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Identification of Eigen-Frequencies and Mode-Shapes of Beams with Continuous Distribution of Mass and Elasticity and for Various Conditions at Supports

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Abstract

In the present article, an equivalent three degrees of freedom (DoF) system of two different cases of inverted pendulums is presented for each separated case. The first case of inverted pendulum refers to an amphi-hinge pendulum that possesses distributed mass and stiffness along its height, while the second case of inverted pendulum refers to an inverted pendulum with distributed mass and stiffness along its height. These vertical pendulums have infinity number of degree of freedoms. Based on the free vibration of the above-mentioned pendulums according to partial differential equation, a mathematically equivalent three-degree of freedom system is given for each case, where its equivalent mass matrix is analytically formulated with reference on specific mass locations along the pendulum height. Using the three DoF model, the first three fundamental frequencies of the real pendulum can be identified with very good accuracy. Furthermore, taking account the 3×3 mass matrix, it is possible to estimate the possible pendulum damages using a known technique of identification mode-shapes via records of response accelerations. Moreover, the way of instrumentation with a local network by three accelerometers is given via the above-mentioned three degrees of freedom.

Keywords: equivalent masses of continuous amphi-hinge vertical pendulum, identification of mode-shapes, distributed mass and stiffness, continuous systems, modal analysis of the continuous beam, inverted pendulum

1. Introduction

An ideal three degrees of freedom system that is equivalent with the modal behavior of an infinity number of degree of freedom of two cases of pendulums is proposed for each case. The first pendulum has hinges at the two ends, while the second pendulum is a cantilever (inverted pendulum). Both pendulums are presented analytically in the present article. This equivalent three DoF system can

be used in instrumentation of such pendulums, where the concept of the concentrated masses is not existing, with a local network of three accelerometers. This issue is a main problem that appears very common during the instrumentation of inverted pendulums or bridge beams or steel stairs [1–3] or wind energy powers [4, 5] in order to identify the real vibration mode-shapes and the fundamental eigen-frequencies of the structure via records of response accelerograms at specific positions due to ambient excitation [6, 7].

2. First case: modal analysis of undamped amphi-hinge vertical pendulum with distributed mass and stiffness

According to the theory of continuous systems [8, 9], consider a straight amphi-hinge vertical pendulum that is loaded by an external continuous dynamic loading $p_z(x, t)$, with reference to a Cartesian three-dimensional reference system $oxyz$, (**Figure 1**).

The vertical pendulum possesses a distributed mass $m(x)$ per unit height, which in the special case of uniform distribution is given as $m(x) = \bar{m}$ in tons per meter (tn/m). Furthermore, according to Bernoulli Technical Bending Theory, the beam has section flexural stiffness $EI_y(x)$, where in the special case of an uniform distribution of the stiffness it is given as $EI_y(x) = EI_y$, where E is the material modulus of elasticity and I_y is the section moment of inertia about y-axis (**Figure 1**). Next, we are examining a such amphi-hinge vertical pendulum that possesses constant value of distributed mass along its height, as well as constant value of distributed section flexural stiffness EI_y . Due to fact that the vertical pendulum mass is continuously distributed, this pendulum/beam has infinity number of degrees of freedom for vibration along the horizontal oz -axis. In order to formulate of the motion equation

Figure 1.
Amphi-hinge vertical pendulum with distributed mass and section flexural stiffness.

of this beam, we consider an infinitesimal part of the vertical beam, at location x from the origin o , that has isolated by two very nearest parallel sections. The infinitesimal length of this part is the dx . On this infinitesimal length, we notice the flexural moment $M(x, t)$, the shear force $Q(x, t)$ with their differential increments, while the axial force $N(x, t)$ is ignored, because it does not affect the horizontal beam vibration along z -axis. Moreover, noted the resulting force $P_z(x, t)$ of the external dynamic loading. Therefore, we can write:

$$P_z(x, t) = p_z(x, t) \cdot dx \quad (1)$$

where the resulting force $P_z(x, t)$ acts at the total beam infinitesimal part. Furthermore, according to D'Alembert principle, the resulting inertia force $F_a(x, t)$ is noted, where:

$$F_a(x, t) = (-\bar{m} \cdot dx) \cdot \partial^2 u_z(x, t) / \partial t^2 \Rightarrow F_a(x, t) = (-\bar{m} \cdot dx) \cdot \ddot{u}_z(x, t) \quad (2)$$

