ($|V_{\pm}\rangle$) for each band (positive and negative) can be accomplished, resulting in:

$$Z = Z_{\pm} = i \int_{-\pi/2}^{\pi/2} dk \langle V_{\pm} | \partial_k V_{\pm} \rangle,$$

Making use of expression (41), the geometric Zak phase results in:

$$Z_{\pm} = \int \frac{(a^2 + b^2)dk}{D_{\pm}^2}, \quad (40)$$

$$\begin{aligned} D_{\pm} &= \sqrt{2n_x^2 + 2n_y^2 + 2n_z^2 \mp 2n_z \sqrt{n_x^2 + n_y^2 + n_z^2}} \\ &= (a^2 + b^2 + c^2 \cos^2(k) + d^2 \sin^2(k) - \sin(2k)cd \\ &\quad \mp (\cos(k)c - \sin(k)d) \\ &\quad \times \sqrt{a^2 + b^2 + c^2 \cos^2(k) + d^2 \sin^2(k) - \sin(2k)cd})^{\frac{1}{2}}. \end{aligned} \quad (41)$$

Note that, for the case of DTQW with non-commuting rotations, the consequences of setting the norm to be fully transverse (i.e., $n_z = 0$) are quite different than in the case of the split-step DTQW. More specific, $n_z = 0$ returns a trivially constant Zak phase $Z = \pi$, since the k -dependence vanishes. For this system, there is no analytic expression for the Zak phase, and the Zak phase landscape can only be obtained by numerical integration. Note that, at the Dirac points indicated in **Figure 5**, the Zak phase is ill defined. A numerical simulation of the Zak phase Φ_{Zak} by numeric integration in Wolfram Mathematica is depicted in **Figure 4**, corresponding to parameter values of the form $\theta_{1,2} = [-\pi, \pi]$ and $\phi = [-\pi, \pi]$ —(a - left) Zak phase for split-step DTQW, given by the analytic expression $Z = \frac{\tan(\theta_2)}{\tan(\theta_1)}$; (b-right) Zak phase for DTQW with non-commuting rotation, obtained by numerical integration of expression Eq. (41).

A brief discussion is in order, it is well known that the Zak phase is Gauge dependent—that is, it depends on the particular choice of origin of the unit cell [74]. Therefore, in general it is not uniquely defined and cannot be considered a topological invariant. Nevertheless, a related topological invariant quantity can be defined in terms of the Zak phase *difference* between two states ($|\psi^1\rangle, |\psi^2\rangle$) differing by a geometric phase only. The Zak phase difference between two such states can be expressed as $\langle \psi^1 | \psi^2 \rangle = e^{i|\Phi_{Zak}^1 - \Phi_{Zak}^2|}$. More explicit, the term geometric invariance refers to geometric properties that do not depend on the choice of origin of the Brillouin zone, and only depend on relative distances between geometric points in the Brillouin zone.

A time-multiplexed experimental scheme, which can be readily implemented to obtain the Zak phase difference between two states at a given time-step N is suggested. For a given choice of origin of the Brillouin zone, the system is characterized by a unitary evolution operator consisting of rotation parameters corresponding to either of the four adjacent Dirac points, where the gap closes. A different geometric phase will be accumulated at each adjacent Dirac point. Such phase difference can be experimentally determined by coherently recombining the

Figure 5. Graphic depiction of the geometric phase Φ acquired along propagation in a closed trajectory. The spin or polarization (yellow arrows) remains perpendicular to the direction of propagation (black arrows). See section 4 and section 5 for further details on topology and holonomy in quantum systems.

states. More specific, in the photonic case, by interfering the states by using a Mach-Zehnder interferometer. A suitable experimental scheme for detection of the Zak phase difference in a photonic system is readily presented in [2, 75].

