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Elliptic Curve over a Local Finite Ring R_n

Abdelhakim Chillali and Lhoussain El Fadil

Abstract

The goal of this chapter is to study some arithmetic proprieties of an elliptic curve defined by a Weierstrass equation on the local ring $R_n = \mathbb{F}_q[X]/(X^n)$, where $n \geq 1$ is an integer. It consists of, an introduction, four sections, and a conclusion. In the first section, we review some fundamental arithmetic proprieties of finite local rings R_n , which will be used in the remainder of the chapter. The second section is devoted to a study the above mentioned elliptic curve on these finite local rings for arbitrary characteristics. A restriction to some specific characteristic cases will then be considered in the third section. Using these studies, we give in the fourth section some cryptography applications, and we give in the conclusion some current research perspectives concerning the use of this kind of curves in cryptography. We can see in the conclusion of research in perspectives on these types of curves.

Keywords: elliptic curve, finite ring, cryptography

1. Introduction

Elliptic curves are especially important in number theory and constitute a major area of current research; for example, they were used in Andrew Wiles's proof of Fermat's Last Theorem. They also find applications in elliptic curve cryptography (ECC), integer factorization, classical mechanics in the description of the movement of spinning tops, to produce efficient codes ... For these reasons, the subject is well known, presented, and worth exploring.

The purpose of cryptography is to ensure the security of communications and data stored in the presence of adversaries [1–3]. It offers a set of techniques for providing confidentiality, authenticity, and integrity services. Cryptology, also known as the science of secrecy, combines cryptography and cryptanalysis. While the role of cryptographers is to design, build, and prove cryptosystems, among other things, the goal of cryptanalysis is to “break” these systems. The history of cryptography has long been the history of secret codes and along all previous times, this has affected the fate of men and nations [4]. In fact, until 1970, the main goal of cryptography was to build a signature encryption systems [5, 6], but thanks to cryptanalysis, the army and the black cabinets of diplomats were able to wage their wars in the shadows controlling the communication networks, especially of their enemies [7, 8]. The internet revolution and the increasingly massive use of information in digital form facilitated communications but in counterparty it weakened the security level of information. Indeed, “open” networks create security holes,

which allow access to the information. Cryptography, or the art of encrypting messages, a science that sits today in the crossroads of mathematics, computer sciences, and some applied physics, has then become a necessity for today's civilization to keep its secrets from adversaries. Confusion is often made between cryptography and cryptology, but the difference exists. Cryptology is the "science of secrecy," and combines two branches on the one hand, cryptography, which makes it possible to encrypt messages, and on the other hand, cryptanalysis, which serves to decrypt them. Our focus in this chapter is to show how some elliptic curves, mathematical objects studied particularly in algebraic geometry [9–12]. You can give several definitions depending on the person you are talking to. Cryptography indeed used elliptic curves for more than 40 years the appearance of the Diffie-Hellman key exchange protocol and the ElGamal cryptogram [13–15]. These cryptographic protocols use in particular group structures, for by applying these methods to groups defined by elliptic curves, a new speciality was born at the end of the 1980: ECC, Elliptic Curve Cryptography. Recall that Diffie-Hellman key exchange which is based on the difficulty of the discrete logarithm problem (DLP) [16–18]. The success of elliptic curves in public key cryptographic systems has then created a new interest in the study of the arithmetic of these geometric objects. The group of points on an elliptical curve is an interesting group in cryptography because there is no known sub-exponential algorithm for sound (DLP) [19–21]. In general, the DLP is difficult to be solved, but not as much as in a generic group as in the case of finite field. We know sub-exponential algorithms to solve it depending on the size of the group to use, which impose criteria for the PLD to be infeasible. The prime number p which is the characteristic of our base ring must then have at least 1024 bits, which offers a security level similar to the one given by a generic order group of 160 bits. Recall that a generic group for the DLP is a group for which there is no a specific algorithm to solve the DLP [22], so that the only available algorithms are those for all groups.

In [23], Elhassani et al. have built an encryption method based on DLP and Lattice. Boulbot et al. in [24] have studied elliptic curves on a non-local ring to compare these curves on local and non-local rings, while in [25], Sahmoudi et al. have studied these types of curves on a family of finite rings in the authors have introduced a cryptosystem on these types of curves, see [26].

In this chapter, d and n are a positive integers and $q = p^d$ is a power of a prime natural number p .

2. The ring $R_n = \mathbb{F}_q[X]/(X^n)$

Let $R_n = \mathbb{F}_q[X]/(X^n)$ be a $\mathbb{F}_q$ -algebra of dimension n , with $(1, \epsilon, \dots, \epsilon^{n-1})$ as a $\mathbb{F}_q$ -basis, where $\epsilon = \bar{X}$, $\epsilon^n = 0$, $\mathbb{F}_q$ is the finite field of order $q = p^r$, and p being a prime integer [27–29].

2.1 Internal laws in R_n

Recall that the two laws "+" and "." are naturally defined on R_n [30, 31]: for every two elements $X = \sum_{i=0}^{n-1} x_i \epsilon^i$ and $Y = \sum_{i=0}^{n-1} y_i \epsilon^i$ in R_n , with $x_1, \dots, x_n, y_1, \dots, y_n$ in $\mathbb{F}_q$,

$$X + Y = \sum_{i=0}^{n-1} z_i \epsilon^i, \text{ where } z_j = x_j + y_j \text{ in } \mathbb{F}_q \quad (1)$$

$$X.Y = \sum_{i=0}^{n-1} z_i \epsilon^i, \text{ where } z_j = \sum_{i=0}^j x_i y_{j-i} \text{ (The cauchy product)} \quad (2)$$

Corollary 2.1 Let $X = \sum_{i=0}^{n-1} x_i \epsilon^i \in R_n$, then $X^2 = \sum_{i=0}^{n-1} x'_i \epsilon^i$ where

$$\forall k \geq 0, \begin{cases} x'_{2k} = x_k^2 + 2 \sum_{i=0}^{k-1} x_i x_{2k-i} \\ x'_{2k+1} = 2 \sum_{i=0}^k x_i x_{2k+1-i} \end{cases} \quad (3)$$

Proof.

By formula (2), we have

$$\forall j \geq 0, x'_j = \sum_{i=0}^j x_i x_{j-i}. \quad (4)$$

$$\text{For } j = 2k, x'_{2k} = \sum_{i=0}^{2k} x_i x_{2k-i}, \quad (5)$$

$$\text{so, } x'_{2k} = x_k^2 + 2 \sum_{i=0}^{k-1} x_i x_{2k-i}. \quad (6)$$

$$\text{Similarly, for } j = 2k + 1, x'_{2k+1} = \sum_{i=0}^{2k+1} x_i x_{2k+1-i}, \quad (7)$$

$$\text{then, } x'_{2k+1} = 2 \sum_{i=0}^k x_i x_{2k+1-i}. \quad (8)$$

Under the same hypotheses of the corollary (2.1) and by an analogous proof, we have the following corollary:

Corollary 2.2 $X^3 = \sum_{i=0}^{n-1} x''_i \epsilon^i$, where

$$\forall k \geq 0, \begin{cases} x''_{2k} = x'_{2k} x_0 + \sum_{l=0}^{k-1} (x'_{2l} x_{2k-2l} + x'_{2l+1} x_{2k-1-2l}) \\ x''_{2k+1} = \sum_{l=0}^k (x'_{2l} x_{2k+1-2l} + x'_{2l+1} x_{2k-2l}) \end{cases} \quad (9)$$

Lemma 2.3 Let $Y = \sum_{i=0}^{n-1} y_i \epsilon^i$ the inverse of $X = \sum_{i=0}^{n-1} x_i \epsilon^i$. Then

$$\begin{cases} y_0 = x_0^{-1} \\ y_j = -x_0^{-1} \sum_{i=0}^{j-1} y_i x_{j-i}, \quad \forall j > 0 \end{cases} \quad (10)$$

Proof.

Let $Y = \sum_{i=0}^{n-1} y_i \epsilon^i$ be the inverse of $X = \sum_{i=0}^{n-1} x_i \epsilon^i$. Then $XY = 1$, by formula (2), we have

$$XY = \sum_{i=0}^{n-1} z_i \epsilon^i, \text{ where } z_j = \sum_{i=0}^j x_i y_{j-i}. \quad (11)$$

So,

$$z_0 = 1 \text{ and } \forall j > 0, z_j = 0, \quad (12)$$

which means that,

$$\begin{cases} y_0 = x_0^{-1} \\ y_j = -x_0^{-1} \sum_{i=0}^{j-1} y_i x_{j-i}, \quad \forall j > 0 \end{cases} \quad (13)$$

Lemma 2.4 *The non inverse elements in R_n are the elements of the form $\sum_{i=1}^{n-1} x_i \epsilon^i$ where $x_i \in \mathbb{F}_q^{n-1}$ for all $1 \leq i \leq n-1$.*

Proof.

Let $X = \sum_{i=0}^{n-1} x_i \epsilon^i \in R_n$. By lemma (2.3), X is invertible in R_n if and only if x_0 is invertible in $\mathbb{F}_q$. As $\mathbb{F}_q$ is a field, this means $x_0 \neq 0$.