Here, we agree that the time derivatives of the displacements are going to symbolize with full stops, while the spatial derivatives of the displacements are going to symbolize with accent. Next, the damping and the second order differential are ignored, so the force equilibrium on the infinitesimal part of the beam along z -axis gives:

$$\begin{aligned} \sum F_z = 0 \Rightarrow Q + P_z(x, t) - \left(Q + \frac{\partial Q}{\partial x} dx \right) + F_a(x, t) = 0 \Rightarrow \\ \frac{\partial Q}{\partial x} = p_z(x, t) - \bar{m} \cdot \ddot{u}_z(x, t) \end{aligned} \quad (3)$$

Moreover, the moment equilibrium with reference to weight center (w.c.) of the infinitesimal part of the beam (see **Figure 1**) gives:

$$\begin{aligned} \sum M_y = 0 \Rightarrow M + Q \cdot \frac{dx}{2} + \left(Q + \frac{\partial Q}{\partial x} dx \right) \cdot \frac{dx}{2} - \left(M + \frac{\partial M}{\partial x} dx \right) = 0 \Rightarrow \\ Q = \frac{\partial M}{\partial x} \end{aligned} \quad (4)$$

According to Euler-Bernoulli Bending theory (where the shear deformations are ignored), it is well-known that the following basic equation is true:

$$M(x, t) = EI_y \cdot \frac{\partial^2 u_z(x, t)}{\partial x^2} \quad (5)$$

Eqs. (4) and (5) are inserted into Eq. (3), so the motion equation without damping for the examined vertical beam is given:

$$\begin{aligned} \frac{\partial^2 M}{\partial x^2} = p_z(x, t) - \bar{m} \cdot \ddot{u}_z(x, t) \Rightarrow \\ \frac{\partial^2}{\partial x^2} \left(EI_y \cdot \frac{\partial^2 u_z(x, t)}{\partial x^2} \right) = p_z(x, t) - \bar{m} \cdot \ddot{u}_z(x, t) \Rightarrow \\ \bar{m} \frac{\partial^2 u_z(x, t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x, t)}{\partial x^4} = p_z(x, t) \Rightarrow \\ \bar{m} \cdot \ddot{u}_z(x, t) + EI_y \cdot u_z''''(x, t) = p_z(x, t) \end{aligned} \quad (6)$$

Eq. (6) is a partial differential equation that describes the motion $u_z(x, t)$ of the vertical beam that is loaded with the external dynamic loading $p_z(x, t)$. In order to arise a unique solution from Eq. (6), the support conditions must be used at the two beam ends. It is worthy to note that the classical case of a beam with distributed mass and section flexural stiffness, under external horizontal excitation (**Figure 2**) on the two supports is mathematically equivalent with the vibration that is described by Eq. (6). Indeed, in the case of **Figure 2**, the total displacement $u_z^{\text{tot}}(x, t)$ of the beam at x -location is given:

$$u_z^{\text{tot}}(x, t) = u_g(t) + u_z(x, t) \quad (7)$$

where $u_g(t)$ is the displacement at the base, same for the two supports.

But, it is known that the inertia forces of the beam are depended by the total displacement $u_z^{\text{tot}}(x, t)$, while the distributed dynamic loading is null, $p_z(x, t) = 0$. Thus, Eq. (3) is transformed into:

$$\begin{aligned} \frac{\partial Q}{\partial x} &= p_z(x, t) - \bar{m} \cdot \frac{\partial^2 u_z^{\text{tot}}(x, t)}{\partial t^2} \Rightarrow \\ \frac{\partial Q}{\partial x} &= 0 - \bar{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} - \bar{m} \cdot \frac{\partial^2 u_z(x, t)}{\partial t^2} \end{aligned} \quad (8)$$

Figure 2.
Amphi-hinge vertical pendulum subjected with the same horizontal ground motion $u_g(t)$ on the two end-supports.

Following, Eqs. (4) and (5) are inserting into Eq. (8), thus we are taken:

$$\begin{aligned} \frac{\partial^2 M}{\partial x^2} &= -\bar{m} \cdot \left(\frac{\partial^2 u_g(t)}{\partial t^2} + \frac{\partial^2 u_z(x, t)}{\partial t^2} \right) \Rightarrow \\ \bar{m} \frac{\partial^2 u_z(x, t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x, t)}{\partial x^4} &= -\bar{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} \end{aligned} \quad (9)$$

By the comparison of Eqs. (6) and (9), we notice that the undamped beam vibration due to horizontal motion of the two supports is mathematically equivalent with the undamped vibration of the same beam where the two supports are fixed and the beam is loaded with the equivalent distributed dynamic loading $p_{eq}(x, t)$:

$$p_{eq}(x, t) = -\bar{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} \quad (10)$$

