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Digit Sums and Infinite Products

Samin Riasat

Abstract

Consider the sequence u_n defined as follows: $u_n = +1$ if the sum of the base b digits of n is even, and $u_n = -1$ otherwise, where we take $b = 2$. Recall that the Woods-Robbins infinite product involves a rational function in n and the sequence u_n . Although several generalizations of the Woods-Robbins product are known in the literature, no other infinite product involving a rational function in n and the sequence u_n was known in closed form until recently. In this chapter we introduce a systematic approach to these products, which may be generalized to other values of b . We illustrate the approach by evaluating a large class of similar infinite products.

Keywords: radix representations, digit sums, Prouhet-Thue-Morse sequence, Woods and Robbins product, closed formulas for infinite products

1. Introduction

Throughout this chapter n will denote a non-negative integer. Let $s_b(n)$ denote the sum of the base b digits of n , and put $u_n = (-1)^{s_2(n)}$. We study infinite products of the form

$$f(b, c) := \prod_{n=1}^{\infty} \left(\frac{n+b}{n+c} \right)^{u_n}. \quad (1)$$

(We show in Section 2 that $f(b, c)$ converges for $b, c \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$).

Plainly $f(b, c) = 1/f(c, b)$ and $f(b, b) = 1$. Up to these relations, it seems that the only known nontrivial value of f is $f(1/2, 1) = \sqrt{2}$, which is the famous Woods-Robbins identity [1, 2]:

$$\prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2} \right)^{u_n} = \frac{1}{\sqrt{2}}. \quad (2)$$

Several infinite products inspired by it were discovered afterwards (see, e.g., [3, 4]). But none of them involve the sequence u_n . Moreover, almost nothing is known (see, e.g., [5, 6]) about the similar product

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{u_n} = f\left(0, \frac{1}{2}\right). \quad (3)$$

Our goal is to study these infinite products in detail. This will allow us to gain a deeper understanding of such products as well as evaluate more products like the Woods-Robbins identity.

The material in this chapter is based on the two papers [7, 8].

2. General properties of the function f

First we establish a general result from [7] on convergence.

Lemma 1.1 Let $R \in \mathbb{C}(X)$ be a rational function such that the values $R(n)$ are defined and nonzero for $n \geq 1$. Then, the infinite product $\prod_n R(n)^{u_n}$ converges if and only if the numerator and the denominator of R have the same degree and same leading coefficient.

Proof. If the infinite product converges, then $R(n)$ must tend to 1 when n tends to infinity. Thus the numerator and the denominator of R have the same degree and the same leading coefficient.

Now suppose that the numerator and the denominator of R have the same leading coefficient and the same degree. Decomposing them in factors of degree 1, it suffices, for proving that the infinite product converges, to show that infinite products of the form

$$\prod_{n=1}^{\infty} \left(\frac{n+b}{n+c} \right)^{u_n} \quad (4)$$

converge for complex numbers b and c such that $n+b$ and $n+c$ do not vanish for any $n \geq 1$. Since the general factor of such a product tends to 1, it is equivalent, grouping the factors pairwise, to proving that the product

$$\prod_{n=1}^{\infty} \left[\left(\frac{2n+b}{2n+c} \right)^{u_{2n}} \left(\frac{2n+1+b}{2n+1+c} \right)^{u_{2n+1}} \right] \quad (5)$$

converges. Since $u_{2n} = u_n$ and $u_{2n+1} = -u_n$, we only need to prove that the infinite product

$$\prod_{n=1}^{\infty} \left(\frac{(2n+b)(2n+1+c)}{(2n+c)(2n+1+b)} \right)^{u_n} \quad (6)$$

converges. Taking (the principal determination of) logarithms, we see that

$$\log \left(\frac{(2n+b)(2n+1+c)}{(2n+c)(2n+1+b)} \right) = O\left(\frac{1}{n^2}\right) \quad (7)$$

which gives the convergence result.

Hence $f(b, c)$ converges for any $b, c \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$. Using the definition of u_n , it follows that for any $b, c, d \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$,

$$1. f(b, b) = 1.$$

$$2. f(b, c)f(c, d) = f(b, d).$$

$$3. f(b, c) = \frac{c+1}{b+1} f\left(\frac{b}{2}, \frac{c}{2}\right) f\left(\frac{c+1}{2}, \frac{b+1}{2}\right).$$

One can ask the natural question: is f the unique function satisfying these properties?

