Chapter 3

Separability and Nonseparability of Elastic States in Arrays of One-Dimensional Elastic Waveguides

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Additional information is available at the end of the chapter

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Abstract

We show that the directional projection of longitudinal waves propagating in a parallel array of N elastically coupled waveguides can be described by a nonlinear Dirac-like equation in a 2^N dimensional exponential space. This space spans the tensor product Hilbert space of the two-dimensional subspaces of N uncoupled waveguides grounded elastically to a rigid substrate (called φ-bits). The superposition of directional states of a φ-bit is analogous to that of a quantum spin. We can construct tensor product states of the elastically coupled system that are nonseparable on the basis of tensor product states of N φ-bits. We propose a system of coupled waveguides in a ring configuration that supports these nonseparable states.

Keywords: one-dimensional elastic waveguides, nonseparability, elastic waves, elastic pseudospin, coupled waveguides

1. Introduction

Quantum bit-based computing platforms can capitalize on exponentially complex entangled states which allow a quantum computer to simultaneously process calculations well beyond what is achievable with serially interconnected transistor-based processors. Ironically, a pair of classical transistors can emulate some of the functions of a qubit. While current manufacturing can fabricate billions of transistors on a chip, it is inconceivable to connect them in the exponentially complex way that would be required to achieve nonseparable quantum superposition analogues. In contrast, quantum systems possess such complexity through the nature of the quantum world. Outside the quantum world, the notion of classical nonseparability [1–3] has been receiving a lot of attention from the theoretical and experimental point of views

in the field of optics. Degrees of freedom of photon states that span different Hilbert spaces can be made to interact in a way that leads to local correlations. Correlation has been achieved between degrees of freedom that include spin angular momentum and orbital angular momentum (OAM) [4–9], OAM, polarization and radial degrees of freedom of a beam of light [10] as well as propagation direction [11, 12]. Recently, we have extended this notion to correlation between directional and OAM degrees of freedom in elastic systems composed of arrays of elastic waveguides [13]. This classical nonseparability lies only in the tensor product Hilbert space of the subspaces associated with these degrees of freedom. This Hilbert space does not possess the exponential complexity of a multiqubit Hilbert space, for instance. It has been suggested theoretically and experimentally that classical systems coupled via nonlinear interactions may have computational capabilities approaching that of quantum computers [14–16].

We demonstrated in Ref. [17] that nonlinear elastic media can be used to produce phonons that can be correlated simultaneously in time and frequency. We have also shown an analogy between the propagation of elastic waves on elastically coupled one-dimensional (1D) wave guides and quantum phenomena [18–21]. More specifically, the projection on the direction of propagation of elastic waves in an elastic system composed of a 1D waveguide grounded to a rigid substrate (denoted φ-bit) is isomorphic to the spin of a quantum particle. The pseudospin states of elastic waves in these systems can be described via a Dirac-like equation and possess 2×1 spinor amplitudes. Unlike the quantum systems, these amplitudes are, however, measurable through the measurement of transmission coefficients. The notion of measurement is an important one as it has been realized that separability is relative to the choice of the partitioning of a multipartite system. Indeed, it is known that given a multipartite physical system, whether quantum or classical, the way to subdivide it into subsystems is not unique [22, 23]. For instance, the states of a quantum system may not appear entangled relative to some decomposition but may appear entangled relative to another partitioning. The criterion for that choice may be the ability to perform observations and measurements of some degrees of freedom of the subsystems [23].

The objective of this paper is to investigate the notion of separability and nonseparability of multipartite classical mechanical systems supporting elastic waves. These systems are composed of 1D elastic waveguides that are elastically coupled along their length to each other and/or to some rigid substrate. The 1D waveguides support spinor-like amplitudes in the two-dimensional (2D) subspace of directional degrees of freedom. The amplitudes of N coupled waveguides span an N-dimensional subspace. Subsequently, the Hilbert space spanned by the elastic modes is a $2N$-dimensional space, comprised of the tensor product of the directional and waveguides subspaces. This representation is isomorphic to the degrees of freedom of photon states in a beam of light. While beams of light cannot be decomposed into subsystems, an elastic system composed of coupled 1D waveguides can. Indeed, the elastic system considered here forms a multipartite system composed of N 1D waveguide subsystems. We show that, since each waveguide possesses two directional degrees of freedom, one can represent the elastic states of the N-waveguide system in the 2^N dimensional tensor product Hilbert space of N 2D spinor subspaces associated with individual waveguides. The elastic modes in this representation obey a 2^N dimensional nonlinear Dirac-like equation. These modes span the same space as that of uncoupled waveguides grounded to a rigid substrate, i.e., N φ-bits. However, the modes' solutions of the nonlinear Dirac equation cannot be expressed as tensor products of the states of N uncoupled grounded waveguides, i.e., φ-bit states.

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In Section 2 of this chapter, we introduce the mathematical formalism that is needed to demonstrate the nonseparability of elastic states of coupled elastic waveguides in an exponentially complex space. Throughout this section, we use illustrations of the concepts in the case of systems composed of small numbers of waveguides. However, the approach is fully scalable and can be generalized to any large number of coupled waveguides. In Section 3, we draw conclusions concerning the applicability of this approach to solve complex problems.

2. Models and methods

We have previously considered systems constituted of N one-dimensional (1D) waveguides coupled elastically along their length [13]. In this section, we summarize the results of these previous investigations to develop a formalism to address our current considerations. The parallelly coupled waveguides can be arranged in any desired way. The propagation of elastic modes is limited to longitudinal modes along the waveguides in the long wavelength limit, i.e., the continuum limit. We consider the representations of the modes of the coupled waveguide systems in two spaces. The first space scales linearly with N. The second space scales as 2^N and leads to a description of the elastic system with exponential complexity. The linear representation enables us to operate easily on the states in the exponential space.

2.1. Representation of elastic states in a space scaling linearly with N

A compact form for the equations of motion of the N coupled waveguides is:

$$\left\{H.I_{N\times N} + \alpha^2 M_{N\times N}\right\} u_{N\times 1} = 0 \tag{1}$$

Here, the propagation of elastic waves in the direction x along the waveguides is modeled by the dynamical differential operator, $H = \frac{\partial^2}{\partial t^2} - \beta^2 \frac{\partial^2}{\partial x^2}$. The parameter β is proportional to the speed of sound in the medium constituting the waveguides and the parameter α^2 characterizes the strength of the elastic coupling between them (here, we consider that the strength is the same for all coupled waveguides). $u_{N\times 1}$ is a vector with components, $u_i, (i = 1, N)$, representing the displacement of the ith waveguide. The coupling matrix operator $M_{N\times N}$ describes the elastic coupling between waveguides which, in the case of N = 3 parallel waveguides in a closed ring arrangement with first neighbor coupling, takes the form:

$$M_{N=3\times N=3} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{2}$$

Eq. (1) takes the form of a generalized Klein-Gordon (KG) equation and its Dirac factorization introduces the notion of the square root of the operator $\left\{H.I_{N\times N} + \alpha^2 M_{N\times N}\right\}$. In this factorization, the dynamics of the system are represented in terms of first derivatives with respect to time, t, and position along the waveguides, x. There are two possible Dirac equations:

$$\left\{U_{N\times N}\otimes\sigma_x\frac{\partial}{\partial t}+\beta U_{N\times N}\otimes\left(-i\sigma_y\right)\frac{\partial}{\partial x}\pm i\alpha U_{2N\times 2N}\sqrt{M_{N\times N}}\otimes\sigma_x\right\}\Psi_{2N\times 1}=0 \tag{3}$$

In Eq. (3), $U_{N\times N}$ and $U_{2N\times 2N}$ are antidiagonal matrices with unit elements. $\sigma_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\sigma_y=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$ are two of the Pauli matrices. $\Psi_{2N\times 1}$ is a $2\ N$ dimensional vector which represents the modes of vibration of the N waveguides projected in the two possible directions of propagation (forward and backward) and $\sqrt{M_{N\times N}}$ is the square root of the coupling matrix. The square root of a matrix is not unique but we will show later that we can pick any form without loss of generality.

We choose components of the $\Psi_{2N\times 1}$ vector in the form of plane waves $\psi_I=a_Ie^{ikx}e^{i\omega t}$ with $I=1,\ldots,2N$ and k and ω being the wave number and angular frequency, respectively, Eq. (3) becomes:

$$\left\{\omega A_{2N\times 2N}+\beta kB_{2N\times 2N}\pm\alpha C_{2N\times 2N}\right\}a_{2N\times 1}=0 \tag{4}$$

where

$$A_{2N\times 2N}=I_{N\times N}\otimes I_{2\times 2} \tag{5a}$$

$$B_{2N\times 2N}=I_{N\times N}\otimes\left(-\sigma_z\right) \tag{5b}$$

$$C_{2N\times 2N}=\sqrt{M_{N\times N}}\otimes\sigma_x \tag{5c}$$

In Eqs. (4) and (5), $\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the third Pauli matrix, $I_{N\times N}$ is the identity matrix of order N and $a_{2\mathrm{Nx}1}$ is a $2\ N$ dimensional vector whose components are the amplitudes a_I. In obtaining Eq. (4), we have multiplied all terms in Eq. (3) on the left by $U_{2N\times 2N}$.

