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Plasma Kinetic Theory

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Abstract

The description of plasma using fluid model is mostly insufficient and requires the consideration of velocity distribution which leads to kinetic theory. Kinetic theory of plasma describes and predicts the condition of plasma from microscopic interactions and motions of its constituents. It provides an essential basis for an introductory course on plasma physics as well as for advanced kinetic theory. Plasma kinetics deals with the relationship between velocity and forces and the study of continua in velocity space. Plasma kinetics mathematical equations provide aid to the readers in understanding simple tools to determine the plasma dynamics and kinetics as described in this chapter. Kinetic theory provides the basics and essential introduction to plasma physics and subsequently advanced kinetic theory. Plasma waves, oscillations, frequencies, and applications are the subjects of kinetic theory. In this chapter, mathematical formulations essential for exploring plasma kinetics are compiled and described simplistically along with a precise discussion on basic plasma parameters in simple language with illustrations in some cases.

Keywords: plasma parameters, kinetic theory, particle distribution

1. Introduction

Plasma is the fourth state of matter, and it is defined as “a quasineutral gas of charged and neutral particles which exhibits collective behavior.” As plasma contains charged particles, these charged particles move around and generate local concentrations of positive or negative charges (collective behavior) which give rise to electric fields. Motion of these charges also generates currents and hence magnetic fields [1]. Therefore, the macroscopic forces acting in plasma are totally different from ordinary gasses and hence remarkable differences in their physical properties are observed. The salient features of the plasma can be understood by investigating the behavior of the electrons, by far the most mobile-charged particle in plasma [2].

Plasma physics deals with the equilibrium and non-equilibrium properties of a statistical system of charged particles. Microscopic degrees of freedom arising from the motion of individual particles describe the system. These statistics therefore theoretically treat the macroscopic behavior of such a system [3].

The knowledge of plasma parameters helps to understand the dynamics of plasma. Electrons being dominant mobile species play an important role in the behavior of the plasma. The most important of these parameters include plasma temperature, electron density, Debye shielding, and Debye length. Plasma is transient in nature. Therefore, the plasma is generally characterized on the basis of instantaneous observations. Charged particles, neutrals, and molecules coexist in plasma under various circumstances. Conditions in the plasma strongly depend on the distribution of charged particles, where electrons being lighter and highly mobile play a dominant role. Therefore, plasma is generally represented through parameters which are derived from the behavior of electrons in the plasma. When an external point charge is introduced or a localized unbalanced charge is formed in the plasma, readjustment of charge density occurs to neutralize the effect by shielding its electric field. The electrons being more mobile than heavier ions move toward or away from the unbalanced charge faster than ions. It gives rise to oscillations which are referred to as *electron oscillations* or *plasma oscillations*. The frequency at which these oscillations take place is called as *plasma frequency* [4]. This phenomenon of shielding or screening a foreign charge or an unbalanced charge inside the plasma is known as *Debye shielding* or sometimes referred to as *Debye screening*. It is specific for plasma. Charges keep accumulating around the foreign or unbalanced charge until the static electric field of the unbalanced charge is screened and the balance is restored. A sphere of charges that is created around the unbalanced charge is called as Debye sphere and its radius is known as *Debye length*. A detailed discussion on two of the most important plasma parameters *electron density* and *plasma temperature* is provided in Sections 2.7 and 2.8.

Comprehensive mathematical details on certain aspects of plasma are found in published literature. The authors aim to provide readers with an overall view of fundamental mathematical relations explaining the kinetics of plasma that are compiled simplistically in one chapter. This can be an easy reference for the researchers interested in plasma kinetics.

2. Plasma kinetic equations

Plasma physics involves phenomena that are related to dynamical processes in statistical mechanics. It is thus very significant to study the properties and structure of the basic kinetic equations governing the dynamical behavior of plasma [5]. The dynamical behavior of a system of N -interacting particles is generally investigated using the Liouville equation. A microscopic distribution function could be used to describe the behavior of such a system. A six-dimensional phase-space distribution function called “Klimontovich distribution function,” which obeys a continuity equation in the phase space, is defined. The system of charged particles can then be described by Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy equations.