2.1 A new function

Properties 1 and 2 above give

$$f(b, c)f(d, e) = \frac{f(b, c)f(c, d)f(d, e)f(d, c)}{f(c, d)f(d, c)} = \frac{f(b, e)f(d, c)}{f(c, c)} = f(b, e)f(d, c). \quad (8)$$

Hence we can rewrite property 3 as

$$f(b, c) = \frac{f\left(\frac{b}{2}, \frac{b+1}{2}\right)}{b+1} / \frac{f\left(\frac{c}{2}, \frac{c+1}{2}\right)}{c+1}. \quad (9)$$

Thus $f(b, c)$ can be computed using only the quantities $h(x) = f\left(\frac{x}{2}, \frac{x+1}{2}\right)$, via

$$f(b, c) = \frac{c+1}{b+1} \cdot \frac{h(b)}{h(c)}. \quad (10)$$

So understanding f is equivalent to understanding h , in the sense that each can be completely evaluated in terms of the other.

Taking $c = b + \frac{1}{2}$ in Eq. (10) gives the functional equation:

$$h(b) = \frac{b+1}{b+\frac{3}{2}} h\left(b + \frac{1}{2}\right) h(2b). \quad (11)$$

Similar questions can be asked for h : is it the unique solution to Eq. (11)? What about monotonic/continuous/smooth solutions?

3. Infinite products

3.1 Direct approach

Theorem 1.1 The following relations hold.

1. For $b, c \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$,

$$\prod_{n=1}^{\infty} \left(\frac{(n+b)(n+\frac{b+1}{2})(n+\frac{c}{2})}{(n+c)(n+\frac{c+1}{2})(n+\frac{b}{2})} \right)^{u_n} = \frac{c+1}{b+1}. \quad (12)$$

2. For $b, c \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$,

$$\prod_{n=0}^{\infty} \left(\frac{(n+b)(n+\frac{b+1}{2})(n+\frac{c}{2})}{(n+c)(n+\frac{c+1}{2})(n+\frac{b}{2})} \right)^{u_n} = 1. \quad (13)$$

3. For $b \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$,

$$\prod_{n=1}^{\infty} \left(\frac{(n+b)(n+\frac{b+1}{4})}{(n+\frac{b+3}{4})(n+\frac{b}{2})} \right)^{u_n} = \frac{b+3}{2(b+1)}. \quad (14)$$

4. For $c \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$,

$$\prod_{n=1}^{\infty} \left(\frac{(n+\frac{1}{2})(n+\frac{c}{2})}{(n+c)(n+\frac{c+1}{2})} \right)^{u_n} = c+1. \quad (15)$$

Proof.

1. This follows immediately using properties 1–3 in Section 2.

2. As above.

3. Take $c = (b + 1)/2$ in Eq. (12).

4. Take $b = 0$ in Eq. (12).

Corollary 1.1 For any positive rational number q , there exist monic polynomials $P, Q \in \frac{1}{4}\mathbb{Z}[X]$, both at most cubic, such that

$$\prod_{n=1}^{\infty} \left(\frac{P(n)}{Q(n)} \right)^{u_n} = q. \quad (16)$$

Furthermore, if q is an integer, then P and Q can be chosen to be at most quadratic.

We still do not know exactly which numbers are given by such infinite products.

3.2 Functional equation approach

Recall the functional Eq. (11):

$$h(b) = \frac{b+1}{b+\frac{3}{2}} h\left(b + \frac{1}{2}\right) h(2b). \quad (17)$$

Taking $b = 0$ in Eq. (11) gives

$$h(0) = \frac{2}{3} h\left(\frac{1}{2}\right) h(0), \quad (18)$$

i.e., $h(1/2) = 3/2$. This shows that

$$\prod_{n=0}^{\infty} \left(\frac{4n+3}{4n+1} \right)^{u_n} = 2. \quad (19)$$

Next, taking $b = 1/2$ in Eq. (11) gives

$$h\left(\frac{1}{2}\right) = \frac{3}{4} h(1)^2; \quad (20)$$

hence $h(1) = \sqrt{2}$, and we recover the Woods-Robbins identity

$$\prod_{n=0}^{\infty} \left(\frac{2n+2}{2n+1} \right)^{u_n} = \sqrt{2}. \quad (21)$$