Writing Eq. (4) as a linear combination of tensor products of $N\times N$ and 2×2 matrix operators:

$$\left\{I_{N\times N}\otimes\left[\omega I_{2\times 2}-\beta k\sigma_z\right]\pm\alpha\sqrt{M_{N\times N}}\otimes\sigma_x\right\}a_{2N\times 1}=0 \tag{6}$$

we seek solutions in the form of tensor products:

$$a_{2N\times 1}=E_{N\times 1}\otimes s_{2\times 1} \tag{7}$$

While the degrees of freedom associated with $E_{N\times 1}$ span an N dimensional Hilbert subspace, the degrees of freedom associated with $s_{2\times 1}$ span a 2D space.

Replacing $a_{2N\times 1}$ from Eq. (7) in Eq. (6) yields:

$$\left\{\left(I_{N\times N}E_{N\times 1}\right)\otimes\left(\left[\omega I_{2\times 2}-\beta k\sigma_z\right]s_{2\times 1}\right)\pm\alpha\left(\sqrt{M_{N\times N}}E_{N\times 1}\right)\otimes\left(\sigma_xs_{2\times 1}\right)\right\}=0. \tag{8}$$

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Choosing $E_{N\times 1}$ to be an eigenvector, e_n, of the matrix $\sqrt{M_{N\times N}}$ with eigen value λ_n Eq. (8) reduces to:

$$e_n \otimes \left\{ \left(\left[\omega I_{2\times 2} - \beta k \sigma_z \right] \pm \alpha \lambda_n \sigma_x \right) s_{2\times 1} \right\} = 0 \tag{9}$$

For nontrivial eigenvectors e_n, the problem in the space of the directions of propagation reduces to finding solutions of

$$\left(\left[\omega I_{2\times 2} - \beta k \sigma_z \right] \pm \alpha \lambda_n \sigma_x \right) s_{2\times 1} = 0 \tag{10}$$

In obtaining Eq. (9), we have also used the fact that e_n is an eigen vector of $I_{N\times N}$ with eigen value 1 and we note that Eq. (9) is the 1D Dirac equation for an elastic system which solutions, $s_{2\times 1}$, have the properties of Dirac spinors [18–21]. The components of the spinor represent the amplitude of the elastic waves in the positive and negative directions along the waveguides, respectively.

Eq. (10), now written in the matrix form, can now be solved for a given λ_n;

$$\begin{pmatrix} \omega_n - \beta k & \pm \alpha \lambda_n \\ \pm \alpha \lambda_n & \omega_n + \beta k \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0 \tag{11}$$

This eigen equation gives the dispersion relation $\omega_n^2 = (\beta k)^2 + (\alpha \lambda_n)^2$ (*vide infra*) and the following eigen vectors projected into the space of directions of propagation:

$$s_{2\times 1} = s_0 \begin{pmatrix} \sqrt{\omega_n + \beta k} \\ \pm \sqrt{\omega_n - \beta k} \end{pmatrix} \tag{12}$$

To determine the eigen vectors of $\sqrt{M_{N\times N}}$, we note that they are identical to the eigen vectors of the coupling matrix $M_{N\times N}$ and the eigen values of $M_{N\times N}$ are also λ_n^2. These properties indicate that we do not have to determine the square root of the coupling matrix to find the solutions $a_{2N\times 1}$. All that is required is to calculate the eigen vectors and the eigen values of the coupling matrix. Hence, the nonuniqueness of $\sqrt{M_{N\times N}}$ does not introduce difficulties in determining the elastic modes of the coupled system in the Dirac representation.

In the case of the coupling matrix, $M_{3\times 3}$, presented in Eq. (2), the eigen values and real eigen vectors are obtained as $\lambda_0^2 = 0,\ \lambda_1^2 = \lambda_2^2 = 3,$ and

$$e_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, e_1 = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} 1 \\ \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} \text{ and } e_2 = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} \frac{-1}{2} \\ 1 \\ \frac{-1}{2} \end{pmatrix} \tag{13}$$

Eq. (3) being linear, its solutions can be written as linear combinations of elastic wave functions in the form:

$$\Psi_{2N\times 1}(n,k) = e_n(N) \otimes s_{2\times 1}(k)\ e^{ikx} e^{i\omega_n(k)t} \tag{14}$$

In Eq. (14), we have expressed the dependencies on the wave number k and the number of waveguides N. The eigen vectors $e_n(N)$ depend on the connectivity of the N waveguides. The space spanned by these solutions scales linearly with the number of waveguides, i.e., as $2N$.

2.2. Representation of elastic states in a space scaling as 2^N

We first illustrate the notion of exponential space in the case of three waveguides. Each guide is connected to a rigid substrate and therefore constitutes a φ-bit. The waveguides are not coupled to each other. The dynamics of the system can be described by a *single* equation which is constructed as follows:

$$\left[\sigma_x \otimes \sigma_x \otimes \sigma_x \frac{\partial}{\partial t} + i\beta\sigma_y \otimes \sigma_x \otimes \sigma_x \frac{\partial}{\partial x_1} + i\beta\sigma_x \otimes \sigma_y \otimes \sigma_x \frac{\partial}{\partial x_2} + i\beta\sigma_x \otimes \sigma_x \otimes \sigma_y \frac{\partial}{\partial x_3} \pm i\alpha I_{2\times 2} \otimes \sigma_x \right.$$
$$\left. \otimes \sigma_x \pm i\alpha\sigma_x \otimes I_{2\times 2} \otimes \sigma_x \pm i\alpha\sigma_x \otimes \sigma_x \otimes I_{2\times 2}\right]\Psi_{8\times 1} = 0 \tag{15}$$

In Eq. (15), we are now defining a positional variable for each waveguide, namely, x_1, x_2, x_3. The quantity α is a measure of the strength of the elastic coupling to the rigid substrate. The solutions are the 8×1 vectors $\Psi_{8\times 1} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \end{pmatrix}$. When seeking solutions in the form of tensor products of spinor solutions for the three waveguides (as indicated by the upper scripts)

$$\Psi_{8\times 1} = \psi^{(1)} \otimes \psi^{(2)} \otimes \psi^{(3)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \end{pmatrix} \otimes \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \psi_1^{(3)} \\ \psi_2^{(3)} \end{pmatrix} = \begin{pmatrix} \psi_1^{(1)}\psi_1^{(2)}\psi_1^{(3)} \\ \psi_1^{(1)}\psi_1^{(2)}\psi_2^{(3)} \\ \psi_1^{(1)}\psi_2^{(2)}\psi_1^{(3)} \\ \psi_1^{(1)}\psi_2^{(2)}\psi_2^{(3)} \\ \psi_2^{(1)}\psi_1^{(2)}\psi_1^{(3)} \\ \psi_2^{(1)}\psi_1^{(2)}\psi_2^{(3)} \\ \psi_2^{(1)}\psi_2^{(2)}\psi_1^{(3)} \\ \psi_2^{(1)}\psi_2^{(2)}\psi_2^{(3)} \end{pmatrix} \tag{16}$$

it is straightforward to show that one recovers from Eq. (15), the six Dirac equations of Eq. (3) with $\sqrt{M_{N=3\times N=3}} = I_{3\times 3}$. The solutions of Eq. (16) are obtained from the spinor solution for individual waveguides (j):

$$\psi^{(j)} = s_0 \begin{pmatrix} \sqrt{\omega + \beta k} \\ \pm\sqrt{\omega - \beta k} \end{pmatrix} e^{ikx} e^{i\omega t} \tag{17}$$

The Hilbert space spanned by the solutions of Eq. (15) is the product space of the three 2D subspaces associated with each waveguide. The states of a system composed of N φ-bits span a space when dimension is 2^N.

The question that arises then concerns the possibility of writing an equation in the exponential Hilbert space for N waveguides coupled to each other. For instance, we wish to obtain the states of the system composed of three waveguides coupled in a ring arrangement from an equation of the form:

$$\left[\sigma_x \otimes \sigma_x \otimes \sigma_x \frac{\partial}{\partial t} + i\beta\sigma_y \otimes \sigma_x \otimes \sigma_x \frac{\partial}{\partial x_1} + i\beta\sigma_x \otimes \sigma_y \otimes \sigma_x \frac{\partial}{\partial x_2} + i\beta\sigma_x \otimes \sigma_x \otimes \sigma_y \frac{\partial}{\partial x_3} \pm i\alpha\varepsilon_{8\times 8}\right]\Psi_{8\times 1} = 0 \quad (18)$$

The matrix $\alpha\varepsilon_{8\times 8}$ represents the coupling between the waveguides in the $2^{N=3}$ space. We are still seeking solutions in the form of tensor products (Eq. (16)). After a lengthy algebraic manipulation, we find that we can reproduce Eq. (3) with the coupling matrix of Eq. (2) if one chooses $\varepsilon_{ij} = 0$ excepting

$$\begin{aligned}
&\varepsilon_{14} = \varepsilon_{41} = \varepsilon_{23} = \varepsilon_{31} = \frac{2\psi_1^{(1)} - \psi_1^{(2)} - \psi_1^{(3)}}{\psi_1^{(1)}}; \varepsilon_{16} = \varepsilon_{61} = \varepsilon_{25} = \varepsilon_{52} = \frac{2\psi_1^{(2)} - \psi_1^{(1)} - \psi_1^{(3)}}{\psi_1^{(2)}};\\
&\varepsilon_{17} = \varepsilon_{71} = \varepsilon_{35} = \varepsilon_{53} = \frac{2\psi_1^{(3)} - \psi_1^{(1)} - \psi_1^{(2)}}{\psi_1^{(3)}}; \varepsilon_{28} = \varepsilon_{82} = \varepsilon_{46} = \varepsilon_{64} = \frac{2\psi_2^{(3)} - \psi_2^{(1)} - \psi_2^{(2)}}{\psi_2^{(3)}};\\
&\varepsilon_{38} = \varepsilon_{83} = \varepsilon_{47} = \varepsilon_{74} = \frac{2\psi_2^{(2)} - \psi_2^{(1)} - \psi_2^{(3)}}{\psi_2^{(2)}}; \varepsilon_{58} = \varepsilon_{85} = \varepsilon_{67} = \varepsilon_{76} = \frac{2\psi_2^{(1)} - \psi_2^{(2)} - \psi_2^{(3)}}{\psi_2^{(1)}}
\end{aligned} \quad (19)$$