2.1. Klimontovich equation

To further formalize the kinetic theory, we introduce Klimontovich's microscopic description and derivation of the Bogoliubov-Born-Green-Kirkwood-Yvon equations [6]. To introduce the Klimontovich equation, we consider a classical system containing N identical particles in a box of volume V ; $n \equiv N/V$ denoting the average number density. Each particle in the box is characterized by an electric charge q and mass m .

In the six-dimensional phase space consisting of the position \mathbf{r} and velocity \mathbf{v} , each of the particle has its own trajectory; for i th particle,

$$X_i(t) \equiv [\mathbf{r}_i(t), \mathbf{v}_i(t)]. \quad (1)$$

The microscopic density of the particles in the phase space may be expressed by the summation of the six-dimensional delta functions as

$$N(X; t) \equiv \left(\frac{1}{n}\right) \sum_{i=1}^n \delta[X - X_i(t)] \quad (2)$$

where $X \equiv (\mathbf{r}, \mathbf{v})$. $N(X; t)$ is known as the *Klimontovich distribution function* which satisfies the continuity equation in the phase space,

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \dot{x} \cdot \frac{\partial N}{\partial X} = 0. \quad (3)$$

In phase-space coordinates, Eq. (3) can be written as

$$\frac{dN}{dt} + \mathbf{v} \cdot \frac{\partial N}{\partial \mathbf{r}} + \dot{\mathbf{v}} \cdot \frac{\partial N}{\partial \mathbf{v}} = 0 \quad (4)$$

where $\dot{\mathbf{v}}$ is the acceleration at point (\mathbf{r}, \mathbf{v}) .

The electromagnetic acceleration is very important in plasma physics,

$$\dot{\mathbf{v}} = \frac{q}{m} \left[\mathbf{E}(\mathbf{r}, t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{r}, t) \right]. \quad (5)$$

The electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ consist of two separate contributions: those applied from the external sources and those produced from the microscopic fine-grained distribution of the charged particles.

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_{ext}(\mathbf{r}, t) + \mathbf{e}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_{ext}(\mathbf{r}, t) + \mathbf{b}(\mathbf{r}, t). \quad (6)$$

The microscopic fine-grained fields $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{b}(\mathbf{r}, t)$ are determined from a solution of Maxwell equations,

$$\nabla \times \mathbf{e} + \frac{1}{c} \frac{\partial \mathbf{b}}{\partial t} = 0, \nabla \times \mathbf{b} - \frac{1}{c} \frac{\partial \mathbf{e}}{\partial t} = \frac{4\pi}{c} qn \int \mathbf{v} N(X; t) d\mathbf{v}, \nabla \cdot \mathbf{e} = 4\pi qn \left[\int N(X; t) d\mathbf{v} - 1 \right], \nabla \cdot \mathbf{b} = 0. \quad (7)$$

For a given $N(X; t)$, the solution to these set of equations can generally be written; the solution, when substituted in Eq. (5), would amount to taking account of both electromagnetic and electrostatic interactions between the particles. The electromagnetic interactions are usually negligible as compared with the electrostatic interactions for a nonrelativistic plasma; hence, the microscopic fields become.

$$e(\mathbf{r}, t) = -qn \frac{\partial}{\partial \mathbf{r}} \int \frac{N(\mathbf{X}'; t)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{X}', \mathbf{b}(\mathbf{r}, t) = \mathbf{0}. \quad (8)$$

Substituting Eq. (8) into Eq. (6), we get an expression for the acceleration in terms of $N(X; t)$. Eq. (3) can be written with the aid of such an expression as

$$\left[\int \frac{\partial}{\partial t} + L(X) - \int V(X, X') N(X'; t) dX' \right] N(X; t) = 0 \quad (9)$$