In the case of the horizontal pendulum/beam free vibration without damping, we consider the first part of Eq. (9) that must be null:

$$\bar{m} \frac{\partial^2 u_z(x, t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x, t)}{\partial x^4} = 0 \quad (11)$$

Furthermore, we ask the unknown spatial time-function $u_z(x, t)$, which is the solution of Eq. (11), must have the form of separated variants:

$$u_z(x, t) = \varphi(x) \cdot q(t) \quad (12)$$

where $\varphi(x)$ is an unknown spatial function and $q(t)$ is an unknown time-function. Eq. (12) has been derived two times with reference to time-dimension t and more two times with reference to spatial-dimension x , so:

$$\frac{\partial^2 u_z(x, t)}{\partial t^2} = \varphi(x) \cdot \ddot{q}(t), \quad \frac{\partial^2 u_z(x, t)}{\partial x^2} = \varphi''(x) \cdot q(t) \quad (13)$$

Eqs. (13) are inserted into Eq. (11), giving:

$$\bar{m} \cdot \varphi(x) \cdot \ddot{q}(t) + EI_y \cdot \varphi''''(x) \cdot q(t) = 0 \quad (14)$$

and, next, divided with the number $\bar{m} \cdot \varphi(x) \cdot q(t)$, thus we are getting:

$$\frac{-\ddot{q}(t)}{q(t)} = \frac{EI_y \cdot \varphi''''(x)}{\bar{m} \cdot \varphi(x)} \quad (15)$$

The left part of Eq. (15) is a time-function, but the right part is a spatial-function. In order to true Eq. (15) for all time values as well as for all spatial positions, the two parts of Eq. (15) must be equal with a constant λ . Thus, Eq. (15) is separated at two following differential equations:

$$\frac{-\ddot{q}(t)}{q(t)} = \lambda \Rightarrow \ddot{q}(t) + \lambda \cdot q(t) = 0 \quad (16)$$

$$\frac{EI_y \cdot \varphi''''(x)}{\bar{m} \cdot \varphi(x)} = \lambda \Rightarrow EI_y \cdot \varphi''''(x) - \lambda \cdot \bar{m} \cdot \varphi(x) = 0 \quad (17)$$

However, the time equation (16) indicates a free vibration of an ideal single degree of freedom system that has eigen-frequency $= \sqrt{\lambda}$. Inserting the eigen-frequency ω into Eq. (17) gives:

$$EI_y \cdot \varphi''''(x) - \omega^2 \cdot \overline{m} \cdot \varphi(x) = 0 \Rightarrow \varphi''''(x) - \frac{\omega^2 \cdot \overline{m}}{EI_y} \cdot \varphi(x) = 0 \quad (18)$$

Next, we set the positive parameter β such as to be equal:

$$\beta^4 = \frac{\omega^2 \cdot \overline{m}}{EI_y} \quad (19)$$

because the parameters $\omega^2, \overline{m}, EI_y$ are always positive. By the mathematic theory it is known that the general solution of Eq. (18) has the following form:

$$\varphi(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \quad (20)$$

where the four unknown parameters C_1, C_2, C_3, C_4 must be calculated. In order to achieve this, four support conditions of the beam have to be used. Indeed, for $x = 0$ and $x = L$ the displacement $u_z(0, t)$ of the amphi-hinge vertical pendulum as well as the flexural moment $M(0, t)$, both are equal zero. The spatial function $\varphi(x)$, which is the solution of Eq. (20), gives the modal elastic line of the beam. Having as known data that the following equation is true:

$$\sinh \beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}, \cosh \beta x = \frac{e^{\beta x} + e^{-\beta x}}{2} \quad (21)$$

The spatial function of the modal elastic line for $x = 0$ is:

$$\begin{aligned} \varphi(0) &= C_1 \sin 0 + C_2 \cos 0 + C_3 \sinh 0 + C_4 \cosh 0 = 0 \Rightarrow \\ C_2 + C_4 &= 0 \end{aligned} \quad (22)$$

and also for $x = 0$, the function of the flexural moment due to examined modal elastic line of the beam is given by Eq. (5):

$$M(0, t) = EI_y \cdot \frac{\partial^2 \varphi(0)}{\partial x^2} = 0 \Rightarrow EI_y \cdot \varphi''(0) = 0 \quad (23)$$