Corollary 2.5 *The ring R_n is local, with maximal ideal $I_n = \epsilon R_n$.*

Notation.

Let $k \geq 2$, we denote:

1.

$$\pi_k : \begin{cases} R_k \rightarrow R_{k-1} \\ \sum_{i=0}^{k-1} x_i \epsilon^i \mapsto \sum_{i=0}^{k-2} x_i \delta^i \end{cases}$$

the projection of R_k on R_{k-1} .

2.

$$k^\pi : \begin{cases} R_k \rightarrow R_1 \\ \sum_{i=0}^{k-1} x_i \epsilon^i \mapsto x_0 \end{cases}$$

the canonical projection of R_k on $R_1 = \mathbb{F}_q$.

Corollary 2.6 *π_k et k^π are two ring homomorphisms.*

Proof.

We have,

$$\begin{aligned} \pi_k \left(\sum_{i=0}^{k-1} x_i \epsilon^i + \sum_{i=0}^{k-1} y_i \epsilon^i \right) &= \pi_k \left(\sum_{i=0}^{k-1} (x_i + y_i) \epsilon^i \right) \\ &= \sum_{i=0}^{k-2} (x_i + y_i) \delta^i \\ &= \pi_k \left(\sum_{i=0}^{k-1} x_i \epsilon^i \right) + \pi_k \left(\sum_{i=0}^{k-1} y_i \epsilon^i \right) \end{aligned} \quad (14)$$

and

$$\begin{aligned} \left(\sum_{i=0}^{k-1} x_i \epsilon^i \right) \left(\sum_{i=0}^{k-1} y_i \epsilon^i \right) &= \sum_{i=0}^{k-1} z_i \epsilon^i, \text{ where } z_j = \sum_{i=0}^j x_i y_{j-i}. \\ \pi_k \left(\sum_{i=0}^{k-1} z_i \epsilon^i \right) &= \sum_{i=0}^{k-2} z_i \delta^i \\ \pi_k \left(\sum_{i=0}^{k-1} x_i \epsilon^i \right) \pi_k \left(\sum_{i=0}^{k-1} y_i \epsilon^i \right) &= \sum_{i=0}^{k-2} z_i \delta^i \end{aligned} \quad (15)$$

Note that in addition, for every $k \geq 1$,

$$k^\pi = \pi_2 \circ \pi_3 \circ \pi_4 \dots \circ \pi_k \quad (16)$$

So, π_k and k^π are two rings morphisms.

Theorem 2.7 Let $n \geq 2$ be an integer,

$a = \tilde{a} + a_{n-1}\epsilon^{n-1}$, $b = \tilde{b} + b_{n-1}\epsilon^{n-1}$, $X = \tilde{X} + x_{n-1}\epsilon^{n-1}$, $Y = \tilde{Y} + y_{n-1}\epsilon^{n-1}$ and $Z = \tilde{Z} + z_{n-1}\epsilon^{n-1}$ be elements of R_n with:

$$Y^2Z = X^3 + aXZ^2 + bZ^3. \quad (17)$$

Then

$$\tilde{Y}^2\tilde{Z} = \tilde{X}^3 + \tilde{a}\tilde{X}\tilde{Z}^2 + \tilde{b}\tilde{Z}^3 + [D - (Ay_{n-1} + Bz_{n-1} + Cx_{n-1})]\epsilon^{n-1} \quad (18)$$

where,

$$A = 2y_0z_0, \quad (19)$$

$$B = y_0^2 - 3z_0^2b_0 - 2z_0a_0x_0, \quad (20)$$

$$C = -(3x_0^2 + a_0z_0^2) \quad (21)$$

and

$$D = b_{n-1}z_0^3 + a_{n-1}x_0z_0^2. \quad (22)$$

Proof.

We have:

$$\begin{aligned} Y^2Z &= (\tilde{Y} + y_{n-1}\epsilon^{n-1})^2(\tilde{Z} + z_{n-1}\epsilon^{n-1}) \\ &= \tilde{Y}^2\tilde{Z} + (y_0^2z_{n-1} + 2y_0z_0y_{n-1})\epsilon^{n-1} \\ X^3 &= (\tilde{X} + x_{n-1}\epsilon^{n-1})^3 \\ &= \tilde{X}^3 + 3x_0^2x_{n-1}\epsilon^{n-1} \end{aligned} \quad (23)$$

$$aXZ^2 = \tilde{a}\tilde{X}\tilde{Z}^2 + (2z_{n-1}z_0a_0x_0 + a_0x_{n-1}z_0^2 + a_{n-1}x_0z_0^2)\epsilon^{n-1}$$

$$bZ^3 = \tilde{b}\tilde{Z}^3 + (b_{n-1}z_0^3 + 3z_0^2z_{n-1}b_0)\epsilon^{n-1}$$

If

$$Y^2Z = X^3 + aXZ^2 + bZ^3, \quad (24)$$

then, $\tilde{Y}^2\tilde{Z} = \tilde{X}^3 + \tilde{a}\tilde{X}\tilde{Z}^2 + \tilde{b}\tilde{Z}^3 + (3x_0^2x_{n-1} + 2z_{n-1}z_0a_0x_0 + a_0x_{n-1}z_0^2 + a_{n-1}x_0z_0^2 - 3z_0^2z_{n-1}b_0 - y_0^2z_{n-1} + 2y_0z_0y_{n-1})\epsilon^{n-1}$ and therefore,

$$\tilde{Y}^2\tilde{Z} = \tilde{X}^3 + \tilde{a}\tilde{X}\tilde{Z}^2 + \tilde{b}\tilde{Z}^3 + [D - (Ay_{n-1} + Bz_{n-1} + Cx_{n-1})]\epsilon^{n-1} \quad (25)$$

where,

$$A = 2y_0z_0, \quad (26)$$

$$B = y_0^2 - 3z_0^2b_0 - 2z_0a_0x_0, \quad (27)$$

$$C = -(3x_0^2 + a_0z_0^2) \quad (28)$$

and

$$D = b_{n-1}z_0^3 + a_{n-1}x_0z_0^2. \quad (29)$$

2.2 Primitive triples

Definition 2.8 Let R be a ring. We say that an element $(x, y, z) \in R^3$ is primitive if: $xR + yR + zR = R$. The set of these primitive triplets will be denoted $\mathcal{P}(R)$.

Remark 2.9 The equality $xR + yR + zR = R$ means that there exists $(\alpha, \beta, \lambda) \in R^3$ such that $1_R = \alpha x + \beta y + \lambda z$.

Proposition 2.10 Let R be a local ring, then $(x, y, z) \in R^3$ is a primitive triple if and only if at least one of the elements x, y , and z is invertible in R .

Proof.

Suppose that x, y and z are not invertible in R , then:

$(x, y, z) \in \mathcal{M}^3$ where $\mathcal{M}$ is the unique maximal ideal of R , hence

$$xR + yR + zR \subset \mathcal{M} \subsetneq R, \quad (30)$$

which contradicts that (x, y, z) is a primitive triple.

Conversely, suppose, for example, that x is invertible in R , then $xR = R$, so $xR + yR + zR = R$.

Remark 2.11 If R is a field, then an element $(x, y, z) \in R^3$ is primitive if and only if $(x, y, z) \neq (0, 0, 0)$.

2.3 The projective plane on a finite ring

Let R is a ring. The projective plane on R is the set of equivalence classes of $\mathcal{P}(R)$ modulo; the equivalence relation $\sim_R$ defined by:

$$(x_1, y_1, z_1) \sim_R (x_2, y_2, z_2) \Leftrightarrow \exists \lambda \in R^\times : (x_2, y_2, z_2) = \lambda(x_1, y_1, z_1). \quad (31)$$

We denote the projective plane on R by $\mathbb{P}^2(R)$, it is the quotient set $\frac{\mathcal{P}(R)}{\sim_R}$, and we write $[x : y : z]$ for the equivalence class of $(x, y, z) \in \mathcal{P}(R)$. Thus, we have:

$$[x_1 : y_1 : z_1] = [x_2 : y_2 : z_2] \Leftrightarrow \exists \lambda \in R^\times : x_2 = \lambda x_1, y_2 = \lambda y_1 \text{ and } z_2 = \lambda z_1. \quad (32)$$

Example 2.12 We Consider the finite ring $\mathbb{F}_2[e] = \{\alpha + \beta e / \alpha \in \mathbb{F}_2 \text{ and } \beta \in \mathbb{F}_2\}$, where e is an indeterminate satisfying $e^2 = 0$. The group of units for this ring is $(\mathbb{F}_2[e])^\times = \{1, 1 + e\}$.