$L(X)$, as a single particle operator is defined by

$$L(X) \equiv \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{q}{m} \left[\mathbf{E}_{ext}(\mathbf{r}, t) + \frac{\mathbf{v}}{c} \times \mathbf{B}_{ext}(\mathbf{r}, t) \right] \cdot \frac{\partial}{\partial \mathbf{v}} \quad (10)$$

And $V(X, X')$ is a two-particle operator arising from the Coulomb interaction defined by

$$V(X, X') \equiv \frac{q^2 n}{m} \left[\frac{\partial}{\partial \mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] \cdot \frac{\partial}{\partial \mathbf{v}} \quad (11)$$

Eq. (9) is known as *Klimontovich equation*. The equation describes the space–time evolution of the microscopic distribution function.

2.2. Liouville distribution

The fine-grained distribution function, which is precise in describing the microscopic conditions of many particles, would not by itself correspond to the coarse-grained quantities in the macroscopic view. There is a need to introduce an averaging process based on the Liouville distribution over the $6N$ -dimensional phase space to establish a connection between them.

In T-space, the microscopic state of the system is expressed by a point.

$\{X_i\} \equiv (X_1, X_2, \dots, X_n)$, called a system point.

By following a normal procedure of the ensemble theory in statistical mechanics, it can assume N replicas which are microscopically identical to the system under consideration. N can be chosen to be very large so that it can approach infinity when it requires. These N replicas are characterized by different microscopic configurations; the system points are scattered over the T-space. Liouville distribution function [7] can then be defined as $D(\{X_i\}; t)$ in the T-space as

$$D(\{X_i\}; t) d\{X_i\} \equiv \lim_{n \rightarrow \infty} \frac{[\text{No. of system points in the infinitesimal volume } d\{X_i\} \text{ in } T\text{-space withing } \{X_i\}]}{N} \quad (12)$$

which by definition satisfies the normalization condition.

$$\int D(\{X_i; t\})d\{X_i\} = 1. \tag{13}$$

The N system points distributed in the T-space do not interact with each other, behaving like an ideal gas. The distribution function $D(\{X_i; t\})$ therefore satisfies a Liouville-type continuity equation

$$\frac{\partial D}{\partial t} + \{\dot{X}_i\} \cdot \frac{\partial D}{\partial \{X_i\}} = 0 \tag{14}$$

The distribution is conserved along a trajectory in the phase-space distribution.

We can now perform a statistical averaging of a fine-grained quantity $A(X, X', \dots; \{X_i(t)\})$ defined at a set of points (X, X', \dots) in the six-dimensional phase space. With the aid of Liouville distribution, we follow this way:

$$\langle A(X, X', \dots; t) \rangle = \int d\{X_i\} D\{X_i; t\} A(X, X', \dots; \{X_i\}) \tag{15}$$

With respect to the conservation property, this average can be transformed equivalently into an average over the initial distribution, such that

$$\langle A(X, X', \dots; t) \rangle = \int d\{X_i(0)\} D\{X_i(0); t\} A(X, X', \dots; \{X_i(0)\}) \tag{16}$$

where $\{X_i(0); t\}$ represents the coordinates of the system points in T-space at t under the condition that it is located at $\{X_i(0)\}$ when $t = 0$.

2.3. BBGKY hierarchy

The distribution functions can be obtained through a statistical average of products of Klimontovich functions. A shorthand notation and numerals 1, 2, 3, ..., etc., in place of X, X', X'' , ..., etc., can be used to simplify the presentations.