Eq. (20) has been derived two times with reference to spatial-dimension x , thus arise:

$$\varphi'(x) = C_1 \cdot \beta \cdot \cos \beta x + C_2 \cdot (-\beta) \cdot \sin \beta x + C_3 \cdot \beta \cdot \cosh \beta x + C_4 \cdot \beta \cdot \sinh \beta x \quad (24)$$

and

$$\varphi''(x) = C_1(-\beta^2) \cdot \sin \beta x + C_2(-\beta^2) \cdot \cos \beta x + C_3\beta^2 \cdot \sinh \beta x + C_4\beta^2 \cdot \cosh \beta x \quad (25)$$

Therefore, Eq. (23) is transformed:

$$EI_y \cdot \beta^2 \cdot (C_4 - C_2) = 0 \quad (26)$$

By Eqs. (22) and (26) directly arise $C_2 = 0$ and $C_4 = 0$, thus the general solution of Eq. (20) is the following:

$$\varphi(x) = C_1 \sin \beta x + C_3 \sinh \beta x \quad (27)$$

In addition, the parameters C_1, C_3 are calculated by the support conditions of the second support of the beam. Therefore, for $x = L$ the vertical displacement $u(L, t) = 0$ be true. Thus, from Eq. (12) arises that $\varphi(L) = 0$ and Eq. (27) gives:

$$\varphi(L) = C_1 \sin \beta L + C_3 \sinh \beta L = 0 \quad (28)$$

In continuous, Eq. (23) gives:

$$M(L, t) = EI_y \cdot \frac{\partial^2 \varphi(L)}{\partial x^2} = 0 \Rightarrow EI_y \cdot \varphi''(L) = 0 \quad (29)$$

where $\varphi''(L)$ is directly getting from Eq. (25) that is equivalent with zero:

$$\varphi''(L) = C_1(-\beta^2) \cdot \sin \beta L + C_3 \beta^2 \cdot \sinh \beta L = 0 \quad (30)$$

However, when Eqs. (28) and (30) are re-written again, we get:

$$C_1 \cdot \sin \beta L + C_3 \cdot \sinh \beta L = 0 \quad (31)$$

$$-C_1 \cdot \sin \beta L + C_3 \cdot \sinh \beta L = 0 \quad (32)$$

And added part to part these two above-mentioned equations arise:

$$2 \cdot C_3 \cdot \sinh \beta L = 0 \quad (33)$$

But, the term $\sinh \beta L$ is not equal with zero, because then vibration is not existing. Therefore, C_3 has to equal with zero, so Eq. (28) is formed:

$$\varphi(L) = C_1 \cdot \sin \beta L = 0 \quad (34)$$

Moreover, by Eq. (34) arise that either $C_1 = 0$ that is impossibility because $\varphi(x) \neq 0$ by Eq. (27), either $\sin \beta L = 0$ that means the following equation must be true:

$$\beta L = n \cdot \pi \quad n = 1, 2, 3, \dots \quad (35)$$

However, Eq. (35) is transformed to Eq. (36):

$$\beta L = n \cdot \pi \Rightarrow \beta^2 L^2 = n^2 \cdot \pi^2 \Rightarrow \beta^2 = \frac{n^2 \cdot \pi^2}{L^2} \quad (36)$$

By the definition of parameter β , we can calculate the eigen-frequency ω :

$$\beta^4 = \frac{\omega^2 \cdot \overline{m}}{EI_y} \Rightarrow \omega^2 = \frac{EI_y \cdot \beta^4}{\overline{m}} \Rightarrow \omega = \beta^2 \cdot \sqrt{\frac{EI_y}{\overline{m}}} \quad (37)$$

Thus, inserting Eq. (36) into Eq. (37), the eigen-frequency ω_n directly arises for each n -value.

$$\omega_n = \frac{n^2 \cdot \pi^2}{L^2} \cdot \sqrt{\frac{EI_y}{\overline{m}}} \quad n = 1, 2, 3, \dots \quad (38)$$

Figure 3.
Modal analysis of continuous amphi-hinge vertical pendulum. The first four mode-shapes.

Therefore, the vibration mode-shape of the examined vertical pendulum arises by Eq. (27)—since previous inserting Eq. (35)—thus:

$$\varphi_n(x) = C_1 \sin \beta x = C_1 \sin \frac{n \cdot \pi \cdot x}{L} \quad n = 1, 2, 3, \dots \quad (39)$$

The value of C_1 is arbitrary, and we usually get it equal to unit. Thus, for each value of parameter n , a mode-shape with its eigen-frequency is resulted. The fundamental (first) mode-shape is resulted for $n = 1$, which shows a half sinusoidal wave, the second mode-shape shows a full sinusoidal wave, etc. (**Figure 3**). The order of the eigen-frequencies is $\omega_1, \omega_2 = 4\omega_1, \omega_3 = 9\omega_1, \omega_4 = 16\omega_1$, etc.