As this ring is local with maximal ideal $e\mathbb{F}_2[e]$, then an element (x, y, z) of $\mathbb{F}_2[e]^3$ is non primitive if and only if $(x, y, z) \in \{0, e\}^3$. As one can see, there are eight elements which are not primitive, and therefore the set $\mathcal{P}(\mathbb{F}_2[e])$ contains $64 - 8 = 56$ primitive triples as given below:

$$\begin{aligned} \mathcal{P}(\mathbb{F}_2[e]) = \{ & (0, 0, 1), (0, 1, 1+e), (0, 1, 0), (0, 1, 1), (0, 1, e), (0, 1, 1+e), (0, e, 1), \\ & (0, e, 1+e), (0, 1+e, 0), (0, 1+e, 1), (0, 1+e, e), (0, 1+e, 1+e), \\ & (1, 0, 0), (1, 0, 1), (1, 0, e), (1, 0, 1+e), (1, 1, 0), (1, 1, 1), (1, 1, e), \\ & (1, 1, 1+e), (1, e, 0), (1, e, 1), (1, e, e), (1, e, 1+e), (1, 1+e, 0), \\ & (1, 1+e, 1), (1, 1+e, e), (1, 1+e, 1+e), (e, 0, 1), (e, 0, 1+e), (e, 1, 0), \\ & (e, 1, 1), (e, 1, e), (e, 1, 1+e), (e, e, 1), (e, e, 1+e), (e, 1+e, 0), (e, 1+e, 1), \\ & (e, 1+e, e), (e, 1+e, 1+e), (1+e, 0, 0), (1+e, 0, 1), (1+e, 0, e), \\ & (1+e, 0, 1+e), (1+e, 1, 0), (1+e, 1, 1), (1+e, 1, e), (1+e, 1, 1+e), \\ & (1+e, e, 0), (1+e, e, 1), (1+e, e, e), (1+e, e, 1+e), (1+e, 1+e, 0), \\ & (1+e, 1+e, 1), (1+e, 1+e, e), (1+e, 1+e, 1+e)\}. \end{aligned} \quad (33)$$

Let (x, y, z) and (x', y', z') be two elements in $\mathcal{P}(\mathbb{F}_2[e])$, then:
 $[x' : y' : z'] = [x : y : z] \Leftrightarrow (x', y', z') = (x, y, z)$ or $(x', y', z') = (x + xe, y + ye, z + ze)$
 so every class in $\mathbb{P}^2(\mathbb{F}_2[e])$ contains two representatives, that is, the projective plane $\mathbb{P}^2(\mathbb{F}_2[e])$ contains exactly the following 28 elements:

$$\begin{aligned} \mathbb{P}^2(\mathbb{F}_2[e]) = \{ & [0 : 1 : 0], [0 : 0 : 1], [0 : 1 : 1], [0 : 1 : e], [0 : 1 : 1+e], [0 : e : 1], [1 : 0 : 0], \\ & [1 : 0 : 1], [1 : 0 : e], [1 : 0 : 1+e], [1 : 1 : 0], [1 : 1 : 1], [1 : 1 : e], [1 : 1 : 1+e], \\ & [1 : e : 0], [1 : e : 1], [1 : e : e], [1 : e : 1+e], [1 : 1+e : 0], [1 : 1+e : 1], \\ & [1 : 1+e : e], [1 : 1+e : 1+e], [e : 0 : 1], [e : 1 : 0], [e : 1 : 1], [e : 1 : e], \\ & [e : 1 : 1+e], [e : e : 1]\}. \end{aligned} \quad (34)$$

3. Elliptic curve over R_n

In this section, we study the elliptic curves defined on finite local rings R_n of characteristic a prime number p ;

1. A projective Weierstrass equation on R_n is an equation of the form:

$$\mathbf{E} : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4X + a_6Z^3 \quad (35)$$

2. A affine Weierstrass equation on R_n is an equation of the form:

$$\mathbf{E}' : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6 \quad (36)$$

where $(a_1, a_2, a_3, a_4, a_6) \in R_n^5$.

3.1 Elliptic curve form

To an affine (or projective) Weierstrass Eqs. (3) and (4), we associate the following quantities:

$$\begin{aligned}
 b_2 &= a_1^2 + 4a_2 \\
 b_4 &= 2a_4 + a_1a_3 \\
 b_6 &= a_3^2 + 4a_6 \\
 b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\
 c_4 &= b_2^2 - 24b_4 \\
 \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\
 j &= \frac{c_4^3}{\Delta} \text{ if } \Delta \neq 0
 \end{aligned} \tag{37}$$

Δ is called the discriminant of $\mathbf{E}$ and j its j - invariant.

Remark 3.1 On the field $R_1 = \mathbb{F}_q$, we denote the discriminant by Δ_0 and the j -invariant by j_0 , while on the ring R_n , $n > 1$ we denote the discriminant by $\Delta_{\varepsilon,n}$ and the j -invariant by $j_{\varepsilon,n}$.

We have $n^\pi(\Delta_{\varepsilon,n}) = \Delta_0$ and $n^\pi(j_{\varepsilon,n}) = j_0$.

Definition 3.2 Let R be a finite ring and let $a = (a_1, a_2, a_3, a_4, a_6) \in R^5$. An elliptic curve on R corresponding to a , which we write $E_a(R)$, is the set of zeros in the projective plane $\mathbb{P}^2(R)$ of the Weierstrass Eq. (3), for which the discriminant Δ is invertible in R .

Remark 3.3 According to the characteristic of the ring R ; $\text{chra}(R)$ we have the following cases:

1. If $\text{char}(R) \neq 2$ and $\text{char}(R) \neq 3$, then:

$$E_{a,b}(R) = \{[X : Y : Z] \in \mathbb{P}^2(R) / Y^2Z = X^3 + aXZ^2 + bZ^3\} \tag{38}$$

for $(a, b) \in R \times R$, with $\Delta = \Delta_{a,b} = -16(4a^3 + 27b^2) \in R^\times$.

2. If $\text{char}(R) = 2$, then $E_{a,b}(R)$ has one of the following forms:

$$E_{a,b}(R) = \{[X : Y : Z] \in \mathbb{P}^2(R) / Y^2Z + XYZ = X^3 + aX^2Z + bZ^3\} \tag{39}$$

for $(a, b) \in R^2$, with $\Delta = \Delta_{a,b} = b \in R^\times$.

Or:

$$E_a(R) = \{[X : Y : Z] \in \mathbb{P}^2(R) / Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_4XZ^2 + a_6Z^3\} \tag{40}$$

for $a = (a_1, a_3, a_4, a_6) \in R^4$, with a_1 non invertible and

$$\Delta = \Delta_a = a_1^3(a_1^3a_6 + a_1^2a_3a_4 + a_1a_4^2 + a_3^3) + a_3^4 \in R^\times. \tag{41}$$

3. If $\text{char}(R) = 3$, then $E_{a,b}(R)$ has one of the following forms:

$$E_{a,b}(R) = \{[X : Y : Z] \in \mathbb{P}^2(R) / Y^2Z = X^3 + aX^2Z + bZ^3\} \tag{42}$$

for $(a, b) \in R^2$, with $\Delta = \Delta_{a,b} = -a^3b \in R^\times$.

Or:

$$E_{a,b}(R) = \{[X : Y : Z] \in \mathbb{P}^2(R) / Y^2Z = X^3 + aXZ^2 + bZ^3\} \tag{43}$$

for $(a, b) \in R^2$, with $\Delta = \Delta_{a,b} = -a^3 \in R^\times$.

Remark 3.4 A projective elliptic curve on a field K has one of the following normal forms (Table 1):

		Normal form
$\text{char}(\mathbf{K}) \neq 2, 3$		$Y^2Z = X^3 + a_4XZ^2 + a_6Z^3$ $\Delta = -16(4a_4^3 + 27a_6^2)$ $j = 1728 \frac{4a_4^3}{4a_4^3 + 27a_6^2}$
$\text{char}(\mathbf{K}) = 3$	$j = 0$	$Y^2Z = X^3 + a_4XZ^2 + a_6Z^3$ $\Delta = -a_4^3$
	$j \neq 0$	$Y^2Z = X^3 + a_2X^2Z + a_6Z^3$ $\Delta = -a_2^3a_6$ $j = -\frac{a_2^3}{a_6}$
$\text{char}(\mathbf{K}) = 2$	$j = 0$	$Y^2Z + a_3YZ^2 = X^3 + a_4XZ^2 + a_6Z^3$ $\Delta = a_3^4$
	$j \neq 0$	$Y^2Z + XYZ = X^3 + a_2X^2Z + a_6Z^3$ $\Delta = a_6$ $j = \frac{1}{a_6}$

Table 1.
 Elliptic curve form on a field.

3.2 Projective coordinates and group law

In this subsection, we give in projective coordinates the formulas for adding the points on an elliptic curve defined by Eq. (3) on the ring R_n , according to the normal form.