Similarly, taking $b = -1/2$ in Eq. (11) gives

$$h\left(-\frac{1}{2}\right) = \frac{1}{2} h(0) h(-1) = \frac{1}{2} f\left(0, \frac{1}{2}\right) f\left(-\frac{1}{2}, 0\right) = \frac{1}{2} f\left(-\frac{1}{2}, \frac{1}{2}\right), \quad (22)$$

i.e.,

$$\prod_{n=1}^{\infty} \left(\frac{(4n-1)(2n+1)}{(4n+1)(2n-1)} \right)^{u_n} = \frac{1}{2}. \quad (23)$$

Taking $b = 1$ in Eq. (11) gives

$$h(1) = \frac{4}{5}h\left(\frac{3}{2}\right)h(2); \quad (24)$$

hence $h(3/2)h(2) = 5\sqrt{2}/4$, and this gives

$$\prod_{n=0}^{\infty} \left(\frac{(4n+3)(2n+2)}{(4n+5)(2n+3)} \right)^{u_n} = \frac{1}{\sqrt{2}}. \quad (25)$$

Taking $b = 3/2$ in Eq. (11) and using the previous result gives

$$h(2)^2h(3) = \frac{3}{\sqrt{2}} \quad (26)$$

which is equivalent to

$$\prod_{n=0}^{\infty} \left(\frac{(2n+2)(n+1)}{(2n+3)(n+2)} \right)^{u_n} = \frac{1}{\sqrt{2}}. \quad (27)$$

These identities can be also combined in pairs to obtain other identities.

4. Some analytical results

We saw in the previous section that some of the infinite products we evaluated were integers, some were rational, and some were quadratic irrational. In the hope of further understanding their nature, we now study the analytical behaviors of f and h .

Lemma 1.2 Let $b, c \in (-1, \infty)$.

1. If $b = c$, then $f(b, c) = 1$.

2. If $b > c$, then

$$\left(\frac{c+1}{b+1} \right)^2 < f(b, c) < 1. \quad (28)$$

3. If $b < c$, then

$$1 < f(b, c) < \left(\frac{c+1}{b+1} \right)^2. \quad (29)$$

Proof. Using properties 1–3 from Section 2, it suffices to prove 2.

Let $b > c > -1$, and put

$$a_n = \log \left(\frac{n+b}{n+c} \right), \quad S_N = \sum_{n=1}^N a_n u_n, \quad U_N = \sum_{n=1}^N u_n. \quad (30)$$

Note that a_n is positive and strictly decreasing to 0. Using $s_2(2n+1) = 1 + s_2(2n)$, it follows that $U_n \in \{-2, -1, 0\}$ and $U_n \equiv n \pmod{2}$, for each n . Using summation by parts,

$$S_N = a_{N+1}U_N + \sum_{n=1}^N U_n(a_n - a_{n+1}). \quad (31)$$

So $-2a_1 < S_N < 0$ for $N \geq 2$. Exponentiating and taking $N \rightarrow \infty$ gives the desired result.

Lemma 1.2 together with Eq. (10) implies the following results.

Theorem 1.2 $h(x)/(x+1)$ is strictly decreasing on $(-1, \infty)$, and $h(x)(x+1)$ is strictly increasing on $(-1, \infty)$.

Proof. Let $-1 < b < c$. By Eqs. (29) and (10),

$$1 < \frac{c+1}{b+1} \cdot \frac{h(b)}{h(c)} < \left(\frac{c+1}{b+1} \right)^2 \quad (32)$$

from which the result follows.

Theorem 1.3 For $b, c \in (-1, \infty)$, $f(b, c)$ is strictly decreasing in b and strictly increasing in c .

Proof. By Eq. (10),

$$\frac{h(c)f(b, c)}{c+1} = \frac{h(b)}{b+1} \quad (33)$$

and

$$\frac{(b+1)f(b, c)}{h(b)} = \frac{c+1}{h(c)} \quad (34)$$

hence the result follows from Theorem 1.2.