7. Conclusions

In this Book Chapter, we reported a review of novel approaches to photonic discrete-time quantum walk (DTQW) platforms. Namely, we discussed implementations via spatial-multiplexing or temporal-multiplexing schemes, and we introduced a novel scheme for implementations based on transverse spatial modes of photons, which are in turn controlled by spatial light modulators (SLMs). While the number of discrete time-steps (n) that can be experimentally implemented via mode multiplexed approaches is typically limited by the mode scaling of the multiplexed technique itself, i.e., $2n + 1$ for spatial mode multiplexing and 2^n for temporal-mode multiplexing, realizations using transverse modes can in principle enable experimental simulation of an arbitrary temporal step n , only limited by the resolution of the SLM itself. We present several relevant applications of DTQWs in quantum simulation. Namely, for the simulation of topological effects, ascribed to each DTQW platform. Specifically, in the context of mode-multiplexed DTQWs, we presented in detail the calculation of the Zak Phase, corresponding to the Berry phase across the Brillouin zone, for the case of the *split-step* DTQW and for the case of DTQW with non-commuting rotations, which are implemented via spatial and temporal mode-multiplexing, respectively.

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Conflicts of interest

The author declares no conflict of interest.

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References

- [1] This review is based on two articles published by the Author [2, 3] under Creative Commons Attribution License which permits unlimited reproduction, upon proper citation.
- [2] G. Puentes, Topology and holonomy in discrete-time quantum walks, *Crystals* **7**, 122 (2017); selected for Cover Story of a Special Issue on Quantum Topology (Open Access Article, distributed under the Creative Commons Attribution License which permits unrestricted use, distribution and reproduction provided proper citation).
- [3] L. Neves and G. Puentes, Photonic Discrete-time Quantum Walks and Applications, *Entropy* **20**, 731 (2018) (Open Access Article, distributed under the Creative Commons Attribution License which permits unrestricted use, distribution and reproduction provided proper citation).
- [4] A.M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, and D. Spielman. Exponential algorithmic speedup by quantum walk. In Proceedings of the 35th ACM Symposium on The Theory of Computation (STOC'03) ACM, pp. 59–68, 2003.
- [5] A. Ambainis. Quantum random walks, a new method for designing quantum algorithms. In SOFSEM 2008: Theory and Practice of Computer Science, Lecture Notes in Computer Science, vol. 4910, pp. 1–4, Springer Berlin/Heidelberg, 2008.
- [6] N. Shenvi, J. Kempe, and R.B. Whaley. A quantum random walk search algorithm. *Phys. Rev. A*, **67**(5): 052307, 2003.
- [7] A. Ambainis. Quantum walk algorithm for element distinctness. In Proceedings of 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS'04), pp. 22–31, 2004
- [8] A. Ambainis. Quantum walks and their algorithmic applications. *International Journal of Quantum Information*, vol. 1 (4), pp. 507–518, 2003.
- [9] M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik. Environment-assisted quantum walks in energy transfer of photosynthetic complexes. *J Chem Phys.*, **129**(17):174106, 2008.
- [10] A.M. Childs. Universal computation by quantum walk. *Phys. Rev. Lett.*, **102**: 180501, 2009.
- [11] A. Tulsı. Faster quantum-walk algorithm for the two-dimensional spatial search. *Phys. Rev. A*, **78**(1): 012310, 2008.
- [12] N.B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon. Universal quantum computation using the discrete-time quantum walk. *Phys. Rev. A*, **81**(4):042330, 2010.
- [13] M. S. Underwood and D.L. Feder. Universal quantum computation by discontinuous quantum walk. *Phys. Rev. A*, **82**(4):042304, 2010.
- [14] Kempe, J. Quantum random walks: an introductory overview. *Cont. Phys.* **2003**, 44, 307, doi:10.1080/00107151031000110776.
- [15] Y. Aharonov, L. Davidovich, and N. Zagury, Quantum random walks. *Phys. Rev. A* **48**, 1687 (1993).
- [16] T. Kitagawa, Matthew A. Broome, Alessandro Fedrizzi, Mark S. Rudner, Erez Berg, Ivan Kassal, Alán Aspuru-Guzik, Eugene Demler, and Andrew G. White, Observation of topologically protected bound states in photonic quantum walks. *Nature Communications* **3**, 882 (2012).
- [17] A. Crespi, R. Osellame, R. Ramponi, V. Giovannetti, R. Fazio, Linda Sansoni,