The Dirac equation of the three coupled waveguides in the exponential space is therefore nonlinear. Generalization to N coupled chains will result in the following nonlinear equation:

$$\left[(\sigma_x)^{\otimes N}\frac{\partial}{\partial t} + i\beta\sigma_y \otimes (\sigma_x)^{\otimes N-1}\frac{\partial}{\partial x_1} + i\beta\sigma_x \otimes \sigma_y \otimes (\sigma_x)^{\otimes N-2}\frac{\partial}{\partial x_2} + \ldots + i\beta(\sigma_x)^{\otimes N-1} \otimes \sigma_y \frac{\partial}{\partial x_N} \pm i\alpha\varepsilon_{2^N\times 2^N}\right]\Psi_{2^N\times 1} = 0 \quad (20)$$

where nonzero components of $\varepsilon_{2^N\times 2^N}$ depend on the $\psi_i^{(j)}, i = 1, 2; j = 1, N$ that appear in the solution $\Psi_{2^N\times 1} = \psi^{(1)} \otimes \psi^{(2)} \otimes \ldots \otimes \psi^{(N)}$. The solutions of the nonlinear Dirac equation for the coupled waveguides span the same space as that of the system of φ-bits, i.e., uncoupled waveguides connected to rigid substrates. The next subsection addresses the question of separability of the coupled waveguide system into a system of uncoupled φ-bits.

2.3. Elastic states in the exponential space

For a system of waveguides that are not coupled, the elastic states, solutions of linear equations of the form of Eq. (15), are tensor products but also linear combinations of tensor products of spinor solution for individual waveguides (see Eq. (17)). It is therefore possible to construct nonseparable states in the exponential space for systems of uncoupled

waveguides. For example, if we consider a system of two uncoupled waveguides, a possible state of the system in the 2^2 space can be constructed in the form of the following linear combination of tensor products:

$$\Psi_{4\times1} = (s_0)^2 \begin{pmatrix} \sqrt{\omega+\beta k} \\ \pm\sqrt{\omega-\beta k} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\omega+\beta k} \\ \pm\sqrt{\omega-\beta k} \end{pmatrix} e^{i2kx} e^{i2\omega t} - \left(s_0'\right)^2 \begin{pmatrix} \sqrt{\omega-\beta k} \\ \pm\sqrt{\omega+\beta k} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\omega-\beta k} \\ \pm\sqrt{\omega+\beta k} \end{pmatrix} e^{-i2kx} e^{i2\omega t} \tag{21}$$

Choosing $s_0 = s_0'$ and writing Eq. (21) at the location $x = 0$, one gets:

$$\Psi_{4\times1}(x=0) = \left\{ \begin{pmatrix} \sqrt{\omega+\beta k}\sqrt{\omega+\beta k} \\ \pm\sqrt{\omega+\beta k}\sqrt{\omega-\beta k} \\ \pm\sqrt{\omega-\beta k}\sqrt{\omega+\beta k} \\ \sqrt{\omega-\beta k}\sqrt{\omega-\beta k} \end{pmatrix} - \begin{pmatrix} \sqrt{\omega-\beta k}\sqrt{\omega-\beta k} \\ \pm\sqrt{\omega-\beta k}\sqrt{\omega+\beta k} \\ \pm\sqrt{\omega+\beta k}\sqrt{\omega-\beta k} \\ \sqrt{\omega+\beta k}\sqrt{\omega+\beta k} \end{pmatrix} \right\} e^{i2\omega t} \tag{22}$$

The bracket takes the form:

$$\left(\sqrt{\omega+\beta k}\sqrt{\omega+\beta k} - \sqrt{\omega-\beta k}\sqrt{\omega-\beta k}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{23}$$

The vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ is not separable into a tensor product of two 2×1 vectors. Considering on the basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, one can write the state given in Eq. (22) in the form of the nonseparable Bell state:

$$\Psi_{4\times1}(x=0) = \left(\sqrt{\omega+\beta k}\sqrt{\omega+\beta k} - \sqrt{\omega-\beta k}\sqrt{\omega-\beta k}\right)\left(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\right) e^{i2\omega t} \tag{24}$$

Since the waveguides are not coupled, it is, however, not possible to manipulate the state of one of the waveguides by manipulating the state of the other one. Simultaneous manipulation of the state of waveguides in the exponential space requires coupling. We now address elastic states in the coupled waveguides system.

For a system of N coupled waveguides, we construct a solution of Eq. (3) that takes the form of a linear combination of solutions given in Eq. (14):

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$$\Psi_{2N\times1}(n,n',k,,k') = \chi_n e_n(N)\otimes s_{2\times1}(k)e^{ikx}e^{i\omega_n(k)t} + \chi_{n'}e_{n'}(N)\otimes s_{2\times1}(k')\, e^{ik'x}e^{i\omega_{n'}(k')t} \tag{25}$$

The n and n' correspond to two different nonzero eigen values, λ_n and $\lambda_{n'}$, i.e., they correspond to two different dispersion relations $\omega_n(k)$ and $\omega_{n'}(k)$. We also choose the wave number k' such that $\omega_{n'}(k') = \omega_n(k) = \omega_0$. These modes are illustrated in **Figure 1** in the case of an N = 9 waveguide system. χ_n and $\chi_{n'}$ are the coefficients of the linear combination.

With $e_n(N) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}$ and $e_{n'}(N) = \begin{pmatrix} A'_1 \\ A'_2 \\ \vdots \\ A'_N \end{pmatrix}$ where the specific values of the components A_I and A'_I are determined by the connectivity and coupling of the waveguides, the state of Eq. (25) can be rewritten as:

$$\Psi_{2N\times1}(n,n',k,,k') = \begin{pmatrix} \chi_n A_1\sqrt{\omega_0+\beta k}e^{ikx} + \chi_{n'}A'_1\sqrt{\omega_0+\beta k'}e^{ik'x} \\ \chi_n A_1\sqrt{\omega_0-\beta k}e^{ikx} + \chi_{n'}A'_1\sqrt{\omega_0-\beta k'}e^{ik'x} \\ \vdots \\ \chi_n A_N\sqrt{\omega_0+\beta k}e^{ikx} + \chi_{n'}A'_N\sqrt{\omega_0+\beta k'}e^{ik'x} \\ \chi_n A_N\sqrt{\omega_0-\beta k}e^{ikx} + \chi_{n'}A'_N\sqrt{\omega_0-\beta k'}e^{ik'x} \end{pmatrix} e^{i\omega_0 t} = \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \\ \vdots \\ \varphi_1^{(N)} \\ \varphi_2^{(N)} \end{pmatrix} \tag{26}$$

Here, we have chosen, for the sake of simplicity, the + of the ± in the $s_{2\times1}$ terms.

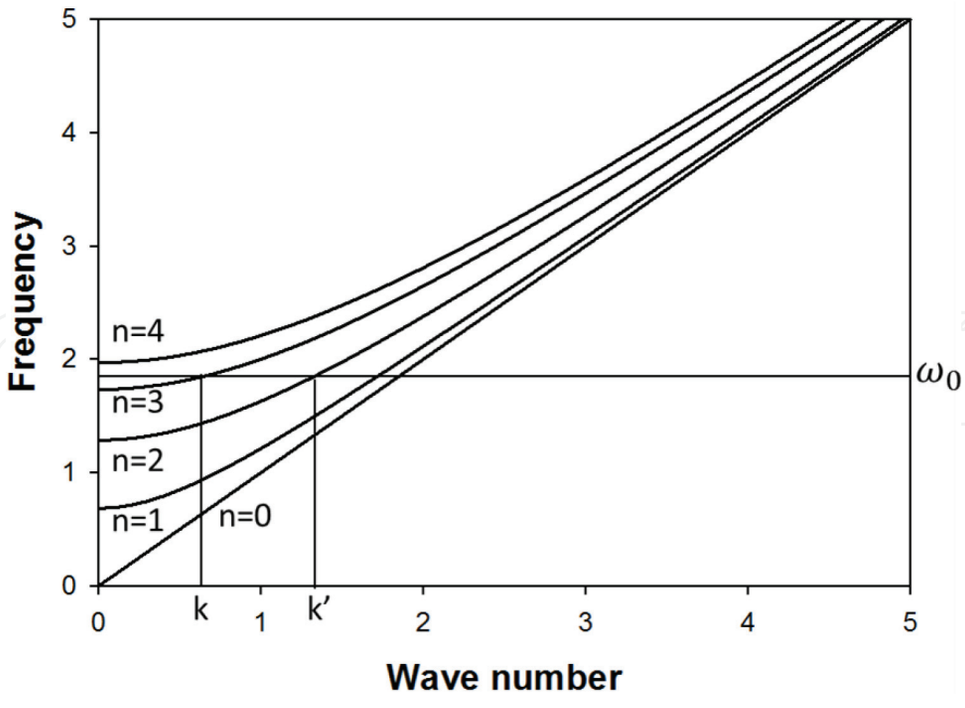

Figure 1. Schematic illustration of the band structure (circular frequency in rad s^{-1} versus the wave number in m^{-1}) for an array of nine elastically coupled waveguides arranged in a ring pattern. The four upper bands are doubly degenerate. We have taken $\beta = 1$ and $\alpha = 1$. The two modes with wave number k and k' ($n = 3$ and $n' = 2$) have the same frequency ω_0.