Using Bosma and Lenstra's theorem see [32], we can deduce the explicit formulas for the commutative additive law of the group $E_a(R_n)$. The results are given in the next theorems following the values of the characteristic of ring R_n [33–36]. Let

$$[X_1 : Y_1 : Z_1] + [X_2 : Y_2 : Z_2] = [X_3 : Y_3 : Z_3]. \quad (44)$$

Theorem 3.5 [Characteristic two case]:

- If $[n^\pi(X_1) : n^\pi(Y_1) : n^\pi(Z_1)] = [n^\pi(X_2) : n^\pi(Y_2) : n^\pi(Z_2)]$, then:

$$X_3 = X_1Y_1Y_2^2 + X_2Y_1^2Y_2 + X_2^2Y_1^2 + X_1X_2^2Y_1 + aX_1^2X_2Y_2 + aX_1X_2^2Y_1 + aX_1^2X_2^2 + bX_1Y_1Z_2^2 + bX_2Y_2Z_1^2 + bX_1^2Z_2^2 + bY_1Z_2^2Z_1 + bY_2Z_1^2Z_2 + bX_1Z_2^2Z_1. \quad (45)$$

$$Y_3 = Y_1^2Y_2^2 + X_2Y_1^2Y_2 + aX_1X_2^2Y_1 + a^2X_1^2X_2^2 + bX_1^2X_2Z_2 + bX_1X_2^2Z_1 + bX_1Y_1Z_2^2 + bX_1^2Z_2^2 + abX_2^2Z_1^2 + abX_1^2Z_2^2 + bY_1Z_1Z_2^2 + bX_1Z_1Z_2^2 + abX_1Z_1Z_2^2 + abX_2Z_1^2Z_2 + b^2Z_1^2Z_2^2 \quad (46)$$

$$Z_3 = X_1^2X_2Y_2 + X_1X_2^2Y_1 + Y_1^2Y_2Z_2 + Y_1Y_2^2Z_1 + X_1^2X_2^2 + Y_1^2X_2Z_2 + X_1^2Y_2Z_2 + aX_1^2Y_2Z_2 + aX_2^2Y_1Z_1 + X_1^2X_2Z_2 + aX_1X_2^2Z_1 + bY_1Z_1Z_2^2 + bY_2Z_1^2Z_2 + bX_1Z_1Z_2^2. \quad (47)$$

- If $[n^\pi(X_1) : n^\pi(Y_1) : n^\pi(Z_1)] \neq [n^\pi(X_2) : n^\pi(Y_2) : n^\pi(Z_2)]$, then:

$$X_3 = X_1Y_2^2Z_1 + X_2Y_1^2Z_2 + X_1^2Y_2Z_2 + X_2^2Y_1Z_1 + aX_1^2X_2Z_2 + aX_1X_2^2Z_1 + bX_1Z_1Z_2^2 + bX_2Z_1^2Z_2. \quad (48)$$

$$Y_3 = X_1^2 X_2 Y_2 + X_1 X_2^2 Y_1 + Y_1^2 Y_2 Z_2 + Y_1 Y_2^2 Z_1 + X_1^2 Y_2 Z_2 + X_2^2 Y_1 Z_1 + a X_1^2 Y_2 Z_2 +$$

$$a X_2^2 Y_1 Z_1 + a X_1^2 X_2 Z_2 + a X_1 X_2^2 Z_1 + b Y_1 Z_1 Z_2^2 + b Y_2 Z_1^2 Z_2 + b X_1 Z_1 Z_2^2 + b X_2 Z_1^2 Z_2. \quad (49)$$

$$Z_3 = X_1^2 X_2 Z_2 + X_1 X_2^2 Z_1 + Y_1^2 Z_2^2 + Y_2^2 Z_1^2 + X_1 Y_1 Z_2^2 + X_2 Y_2 Z_1^2 + a X_1^2 Z_2^2 + a X_2^2 Z_1^2. \quad (50)$$

Theorem 3.6 [Characteristic three case]:

• If $[n^\pi(X_1) : n^\pi(Y_1) : n^\pi(Z_1)] = [n^\pi(X_2) : n^\pi(Y_2) : n^\pi(Z_2)]$, then:

$$X_3 = Y_1 Y_2^2 X_1 + Y_1^2 Y_2 X_2 + 2a X_1^2 X_2 Y_2 + 2a X_1 X_2^2 Y_1 + 2Z_1 Z_2^2 ab Y_1 + 2Z_1^2 Z_2 ab Y_2. \quad (51)$$

$$Y_3 = Y_1^2 Y_2^2 + 2a^2 X_1^2 X_2^2 + a^2 b X_1 Z_1 Z_2^2 + a^2 b X_2 Z_1^2 Z_2. \quad (52)$$

$$Z_3 = a X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) + a (X_1 Y_2 + X_2 Y_1) (X_1 Z_2 + X_2 Z_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1). \quad (53)$$

• If $[n^\pi(X_1) : n^\pi(Y_1) : n^\pi(Z_1)] \neq [n^\pi(X_2) : n^\pi(Y_2) : n^\pi(Z_2)]$, then:

$$X_3 = 2X_1 Y_2 Y_1 Z_2 + X_1 Y_2^2 Z_1 + 2X_2 Y_1^2 Z_2 + X_2 Y_1 Y_2 Z_1 + 2a X_1^2 X_2 Z_2 + a X_1 X_2^2 Z_1. \quad (54)$$

$$Y_3 = 2Y_1^2 Y_2 Z_2 + Y_1 Y_2^2 Z_1 + 2a X_1 X_2 Y_1 Z_2 + a X_1 X_2 Y_2 Z_1 + 2a X_1^2 Y_2 Z_2 + a X_2^2 Y_1 Z_1. \quad (55)$$

$$Z_3 = 2Y_1^2 Z_2^2 + Y_2^2 Z_1^2 + a X_1^2 Z_2^2 + 2a X_2^2 Z_1^2. \quad (56)$$

Theorem 3.7 [The case where the characteristic is different from two and from three]:

• If $[n^\pi(X_1) : n^\pi(Y_1) : n^\pi(Z_1)] = [n^\pi(X_2) : n^\pi(Y_2) : n^\pi(Z_2)]$, then:

$$X_3 = Y_1^2 X_2 Z_2 - Z_1 X_1 Y_2^2 a (Z_1 X_2 + X_1 Z_2) (Z_1 X_2 - X_1 Z_2) + (2Y_1 Y_2 - 3b Z_1 Z_2) (Z_1 X_2 - X_1 Z_2) \quad (57)$$

$$Y_3 = Y_1 Y_2 (Z_2 Y_1 - Z_1 Y_2) - a (X_1 Y_1 Z_2^2 - Z_1^2 X_2 Y_2) + (-2a Z_1 Z_2 - 3X_1 X_2) (X_2 Y_1 - X_1 Y_2) - 3b Z_1 Z_2 (Z_2 Y_1 - Z_1 Y_2) \quad (58)$$

$$Z_3 = (Z_1 Y_2 + Z_2 Y_1) (Z_2 Y_1 - Z_1 Y_2) + (3X_1 X_2 + a Z_1 Z_2) (Z_1 X_2 - X_1 Z_2) \quad (59)$$

• If $[n^\pi(X_1) : n^\pi(Y_1) : n^\pi(Z_1)] \neq [n^\pi(X_2) : n^\pi(Y_2) : n^\pi(Z_2)]$, then:

$$X_3 = (Y_1 Y_2 - 6b Z_1 Z_2) (X_2 Y_1 + X_1 Y_2) + (a^2 Z_1 Z_2 - 2a X_1 X_2) (Z_1 Y_2 + Z_2 Y_1) - 3b (X_1 Y_1 Z_2^2 + Z_1^2 X_2 Y_2) - a (Y_1 Z_1 X_2^2 + X_1^2 Y_2 Z_2) \quad (60)$$

$$Y_3 = Y_1^2 Y_2^2 + 3a X_1^2 X_2^2 + (-a^3 - 9b^2) Z_1^2 Z_2^2 - a^2 (Z_1 X_2 + X_1 Z_2)^2 - 2a^2 Z_1 X_1 Z_2 X_2 + (9b X_1 X_2 - 3ab Z_1 Z_2) (Z_1 X_2 + X_1 Z_2) \quad (61)$$

$$Z_3 = (Y_1 Y_2 + 3b Z_1 Z_2) (Z_1 Y_2 + Z_2 Y_1) + (3X_1 X_2 + 2a Z_1 Z_2) (X_2 Y_1 + X_1 Y_2) + a (X_1 Y_1 Z_2^2 + Z_1^2 X_2 Y_2). \quad (62)$$

4. Elliptic curve on R_n where $\text{char}(\mathbf{R}_n) \neq 2, 3$

The objective of this chapter is to study elliptic curves defined by a Weierstrass equation with coefficients in a ring R_n such that $\text{char}(\mathbf{R}_n) \neq 2, 3$. We denote it by $E_{a,b}^n$. Let

$$k_\theta : \begin{cases} \mathbb{F}_q^{k-1} \rightarrow E_{a,b}^k \\ (x_1, x_2, \dots, x_{k-1}) \mapsto \left[\sum_{i=1}^{k-1} x_i \epsilon^i : 1 : \sum_{i=3}^{k-1} z_i \epsilon^i \right] \end{cases} \quad (63)$$

we denote, $k_G = k_\theta(\mathbb{F}_q^{k-1})$.

4.1 The morphisms π^k et θ_k

Lemma 4.1 *The application*

$$\pi^k : \begin{cases} E_{a,b}^k \rightarrow E_{\pi_k(a), \pi_k(b)}^{k-1} \\ [X : Y : Z] \mapsto [\pi_k(X) : \pi_k(Y) : \pi_k(Z)] \end{cases} \quad (64)$$

is a surjective group homomorphism.

Proof.