- F. De Nicola, F. Sciarrino, and P. Mataloni, Anderson localization of entangled photons in an integrated quantum walk. *Nature Photon.* **7**, 322 (2013).
- [18] Maximilian Genske, Wolfgang Alt, Andreas Steffen, Albert H. Werner, Reinhard F. Werner, Dieter Meschede, and Andrea Alberti, Electric quantum walks with individual atoms. *Phys. Rev. Lett.* **110**, 190601 (2013).
- [19] T. Kitagawa, M. S. Rudner, E. Berg, and E. Demler, Exploring topological phases with quantum walks. *Phys. Rev. A* **82**, 033429 (2010).
- [20] H. Obuse and N. Kawakami, Topological phases and delocalization of quantum walks in random environments. *Phys. Rev. B* **84**, 195139 (2011).
- [21] C. Cedzich, F. Grunbaum, C. Stahl, L. Velázquez, A. Werner, and R. Werner, Bulk-edge correspondence of one-dimensional quantum walks. *J. Phys. A: Math. Theor.* **49**, 21LT01 (2016).
- [22] Antoni Wojcik, Tomasz Luczak, Paweł Kurzynski, Andrzej Grudka, Tomasz Gdala, and Malgorzata Bednarska-Bzdega, Trapping a particle of a quantum walk on the line. *Phys. Rev. A* **85**, 012329 (2012).
- [23] Y. Shikano, K. Chisaki, E. Segawa, and N. Konno, Emergence of randomness and arrow of time in quantum walks. *Phys. Rev. A* **81**, 062129 (2010).
- [24] J. K. Asbóth, Symmetries, topological phases, and bound states in the one-dimensional quantum walk. *Phys. Rev. B* **86**, 195414 (2012).
- [25] H. Obuse, J. K. Asboth, Y. Nishimura, N. Kawakami, Unveiling hidden topological phases of a one-dimensional Hadamard quantum walk. *Phys. Rev. B* **2015**, 92, 045424.
- [26] S. Moulieras, M. Lewenstein, and G. Puentes, Entanglement engineering and topological protection in discrete-time quantum walks. *J. Phys. B* **46**, 104005 (2013).
- [27] C. Beenakker and L. Kouwenhoven, A road to reality with topological superconductors. *Nature Phys.* **12**, 618 (2016).
- [28] Meng Xiao, Guancong Ma, Zhiyu Yang, Ping Sheng, Z. Q. Zhang, and C. T. Chan, Geometric phase and band inversion in periodic acoustic systems. *Nature Phys.* **11**, 240 (2015).
- [29] S. D. Huber, Topological mechanics. *Nature Phys.* **12**, 621 (2016).
- [30] V. Peano, C. Brendel, M. Schmidt, F. Marquardt, Topological phases of sound and light. *Phys. Rev. X* **5**, 031011 (2015).
- [31] L. Lu, J. Joannopoulos, M. Solijacic, Topological states in photonic systems. *Nature Phys.* **12**, 626 (2016).
- [32] S. Bose, Quantum communication through an unmodulated spin chain. *Phys. Rev. Lett.* **91**, 207901 (2003).
- [33] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Perfect state transfer in quantum spin networks. *Phys. Rev. Lett.* **92**, 187902 (2004).
- [34] M. B. Plenio, and S. F. Huelga, Dephasing-assisted transport: quantum networks and biomolecules. *New. J. Phys.* **10**, 113019 (2008).
- [35] J. Spring *et al.*, Boson sampling on a photonic chip. *Science* **339**, 798–801 (2013).
- [36] Matthew A. Broome, Alessandro Fedrizzi, Saleh Rahimi-Keshari, Justin Dove, Scott Aaronson, Timothy C. Ralph, Andrew G. White, Photonic boson sampling in a tunable circuit. *Science* **339**, 6121 (2013).