The first two terms in Eq. (26) form a 2×1 spinor, $\varphi^{(1)} = \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \end{pmatrix}$, which corresponds to the first waveguide, the next two terms form a spinor $\varphi^{(2)}$ for the second waveguide, etc. We can then construct a solution of the nonlinear Dirac Eq. (20) in the exponential space as the tensor product:

$$\Phi_{2^N \times 1} = \varphi^{(1)} \otimes \varphi^{(2)} \otimes \ldots \otimes \varphi^{(N)} \tag{27}$$

Since Eq. (20) is nonlinear, linear combinations of tensor product solutions of the form above are not solutions. Solutions of the nonlinear Dirac equation always take the form of a tensor product when the spinor wave functions $\varphi^{(j)}$ are expressed on the basis of 2×1 vectors. $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If one desires to express $\Phi_{2^N \times 1}$ as a nonseparable state, one has to define a new basis in which this wave function cannot be expressed as a tensor product. This is done in Section 2.5. However, prior to demonstrating this, we illustrate in the next subsection how one can manipulate states of the form $\Phi_{2^N \times 1}$ in the exponential space.

2.4. Operating on exponentially-complex tensor product elastic states

In this subsection, we expand tensor product states of the form given in Eq. (27) in linear combinations of tensor products of pure states in the exponential space. We illustrate this expansion in the case of three parallel waveguides elastically coupled to each other. Each waveguide is also coupled elastically to a rigid substrate. We treat the case where the strength of all the couplings is the same. In that case, the coupling matrix is:

$$M_{N=3 \times N=3} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

This matrix has three nonzero eigen values $\lambda_0^2 = 1,$ and $\lambda_1^2 = \lambda_2^2 = 4$ corresponding to two dispersion relations $\omega_n^2 = (\beta k)^2 + (\alpha \lambda_n)^2$ with cutoff frequencies. The second band is doubly degenerate. The eigen vectors are also given in Eq. (13). We now consider an elastic mode in the linear space that is a linear combination of these eigen modes (see Eq. (26)):

$$\Psi_{6\times 1}(n, n', k, , k') = \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \\ \varphi_1^{(2)} \\ \varphi_2^{(2)} \\ \varphi_1^{(3)} \\ \varphi_2^{(3)} \end{pmatrix} = \begin{pmatrix} \chi_n A_1 \sqrt{\omega_0 + \beta k} e^{ikx} + \chi_{n'} A_1' \sqrt{\omega_0 + \beta k'} e^{ik'x} \\ \chi_n A_1 \sqrt{\omega_0 - \beta k} e^{ikx} + \chi_{n'} A_1' \sqrt{\omega_0 - \beta k'} e^{ik'x} \\ \chi_n A_2 \sqrt{\omega_0 + \beta k} e^{ikx} + \chi_{n'} A_2' \sqrt{\omega_0 + \beta k'} e^{ik'x} \\ \chi_n A_2 \sqrt{\omega_0 - \beta k} e^{ikx} + \chi_{n'} A_2' \sqrt{\omega_0 - \beta k'} e^{ik'x} \\ \chi_n A_3 \sqrt{\omega_0 + \beta k} e^{ikx} + \chi_{n'} A_3' \sqrt{\omega_0 + \beta k'} e^{ik'x} \\ \chi_n A_3 \sqrt{\omega_0 - \beta k} e^{ikx} + \chi_{n'} A_3' \sqrt{\omega_0 - \beta k'} e^{ik'x} \end{pmatrix} e^{i\omega_0 t} \tag{28}$$

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In Eq. (28), the A_I's can be the components of the eigen vector e_0 and the A'_I's can be linear combinations of the components of the eigen vectors e_1 and e_2. We can calculate the tensor product of the spinor components in the form of Eq. (27)

$$\Phi_{2^3\times 1} = \varphi^{(1)} \otimes \varphi^{(2)} \otimes \varphi^{(3)} \tag{29}$$

Eq. (29) can be rewritten after some algebraic manipulations in the form of the linear combination:

$$\begin{aligned}\Phi_{2^3\times 1} = \Bigg\{ &\begin{pmatrix}\zeta_1\\ \xi_1\end{pmatrix}\otimes\begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix}\otimes\begin{pmatrix}\zeta_3\\ \xi_3\end{pmatrix} + \begin{pmatrix}\zeta_1\\ \xi_1\end{pmatrix}\otimes\begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix}\otimes\begin{pmatrix}\zeta'_3\\ \xi'_3\end{pmatrix} + \begin{pmatrix}\zeta_1\\ \xi_1\end{pmatrix}\otimes\begin{pmatrix}\zeta'_2\\ \xi'_2\end{pmatrix}\otimes\begin{pmatrix}\zeta_3\\ \xi_3\end{pmatrix} \\ &+ \begin{pmatrix}\zeta'_1\\ \xi'_1\end{pmatrix}\otimes\begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix}\otimes\begin{pmatrix}\zeta_3\\ \xi_3\end{pmatrix} + \begin{pmatrix}\zeta_1\\ \xi_1\end{pmatrix}\otimes\begin{pmatrix}\zeta'_2\\ \xi'_2\end{pmatrix}\otimes\begin{pmatrix}\zeta'_3\\ \xi'_3\end{pmatrix} + \begin{pmatrix}\zeta'_1\\ \xi'_1\end{pmatrix}\otimes\begin{pmatrix}\zeta_2\\ \xi_2\end{pmatrix}\otimes\begin{pmatrix}\zeta'_3\\ \xi'_3\end{pmatrix} \\ &+ \begin{pmatrix}\zeta'_1\\ \xi'_1\end{pmatrix}\otimes\begin{pmatrix}\zeta'_2\\ \xi'_2\end{pmatrix}\otimes\begin{pmatrix}\zeta_3\\ \xi_3\end{pmatrix} + \begin{pmatrix}\zeta'_1\\ \xi'_1\end{pmatrix}\otimes\begin{pmatrix}\zeta'_2\\ \xi'_2\end{pmatrix}\otimes\begin{pmatrix}\zeta'_3\\ \xi'_3\end{pmatrix}\Bigg\} e^{i3\omega_0 t}\end{aligned} \tag{30}$$

In Eq. (30), we have defined

$$\begin{pmatrix}\zeta_I\\ \xi_I\end{pmatrix} = \chi_n A_I e^{ikx}\begin{pmatrix}\sqrt{\omega_0+\beta k}\\ \sqrt{\omega_0-\beta k}\end{pmatrix} = \chi_n A_I e^{ikx} s_{2\times 1} \tag{31a}$$

$$\begin{pmatrix}\zeta'_I\\ \xi'_I\end{pmatrix} = \chi_{n''} A'_1 e^{ik''x}\begin{pmatrix}\sqrt{\omega_0+\beta k'}\\ \sqrt{\omega_0-\beta k'}\end{pmatrix} = \chi_{n''} A'_1 e^{ik''x} s'_{2\times 1} \tag{31b}$$

The tensor product of Eq. (30) then reduces to

$$\begin{aligned}\Phi_{2^3\times 1} = \Big\{ &(\chi_n)^3 A_1A_2A_3 e^{i3kx} s_{2\times1}\otimes s_{2\times1}\otimes s_{2\times1} + (\chi_n)^2\chi_{n'}A_1A_2A'_3 e^{i2kx}e^{ik'x} s_{2\times1}\otimes s_{2\times1}\otimes s'_{2\times1} \\ &+ (\chi_n)^2\chi_{n'}A_1A'_2A_3 e^{i2kx}e^{ik'x} s_{2\times1}\otimes s'_{2\times1}\otimes s_{2\times1} + (\chi_n)^2\chi_{n'}A'_1A_2A_3 e^{i2kx}e^{ik'x} s'_{2\times1}\otimes s_{2\times1}\otimes s_{2\times1} \\ &+ (\chi_{n'})^2\chi_n A_1A'_2A'_3 e^{ikx}e^{i2k'x} s_{2\times1}\otimes s'_{2\times1}\otimes s'_{2\times1} + (\chi_{n'})^2\chi_n A'_1A_2A'_3 e^{ikx}e^{i2k'x} s'_{2\times1}\otimes s_{2\times1}\otimes s'_{2\times1} \\ &+ (\chi_{n'})^2\chi_n A'_1A'_2A_3 e^{ikx}e^{i2k'x} s'_{2\times1}\otimes s'_{2\times1}\otimes s_{2\times1} + (\chi_{n'})^3 A'_1A'_2A'_3 e^{i3k'x} s'_{2\times1}\otimes s'_{2\times1}\otimes s'_{2\times1}\Big\} e^{i3\omega_0 t}\end{aligned} \tag{32}$$

The spinors $s_{2\times1}$ and $s'_{2\times1}$ can be expressed on the basis $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$:

$$s_{2\times1} = s_1|0\rangle + s_2|1\rangle \tag{33a}$$

$$s'_{2\times1} = s'_1|0\rangle + s'_2|1\rangle \tag{33b}$$

With $s_1 = \sqrt{\omega_0+\beta k}$, $s_2 = \sqrt{\omega_0-\beta k}$, $s'_1 = \sqrt{\omega_0+\beta k'}$ and $s'_2 = \sqrt{\omega_0-\beta k'}$. Inserting Eqs. (33a) and (33b) into Eq. (32), we can express the tensor product $\Phi_{2^3\times1}$ on the basis $\{|0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|1\rangle,$

$|0\rangle|1\rangle|0\rangle, |1\rangle|0\rangle|0\rangle, |0\rangle|1\rangle|1\rangle, |1\rangle|0\rangle|1\rangle, |1\rangle|1\rangle|0\rangle, |1\rangle|1\rangle|1\rangle\}$. In defining the basis vectors for the exponential space, we have omitted the symbols $\otimes$. It is also implicit that the left, middle, and right elements in the tensor product $|a\rangle|b\rangle|c\rangle$ correspond to the first, second, and third waveguides, respectively.