π^k is well defined because π_k is a morphism of rings. According to theorem (2.7), we have $AY_{k-1} + BZ_{k-1} + CX_{k-1} = D \bmod p$, with

$$A = 2y_0z_0, \quad (65)$$

$$B = y_0^2 - 3z_0^2b_0 - 2z_0a_0x_0, \quad (66)$$

$$C = -(3x_0^2 + a_0z_0^2) \quad (67)$$

and

$$D = b_{n-1}z_0^3 + a_{n-1}x_0z_0^2. \quad (68)$$

The coefficients A , B and $-C$ are the partial derivatives of the function

$$F(X, Y, Z) = Y^2Z - X^3 - a_0XZ^2 - b_0Z^3 \quad (69)$$

calculated starting from (x_0, y_0, z_0) , which are not all equal to zero and deducing the existence of $[x_{k-1} : y_{k-1} : z_{k-1}]$. Hence, π^k est surjectif.

Using corollary (2.6), we deduce that π^k is a group homomorphism.

Lemma 4.2 *For all $k \geq 2$,*

$$\text{Ker}(\pi^k) = \{ [l\epsilon^{k-1} : 1 : 0] \mid l \in \mathbb{F}_q \}. \quad (70)$$

Proof.

We have:

$$\text{Ker}(\pi^k) = \{ P \in E_{a,b}^k \mid \pi^k(P) = [0 : 1 : 0] \}. \quad (71)$$

Then, $P = [x_{k-1}\epsilon^{k-1} : 1 + y_{k-1}\epsilon^{k-1} : z_{k-1}\epsilon^{k-1}] = [x_{k-1}\epsilon^{k-1} : 1 : z_{k-1}\epsilon^{k-1}]$.

As $P \in E_{a,b}^k$, we have

$$\begin{aligned} z_{k-1}\epsilon^{k-1} &= (x_{k-1}\epsilon^{k-1})^3 + ax_{k-1}\epsilon^{k-1}(z_{k-1}\epsilon^{k-1})^2 + b(z_{k-1}\epsilon^{k-1})^3 \\ &= 0, \end{aligned} \quad (72)$$

so, $z_{k-1} = 0$.

This yields $\text{Ker}(\pi^k) = \{ [l\epsilon^{k-1} : 1 : 0] \mid l \in \mathbb{F}_q \}$.

Lemma 4.3 *The application*

$$\theta_k : \begin{cases} \mathbb{F}_q \rightarrow E_{a,b}^k \\ l \mapsto [l\epsilon^{k-1} : 1 : 0] \end{cases} \quad (73)$$

is an injective group homomorphism.

Proof.

The application θ_k is injective by construction.

Let $[l\epsilon^{k-1} : 1 : 0]$ and $[h\epsilon^{k-1} : 1 : 0]$ be two elements in $E_{a,b}^k$, then:

$$\begin{aligned} k^\pi(l\epsilon^{k-1}) &= k^\pi(h\epsilon^{k-1}) \\ k^\pi(1) &= k^\pi(1) \\ k^\pi(0) &= k^\pi(0). \end{aligned} \quad (74)$$

so, using theorem (3.7),

$$\begin{aligned} X_3 &= (l + h)\epsilon^{k-1} \\ Y_3 &= 1 \\ Z_3 &= 0. \end{aligned} \quad (75)$$

This yields

$$\theta_k(l + h) = \theta_k(l) + \theta_k(h). \quad (76)$$

Thus, θ_k is an injective group homomorphism.

4.2 Main applications

In this subsection, we consider a prime p which does not divide N , where $N = \#E_{k^\pi(a), k^\pi(b)}^1$.

Corollary 4.4 *Let $P \in E_{a,b}^k$, then*

$$NP = [0 : 1 : 0] \Leftrightarrow P \in E_{k^\pi(a), k^\pi(b)}^1. \quad (77)$$

Proof.

If $P \in E_{k^\pi(a), k^\pi(b)}^1$, then $NP = [0 : 1 : 0]$.

Let $P = [x_0 + X : y_0 + Y : z_0 + Z] \in E_{a,b}^k$ and $Q = [x_0 : y_0 : z_0] \in E_{k^\pi(a), k^\pi(b)}^1$.

If $NP = [0 : 1 : 0]$, then $N(P - Q) = [0 : 1 : 0]$.

So, $P - Q = k_\theta(l_1, l_2, \dots, l_{k-1})$.

We deduce that $Nl_i \equiv 0 \pmod{p}$, $i = 1, 2, \dots, k-1$, where $\text{pgcd}(N, p) = 1$, which proves that $l_i = 0$ et $P = Q$.

Corollary 4.5

$$\forall P \in E_{a,b}^k, \text{ we have } pNP = [0 : 1 : 0]. \quad (78)$$

Proof.

$\forall P \in E_{a,b}^k, NP \in k_G$, so $pNP = [0 : 1 : 0]$.

Lemma 4.6 *If p do not divide N , then there exists a unique homomorphism*

$$v_k : E_{a_0,b_0}^1 \rightarrow E_{a,b}^k \tag{79}$$

for which the following diagram is commutative; (named Diagram(d)).

$$\begin{array}{ccc}
 E_{a,b}^k & \xrightarrow{\pi_k} & E_{a_0,b_0}^1 \\
 \searrow [p] & & \swarrow v_k \\
 & E_{a,b}^k &
 \end{array}$$

Proof.

Let $P \in k_G$, we have $pP = [0 : 1 : 0]$.

Then

$$k_G \subset \ker([p]). \tag{80}$$

Hence, there is a unique homomorphism

$$v_k : E_{a_0,b_0}^1 \rightarrow E_{a,b}^k \tag{81}$$

which makes the diagram(d) commutative.

Theorem 4.7 *If p do not divide N , then there exists a unique homomorphism*

$$s_k : E_{a_0,b_0}^1 \rightarrow E_{a,b}^k \tag{82}$$

such that $\pi_k \circ s_k = id_{E_{a_0,b_0}^1}$.

Proof.

Let $N' \in \mathbb{Z}$ as it exists $t \in \mathbb{Z}$ checking $1 - NN' = tp$. Then,

$$[1 - NN'] = [t]o[p]. \tag{83}$$

According to Lemma (4.6), there is a unique homomorphism

$$s_k : E_{a_0,b_0}^1 \rightarrow E_{a,b}^k \tag{84}$$

which makes the following diagram commutative; (named Diagram(d')):

$$\begin{array}{ccc}
 E_{a,b}^k & \xrightarrow{\pi_k} & E_{a_0,b_0}^1 \\
 \searrow [tp] & & \swarrow s_k \\
 & E_{a,b}^k &
 \end{array}$$

Let $P \in E_{a_0,b_0}^1$, then there exists $P' \in E_{a,b}^k$ such that $\pi_k(P') = P$. So,

$$\begin{aligned}
 \pi_k \circ s_k(P) &= \pi_k \circ s_k \circ \pi_k(P') \\
 &= \pi_k([1 - NN'](P')) \\
 &= \pi_k(P' - NN'P') \\
 &= P - NN'P \\
 &= P.
 \end{aligned} \tag{85}$$

Theorem 4.8 *If p do not divide N , then $E_{a,b}^k \cong E_{a_0,b_0}^1 \times k_G$.*

Proof.

The isomorphism

$$f_k : \begin{cases} E_{a_0,b_0}^1 \times k_G \rightarrow E_{a,b}^k \\ (P, Q) \mapsto s_k(P) + Q \end{cases} \tag{86}$$

admits an inverse application

$$F_k : \begin{cases} E_{a,b}^k \rightarrow E_{a_0,b_0}^1 \times k_G \\ P \mapsto (\pi_k(P), NN'P) \end{cases}. \tag{87}$$

Indeed,

$$\begin{aligned}
 f_k \circ F_k(P) &= f_k((\pi_k(P), NN'P)) \\
 &= s_k \circ \pi_k(P) + NN'P \\
 &= (1 - NN')P + NN'P \\
 &= P.
 \end{aligned} \tag{88}$$

Likewise,

$$\begin{aligned}
 F_k \circ f_k(P, Q) &= F_k(s_k(P) + Q) \\
 &= (\pi_k(s_k(P) + Q), NN'(s_k(P) + Q)).
 \end{aligned} \tag{89}$$

So,

$$\begin{aligned}
 \pi_k(s_k(P) + Q) &= \pi_k(s_k(P)) + \pi_k(Q) \\
 &= P + [0 : 1 : 0] \\
 &= P.
 \end{aligned}$$

$$\begin{aligned}
 NN'(s_k(P) + Q) &= NN'(s_k(P)) + NN'Q \\
 &= NN'(1 - NN')P' + NN'Q \\
 &= N'tpNP' + NN'Q \\
 &= [0 : 1 : 0] + NN'Q \\
 &= NN'Q.
 \end{aligned} \tag{90}$$

As, $pQ = [0 : 1 : 0]$, then we have

$$\begin{aligned} NN'Q &= (1 - tp)Q \\ &= Q - tpQ \\ &= Q. \end{aligned} \quad (91)$$

We conclude,

$$F_{k \text{ of } k}(P, Q) = (P, Q). \quad (92)$$

Corollary 4.9 *If p do not divide N , then $E_{a,b}^k \cong E_{a_0,b_0}^1 \times \mathbb{F}_q^{k-1}$.*

Proof.

We have, $k_G \cong \mathbb{F}_q^{k-1}$, see [27, 30, 33].