3. The equivalent three degrees of freedom system for amphi-hinges pendulum

At vertical pendulums or beams where the fundamental horizontal mode-shape does not activate the 90% of the total beam mass, we ask to consider the three first mode-shapes. Thus, for this purpose, we must define an ideal equivalent three degrees of freedom beam, which is going to give the first three frequencies and the first three mode-shapes of the examined beam. Therefore, which is the ideal three degrees of freedom system, where its three mode-shapes coincide with the real first three frequencies of the vertical pendulum with distributed mass and flexural stiffness?

In order to answer the above-mentioned question, consider a weightless vertical pendulum with height L and constant section in elevation, where carry three concentrated masses that each one has the same mass-value m_{eq} , located per distance $0.25L$, between one to one, and each one mass possesses an horizontal degree of freedom (**Figure 4**).

Figure 4.
 The equivalent three-degree of freedom system on amphi-hinge vertical pendulum.

The pendulum displacement vector $\mathbf{u}$ of the three degrees of freedom, as well as the diagonal beam mass matrix $\mathbf{m}$ is written:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \mathbf{m} = \begin{bmatrix} m_{eq} & 0 & 0 \\ 0 & m_{eq} & 0 \\ 0 & 0 & m_{eq} \end{bmatrix} \quad (40)$$

Furthermore, the pendulum flexibility matrix $\mathbf{f}$ can be calculated using a suitable method (**Figure 4**), and the inverse matrix gives the stiffness matrix $\mathbf{k}$ of the three degrees of freedom beam.

$$\mathbf{f} = \begin{bmatrix} D_{1,1} & D_{1,2} & D_{1,3} \\ D_{2,1} & D_{2,2} & D_{2,3} \\ D_{3,1} & D_{3,2} & D_{3,3} \end{bmatrix} = \frac{L^3}{48EI_y} \cdot \begin{bmatrix} 1 & 0.6875 & 0.6875 \\ 0.6875 & 0.5625 & 0.4375 \\ 0.6875 & 0.4375 & 0.5625 \end{bmatrix} \quad (41)$$

$$\mathbf{k} = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \\ k_{3,1} & k_{3,2} & k_{3,3} \end{bmatrix} = \frac{48EI_y}{L^3} \cdot \begin{bmatrix} 18.285714 & -12.571429 & -12.571429 \\ -12.571429 & 13.142857 & 5.142857 \\ -12.571429 & 5.142857 & 13.142857 \end{bmatrix} \quad (42)$$

The equations of motion for the case of the free undamped vibration of the ideal beam is given as:

$$\mathbf{m} \ddot{\mathbf{u}}(t) + \mathbf{k} \mathbf{u}(t) = \mathbf{0} \quad (43)$$

The eigen-problem is written as:

$$(\mathbf{k} - \omega_n^2 \mathbf{m}) \boldsymbol{\varphi}_n = \mathbf{0} \quad n = 1, 2, 3. \quad (44)$$

where the eigen-frequencies ω_n and the three mode-shapes $\boldsymbol{\varphi}_n$ are known by Eq. (38) and (39) and **Figure 4**. Therefore, the unique unknown parameter is the mass m_{eq} . Thus,

$$\det(\mathbf{k} - \omega_1^2 \mathbf{m}) = 0 \Rightarrow \quad (45)$$

$$m_{eq}^3 + A \cdot m_{eq}^2 + B \cdot m_{eq} + C = 0 \quad (46)$$

where

$$A = -\frac{k_{11} + k_{22} + k_{33}}{\omega_1^2}, B = \frac{k_{11}k_{33} + k_{11}k_{22} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2}{\omega_1^4}$$

$$C = -\frac{k_{11}k_{22}k_{33} + 2k_{12}k_{13}k_{23} - k_{11}k_{23}^2 - k_{22}k_{13}^2 - k_{33}k_{12}^2}{\omega_1^6}$$

The numerical solution of Eq. (46) gives three roots for parameter m_{eq} , where only the first root is acceptable, because the other two values rejected since do not have natural meaning (appear values greater from the total pendulum mass $\bar{m}L$). Thus, the only one acceptable root is given as:

$$m_{eq} = 0.24984748 \cdot (\bar{m}L) \quad (47)$$

Therefore, inserting the ideal equivalent mass m_{eq} by Eq. (47) at three degrees of freedom system of **Figure 4**, the three eigen-frequencies coincide with the real values of the initial vertical pendulum that has distributed mass and flexural stiffness.