We find

$$\Phi_{2^3\times 1} = \{T_1|0\rangle|0\rangle|0\rangle + T_2|0\rangle|0\rangle|1\rangle + \ldots + T_8|1\rangle|1\rangle|1\rangle\}e^{i3\omega_0 t} \tag{34}$$

with

$$T_1 = Q_1 s_1 s_1 s_1 + Q_2 s_1 s_1 s_1' + Q_3 s_1 s_1' s_1 + Q_4 s_1' s_1 s_1 + Q_5 s_1 s_1' s_1' + Q_6 s_1' s_1 s_1' + Q_7 s_1' s_1' s_1 + Q_8 s_1' s_1' s_1' \tag{35a}$$

$$T_2 = Q_1 s_1 s_1 s_2 + Q_2 s_1 s_1 s_2' + Q_3 s_1 s_1' s_2 + Q_4 s_1' s_1 s_2 + Q_5 s_1 s_1' s_2' + Q_6 s_1' s_1 s_2' + Q_7 s_1' s_1' s_2 + Q_8 s_1' s_1' s_2' \tag{35b}$$

$$T_3 = Q_1 s_1 s_2 s_1 + Q_2 s_1 s_2 s_1' + Q_3 s_1 s_2' s_1 + Q_4 s_1' s_2 s_1 + Q_5 s_1 s_2' s_1' + Q_6 s_1' s_2 s_1' + Q_7 s_1' s_2' s_1 + Q_8 s_1' s_2' s_1' \tag{35c}$$

$$T_4 = Q_1 s_2 s_1 s_1 + Q_2 s_2 s_1 s_1' + Q_3 s_2 s_1' s_1 + Q_4 s_2' s_1 s_1 + Q_5 s_2 s_1' s_1' + Q_6 s_2' s_1 s_1' + Q_7 s_2' s_1' s_1 + Q_8 s_2' s_1' s_1' \tag{35d}$$

$$T_5 = Q_1 s_1 s_2 s_2 + Q_2 s_1 s_2 s_2' + Q_3 s_1 s_2' s_2 + Q_4 s_1' s_2 s_2 + Q_5 s_1 s_2' s_2' + Q_6 s_1' s_2 s_2' + Q_7 s_1' s_2' s_2 + Q_8 s_1' s_2' s_2' \tag{35e}$$

$$T_6 = Q_1 s_2 s_1 s_2 + Q_2 s_2 s_1 s_2' + Q_3 s_2 s_1' s_2 + Q_4 s_2' s_1 s_2 + Q_5 s_2 s_1' s_2' + Q_6 s_2' s_1 s_2' + Q_7 s_2' s_1' s_2 + Q_8 s_2' s_1' s_2' \tag{35f}$$

$$T_7 = Q_1 s_2 s_2 s_1 + Q_2 s_2 s_2 s_1' + Q_3 s_2 s_2' s_1 + Q_4 s_2' s_2 s_1 + Q_5 s_2 s_2' s_1' + Q_6 s_2' s_2 s_1' + Q_7 s_2' s_2' s_1 + Q_8 s_2' s_2' s_1' \tag{35g}$$

$$T_8 = Q_1 s_2 s_2 s_2 + Q_2 s_2 s_2 s_2' + Q_3 s_2 s_2' s_2 + Q_4 s_2' s_2 s_2 + Q_5 s_2 s_2' s_2' + Q_6 s_2' s_2 s_2' + Q_7 s_2' s_2' s_2 + Q_8 s_2' s_2' s_2' \tag{35h}$$

with

$$Q_1 = (\chi_n)^3 A_1 A_2 A_3 e^{i3kx}; \quad Q_2 = (\chi_n)^2 \chi_{n'} e^{i2kx} e^{ik'x} A_1 A_2 A_3'; \quad Q_3 = (\chi_n)^2 \chi_{n'} e^{i2kx} e^{ik'x} A_1 A_2' A_3;$$
$$Q_4 = (\chi_n)^2 \chi_{n'} e^{i2kx} e^{ik'x} A_1' A_2 A_3; \quad Q_5 = (\chi_{n'})^2 \chi_n e^{ikx} e^{i2k'x} A_1 A_2' A_3'; \quad Q_6 = (\chi_{n'})^2 \chi_n e^{ikx} e^{i2k'x} A_1' A_2 A_3';$$
$$Q_7 = (\chi_{n'})^2 \chi_n e^{ikx} e^{i2k'x} A_1' A_2' A_3; \quad Q_8 = (\chi_{n'})^3 A_1' A_2' A_3' e^{i3k'x}.$$

In a true quantum system composed of three spins for instance, states can be created in the form of linear combinations like $m_1|0\rangle|0\rangle|0\rangle + m_2|0\rangle|0\rangle|1\rangle + \ldots + m_8|1\rangle|1\rangle|1\rangle$. For the quantum system, the linear coefficients m_1, m_2,…, m_8 are independent. The classical elastic analogue, introduced here, states which are given in Eq. (34) possesses linear coefficients T_1,…, T_8 are

interdependent. While somewhat restrictive compared to true quantum systems, the coefficients T_I depend on an extraordinary number of degrees of freedom which allows exploration of a large volume of the exponential tensor product space. In the case of the three waveguides, these degrees of freedom include (a) the components (or linear combinations) of the eigen vectors of the coupling matrix through the choice of the eigen modes or the application of a rotational operation that creates cyclic permutations of the eigen vector components, (b) the linear coefficients χ_n and $\chi_{n'}$ used to form the multiband linear superposition of states in the linear space, (c) the frequency and therefore wave number which affect the spinor states and the phase factors e^{ikx} and $e^{ik'x}$, and (d) a phase added to the terms e^{ikx} and $e^{ik'x}$.

In the case of $N > 3$, Eq. (21) can be extended to linear combinations of more than two modes with the same frequency, leading to additional freedom in the control of the T_I. Furthermore, the elastic coefficients β of the waveguides and the coupling elastic coefficient α could also be modified by using constitutive materials with tunable elastic properties via, for instance, the piezoelectric, magneto-elastic or photoelastic effects [24–26]. Also note that in all the examples we considered, the coupling of the waveguides had the same strength. Tunability of the coupling elastic medium would lead to the ability to modify the connectivity of the waveguides and therefore the coupling matrix. Exploration of the elastic modes given in Eq. (34) can be realized by varying any number of these variables. We illustrate in **Figure 2** an example of operation in a very simple case. **Figure 2b** shows that by varying a single parameter one may achieve a wide variety of states. For instance, one can obtain states with $T_1 > 0$ and $T_2 > 0$ or $T_1 > 0$ and $T_2 = 0$ or $T_1 > 0$ and $T_2 < 0$ or $T_1 = 0$ and $T_2 = 0$. Another interesting example occurs at $\chi_{n'}$~0.6, there only T_5 and T_8 are different from zero. Then, $\Phi_{2^3\times 1} = \{T_5(0.6)|0\rangle|1\rangle|1\rangle + T_8(0.6)|1\rangle|1\rangle|1\rangle\}e^{i3\omega_0 t}$ can be written as the tensor product state $\{(T_5(0.6)|0\rangle + T_8(0.6)|1\rangle)|1\rangle|1\rangle\}e^{i3\omega_0 t}$. A similar state is also obtained for $\chi_{n'} \sim 0.25$. This is the state $\{(T_1(0.25)|0\rangle + T_4(0.25)|1\rangle)|0\rangle|0\rangle\}e^{i3\omega_0 t}$. Varying $\chi_{n'}$ can be visualized as a matrix operator. For example, in this latter case, one can define the operation:

$$\begin{pmatrix} q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1(\chi_{n'}) \\ T_2(\chi_{n'}) \\ T_3(\chi_{n'}) \\ T_4(\chi_{n'}) \\ T_5(\chi_{n'}) \\ T_6(\chi_{n'}) \\ T_7(\chi_{n'}) \\ T_8(\chi_{n'}) \end{pmatrix} = \begin{pmatrix} T_1(0.25) \\ T_2(0.25) \\ T_3(0.25) \\ T_4(0.25) \\ T_5(0.25) \\ T_6(0.25) \\ T_7(0.25) \\ T_8(0.25) \end{pmatrix} = \begin{pmatrix} T_1(0.25) \\ 0 \\ 0 \\ T_4(0.25) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{36}$$

with $q_{11} = \frac{T_1(0.25)}{T_1(\chi_{n'})}$ and $q_{44} = \frac{T_4(0.25)}{T_4(\chi_{n'})}$. Another interesting state occurs at $\chi_{n'} = 0.5$. Here, we have $T_1 = T_5$, $T_2 = T_3$, $T_4 = T_8$ and $T_6 = T_7$. We also have $T_3 = -T_5$ and $T_4 = -T_6$ This state can be written as the tensor product $(T_1(0.5)|0\rangle - T_6(0.5)|1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)$.