The Klimontovich Eq. (9) can therefore be written as

$$\left[\frac{\partial}{\partial t} + L(1) \right] N(1; t) = \int V(1, 2) N(1; t) N(2; t) d2. \tag{17}$$

The Liouville average of this equation can be obtained by using the methods in Eq. (16). The averaging process commutes with differential operators involved in Eq. (17). Now, with the aid of a single-particle distribution function

$$\langle N(X; t) \rangle = f_1(X; t) \tag{18}$$

The average of the second term defines the two-particle distribution function:

$$f_2(X, X'; t); \langle N(X; t)N(X'; t) \rangle = \frac{1}{n} \delta(X - X') f_1(X; t) + f_2(X, X'; t). \tag{19}$$

$$\left[\frac{\partial}{\partial t} + L(1) \right] f_1(1; t) = \int V(1, 2) \left\{ \frac{1}{n} \delta(1 - 2) f_1(1; t) + f_2(1, 2; t) \right\} d2 \tag{20}$$

For an arbitrary function $y(1, 2, \dots; t)$, it can be proved from symmetry considerations that

$$\int V(1, 2) \delta(1 - 2) y(1, 2, \dots; t) d2 = 0 \tag{21}$$

Consequently,

$$\left[\frac{\partial}{\partial t} + L(1) \right] f_1(1; t) = \int V(1, 2) f_2(1, 2; t) d2 \tag{22}$$

It can start from an equation as well

$$\left[\frac{\partial}{\partial t} + L(1) + L(2) \right] N(1; t)N(2; t) = \int [V(1, 3) + V(2, 3)] N(1; t)N(2; t)N(3; t) d3 \tag{23}$$

Eq. (23) can be derived from a combination of Klimontovich equations. After averaging this equation with respect to the Liouville distribution and Eqs. (16) and (17), an equation involving f_1, f_2 , and f_2 can be obtained. This equation can then be simplified with the aid of Eqs. (21) and (22), the result yields

$$\left\{ \frac{\partial}{\partial t} + L(1) + L(2) - \frac{1}{n} [V(1, 2) + V(2, 1)] \right\} f_2(1, 2; t) = \int [V(1, 3) + V(2, 3)] f_3(1, 2, 3; t) d3. \tag{24}$$

Similarly, it is considered that a Klimontovich equation for a product of an arbitrary number of the Klimontovich functions performs a statistical average of the equation. We therefore obtain the BBGKY hierarchy equations expressed as

$$\left[\sum_{i=1}^s L(i) - \frac{1}{n} \sum_{i \neq j}^s V(i, j) \right] f_s(1, \dots, s; t) = \sum_{i=1}^s \int [V(i, s+1) f_{s+1}(1, \dots, s+1; t)] d(s+1). \tag{25}$$

The set of equations in Eq. (25) is the basis for the kinetic theory of plasmas.

2.4. Vlasov's equation

For identical non-interacting particles, Liouville's equation can be written in T-space. Introducing two properties of identical non-interacting particles such as the distribution function f and the Hamiltonian function q simplifies the problem. The distribution function written as a function of $6N$ variables and time factorizes to a product of N functions, each involving only the coordinates and momenta of one particle, and time.

The Hamiltonian function of N non-interacting particles is the sum of N terms, each involving only the coordinates and momenta of one particle. For identical particles, the terms of the Hamiltonian are also identical. For weakly or occasionally interacting particles, the decomposition of f into a product of factors and of q into a sum of terms is identical.

For a collection of N identical interacting particles, Liouville's theorem can be written in the μ -space as

$$\frac{\partial f}{\partial t} + \sum_1^3 \left[\left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial x_j}{\partial t} \right) + \left(\frac{\partial f}{\partial p_j} \right) \left(\frac{\partial p_j}{\partial t} \right) \right] = \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t} \right)_{int} \quad (26)$$

The right-hand side of Eq. (26) can be evaluated using the exact form of the interaction terms in the $6N+1$ Hamiltonian function variable. It is assumed that the Hamiltonian expression involves no magnetic terms. Under such conditions, $p = Mv$. If the coordinate system in the μ -space is changed from x, y, z, p_x, p_y, p_z to x, y, z, v_x, v_y, v_z , then the corresponding volume elements will be in the ratio M^3 . Hence, if the figurative points density in the (r, p) space is constant according to Eq. (26),

$$\frac{\partial f}{\partial t} + \sum_1^3 \left[\left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial x_j}{\partial t} \right) + \left(\frac{\partial f}{\partial v_j} \right) \left(\frac{\partial v_j}{\partial t} \right) \right] = \frac{\partial f}{\partial t} = 0 \quad (27)$$