Corollary 4.10 *If p do not divide N , then*

$$\begin{aligned} E_{a,b}^k &\cong C_N \times \mathbb{F}_q^{k-1}, \text{ with } C_N \text{ cyclic} \\ \text{or} \\ E_{a,b}^k &\cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \mathbb{F}_q^{k-1}, \text{ where } n_2 | (n_1 \wedge p - 1). \end{aligned} \quad (93)$$

Proof.

We have

$$\begin{aligned} E_{a_0,b_0}^1 &\cong C_N, \text{ with } C_N \text{ cyclic} \\ \text{or} \\ E_{a_0,b_0}^1 &\cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}, \text{ where } n_2 | (n_1 \wedge p - 1). \end{aligned} \quad (94)$$

And

$$E_{a,b}^k \cong E_{a_0,b_0}^1 \times \mathbb{F}_q^{k-1}. \quad (95)$$

Corollary 4.11 *If p do not divide N , then $(\sqrt{q} - 1)^2 q^{k-1} \leq \#(E_{a,b}^k) \leq (\sqrt{q} + 1)^2 q^{k-1}$.*

Proof.

According to Haas' theorem, we have:

$$|q + 1 - N| \leq 2\sqrt{q} \quad (96)$$

so

$$(\sqrt{q} - 1)^2 q^{k-1} \leq \#(E_{a,b}^k) \leq (\sqrt{q} + 1)^2 q^{k-1}. \quad (97)$$

5. Applications

In this section, we are interested in ECC using elliptic curves over the ring R_n .

5.1 The discrete logarithm on $E_{a,b}^n$

The discrete logarithm problem that we denote DLP, (Discrete logarithm problem), is a generally difficult problem which depends on the considered group $\mathcal{G}$. In many situations, due to the asymmetry existing between problems concerning the

calculation of logarithms and calculation of powers which is more easier and so of great interest in cryptograph, the above mentioned makes Diffie and Hellman were the first to build a cryptosystem from this situation [37, 38].

Definition 5.1 Let $\mathcal{G}$ be a finite cyclic group of order ρ and s, r two elements of $\mathcal{G}$. We call discrete logarithm of base s of r , the only element m in $[[0, \rho - 1]]$ such that $s^m = r$. The discrete logarithm for elliptic curves is defined in an analogous way to be, the only element m in $[[0, \rho - 1]]$ such that $mP = Q$, where P , and Q are two points of an additive subgroup $\mathcal{G}$ of $\mathbf{E}_{a,b}^n$.

By using the isomorphism proved given in theorem (4.8), we get the results gathered in the next theorem:

Theorem 5.2 If p does not divide N , then.

- $\#\mathbf{E}_{a,b}^n = p^{d(n-1)} \times N$.
- The problem of the discrete logarithm on the elliptic curve $\mathbf{E}_{a,b}^n$ is equivalent to that of $\mathbf{E}_{a_0,b_0}^1$.
- If the problem of the discrete logarithm on $\mathbf{E}_{a,b}^n$ is trivial, then it is also trivial on the elliptic curve $\mathbf{E}_{a_0,b_0}^1$.

5.2 Cryptography based on elliptic curves $\mathbf{E}_{a,b}^n$

Elliptic curve cryptography (ECC) is public key cryptography, which relies on the use of curves over finite fields. Essentially, there are two families of these curves which are used in cryptography. The first uses elliptic curves on a finite field $\mathbb{F}_{p^d}$, where p is a large prime number. This family is the best choice for a high software level when implementing ECC. The second family uses elliptic curves on a binary field $\mathbb{F}_{2^d}$ where d is a large positive integer, this family is more appropriate at the material level point of view when implementing ECC. Another family which is also interesting in ECC implementations is the family of elliptic curves on the previously seen rings $\mathbf{R}_n$. The most important advantage presented by the use of elliptic curves in cryptography (ECC) consists in the high security they provide for wireless applications compared to other asymmetric key cryptosystems, also their small key size. Indeed, a 160-bit key for (ECC) can replace a 1024-bit key for (RSA). Given d ; a large integer, $P \in \mathbf{E}_{a,b}^n$ and $Q \in \mathcal{G} \subset \mathbf{E}_{a,b}^n$. The discrete elliptical logarithm problem (DLEP) consists in finding $k \in \mathbb{Z}$ such that $Q = [k]P$, where

$$[k]P = \underbrace{P + P + \dots + P}_{k \text{ times}} = kP. \quad (98)$$

This is in fact a difficult problem, whose resolution is exponential.

5.3 Elliptical Diffie-Hellman cryptosystem

Recall that Alice and Bob can publicly agree on a common secret (that we describe below).

1. They choose on a large integer d , $\mathbf{E}_{a,b}^n$ and $P \in \mathbf{E}_{a,b}^n$.
2. Alice chooses $t \in \mathbb{Z}$ and calculates tP .

3. Bob chooses $s \in \mathbb{Z}$ and calculates sP .
4. Alice let public tP and keep private t .
5. Bob let public sP and keep private s .
6. Then, Alice and Bob build their common secret key $K = tsP = stP$.

Remark 5.3

1. Unlike the classic Diffie-Hellman algorithm, we do not ask that P be a generator of $E_{a,b}^n$. The analogue of the subgroup $\mathbb{F}_p^*$ of order $p - 1$, is the cyclic subgroup of $E_{a,b}^n$, generated by the point P .
2. As soon as we have a group $E_{a,b}^n$, and an element $P \in E_{a,b}^n$ of finite order we can consider a Diffie-Hellman system on $\mathcal{G} = \langle P \rangle$ which is cyclic. For this construction to have a cryptographic interest, $\log_p(tP) = t$ must be not easy to calculate.
3. $E_{a,b}^n$, is not always cyclical.
4. If, Oscar (program) is giving $d, E_{a,b}^n, tP$ and sP , then it is able to solve the discrete elliptical logarithm problem and find t or s .

5.4 Elliptical ElGamal cryptosystem

Let $P_m \in E_{a,b}^n$ be the point representing the message m , to encrypt P_m :

1. Key generation algorithm

- Bob chooses the private key $t \in \mathbb{Z}$ known only to him.
- $d \in \mathbb{N}, P \in E_{a,b}^n$ and $R = [t]P$ are public.

2. Encryption algorithm

- Alice Randomly chooses $k \in \mathbb{Z}$;
- She calculates $c_1 = [k]P \in E_{a,b}^n$;
- She also calculates $c_2 = P_m + [k]R$;
- Then, he makes public c_1, c_2 , or $C = (c_1; c_2)$.

3. To decrypt received message (c_1, c_2) , Bob calculates:

$$P_m = c_2 - [k]R = c_2 - [k][t]P = c_2 - [t]c_1. \quad (99)$$

Now, Oscar encounters the discrete elliptic logarithm problem, because to decipher the message P_m he must know t (i.e.; calculate t such that $R = [t]P$).

5.5 Coding example

Let d be a positive integer, we consider the quotient ring $\mathbf{R}_2 = \frac{\mathbb{F}_{2^d}[X]}{(X^2)}$, where $\mathbb{F}_{2^d}$ is the finite field of order 2^d .

Then the ring $\mathbf{R}_2$ is identified with the ring $\mathbb{F}_{2^d}[\varepsilon]$, where $\varepsilon^2 = 0$, i.e.,

$$\mathbf{R}_2 = \{a_0 + a_1 \cdot \varepsilon \mid a_0, a_1 \in \mathbb{F}_{2^d}\}. \quad (100)$$

We consider the elliptic curve on the ring $\mathbf{R}_2$ given by the equation:

$$Y^2Z + XYZ = X^3 + aX^2Z + bZ^3. \quad (101)$$

where a, b in $\mathbf{R}_2$ and b is invertible in $\mathbf{R}_2$. Each element of $\mathbf{E}_{a,b}^2$ is of the form; $[X : Y : 1]$ or $[x\varepsilon : 1 : 0]$, with $x \in \mathbb{F}_{2^d}$. Write:

$$\mathbf{E}_{a,b}^2 = \{[X : Y : 1] \in \mathbb{P}_2^2 \mid Y^2 + XY = X^3 + aX^2 + b\} \cup \{[x\varepsilon : 1 : 0] \mid x \in \mathbb{F}_{2^d}\}. \quad (102)$$

Let $\mathbf{E}_{a,b}^2$ be the elliptic curve over $\mathbf{R}_2$ and consider the irreducible polynomial $T(X) = 1 + X + X^3$ in $\mathbb{F}_2[X]$. Let α be such that $T(\alpha) = 0$ in $\mathbb{F}_8 = \frac{\mathbb{F}_2[X]}{(T(X))}$, then $(1, \alpha, \alpha^2)$ is a vector space base of $\mathbb{F}_8$ over $\mathbb{F}_2$.

$$\mathbb{F}_8 = \{0, 1, \alpha, \alpha^2, \alpha + 1, \alpha^2 + \alpha, \alpha^2 + 1, \alpha^2 + \alpha + 1\} \quad (103)$$

★ Put:

$$a = 1 + \alpha; \quad (104)$$

$$b = 1 + \alpha^2\varepsilon. \quad (105)$$

We have: $\mathbf{R}_2 = \mathbb{F}_8[\varepsilon]$ and $\mathbf{E}_{a,b}^2 : Y^2 + XY = X^3 + (1 + \alpha)X^2 + (1 + \alpha^2\varepsilon)$. Consider $P \in \mathbf{E}_{a,b}^2$ of order l , and consider the subgroup $\mathcal{G} = \langle P \rangle$, generated by P , to encrypt and decrypt our messages.