This simple example indicates the large variability in T_I's (i.e., of states) that we can achieve with a single variable. The large number of available variables will lead to even more flexibility in defining states and operators in the exponential tensor product space.

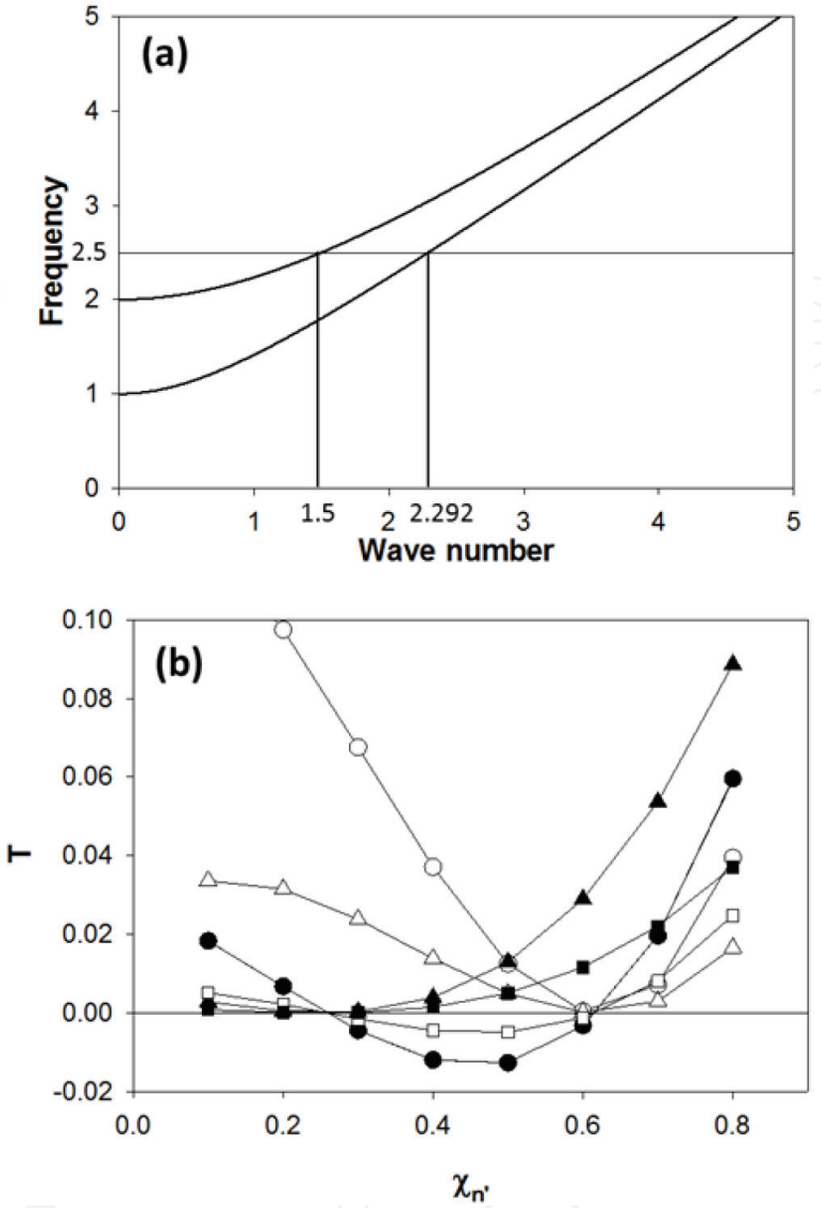

Figure 2. (a) Schematic illustration of the band structure for an array of three elastically coupled waveguides arranged in a ring pattern. Each of the waveguides is also grounded elastically to a rigid substrate. The upper band is doubly degenerate. We have taken $\beta = 1$ and $\alpha = 1$. We highlight the frequency $\omega_0 = 2.5$ corresponding to the wave numbers $k = 2.292$ and $k' = 1.5$. (b) Calculated values of T_1 (open circles), T_2 and T_3 (closed circles), T_4 (open triangles), T_5 (closed triangles), T_6 and T_7 (open squares) and T_8 (closed squares) as functions of $\chi_{n'}$ for $\chi_n = 0.4$. We have fixed $x = 0$.

2.5. Nonseparability of states in exponentially complex space

States given in Eq. (27) are tensor products on the basis $\{|0\rangle\ldots|0\rangle|0\rangle, |0\rangle\ldots|0\rangle|1\rangle, |0\rangle\ldots|1\rangle|0\rangle, \ldots, |1\rangle\ldots|1\rangle|1\rangle\}$. They are therefore always separable in that basis. Consequently these states cannot be written as nonseparable Bell states. However, we might be able to identify a basis in which Eq. (27) is not separable.

Since the Dirac equation (Eq. (15)) for the uncoupled waveguides is linear, its solutions can be a tensor product of a linear combination of N different individual spinors. For instance, we can write:

$$\Psi_{2^N\times 1} = \begin{pmatrix} \rho_1\sqrt{\omega_1+\beta k_1}e^{ik_1x} + \mu_1\sqrt{\omega_1-\beta k_1}e^{-ik_1x} \\ \rho_1\sqrt{\omega_1-\beta k_1}e^{ik_1x} + \mu_1\sqrt{\omega_1+\beta k_1}e^{-ik_1x} \end{pmatrix} e^{i\omega_1 t} \otimes \ldots \\ \otimes \begin{pmatrix} \rho_N\sqrt{\omega_N+\beta k_N}e^{ik_Nx} + \mu_N\sqrt{\omega_N-\beta k_N}e^{-ik_Nx} \\ \rho_N\sqrt{\omega_N-\beta k_N}e^{ik_Nx} + \mu_N\sqrt{\omega_N+\beta k_N}e^{-ik_Nx} \end{pmatrix} e^{i\omega_N t} \tag{37}$$

ρ_I and μ_I, $I = 1, \ldots, N$ are linear coefficients.

The state of the coupled waveguide system will be separable in the exponential space into the state of φ-bits if

$$\Phi_{2^N\times 1} = \Psi_{2^N\times 1} \tag{38}$$

A necessary condition for satisfying Eq. (38) is that $N\omega_0 = \omega_1 + \ldots + \omega_N$.

Furthermore, the first two terms in the column vector $\Phi_{2^N\times 1}$ of the coupled system are $\varphi_1^{(1)}\varphi_1^{(2)}\ldots\varphi_1^{(N)}$ and $\varphi_1^{(1)}\varphi_1^{(2)}\ldots\varphi_2^{(N)}$. Their ratio is simply equal to $r_N^c = \frac{\varphi_1^{(N)}}{\varphi_2^{(N)}}$. Similarly, the ratio of the first two terms in the vector $\Psi_{2^N\times 1}$ of the φ-bit system is given by $r_N^u = \frac{\rho_N\sqrt{\omega_N+\beta k_N}e^{ik_Nx}+\mu_N\sqrt{\omega_N-\beta k_N}e^{-ik_Nx}}{\rho_N\sqrt{\omega_N-\beta k_N}e^{ik_Nx}+\mu_N\sqrt{\omega_N+\beta k_N}e^{-ik_Nx}}$. Anecessary condition for Eq. (38) to be satisfied is that

$$r_N^C = r_N^u \tag{39}$$

This leads to

$$\frac{\chi_n A_N\sqrt{\omega_0+\beta k}e^{ikx} + \chi_{n'}A'_N\sqrt{\omega_0+\beta k'}e^{ik'x}}{\chi_n A_N\sqrt{\omega_0-\beta k}e^{ikx} + \chi_{n'}A'_N\sqrt{\omega_0-\beta k'}e^{ik'x}} = \frac{\rho_N\sqrt{\omega_N+\beta k_N}e^{ik_Nx} + \mu_N\sqrt{\omega_N-\beta k_N}e^{-ik_Nx}}{\rho_N\sqrt{\omega_N-\beta k_N}e^{ik_Nx} + \mu_N\sqrt{\omega_N+\beta k_N}e^{-ik_Nx}}$$

which can be rewritten as:

$$\frac{ue^{ikx} + u'e^{ik'x}}{ve^{ikx} + v'e^{ik'x}} = \frac{\gamma e^{ik_Nx} + \gamma' e^{-ik_Nx}}{\delta e^{ik_Nx} + \delta' e^{-ik_Nx}}$$

This condition takes the more compact form:

$$Pe^{i(k+k_N)x} + Qe^{i(k-k_N)x} + Re^{i(k'+k_N)x} + Se^{i(k'-k_N)x} = 0 \tag{40}$$

P, Q, R, S are real. Eq. (40) is true for all values of position x. At $x = 0$, we obtain the relation $P + Q + R + S = 0$. Inserting that relation into Eq. (27) and eliminating Q yields:

$$i2P\sin k_Nx = \{-(R+S)\cos k_Nx[\cos(k'-k)x - 1] + (R-S)\sin(k'-k)\sin k_Nx\} \\ + i\{(R+S)\sin(k'-k)x + \sin k_Nx[(R-S)\cos(k'-k)x + (R+S)]\}$$

For this condition to be satisfied, one needs the real part of the right-hand side of the equation to be equal to zero. This can be achieved for all x's by setting $k' = k$. In this case, equating the imaginary parts leads to $R = -P$. However, when $k' \neq k$, Eq. (39) and therefore Eq. (38) are not

satisfied. When $k' \neq k$ which corresponds to considering a linear combination of multiband states, $\Phi_{2^N \times 1}$ is not separable into the tensor product of individually uncoupled φ-bit waveguides. Therefore, we conclude that there are a large number of solutions of the nonlinear Dirac equation (Eq. (20)) representing states of arrangements of elastically coupled 1-D waveguides that are not separable in the 2^N dimensional tensor product Hilbert space of individual φ-bits.