If magnetic terms are included in the Hamiltonian function, then $p = Mv + qA$. When the coordinates are changed, the ratio of the corresponding volume elements can be calculated by using Jacobian. The Jacobian is a determinant calculated by taking the partial derivative of any coordinate of one system with respect to all the coordinates of the second system. For instance, in the physical space, the Jacobian is

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x_1}{\partial x_2} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial y_2} & \frac{\partial y_1}{\partial z_2} \\ \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial y_2} & \frac{\partial z_1}{\partial z_2} \end{vmatrix}$$

The value of the determinant is equal to the ratio of the corresponding volume elements. The ratio of the volume elements, M^3 , is constant even in the presence of magnetic forces.

$\frac{\partial v_j}{\partial t} = a_j = F_j/M$, where a = acceleration and F = external force. Introducing an operator ∇_v

$$\nabla_v = i \frac{\partial}{\partial v_x} + j \frac{\partial}{\partial v_y} + k \frac{\partial}{\partial v_z} \quad (28)$$

where i, j , and k are the unit vectors in the v_x, v_y , and v_z directions, respectively. In a vectorial form, Eq. (27) now becomes

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{M} \cdot \nabla_{\mathbf{v}} f = 0 \quad (29)$$

Eq. (29) is known as *Vlasov's equation*. F is the sum of the electric, magnetic, and gravitational forces resulting from external fields and the macroscopic forces resulting from the plasma itself. If we consider the viscous-like forces, Vlasov's equation becomes

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \nabla_{\mathbf{v}} \cdot \left[\frac{\mathbf{F}}{M} f \right] = 0 \quad (30)$$

2.5. Maxwell's equations

Maxwell's equations express the relations between electric and magnetic fields in a medium. Consider a current I flowing in an element of length dl . The magnetic flux density $d\mathbf{B}$ produced by this current at a point \mathbf{P} , a distance r from the element, and making an angle θ with its axis is known as Ampere's law

$$d\mathbf{B} = K I dl \sin \theta / r^2 \quad (31)$$

where K is a constant of proportionality defined as

$$K = \mu / 4\pi \quad (32)$$

where μ = permeability of the medium. For vacuum, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m. The total magnetic flux density \mathbf{B} produced at point \mathbf{P} by the current flowing in a long conductor is

$$\mathbf{B} = \frac{\mu I}{4\pi} \int \frac{\sin \theta}{r^2} dl \quad (33)$$

For infinite and linear conductor, $\mathbf{B} = \frac{\mu I}{2\pi r}$; where r = radial distance from \mathbf{P} to the linear conductor as shown in **Figure 1**.

Figure 1. Flux density near a straight wire in which current I flows where $\sin \theta dl = r d\theta$ and $r_0 = r \sin \theta$.

If \mathbf{B} is integrated around the path that encloses the wire, then

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu I \quad (34)$$

A magnetic field vector \mathbf{H} is introduced to make the equation independent of the medium. It is such that

$$\mathbf{B} = \mu \mathbf{H} \quad (35)$$

Introducing Eq. (35) into Eq. (34) and the current I by the surface integral of the conduction current density \mathbf{J} over the area described by the path of integration of \mathbf{H} ,

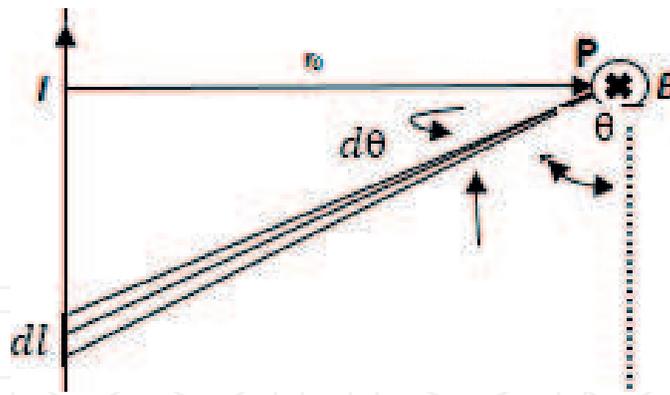


Figure 1. Flux density near a straight wire in which current I flows where $\sin\theta dl = r d\theta$ and $r_0 = r \sin\theta$.