4. Discussion about the amphi-hinge vertical pendulums

By the previous mathematic analysis, an ideal three degrees of freedom system that is equivalent with the modal behavior of the amphi-hinge vertical pendulum with distributed mass and flexural stiffness along its height has been presented. This ideal three degrees of freedom system can be used in instrumentation of a such vertical tower, which does not possess concentrated masses. In the framework of the identification of mode-shapes of an amphi-hinge pendulum, the equivalent mass by Eq. (47) permits to locate accelerometers per 0.25L (as shown at **Figure 4**) and there measure the response horizontal acceleration histories, in order to calculate the real first three mode-shapes of the vertical pendulum in order to avoid the instability and resonance-vibrations between the examined vertical pendulum and the supported rockets or space bus before the last launched.

5. Second case: modal analysis of undamped inverted pendulum with distributed mass and stiffness

According to the theory of continuous systems [8, 9], consider a straight vertical cantilever (inverted pendulum) that is loaded by an external continuous dynamic loading $p_z(x, t)$, with reference to a Cartesian three-dimensional reference system xyz , (**Figure 5**).

The vertical inverted pendulum possesses a distributed mass $m(x)$ per unit height, which in the special case of uniform distribution is given as $m(x) = \bar{m}$ in tons per meter (tn/m). Furthermore, according to Bernoulli Technical Bending Theory, this cantilever has section flexural stiffness $EI_y(x)$, where in the special case of an uniform distribution of the stiffness it is given as $EI_y(x) = EI_y$, where E is the

Figure 5.
 Vertical cantilever (inverted pendulum) with distributed mass and section flexural stiffness.

material modulus of elasticity and I_y is the section moment of inertia about y-axis (**Figure 1**). Next, we are examining such an inverted pendulum that possesses constant value of distributed mass along its height, as well as constant value of distributed section flexural stiffness EI_y . Due to fact that the cantilever mass is continuously distributed, this inverted pendulum/beam has infinity number of degrees of freedom for vibration along the horizontal oz -axis. In order to formulate of the motion equation of this beam, we consider an infinitesimal part of the vertical beam, at location x from the origin o , that has isolated by two very nearest parallel sections.

The infinitesimal length of this part is the dx . On this infinitesimal length, we notice the flexural moment $M(x,t)$, the shear force $Q(x,t)$ with their differential increments, while the axial force $N(x,t)$ is ignored, because it does not affect the horizontal cantilever vibration along z -axis. Moreover, noted the resulting force $P_z(x,t)$ of the external dynamic loading. Eqs. (1)–(20) describe mathematically the modal analysis of the inverted pendulum. Afterward, the four unknown parameters C_1, C_2, C_3, C_4 of Eq. (20) must be calculated taking account the conditions of the cantilever. In order to achieve this, four support conditions of the cantilever have to be used. Indeed, for the end at $x = 0$ the displacement $u_z(0,t)$ and the slope $u'_z(0,t)$ of displacement profile must be zero. So, Eq. (20) gives:

$$\varphi(0) = C_1 \sin 0 + C_2 \cos 0 + C_3 \sinh 0 + C_4 \cosh 0 = 0 \Rightarrow C_2 + C_4 = 0 \quad (48)$$

and

$$\varphi'(0) = \beta(C_1 + C_3) = 0 \Rightarrow C_1 + C_3 = 0 \quad (49)$$

Moreover, for the free end at $x = L$ of the cantilever, the flexural moment $M(L,t)$ as well as the shear force $Q(L,t)$ must be both zero. So, from Eqs. (20) and (5) and afterward from Eq. (48) we take:

$$M(L, t) = EI_y \cdot \frac{\partial^2 \varphi(L)}{\partial x^2} = 0 \Rightarrow EI_y \cdot \varphi''(L) = 0 \Rightarrow$$

$$C_1(\sin \beta L + \sinh \beta L) + C_2(\cos \beta L + \cosh \beta L) = 0 \quad (50)$$

and

$$Q(L, t) = EI_y \cdot \frac{\partial^3 \varphi(L)}{\partial x^3} = 0 \Rightarrow EI_y \cdot \varphi'''(L) = 0 \Rightarrow$$

$$C_1(\cos \beta L + \cosh \beta L) + C_2(-\sin \beta L + \sinh \beta L) = 0 \quad (51)$$