1. Coding of elements of $\mathcal{G}$

We will give a code to each element $Q = mP$, where $m \in \{1, 2, \dots, l\}$, defined as follows:

If $Q = [x_0 + x_1\varepsilon : y_0 + y_1\varepsilon : Z]$, where $x_i, y_i \in \mathbb{F}_8$ for $i = 0, 1$, and $Z = 0$ or 1 , then we set:

$$x_i = c_{0i} + c_{1i}\alpha + c_{2i}\alpha^2; \quad (106)$$

$$y_i = d_{0i} + d_{1i}\alpha + d_{2i}\alpha^2, \quad (107)$$

where α is the primitive root of the irreducible polynomial $T(X) = 1 + X + X^3$, and $c_{ij}, d_{ij} \in \mathbb{F}_2$.

So, we code Q as follows:

If $Z = 1$: $Q = c_{00}c_{10}c_{20}c_{01}c_{11}c_{21}d_{00}d_{10}d_{20}d_{01}d_{11}d_{21}1$.

If $Z = 0$: $Q = 00c_{01}c_{11}c_{21}d_{01}d_{11}d_{21}10000$.

2. Example: with the same a and b ;

$$a = 1 + \alpha; \quad (108)$$

$$b = 1 + \alpha^2 \varepsilon. \quad (109)$$

The elliptic curve $E_{a,b}^2$ contains 112 elements to know:

$$\begin{aligned} E_{a,b}(\mathbb{A}_2) = \{ & [(\alpha^2 + \alpha + 1)\varepsilon : 1 : 0], [(\alpha^2 + 1)\varepsilon : 1 : 0], [(1 + \alpha)\varepsilon : 1 : \\ & 0], [(\alpha^2 + \alpha)\varepsilon : 1 : 0], [\alpha^2 \varepsilon : 1 : 0], [\alpha \varepsilon : 1 : 0], [\varepsilon + \alpha^2 + 1 : \alpha \varepsilon + \alpha : \\ & 1], [\alpha^2 + \alpha : (\alpha^2 + \alpha + 1)\varepsilon : 1], [\alpha^2 \varepsilon : \varepsilon + 1 : 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 + 1 : \\ & (\alpha^2 + \alpha)\varepsilon + \alpha : 1], [(1 + \alpha)\varepsilon + \alpha^2 : (\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + 1 : 1], [(\alpha^2 + \alpha)\varepsilon + 1 + \alpha : \\ & (\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 : 1], [(1 + \alpha)\varepsilon + \alpha^2 + 1 : \alpha : 1], [\alpha^2 \varepsilon + \alpha^2 + \alpha + 1 : \varepsilon + 1 : \\ & 1], [1 + \alpha : (\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha + 1 : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha : (\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha : \\ & 1], [\alpha \varepsilon + \alpha^2 : (\alpha^2 + 1)\varepsilon + \alpha^2 + 1 : 1], [(1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : (1 + \alpha)\varepsilon + 1 : \\ & 1], [\alpha^2 \varepsilon : (1 + \alpha)\varepsilon + 1 : 1], [\alpha^2 \varepsilon + 1 + \alpha : \alpha^2 \varepsilon + \alpha^2 + \alpha + 1 : 1], [\alpha \varepsilon + \alpha^2 + 1 : \\ & \varepsilon + \alpha : 1], [(1 + \alpha)\varepsilon + \alpha^2 + \alpha : (\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha : 1], [\alpha^2 \varepsilon + \alpha^2 + \alpha + 1 : \\ & (\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha : 1], [(\alpha^2 + 1)\varepsilon + 1 + \alpha : (\alpha^2 + \alpha)\varepsilon + \alpha^2 : 1], [\alpha \varepsilon + \alpha^2 + \alpha : \\ & 0 : 1], [\alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : 1], [\varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : \\ & 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 : \\ & (1 + \alpha)\varepsilon + 1 : 1], [\alpha^2 \varepsilon + \alpha^2 + 1 : (\alpha^2 + \alpha + 1)\varepsilon + \alpha : 1], [(1 + \alpha)\varepsilon + \alpha^2 + \alpha : \\ & (\alpha^2 + \alpha)\varepsilon : 1], [(\alpha^2 + \alpha + 1)\varepsilon + 1 + \alpha : \varepsilon + \alpha^2 : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \\ & \alpha^2 + \alpha + 1 : 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : 1], [(\alpha^2 + 1)\varepsilon + 1 + \alpha : \\ & (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : 1], [\varepsilon + \alpha^2 : \alpha \varepsilon + 1 : 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha + 1 : \alpha \varepsilon + 1 : \\ & 1], [\alpha^2 \varepsilon : \alpha \varepsilon + 1 : 1], [\varepsilon + \alpha^2 + \alpha : \varepsilon : 1], [(1 + \alpha)\varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : \\ & 1], [\alpha^2 \varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : 1], [\alpha \varepsilon + \alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : \\ & 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha + 1 : \alpha^2 \varepsilon + 1 : 1], [(1 + \alpha)\varepsilon + \alpha^2 : \alpha^2 \varepsilon + 1 : 1], [\alpha^2 \varepsilon : \\ & \alpha^2 \varepsilon + 1 : 1], [\alpha^2 + \alpha + 1 : (\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha : 1], [\alpha^2 \varepsilon : 1 : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 : \\ & 1 : 1], [\varepsilon + \alpha, (1 + \alpha)\varepsilon + \alpha^2 + \alpha : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha + 1 : (1 + \alpha)\varepsilon + \alpha^2 + \alpha : \\ & 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha : (1 + \alpha)\varepsilon + \alpha^2 + \alpha : 1], [(\alpha^2 + \alpha)\varepsilon + 1 + \alpha : \varepsilon + \alpha^2 + \alpha + 1 : \\ & 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha : (1 + \alpha)\varepsilon : 1], [\varepsilon + \alpha^2 + \alpha + 1 : \alpha^2 \varepsilon + \alpha^2 + \alpha : \\ & 1], [\varepsilon + 1 + \alpha : \alpha \varepsilon + \alpha^2 + \alpha + 1 : 1], [\alpha : \alpha \varepsilon + \alpha^2 + \alpha : 1], [\alpha \varepsilon + \alpha^2 + \alpha + 1 : \\ & \alpha \varepsilon + \alpha^2 + \alpha : 1], [\alpha \varepsilon + \alpha^2 + \alpha + 1 : 1 : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha : \alpha^2 \varepsilon + \alpha^2 + \alpha : \\ & 1], [\alpha^2 : \varepsilon + \alpha^2 + 1 : 1], [(1 + \alpha)\varepsilon + 1 + \alpha : (\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha + 1 : \\ & 1], [\alpha^2 + \alpha : (\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha : 1], [\varepsilon + \alpha^2 + \alpha : \alpha^2 + \alpha : 1], [\alpha \varepsilon + 1 + \alpha : \\ & \alpha^2 + \alpha + 1 : 1], [\alpha^2 : \varepsilon + 1 : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : (\alpha^2 + \alpha + 1)\varepsilon + 1 : \\ & 1], [\alpha^2 \varepsilon : (\alpha^2 + \alpha + 1)\varepsilon + 1 : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 + 1 : (\alpha^2 + 1)\varepsilon + \alpha : 1], [\varepsilon + \alpha^2 : \\ & (1 + \alpha)\varepsilon + \alpha^2 + 1 : 1], [\varepsilon + 1 + \alpha : (1 + \alpha)\varepsilon + \alpha^2 : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 : \\ & (\alpha^2 + \alpha)\varepsilon + \alpha^2 + 1 : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha : (\alpha^2 + 1)\varepsilon : 1], [\varepsilon + \alpha^2 + \alpha + 1 : \\ & (\alpha^2 + 1)\varepsilon + 1 : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + 1 : \alpha^2 \varepsilon + \alpha : 1], [\alpha^2 \varepsilon + \alpha^2 + \alpha : \\ & \alpha \varepsilon : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha : \alpha^2 \varepsilon + \alpha^2 + \alpha : 1], [(\alpha^2 + \alpha)\varepsilon + \alpha : \alpha \varepsilon + \alpha^2 : \\ & 1], [(\alpha^2 + 1)\varepsilon + \alpha : \alpha \varepsilon + \alpha^2 : 1], [(1 + \alpha)\varepsilon + \alpha : \alpha \varepsilon + \alpha^2 : 1], [(1 + \alpha)\varepsilon + \alpha : \\ & \varepsilon + \alpha^2 + \alpha : 1], [(1 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : \alpha^2 + \alpha : 1], [\alpha^2 + \alpha + 1 : \\ & (\alpha^2 + \alpha)\varepsilon + 1 : 1], [\alpha^2 \varepsilon : (\alpha^2 + \alpha)\varepsilon + 1 : 1], [\alpha^2 \varepsilon + \alpha : \alpha \varepsilon + \alpha^2 : \\ & 1], [\alpha \varepsilon + 1 + \alpha : \alpha \varepsilon + \alpha^2 : 1], [\alpha \varepsilon + \alpha : \alpha \varepsilon + \alpha^2 : 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 : \alpha^2 + 1 : \\ & 1], [\alpha \varepsilon + \alpha : \alpha^2 + \alpha : 1], [(1 + \alpha)\varepsilon + 1 + \alpha : \alpha^2 \varepsilon + \alpha^2 : 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha : \\ & \varepsilon + \alpha^2 + \alpha : 1], [\alpha \varepsilon + \alpha^2 + \alpha : \alpha \varepsilon + \alpha^2 + \alpha : 1], [\alpha^2 \varepsilon + \alpha^2 : \alpha \varepsilon + \alpha^2 + 1 : \\ & 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha + 1 : (\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha : 1], [(\alpha^2 + 1)\varepsilon + \alpha : \\ & (\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 + \alpha : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha^2 : \alpha^2 \varepsilon + \alpha^2 + 1 : 1], [\alpha^2 \varepsilon : \end{aligned} \quad (111)$$