$$\int \mathbf{H} \cdot d\mathbf{l} = \iint_s \mathbf{J} \cdot d\mathbf{S} \quad (36)$$

The total current density = conduction current density σE ; σ = conductivity of the wire and E = electric field. $\partial D / \partial t$ = displacement current density; D = electric flux density. Applying Stokes' theorem, Eq. (36) can now be written in a general form as

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (37)$$

According to Faraday's law, the total e.m.f (V) induced in a closed loop as a function of the total flux Φ_m / dt producing the e.m.f is given as.

$$V = - d\Phi_m / dt \quad (38)$$

If the total flux linkage Φ_t is $n\Phi_m$, then $V = -d\Phi_t / dt$. The total flux through the circuit is equal to the integral of \mathbf{B} over the area bounded by the circuit. Therefore,

$$V = - \frac{d}{dt} \iint_s \mathbf{B} \cdot d\mathbf{S} \quad (39)$$

The change in magnetic field produces an electric field E , thus

$$V = \int \mathbf{E} \cdot d\mathbf{l} \text{ or } \mathbf{E} = -\nabla V \quad (40)$$

Combining Eqs. (39) and (40), the induced e.m.f is

$$\int \mathbf{E} \cdot d\mathbf{l} = - \iint \left(\frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} \quad (41)$$

Applying Stokes' theorem on Eq. (41),

$$\nabla \times \mathbf{E} = -\left(\frac{\partial \mathbf{B}}{\partial t}\right) \quad (42)$$

Gauss' law states that the surface integral of the normal component of the electric flux density \mathbf{D} over any closed surface equals the total enclosed charge q [8]. \mathbf{D} is proportional to the electric field with permittivity ϵ of the medium as the constant of proportionality ($\mathbf{D} \propto \epsilon \mathbf{E}$). $\epsilon = \epsilon_0$ in free space.

Replacing q with the integral of the charge density ρ_e over the volume enclosed by the surface \mathbf{S} , the vectorial form

$$\nabla \cdot \mathbf{D} = \rho_e \quad (43)$$

For magnetic field, the integral of \mathbf{B} over a closed surface is always equal to zero, thus

$$\nabla \cdot \mathbf{B} = 0 \quad (44)$$

Eqs. (37) and (42)–(44) are known as Maxwell's equations. The equations are satisfied in all plasma physics phenomena.

2.6. Liouville's theorem

In relation with the Boltzmann approach, most of the problems of statistical mechanics are best studied in multidimensional spaces called "phase spaces." Consider a μ -space, which is a six-dimensional and makes use of coordinates and the three components of momentum x, y, z, p_x, p_y, p_z or any set of Lagrangian coordinates for a point together with the associated generalized momenta. In this space, each plasma particle is represented by a point. If only one degree of freedom exists, the μ -space can be represented on a plane of **Figure 2**.

Figure 2. Trajectory of an oscillating point in the μ -space.

In the μ -space, the distribution function f is a function of seven variables. Thus, the probability of finding a particle in a given volume element depends only on the coordinates and momenta of this particle, not on those of the other particles. This is a simplified version of Liouville's theorem for a large number of non-interacting particles.