However, re-writing Eqs. (50) and (51) again, we get the matrix form:

$$\begin{bmatrix} (\sin \beta L + \sinh \beta L) & (\cos \beta L + \cosh \beta L) \\ (\cos \beta L + \cosh \beta L) & (-\sin \beta L + \sinh \beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (52)$$

Eq. (52) is a real eigenvalue problem. In order to calculate the eigenvalues, parameters C_1 and C_2 must not both equal zero. Thus, the determinant of the matrix by Eq. (52) must be zero, where it drives to the following frequency equation:

$$1 + (\cos \beta L) \cdot (\cosh \beta L) = 0 \quad (53)$$

However, Eq. (54) can be solved numerically only, where we obtain the first four roots ($n = 1, 2, 3, 4$):

$$\beta_1 L = 1.8751, \quad \beta_2 L = 4.6941, \quad \beta_3 L = 7.8548 \text{ and } \beta_4 L = 10.996 \quad (54)$$

By the definition of parameter β , we can calculate the first four circular eigen-frequency ω using Eq. (36):

$$\beta^4 = \frac{\omega^2 \cdot \overline{m}}{EI_y} \Rightarrow \omega^2 = \frac{EI_y \cdot \beta^4}{\overline{m}} \Rightarrow \omega = \beta^2 \cdot \sqrt{\frac{EI_y}{\overline{m}}} \quad (55)$$

Thus, inserting Eq. (54) into Eq. (55), the first four eigen-frequency ω_n is directly arise for each n -value.

$$\omega_1 = \frac{3.516}{L^2} \cdot \sqrt{\frac{EI_y}{\overline{m}}}, \quad \omega_2 = \frac{22.03}{L^2} \cdot \sqrt{\frac{EI_y}{\overline{m}}},$$

$$\omega_3 = \frac{61.70}{L^2} \cdot \sqrt{\frac{EI_y}{\overline{m}}}, \quad \omega_4 = \frac{120.9}{L^2} \cdot \sqrt{\frac{EI_y}{\overline{m}}} \quad (56)$$

Therefore, the vibration mode-shape of the examined vertical inverted pendulum arises by Eq. (20)—since previous inserting Eq. (56)—thus:

$$\varphi_n(x) = C_1 \left[\cosh \beta_n x - \cos \beta_n x - \frac{\cosh \beta_n L + \cos \beta_n L}{\sinh \beta_n L + \sin \beta_n L} (\sinh \beta_n x - \sin \beta_n x) \right] \quad (57)$$

The value of C_1 is an arbitrary constant, and we usually get it equal to unit. Thus, for each value of parameter n , a mode-shape with its eigen-frequency is resulted. The fundamental (first) mode-shape is resulted for $n = 1$, etc. (**Figure 6**).

Figure 6.
 Modal analysis of continuous inverted pendulum. The first four mode-shapes.

6. The equivalent three degrees of freedom system of inverted pendulum

At inverted pendulums cantilevers, where the fundamental horizontal mode-shape does not activate the 90% of the total cantilever mass, we ask to consider the three first mode-shapes. Thus, for this purpose, we must define an ideal equivalent three degrees of freedom beam, which is going to give the three mode-shapes of the examined beam. Therefore, which is the ideal three degrees of freedom system, where its three eigen-frequencies coincide with the real first three frequencies of the inverted pendulum with distributed mass and flexural stiffness?

In order to answer the above-mentioned question, consider a weightless vertical inverted pendulum with height L and constant section in elevation, where carry three concentrated masses that each one has the same mass-value m_{eq} , located per distance $0.333L$, between one to one, and each one mass possesses an horizontal degree of freedom (**Figure 7**).

The inverted pendulum displacement vector $\mathbf{u}$ of the three degrees of freedom, as well as the diagonal beam mass matrix $\mathbf{m}$ are written:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \mathbf{m} = \begin{bmatrix} m_{eq} & 0 & 0 \\ 0 & m_{eq} & 0 \\ 0 & 0 & m_{eq} \end{bmatrix} \quad (58)$$

Furthermore, the inverted pendulum flexibility matrix $\mathbf{f}$ can be calculated using a suitable method (**Figure 7**), and the inverse matrix gives the stiffness matrix $\mathbf{k}$ of the three degrees of freedom beam.