- [37] Max Tillmann, Borivoje Dakic, René Heilmann, Stefan Nolte, Alexander Szameit and Philip Walther, Experimental boson sampling. *Nature Photon.* **7**, 540 (2013).
- [38] A. Crespi *et al.*, Integrated multimode interferometers with arbitrary designs for photonic boson sampling. *Nature Photon.* **7**, 545 (2013).
- [39] N. Spagnolo *et al.*, Efficient experimental validation of photonic boson sampling against the uniform distribution. *Nature Photonics* **8**, 615 (2014).
- [40] J. Carolan *et al.*, On the experimental verification of quantum complexity in linear optics. *Nature Photonics* **8**, 621 (2014).
- [41] A. M. Childs, Universal computation by quantum walk. *Phys. Rev. Lett.* **102**, 180501 (2009).
- [42] A Aiello, G Puentes, D Voigt, JP Woerdman, Maximally entangled mixed-state generation via local operations. *Phys. Rev. A* **75** (6), 062118 (2007).
- [43] G. Puentes, D. Voigt, A. Aiello, and J. P. Woerdman, Universality in depolarized light scattering. *Opt. Lett.* **30** (23), 3216–3218 (2006).
- [44] A. Peruzzo *et al.*, Quantum walks of correlated photons. *Science* **329**, 1500–1503 (2010).
- [45] K. Poulios *et al.*, Quantum walks of correlated photon pairs in two-dimensional waveguide arrays. *Phys. Rev. Lett.* **112**, 143604 (2014).
- [46] Andreas Schreiber, Aurel Gabris, Peter P. Rohde, Kaisa Laiho, Martin Stefanak, Vaclav Potocek, Craig Hamilton, Igor Jex, Christine Silberhorn, A 2D quantum walk simulation of two-particle dynamics. *Science* **336**, pp. 55–58 (2012).
- [47] Broome, M.A.; Fedrizzi, A.; Lanyon, B.P.; Kassal, I.; Aspuru-Guzik, A.; White, A.G. Discrete Single-Photon Quantum Walks with Tunable Decoherence. *Phys. Rev. Lett.* **2010**, *104*, 153602, doi:10.1103/PhysRevLett.104.153602.
- [48] A. Schreiber, K. N. Cassemiro, V. Potocek, A. Gabris, I. Jex, and Ch. Silberhorn, Decoherence and disorder in quantum walks: from ballistic spread to localization. *Phys. Rev. Lett.* **106**, 180403 (2011).
- [49] M. Nielsen, and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).
- [50] A. Schreiber, K. N. Cassemiro, V. Potocek, A. Gabris, P. J. Mosley, E. Andersson, I. Jex, and Ch. Silberhorn, Photons walking the line: a quantum walk with adjustable coin operations. *Phys. Rev. Lett.* **104**, 050502 (2010).
- [51] F. Zahringer, G. Kirchmair, R. Gerritsma, E. Solano, R. Blatt, and C. F. Roos, Realization of a quantum walk with one and two trapped ions. *Phys. Rev. Lett.* **104**, 100503 (2010).
- [52] Maximilian Genske, Wolfgang Alt, Andreas Steffen, Albert H. Werner, Reinhard F. Werner, Dieter Meschede, and Andrea Alberti, Electric Quantum Walks with Individual Atoms. *Phys. Rev. Lett.* **110**, 190601 (2013).
- [53] Fruchart, M. *Phys. Rev. B* **2015**, *92*, 045424.
- [54] M. S. Rudner, N. H. Lindner, E. Berg, M. Levin, Anomalous edge states and the bulk-edge correspondence for periodically driven two-dimensional systems. *Phys. Rev. X* **2013**, *3*, 031005.
- [55] Francisco, D.; Iemmi, C.; Paz, J.P.; Ledesma, S. Simulating a quantum walk with classical optics. *Phys. Rev. A* **2006**, *74*, 052327, doi:10.1103/PhysRevA.74.052327.