We illustrate the notion of nonseparability of exponentially complex states of a coupled system composed of $N = 2$ waveguides on a basis in the exponential Hilbert space of two individual φ-bits. The waveguides are coupled to each other but also to a rigid substrate such that the coupling matrix, $M_{N\times N}$, takes the form:

$$M_{2\times 2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The eigen values and real eigen vectors of this coupling matrix are $\lambda_0^2 = 1,$ and $\lambda_1^2 = 3$ and

$$e_0 = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e_1 = \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{41}$$

Following the procedure of Section 2.4, we construct a tensor product state in the 2^2 exponential space:

$$\begin{aligned} \Phi_{2^2\times 1} = \Big\{ &(\chi_n)^2 A_1 A_2 e^{i2kx} s_{2\times 1} \otimes s_{2\times 1} + \chi_n \chi_{n'} A_1 A_2' e^{ikx} e^{ik'x} s_{2\times 1} \otimes s'_{2\times 1} + \chi_{n'} \chi_n A_1' A_2 e^{ik'x} e^{ikx} s'_{2\times 1} \otimes s_{2\times 1} \\ &+ (\chi_{n'})^2 A_1' A_2' e^{i2k'x} s'_{2\times 1} \otimes s'_{2\times 1} \Big\} e^{i2\omega_0 t} \end{aligned} \tag{42}$$

Eq. (42) is equivalent to Eq. (32) but for two coupled waveguides.

On the basis, $\eta_1 = e^{i2\omega_0 t} e^{i2kx} s_{2\times 1} \otimes s_{2\times 1}$, $\eta_2 = e^{i2\omega_0 t} e^{ikx} e^{ik'x} s_{2\times 1} \otimes s'_{2\times 1}$, $\eta_3 = e^{i2\omega_0 t} e^{ik'x} e^{ikx} s'_{2\times 1} \otimes s_{2\times 1}$, and $\eta_4 = e^{i2\omega_0 t} e^{i2k'x} s'_{2\times 1} \otimes s'_{2\times 1}$, Eq. (42) can be rewritten as:

$$\Phi_{2^2\times 1} = \{ a_{11}\eta_1 + a_{12}\eta_2 + a_{21}\eta_3 + a_{22}\eta_4 \} \tag{43}$$

with $a_{11} = (\chi_n)^2 A_1 A_2 = \frac{1}{2}(\chi_n)^2$, $a_{12} = \chi_n \chi_{n'} A_1 A_2' = -\frac{1}{2}\chi_n \chi_{n'}$, $a_{21} = \chi_{n'} \chi_n A_1' A_2 = \frac{1}{2}\chi_{n'} \chi_n$, and $a_{22} = (\chi_{n'})^2 A_1' A_2' = -\frac{1}{2}(\chi_{n'})^2$. It is then easy to demonstrate that $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} \frac{1}{2}(\chi_n)^2 & -\frac{1}{2}\chi_n \chi_{n'} \\ \frac{1}{2}\chi_{n'} \chi_n & -\frac{1}{2}(\chi_{n'})^2 \end{vmatrix} = 0$, which indicates that the state $\Phi_{2^2\times 1}$ is separable on the basis $\{\eta_1, \eta_2, \eta_3, \eta_4\}$. At this stage, there is nothing surprising as the state $\Phi_{2^2\times 1}$ was constructed as a tensor product. We now try to express the state given in Eq. (42) on a basis of two individually uncoupled φ-bits. Considering the Hilbert space of the first φ-bit, $H^{(1)}$, we use the spinor solutions for uncoupled waveguides given in Eq. (17) to construct the orthonormal basis

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$$\psi_1^{(1)} = \frac{1}{\sqrt{2\omega_1}} \begin{pmatrix} \sqrt{\omega_1 + \beta_1 k_1} \\ \sqrt{\omega_1 - \beta_1 k_1} \end{pmatrix} e^{ik_1 x} e^{i\omega_1 t} = s_1^{(1)}(k_1) e^{ik_1 x} e^{i\omega_1 t} \tag{44a}$$

$$\psi_2^{(1)} = \frac{1}{\sqrt{2\omega_1}} \begin{pmatrix} \sqrt{\omega_1 - \beta_1 k_1} \\ -\sqrt{\omega_1 + \beta_1 k_1} \end{pmatrix} e^{-ik_1 x} e^{i\omega_1 t} = s_2^{(1)}(k_1) e^{-ik_1 x} e^{i\omega_1 t} \tag{44b}$$

Similarly, we define the orthonormal basis in the Hilbert space, $H^{(2)}$, of the second φ-bit,

$$\psi_1^{(2)} = \frac{1}{\sqrt{2\omega_2}} \begin{pmatrix} \sqrt{\omega_2 + \beta_2 k_2} \\ \sqrt{\omega_2 - \beta_2 k_2} \end{pmatrix} e^{ik_2 x} e^{i\omega_2 t} = s_1^{(2)}(k_2) e^{ik_2 x} e^{i\omega_2 t} \tag{45a}$$

$$\psi_2^{(2)} = \frac{1}{\sqrt{2\omega_2}} \begin{pmatrix} \sqrt{\omega_2 - \beta_2 k_2} \\ -\sqrt{\omega_2 + \beta_2 k_2} \end{pmatrix} e^{-ik_2 x} e^{i\omega_2 t} = s_2^{(2)}(k_2) e^{-ik_2 x} e^{i\omega_2 t} \tag{45b}$$

In these equations, we have used $s_1^{(1)}(k_1)$, $s_2^{(1)}(k_1)$, $s_1^{(2)}(k_2)$, and $s_2^{(2)}(k_2)$ as short-hands for the spinor parts of the basis functions.

The basis in the tensor product space $H^{(1)} \otimes H^{(2)}$ is given by the four functions:

$$\tau_1 = \psi_1^{(1)} \otimes \psi_1^{(2)}, \tau_2 = \psi_1^{(1)} \otimes \psi_2^{(2)}, \tau_3 = \psi_2^{(1)} \otimes \psi_1^{(2)}, \tau_4 = \psi_2^{(1)} \otimes \psi_2^{(2)} \tag{46}$$

We have

$$\tau_1 = s_1^{(1)}(k_1) \otimes s_1^{(2)}(k_2) e^{i(k_1+k_2)x} e^{i(\omega_1+\omega_2)t} \tag{47a}$$

$$\tau_2 = s_1^{(1)}(k_1) \otimes s_2^{(2)}(k_2) e^{i(k_1-k_2)x} e^{i(\omega_1+\omega_2)t} \tag{47b}$$

$$\tau_3 = s_2^{(1)}(k_1) \otimes s_1^{(2)}(k_2) e^{i(-k_1+k_2)x} e^{i(\omega_1+\omega_2)t} \tag{47c}$$

$$\tau_4 = s_2^{(1)}(k_1) \otimes s_2^{(2)}(k_2) e^{i(-k_1-k_2)x} e^{i(\omega_1+\omega_2)t} \tag{47d}$$

It is straightforward to show that $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ form an orthogonal basis. That is, $\tau_i^\dagger \tau_j = 0$ if $i \neq j$, where $\tau_i^\dagger$ is the Hermitian conjugate of τ_i.

We want now to express the state $\Phi_{2^2 \times 1}$ in the τ basis:

$$\Phi_{2^2 \times 1} = \{b_{11}\tau_1 + b_{12}\tau_2 + b_{21}\tau_3 + b_{22}\tau_4\} \tag{48}$$

For this, we now need to expand the basis vectors $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ on the basis $\{\tau_1, \tau_2, \tau_3, \tau_4\}$

We define the expansions:

$$\eta_1 = c_{11}\tau_1 + c_{12}\tau_2 + c_{13}\tau_3 + c_{14}\tau_4 \tag{49a}$$

$$\eta_2 = c_{21}\tau_1 + c_{22}\tau_2 + c_{23}\tau_3 + c_{24}\tau_4 \tag{49b}$$

$$\eta_3 = c_{31}\tau_1 + c_{32}\tau_2 + c_{33}\tau_3 + c_{34}\tau_4 \tag{49c}$$

$$\eta_4 = c_{41}\tau_1 + c_{42}\tau_2 + c_{43}\tau_3 + c_{44}\tau_4 \tag{49d}$$

Note that the c_{ij}'s are functions of k_1, k_2, x, and t.