In the six-dimensional μ -space, the particles are conservative just as they are in the ordinary space. Hence, the conservation theorem may be applied in the phase space,

$$\frac{\partial f}{\partial t} + \nabla_6 \cdot (f \mathbf{v}_6) = 0 \quad (45)$$

where ∇_6 = six-dimensional divergence and \mathbf{v}_6 is a six-dimensional velocity vector whose components are exact time derivatives of the six coordinates of the μ -space. We now write ∇ temporarily x_1, x_2, x_3 instead of x, y, z ,

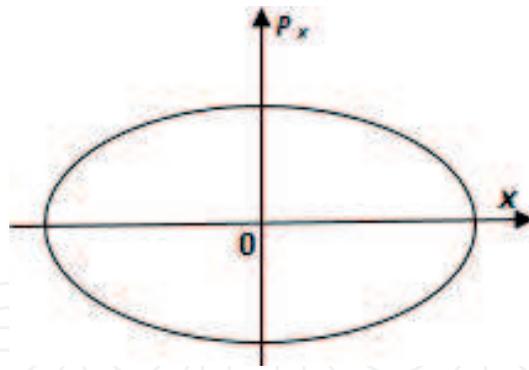


Figure 2. Trajectory of an oscillating point in the μ -space.

$$\frac{\partial B}{\partial t} + \sum_1^3 \left[\frac{\partial(f\dot{x}_j)}{\partial x_j} + \frac{\partial(f\dot{p}_j)}{\partial p_j} \right] = 0 \tag{46}$$

or

$$\frac{\partial f}{\partial t} + \sum_1^3 \left[\frac{\partial f}{\partial x_j} \dot{x}_j + \frac{\partial f}{\partial p_j} \dot{p}_j + f \left(\frac{\partial \dot{x}_j}{\partial x_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right) \right] = 0 \tag{47}$$

Hamilton’s canonical equations yield

$$\frac{\partial q_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial}{\partial q_j} \left(\frac{\partial q}{\partial p_j} \right) - \frac{\partial}{\partial p_j} \left(\frac{\partial q}{\partial q_j} \right) = \frac{\partial^2 q}{\partial p_j \partial q_j} - \frac{\partial^2 q}{\partial q_j \partial p_j} = 0 \tag{48}$$

where q = Hamiltonian. Eq. (47) now becomes

$$\frac{\partial f}{\partial t} + \sum_1^3 \left[\left(\frac{\partial f}{\partial x_j} \right) \left(\frac{\partial x_j}{\partial t} \right) + \left(\frac{\partial f}{\partial p_j} \right) \left(\frac{\partial p_j}{\partial t} \right) \right] = \frac{\partial f}{\partial t} = 0 \tag{49}$$

Eq. (49) is known as *Liouville’s equation*.

2.7. Boltzmann’s equation

The Boltzmann equation provides the statistical analysis of all the individual positions and momenta of each particle in the fluid (macro-system) at an instant, that is, number of particles in a particular level and their distribution among different levels [9]. It gives relative number of atoms in different excitation states as a function of temperature and refers to certain number of atoms or ions in a particular excitation state with respect to the ground state. Boltzmann equation gives a mathematical description of the state of a system and how it changes. It describes a quantity called the distribution function, f , which depends on a position, velocity, and the time. The function f determines the average number of particles having velocities

within a small range from ν to $\nu + \Delta\nu$ and coordinates within a small range from r to $r + \Delta r$ in time Δt .

In a hot dense gas, the atoms constantly experience collisions with each other, which lead to excitation to the different possible energy levels. The collisional excitation follows radiative de-excitation in timescales of the order of nanoseconds. For a constant temperature and pressure, a dynamic equilibrium is established between collisional excitations and radiative de-excitations, which lead to particular distribution of the atoms among different energy levels. Most of the atoms are at low-lying levels. The number of atoms at higher levels decreases exponentially with energy level. At low temperature, the faster the population drops at the higher levels. Only at very high temperatures, high-lying energy levels are occupied by an appreciable number of atoms. Boltzmann's equation gives the distribution of the atoms among the various energy levels as a function of energy and temperature.