- [56] Lazarev, G.; Hermerschmidt, A.; Krueger, S.; Osten, S. LCOS spatial light modulators: trends and applications. In *Optical Imaging and Metrology: Advanced Technologies*, 1st ed.; Osten, W., Reingand, N., Eds.; Wiley-VCH: Weinheim, Germany, 2012; pp. 1–29; ISBN: 978–3–527–41064-4.
- [57] G. Puentes, C. La Mela, S. Ledesma, C. Iemmi, J.P. Paz, M. Saraceno, Optical simulation of quantum algorithms using programmable liquid-crystal displays, *Physical Review A* **69**, 042319 (2004).
- [58] Marques, B.; Matoso, A.A.; Pimenta, W.M.; Gutiérrez-Esparza, A.J.; Santos, M. F.; Pádua, S. Experimental simulation of decoherence in photonics qudits. *Sci. Rep.* **2015**, 5, 16049, doi:10.1038/srep16049.
- [59] Solís-Prosser, M.A.; Fernandes, M. F.; Jiménez, O.; Delgado, A.; Neves, L. Experimental Minimum-Error Quantum-State Discrimination in High Dimensions. *Phys. Rev. Lett.* **2017**, 118, 100501, doi:10.1103/PhysRevLett.118.100501.
- [60] Bouchard, F.; Fickler, R.; Boyd, R. W.; Karimi, E. High-dimensional quantum cloning and applications to quantum hacking. *Sci. Adv.* **2017**, 3, e1601915, doi:10.1126/sciadv.1601915.
- [61] Solís-Prosser, M.A.; Arias, A.; Varga, J.J.M.; Rebón, L.; Ledesma, S.; Iemmi, C.; Neves, L. Preparing arbitrary pure states of spatial qudits with a single phase-only spatial light modulator. *Opt. Lett.* **2013**, 38, 4762, doi:10.1364/OL.38.004762.
- [62] Moreno, I.; Velásquez, P.; Fernández-Pousa, C.R.; Sánchez-López, M.M.; Mateos, F. Jones matrix method for predicting and optimizing the optical modulation properties of a liquid-crystal display. *J. Appl. Phys.* **2003**, 94, 3697, doi:10.1063/1.1601688.
- [63] Davis, J.A.; McNamara, D.E.; Cottrell, D.M.; Sonehara, T. Two-dimensional polarization encoding with a phase-only liquid-crystal spatial light modulator. *Appl. Opt.* **2000**, 39, 1549, doi:10.1364/AO.39.001549.
- [64] Moreno, I.; Martínez, J.L.; Davis, J. A. Two-dimensional polarization rotator using a twisted-nematic liquid-crystal display. *Appl. Opt.* **2007**, 46, 881, doi: 10.1364/AO.46.000881.
- [65] M. V. Berry, Classical adiabatic angles and quantal adiabatic phase. *J. Phys. A* **18**, 15 (1985).
- [66] J. Hannay, Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian. *J. Phys. A* **18**, 221 (1985).
- [67] Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, Experimental observation of the quantum Hall effect and Berry's phase in graphene. *Nature* **438** 201 (2005).
- [68] P. Delplace, D. Ullmo, G. Montambaux, Zak phase and the existence of edge states in graphene. *Phys. Rev. B* **2011**, 84, 195452.
- [69] C. L. Kane, and E. J. Mele, Z 2 topological order and the quantum spin Hall effect, *Phys. Rev. Lett.* **95**, 146802 (2005); B. Bernevig *et al.*, Quantum spin Hall effect and topological phase transition in HgTe quantum wells, *Science* **314**, 1757 (2006); M. Köning *et al.*, Quantum spin Hall insulator state in HgTe quantum wells, *Science* **318**, 766 (2007).
- [70] G. Delacretaz, E. R. Grant, R.L. Whetten, L. Wöste, and J. W. Zwanziger, Fractional quantization of molecular pseudorotation in Na 3. *Phys. Rev. Lett.* **56**, 2598 (1986).
- [71] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. I. Seo, A. H. MacDonald, B. A. Bernevig, A. Yazdani, Observation of Majorana fermions in ferromagnetic atomic chains on a superconductor. *Science* **346**, 602 (2014).

[72] J. Provost and G. Vallee,
Riemannian structure on manifolds of
quantum states. *Comm. Math. Phys* **76**,
289 (1980).

[73] G. Puentes, Spontaneous parametric
downconversion and quantum walk
topology, *JOSA B* **33**, 461–467 (2016).

[74] M. Atala, M. Aidelsburger, J.
Barreiro, D. Abanin, T. Kitagawa, E.
Demler, I. Bloch, Direct measurement of
the Zak phase in topological Bloch
bands. *Nature Phys.* **9**, 795 (2013).

[75] J. C. Loredó, M. A. Broome, D. H.
Smith, and A. G. White, Observation of
entanglement-dependent two-particle
holonomic phase. *Phys. Rev. Lett.* **112**,
143603 (2014).