$$\begin{aligned} &(\alpha^2 + 1)\varepsilon + 1 : 1], [\alpha^2\varepsilon + \alpha : (\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha : 1], [1 + \alpha : (\alpha^2 + 1)\varepsilon + \alpha^2 : \\ &1], [(\alpha^2 + \alpha + 1)\varepsilon + 1 + \alpha : (\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : 1], [\alpha^2\varepsilon + \alpha^2 : (\alpha^2 + \alpha)\varepsilon + 1 : \\ &1], [\alpha^2 + 1 : (1 + \alpha)\varepsilon + \alpha : 1], [(\alpha^2 + 1)\varepsilon + \alpha^2 + \alpha : \alpha^2\varepsilon : 1], [\alpha^2\varepsilon + 1 + \alpha : \alpha^2 : \\ &1], [0 : 1 : 0], [\alpha : \alpha\varepsilon + \alpha^2 : 1], [\varepsilon + \alpha : \alpha\varepsilon + \alpha^2 : 1], [(\alpha^2 + \alpha + 1)\varepsilon + \alpha : \alpha\varepsilon + \alpha^2 : \\ &1], [(\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha + 1 : \varepsilon + \alpha^2 + \alpha : 1], [\alpha^2\varepsilon + \alpha^2 + \alpha : (\alpha^2 + \alpha)\varepsilon + \alpha^2 + \alpha : \\ &1], [\alpha\varepsilon + \alpha^2 : (\alpha^2 + \alpha + 1)\varepsilon + 1 : 1], [\varepsilon : 1 : 0], [(\alpha^2 + 1)\varepsilon + \alpha^2 : (\alpha^2 + 1)\varepsilon + 1 : 1]\} \end{aligned} \tag{112}$$

We consider: $P = [\alpha : \alpha + \alpha^2 + \alpha\varepsilon : 1] = 0100000110101$, then $\mathcal{G} = \langle P \rangle$ is of order 28. We attach to each point $Q \in \mathcal{G}$ a letter of the alphabet and a code. We collect the results in the following **Table 2**:

	mP	Code for mP	Symbol
1	$[\alpha : \alpha + \alpha^2 + \alpha\varepsilon : 1]$	0100000110101	<i>a</i>
2	$[1 + \alpha + \varepsilon : \alpha^2 + (1 + \alpha)\varepsilon : 1]$	1101000011101	<i>b</i>
3	$[\alpha^2 + \alpha\varepsilon : 1 + \alpha^2 + (\alpha^2 + 1)\varepsilon : 1]$	0010101011011	<i>c</i>
4	$[1 + \alpha + \alpha^2 + (1 + \alpha)\varepsilon : \alpha + \alpha^2 : 1]$	1111100110001	<i>d</i>
5	$[\alpha + \alpha^2 + (1 + \alpha + \alpha^2)\varepsilon : (1 + \alpha)\varepsilon : 1]$	0111110001101	<i>e</i>
6	$[1 + \alpha^2 + (1 + \alpha + \alpha^2)\varepsilon : \alpha + \alpha^2\varepsilon : 1]$	1011110100011	<i>f</i>
7	$[\alpha^2\varepsilon : 1 + (1 + \alpha^2)\varepsilon : 1]$	0000011001011	<i>g</i>
8	$[1 + \alpha^2 + (1 + \alpha^2)\varepsilon : 1 + \alpha + \alpha^2 + (1 + \alpha)\varepsilon : 1]$	1011011111101	<i>h</i>
9	$[\alpha + \alpha^2 + \alpha\varepsilon : \alpha + \alpha^2 + \alpha\varepsilon : 1]$	0110100110101	<i>i</i>
10	$[1 + \alpha + \alpha^2 + \alpha\varepsilon : 1 : 1]$	1110101000001	<i>j</i>
11	$[\alpha^2 + \alpha^2\varepsilon : 1 + (\alpha + \alpha^2)\varepsilon : 1]$	0010011000111	<i>k</i>
12	$[1 + \alpha + (\alpha + \alpha^2)\varepsilon : 1 + \alpha + \alpha^2 + \varepsilon : 1]$	1100111111001	<i>l</i>
13	$[\alpha + (1 + \alpha)\varepsilon : \alpha^2 + \alpha\varepsilon : 1]$	0101100010101	<i>m</i>
14	$[\alpha^2\varepsilon : 1 : 0]$	0000100010000	<i>n</i>
15	$[\alpha + (1 + \alpha)\varepsilon : \alpha + \alpha^2 + \varepsilon : 1]$	0101100111001	<i>o</i>
16	$[1 + \alpha + (\alpha + \alpha^2)\varepsilon : \alpha^2 + (1 + \alpha + \alpha^2)\varepsilon : 1]$	1100110011111	<i>p</i>
17	$[\alpha^2 + \alpha^2\varepsilon : 1 + \alpha^2 + \alpha\varepsilon : 1]$	0010011010101	<i>q</i>
18	$[1 + \alpha + \alpha^2 + \alpha\varepsilon : \alpha + \alpha^2 + \alpha\varepsilon : 1]$	1110100110101	<i>r</i>
19	$[\alpha + \alpha^2 + \alpha\varepsilon : 0 : 1]$	0110100000001	<i>s</i>
20	$[1 + \alpha^2 + (1 + \alpha^2)\varepsilon : \alpha + (\alpha + \alpha^2)\varepsilon : 1]$	1011010100111	<i>t</i>
21	$[\alpha^2\varepsilon : 1 + \varepsilon : 1]$	0000011001001	<i>u</i>
22	$[1 + \alpha^2 + (1 + \alpha + \alpha^2)\varepsilon : 1 + \alpha + \alpha^2 + (1 + \alpha)\varepsilon : 1]$	1011111111101	<i>v</i>
23	$[\alpha + \alpha^2 + (1 + \alpha + \alpha^2)\varepsilon : \alpha + \alpha^2 + \alpha^2\varepsilon : 1]$	0111110110011	<i>w</i>
24	$[1 + \alpha\alpha^2 + (1 + \alpha)\varepsilon : 1 + (1 + \alpha)\varepsilon : 1]$	1111101001101	<i>x</i>
25	$[\alpha^2 + \alpha\varepsilon : 1 + (1 + \alpha + \alpha^2)\varepsilon : 1]$	0010101001111	<i>y</i>
26	$[1 + \alpha + \varepsilon : 1 + \alpha + \alpha^2 + \alpha\varepsilon : 1]$	1101001110101	<i>z</i>
27	$[\alpha : \alpha^2 + \alpha\varepsilon : 1]$	0100000010101	<i>space</i>
28	$[0 : 1 : 0]$	0000000010000	,

Table 2.
Points coding.

5.6 Encryption and decryption procedures

- The encryption of our message “for the elliptical curve”, is;

```
01111101100110111110001101010000001010100000
11001001011010000000101111100011010100000010
10110110101001111011011111101011111000110101
00000010101011111000110111001111110011100111
11100101101001101011100110011111101101010011
10110100110101001010101101101000000101010010
10101101100000110010011110100110101101111111
110101111100011010000000010000
```

(113)

- Decryption of this message:

```
01000001101011101000011101111110011000101111
10001101110011111100110110111111010100000110
10101011000101010110100110101111110011000101
00000010101101101010011101000001101011111100
1100010101100010101010110011100111101001101010
110100110101
```

(114)

is: hello to abdelhakim.

6. Conclusion

The results obtained are very important from theoretical points of view because to study an elliptic curve on a finite local ring it suffices to study these curves on finite fields, for the applications of these curves they can be applied in cryptography to reinforce security and we can use them in cryptanalysis to solve the PDL on special curves. This results are very imploring and give applications in different fields such as classical mechanics, number theory, cryptology, information theory ... and we can quote here:

1. The generalization of Hass's theorem, corollary 4.9.
2. The result of the corollary 4.11, then in [24], we have the result of the Proposition 3.12.

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Author details

Abdelhakim Chillali* and Lhoussain El Fadil
Sidi Mohamed Ben Abdellah University, Morocco

*Address all correspondence to: abdelhakim.chillali@usmba.ac.ma

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