Theorem 1.4 For $x \in (-2, \infty)$,

$$1 < h(x) < \left(\frac{x+3}{x+2} \right)^2. \quad (35)$$

Proof. This follows from taking $b = x/2$ and $c = (x+1)/2$ in Eq. (10), then using Eq. (29).

We now prove some results on differentiability.

Theorem 1.5 $h(x)$ is smooth on $(-2, \infty)$.

Proof. Take $b = x/2$ and $c = (x+1)/2$ in Eq. (30). Then the sequence S_n of smooth functions on $(-2, \infty)$ converges pointwise to $\log h$.

Differentiating with respect to x gives

$$S'_N = \sum_{n=1}^N \frac{u_n}{(2n+x)(2n+1+x)} = \sum_{n=1}^N u_n \left(\frac{1}{2n+x} - \frac{1}{2n+1+x} \right). \quad (36)$$

Hence

$$\begin{aligned} |S'_N - S'_M| &\leq \sum_{n=M+1}^N \left(\frac{1}{2n+x} - \frac{1}{2n+1+x} \right) \\ &\leq \sum_{n=M+1}^N \left(\frac{1}{2n-1+x} - \frac{1}{2n+1+x} \right) \\ &= \frac{1}{2M+1+x} - \frac{1}{2N+1+x} \\ &< \frac{1}{2M-1} \rightarrow 0 \end{aligned} \quad (37)$$

as $M \rightarrow \infty$, for any $x \in (-2, \infty)$ and $N > M$. Thus S'_n converges uniformly on $(-2, \infty)$, which shows that $\log h$, hence h , is differentiable on $(-2, \infty)$.

Now suppose that derivatives of h up to order k exist for some $k \geq 1$. Note that

$$S_N^{(k+1)} = (-1)^k k! \sum_{n=1}^N u_n \left(\frac{1}{(2n+x)^{k+1}} - \frac{1}{(2n+1+x)^{k+1}} \right). \quad (38)$$

As before,

$$\begin{aligned} \left| S_N^{(k+1)} - S_M^{(k+1)} \right| &\leq k! \sum_{n=M+1}^N \left(\frac{1}{(2n+x)^{k+1}} - \frac{1}{(2n+1+x)^{k+1}} \right) \\ &\leq k! \sum_{n=M+1}^N \left(\frac{1}{(2n-1+x)^{k+1}} - \frac{1}{(2n+1+x)^{k+1}} \right) \\ &= \frac{k!}{(2M+1+x)^{k+1}} - \frac{k!}{(2N+1+x)^{k+1}} \\ &< \frac{k!}{(2M-1)^{k+1}} \rightarrow 0 \end{aligned} \quad (39)$$

as $M \rightarrow \infty$, for any $x \in (-2, \infty)$ and $N > M$. Hence $S_n^{(k+1)}$ converges uniformly on $(-2, \infty)$, i.e., $h^{(k)}$ is differentiable on $(-2, \infty)$.

Therefore, by induction, h has derivatives of all orders on $(-2, \infty)$.

Theorem 1.6 Let $a \geq 0$. Then

$$\log h(x) = \log h(a) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{n=2}^{\infty} \frac{u_n}{(n+a)^k} \right) (x-a)^k \quad (40)$$

for $x \in [a-1, a+1]$.

Proof. Let $H(x) = \log h(x)$. By Theorem 1.5,

$$H^{(k+1)}(x) = (-1)^k k! \sum_{n=2}^{\infty} \frac{u_n}{(n+x)^{k+1}}. \quad (41)$$

Hence

$$|H^{(k+1)}(x)| \leq k! \sum_{n=2}^{\infty} \frac{1}{|n+x|^{k+1}} \leq k! \sum_{n=2}^{\infty} \frac{1}{(n+a-1)^{k+1}} \quad (42)$$

for $x \in [a-1, a+1]$. So by Taylor's inequality, the remainder for the Taylor polynomial for $H(x)$ of degree k is absolutely bounded above by

$$\frac{1}{k+1} \left(\sum_{n=2}^{\infty} \frac{1}{(n+a-1)^{k+1}} \right) |x-a|^{k+1} \quad (43)$$

which tends to 0 as $k \rightarrow \infty$, since $a \geq 0$ and $|x-a| \leq 1$. Therefore $H(x)$ equals its Taylor expansion about a for x in the given range.