$$\mathbf{f} = \begin{bmatrix} D_{1,1} & D_{1,2} & D_{1,3} \\ D_{2,1} & D_{2,2} & D_{2,3} \\ D_{3,1} & D_{3,2} & D_{3,3} \end{bmatrix} = \frac{L^3}{3EI_y} \cdot \begin{bmatrix} 1 & 0.5185 & 0.1481 \\ 0.5185 & 0.2963 & 0.0926 \\ 0.1481 & 0.0926 & 0.0370 \end{bmatrix} \quad (59)$$

Figure 7.
The equivalent three-degree of freedom system on inverted pendulum.

$$\mathbf{k} = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \\ k_{3,1} & k_{3,2} & k_{3,3} \end{bmatrix} = \frac{3EI_y}{L^3} \cdot \begin{bmatrix} 14.53846 & -33.23077 & 24.92308 \\ -33.23077 & 91.38462 & -95.53846 \\ 24.92308 & -95.53846 & 166.15385 \end{bmatrix} \quad (60)$$

The equations of motion for the case of the free undamped vibration of the ideal beam is given as:

$$\mathbf{m} \ddot{\mathbf{u}}(t) + \mathbf{k} \mathbf{u}(t) = \mathbf{0} \quad (61)$$

The eigen-problem is written as:

$$(\mathbf{k} - \omega_n^2 \mathbf{m}) \boldsymbol{\varphi}_n = \mathbf{0} \quad n = 1, 2, 3. \quad (62)$$

where the eigen-frequencies ω_n and the three mode-shapes $\boldsymbol{\varphi}_n$ are known by Eq. (56) and **Figure 6**. Therefore, the unique unknown parameter is the mass m_{eq} . Thus,

$$\det(\mathbf{k} - \omega_1^2 \mathbf{m}) = 0 \Rightarrow \quad (63)$$

$$m_{eq}^3 + A \cdot m_{eq}^2 + B \cdot m_{eq} + C = 0 \quad (64)$$

where

$$A = -\frac{k_{11} + k_{22} + k_{33}}{\omega_1^2}, B = \frac{k_{11}k_{33} + k_{11}k_{22} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2}{\omega_1^4}$$

$$C = -\frac{k_{11}k_{22}k_{33} + 2k_{12}k_{13}k_{23} - k_{11}k_{23}^2 - k_{22}k_{13}^2 - k_{33}k_{12}^2}{\omega_1^6}$$

The numerical solution of Eq. (64) gives three roots for parameter m_{eq} , where only the first root is acceptable, because the other two values rejected since do not have natural meaning (appear values greater from the total pendulum mass $\bar{m}L$). Thus, the only one acceptable root is given:

$$m_{eq} = 0.1868388 \cdot (\overline{m}L) \quad (65)$$

Therefore, inserting the ideal equivalent mass m_{eq} by Eq. (65) at three degrees of freedom system of **Figure 7**, the three eigen-frequencies coincide with the real values of the initial vertical inverted pendulum that has distributed mass and flexural stiffness.

On the contrary to above-mentioned about the examined cantilever, it is worth noting that in the case where ask an ideal single degree of freedom system that has eigen-frequency equal to fundamental eigen-frequency of the real cantilever we write the following equations:

- Eigen-frequency of single degree of freedom system with k the cantilever lateral stiffness and $m_{eq, sdof}$ the concentrated mass at the top of the cantilever:

$$\omega^2 = \frac{k}{m_{eq, sdof}} = \frac{3EI_y/L^3}{m_{eq, sdof}} \quad (66)$$

- Fundamental (first) eigen-frequency (see Eq. (56)) of the real cantilever with continuous distribution of mass and flexural stiffness:

$$\omega = \frac{3.516}{L^2} \cdot \sqrt{\frac{EI_y}{\overline{m}}} \quad (67)$$

- Thus, inserting Eq. (67) into Eq. (66) the equivalent concentrated mass $m_{eq, sdof}$ at the top of the cantilever is given [9]:

$$m_{eq, sdof} = 0.2426742 \cdot (\overline{m}L) \quad (68)$$

7. Conclusions

The present article has presented a mathematic ideal three degrees of freedom system that is equivalent with the modal behavior of two cases of pendulums. First, an amphi-hinge vertical pendulum with distributed mass and flexural stiffness along its height has been examined. Second, an inverted pendulum that can be simulates a tower (or cantilever) has been examined too. For each case, an ideal three degrees of freedom system has been proposed that can be used in instrumentation of such a vertical tower, which does not possess concentrated masses. In the framework of the identification of mode-shapes of the above-mentioned pendulums, the equivalent mass by Eqs. (47) and (65) permits to locate accelerometers (as shown at **Figures 4** and **7**) and measures the response acceleration histories, in order to calculate the real first three frequencies.

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