We can find the coefficients c_{ij} by exploiting the orthogonality of the $\tau_i's$. For instance, we can multiply Eq. (49a) to the left by the Hermitian conjugate $\tau_1^\dagger$, leading to

$$\tau_1^\dagger\eta_1 = c_{11}\tau_1^\dagger\tau_1 + c_{12}\tau_1^\dagger\tau_2 + c_{13}\tau_1^\dagger\tau_3 + c_{14}\tau_1^\dagger\tau_4 = c_{11}\tau_1^\dagger\tau_1 \tag{50}$$

or

$$\begin{aligned} c_{11}(k_1,k_2,x,t) = &\left(s_1^{(1)}(k_1)\otimes s_1^{(2)}(k_2)\right)^\dagger (s_{2\times1}\otimes s_{2\times1})e^{-i(k_1+k_2)x}e^{i2kx}e^{-i(\omega_1+\omega_2)t}e^{i2\omega_0 t}/ \\ &\left(s_1^{(1)}(k_1)\otimes s_1^{(2)}(k_2)\right)^\dagger \left(s_1^{(1)}(k_1)\otimes s_1^{(2)}(k_2)\right) \end{aligned} \tag{51}$$

We can obtain all other c_{ij}'s in a similar fashion. Eqs. (49a)–(49d) can be rewritten in the form:

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_1\end{pmatrix} = \eta_{4\times1} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22} & c_{23} & c_{24}\\ c_{31} & c_{32} & c_{33} & c_{34}\\ c_{41} & c_{42} & c_{43} & c_{44}\end{pmatrix}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\end{pmatrix} = C_{4\times4}\tau_{4\times1} \tag{52}$$

The matrix $C_{4\times4}$ can be diagonalized. Let d_1, d_2, d_3, and d_4 be the four eigen values of $C_{4\times4}$ with their associated eigen vectors $\begin{pmatrix}v_{11}\\ v_{12}\\ v_{13}\\ v_{14}\end{pmatrix}$, $\begin{pmatrix}v_{21}\\ v_{22}\\ v_{23}\\ v_{24}\end{pmatrix}$, $\begin{pmatrix}v_{31}\\ v_{32}\\ v_{33}\\ v_{34}\end{pmatrix}$ and $\begin{pmatrix}v_{41}\\ v_{42}\\ v_{43}\\ v_{44}\end{pmatrix}$. We can construct the following matrix out of the four eigen vectors:

$$V_{4\times4} = \begin{pmatrix} v_{11} & v_{21} & v_{31} & v_{41}\\ v_{12} & v_{22} & v_{32} & v_{42}\\ v_{13} & v_{23} & v_{33} & v_{43}\\ v_{14} & v_{24} & v_{34} & v_{44}\end{pmatrix}$$

On the new basis $\{\tilde{\tau}_1,\tilde{\tau}_2,\tilde{\tau}_3,\tilde{\tau}_4\}$ constructed by using the relation $\tilde{\tau}_{4\times1} = V_{4\times4}^{-1}\tau_{4\times1}V_{4\times4}$, the matrix that couples the η basis and the τ basis takes the form: $\tilde{C}_{4\times4} = \begin{pmatrix} d_1 & 0 & 0 & 0\\ 0 & d_2 & 0 & 0\\ 0 & 0 & d_3 & 0\\ 0 & 0 & 0 & d_4\end{pmatrix}$ so $\eta_1 = d_1\tilde{\tau}_1$, $\eta_2 = d_2\tilde{\tau}_2$, $\eta_3 = d_3\tilde{\tau}_3$ and $\eta_4 = d_4\tilde{\tau}_4$. On the $\tilde{\tau}$ basis, Eq. (43) can be rewritten as

$$\Phi_{2^2\times1} = \{a_{11}d_1\tilde{\tau}_1 + a_{12}d_2\tilde{\tau}_2 + a_{21}d_3\tilde{\tau}_3 + a_{22}d_4\tilde{\tau}_4\} \tag{53}$$

Then on the basis $\tilde{\tau}$, we can investigate the separability or nonseparability of $\Phi_{2^2\times1}$ by calculating the determinant of the linear coefficients in Eq. (53), that is

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$$\det\begin{pmatrix} a_{11}d_1 & a_{12}d_2 \\ a_{21}d_3 & a_{22}d_4 \end{pmatrix} = \begin{vmatrix} \frac{1}{2}(\chi_n)^2 d_1 & -\frac{1}{2}\chi_n\chi_{n'}d_2 \\ \frac{1}{2}\chi_{n'}\chi_n d_3 & -\frac{1}{2}(\chi_{n'})^2 d_4 \end{vmatrix} = -\frac{1}{4}(\chi_n)^2(\chi_{n'})^2[d_1d_4 - d_2d_3] \tag{54}$$

Only in the unlikely event of degenerate eigen values, d_1, d_2, d_3, and d_4, would this determinant be equal to zero. A nonzero determinant given in Eq. (54) indicates that the state $\Phi_{2^2\times 1}$ is nonseparable on the basis $\{\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4\}$.

The existence of nonseparable solutions to the nonlinear Dirac equation raises the possibility of exploiting these solutions for storing and manipulating data within the 2^N dimensional tensor product Hilbert space. The exploration of algorithms for exploiting these solutions is beyond the scope of this chapter; however, we note that these solutions may well be observed in physical systems including elastic waveguides which are embedded in a coupling matrix. The manipulation of the system could be achieved either by externally altering the parameters of the system, i.e., the elastic properties of the waveguides, or by changing the frequency and wavenumber of input waves. These possibilities are illustrated for a five-waveguide system driven by transducers in Section 2.6.

2.6. Physical realization and actuation

Figure 3 illustrates a possible realization of a five waveguide system. The parallel elastic waveguides are embedded in an elastic medium which couples them elastically. The waveguides are arranged in a ring pattern.

Modes of the form given in Eq. (21) can be excited with N transducers attached to the input ends of the N waveguides and connected to N phase-locked signal generators to excite the appropriate eigen vectors e_n and $e_{n'}$. These modes can be excited by applying a superposition of signals on the transducers with the appropriate phase, amplitude and frequency relations. The frequencies $\omega_n(k)$ and $\omega_{n'}(k')$ are used to control the spinor parts of the wave functions

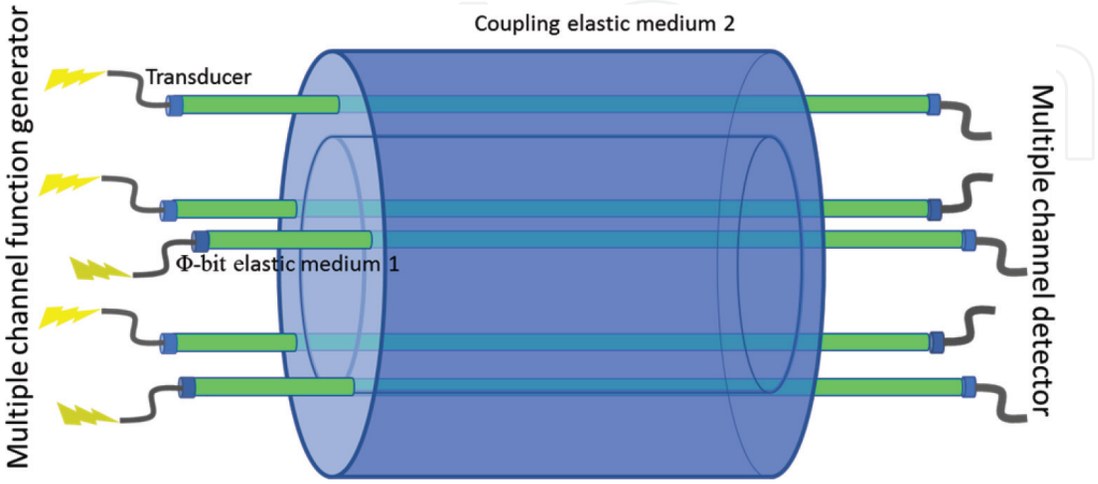

Figure 3. Schematic illustration of a five waveguide system. The waveguides are composed of an elastic medium 1 in which mass density and elastic stiffness determine the physical parameter β. The waveguides are embedded in an elastic medium 2 in which mass density and stiffness relate to the parameter α. The waveguides are actuated via transducers (see the text for details).

$s_{2\times1}(k)$ and $s_{2\times1}(k')$. The spinor components which represent a quasistanding wave can be quantified by measuring the transmission coefficient (normalized transmitted amplitude) along any one of the waveguides. It is then possible to operate on the eigen vectors e_n and $e_{n'}$ without affecting the spinor states or vice versa. For instance, one could apply a rotation that permutes cyclically the components of e_n by changing the phase of the signal generators. Such an operation could be quantified by measuring the phase of the transmission amplitude at the output end of the waveguides.

3. Conclusions

We have shown that the directional projection of elastic waves supported by a parallel array of N elastically coupled waveguides can be described by a nonlinear Dirac-like equation in a 2^N dimensional exponential space. This space spans the tensor product Hilbert space of the two-dimensional subspaces of N uncoupled waveguides grounded elastically to a rigid substrate (which we called φ-bits). We demonstrate that we can construct tensor product states of the elastically coupled system that are nonseparable on the basis of tensor product states of N uncoupled φ-bits. A φ-bit exhibits superpositions of directional states that are analogous to those of a quantum spin, hence it acts as a pseudospin. Since parallel arrays of coupled waveguides span the same exponentially complex space as that of uncoupled pseudospins, the type of elastic systems described here may serve as a simulator of interacting spin networks. The possibility of tuning the elastic coefficients and the elastic coupling constants of the waveguides would allow us to explore the properties of spin networks with variable connectivity and coupling strength. The mapping between the $2N$ dimensional and the 2^N dimensional representations of the elastic system leads to the capacity for exploring an exponentially scaling space by handling a linearly growing number of waveguides (i.e., preparation, manipulation, and measurement of these states). The scalability of the elastic system, the coherence of elastic waves at room temperature, and the ability to measure classical superpositions of states may offer an attractive way for addressing exponentially complex problem through the analogy with quantum systems.

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Conflict of interest

The authors declare that they have no affiliations with or involvement in any organization or entity with any financial interest or nonfinancial interest in the subject matter or materials discussed in this manuscript.

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