5. Further remarks on $h(0)$

As mentioned in Section 1, not much is known about the quantity $h(0) \approx 1.62816$. We give the following explanation as to why $h(0)$ might behave specially in a sense.

Note that the only way nontrivial cancelation occurs in Eq. (11) is when $b = 0$. Likewise, nontrivial cancelation occurs in Eq. (10) or property 3 in Section 2 only for $(b, c) = (0, 1/2)$ and $(1/2, 0)$. That is, the victim of any such cancelation is always $h(0)$ or $h(0)^{-1}$. So we must look for other ways to study $h(0)$.

Using the two known values $h(1/2) = 3/2$ and $h(1) = \sqrt{2}$, the following expressions for $h(0)$ are obtained from Theorem 1.6 by choosing various values for x and a .

- $x = 0$ and $a = 1$:

$$h(0) = \sqrt{2} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \frac{u_n}{(n+1)^k} \right) \quad (44)$$

- $x = 1$ and $a = 0$:

$$h(0) = \sqrt{2} \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{n=2}^{\infty} \frac{u_n}{n^k} \right) \quad (45)$$

- $x = 0$ and $a = 1/2$:

$$h(0) = \frac{3}{2} \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \frac{u_{2n+1}}{(2n+1)^k} \right) \quad (46)$$

- $x = 1/2$ and $a = 0$:

$$h(0) = \frac{3}{2} \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{n=2}^{\infty} \frac{u_{2n}}{(2n)^k} \right) \quad (47)$$

The Dirichlet series appearing in the above expressions were studied in [9]. We think that these identities and the results from Section 4 might help in shedding some light on the nature of $h(0)$.

6. Conclusions and future developments

We evaluated infinite products involving the digit sum function $s_b(n)$ by splitting the product based on the congruence classes modulo b . We illustrated two approaches for doing so, one by direct computation and another using functional equations. For $b = 2$ we proved some analytical results to aid us in understanding the behavior of these products. Many open questions still remain.

Although we only considered the base $b = 2$, many of the results above easily generalize to other bases. One possible direction toward a generalization is to take $u_n = (-1)^{s_b(n)}$. Another is $u_n = \omega_b^{s_b(n)}$, where ω_b is a primitive b -th root of unity. We leave these as work to be done in the future.

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References

- [1] Robbins D. Solution to problem E 2692. *American Mathematical Monthly*. 1979;**86**:394-395
- [2] Woods DR. Elementary problem proposal E 2692. *American Mathematical Monthly*. 1978;**85**:48
- [3] Allouche J-P, Shallit J. Infinite products associated with counting blocks in binary strings. *Journal of the London Mathematical Society*. 1989;**39**: 193-204
- [4] Allouche J-P, Sondow J. Infinite products with strongly B -multiplicative exponents. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Biologica/Errata: Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Biologica*. 2008/2010;**28/32**:35-53/253
- [5] Allouche J-P. Thue, combinatorics on words, and conjectures inspired by the Thue-Morse sequence. *Journal de Théorie des Nombres de Bordeaux*. 2015;**27**(2):375-388
- [6] Allouche J-P, Shallit J. The ubiquitous Prouhet-Thue-Morse sequence. *Sequences and their Applications*. In: Ding C, Helleseht T, Niederreiter H, editors. *Proceedings of SETA'98*. Springer Verlag. 1999; 1-16
- [7] Allouche J-P, Riasat S, Shallit J. More infinite products: Thue-Morse and the gamma function. *The Ramanujan Journal*. 2019;**49**:115-128. DOI: 10.1007/s11139-017-9981-7
- [8] Riasat S. Infinite products involving binary digit sums. In: Kilgour D, Kunze H, Makarov R, Melnik R, Wang X, editors. *Recent Advances in Mathematical and Statistical Methods. AMMCS 2017*. Springer Proceedings in Mathematics & Statistics. Vol. 259. Cham: Springer; 2017
- [9] Allouche J-P, Cohen H. Dirichlet series and curious infinite products. *The Bulletin of the London Mathematical Society*. 1985;**17**:531-538