Let us consider a system at local thermal equilibrium (LTE) with a constant volume consisting of " N " atoms and each of atoms has " m " possible energy levels. Suppose there are " N_j " atoms in energy level " E_j ." The total number N of atoms is given as

$$N = \sum_{i=1}^m N_i \quad (50)$$

The total energy " E " of the system can be written as

$$E = \sum_{i=1}^m N_i E_i \quad (51)$$

A number of ways in which " N_1 " atoms from total atoms " N " can occupy the first level are $\binom{N}{N_1}$. In the same manner, the total number of ways to arrange " N_2 " atoms from the remaining " $N-1$ " atoms is $\binom{N-1}{N_2}$ and so on. Thus, the total number of microstates " X " in the system, that is, the number of ways to arrange " N " atoms of the system, is given as

$$X = \binom{N}{N_1} \binom{N-1}{N_2} \binom{N-2}{N_3} \dots \binom{N-(m-1)}{N_m} \quad (52)$$

By solving the above binomial

$$X = \frac{N!}{N_1! N_2! \dots N_j! \dots N_m!} \quad (53)$$

$$X = \frac{N!}{\prod_{i=1}^m N_i!} \quad (54)$$

Taking log on both sides of Eq. (53),

$$\ln X = \ln N! - \ln N_1! - \ln N_2! \dots - \ln N_j! \dots - \ln N_m! \quad (55)$$

By applying Stirling's approximations to the factorials of all variables,

$$\ln X \cong \ln N! - (N_1 \ln N_1 - N_1) - (N_2 \ln N_2 - N_2) \dots \quad (56)$$

$$\ln X \cong \ln N! - \sum_{i=1}^m N_i \ln N_i + N \quad (57)$$

Let us maximize the $\ln X$ with respect to one microstate " N_j ," in a manner that is consistent with constraints of Eqs. (50) and (51). Lagrangian multiplier for the most probable occupation of the j th level is given as

$$\frac{\partial \ln X}{\partial N_j} + \lambda \frac{\partial N}{\partial N_j} + \mu \frac{\partial E}{\partial N_j} = 0 \quad (58)$$

where μ and λ are Lagrangian's multipliers.

By adding values from Eqs. (50), (51), and (56) in Eq. (58),

$$\begin{aligned} & \frac{\partial [\ln N! - (N_1 \ln N_1 - N_1) - (N_2 \ln N_2 - N_2) \dots]}{\partial N_j} + \lambda \frac{\partial (N_1 + \dots + N_j \dots)}{\partial N_j} \\ & + \mu \frac{\partial (E_1 N_1 + \dots + E_j N_j \dots)}{\partial N_j} = 0 \end{aligned} \quad (59)$$

On operating differential,

$$-\ln N_j + \lambda + \mu E_j = 0 \quad (60)$$

$$\lambda + \mu E_j = \ln N_j \quad (61)$$

$$N_j = e^{\lambda + \mu E_j} \quad (62)$$

$$N_j = e^{\lambda} e^{\mu E_j} \quad (63)$$

$$N_j = C e^{\mu E_j} \quad (64)$$

In general form by multiplying both sides by " N_j ," Eq. (60) can be written as

$$-\sum_{i=1}^m N_i \ln N_i + N \lambda + \mu E = 0 \quad (65)$$

From Eqs. (57) and (65),

$$\ln X - \ln N! - N + N \lambda + \mu E = 0 \quad (66)$$

$$\ln X = \ln N! - (\lambda + 1)N - \mu E \quad (67)$$

The change in the internal energy of the system in terms of thermodynamics equation can be written as

$$dE = TdS - PdV \quad (68)$$

where “ T ” is the temperature, “ P ” is the pressure, and “ S ” is the entropy of the system.

The change in the internal energy at a constant volume is given as

$$\left(\frac{\partial E}{\partial S}\right)_V = T \quad (69)$$

$$\left(\frac{\partial S}{\partial E}\right)_V = \frac{1}{T} \quad (70)$$

Boltzmann’s equation for entropy is

$$S = k \ln X \quad (71)$$

$$S = k \ln X \quad (72)$$

From Eqs. (67) and (72),

$$S = k[\ln N! - (\lambda + 1)N - \mu E] \quad (73)$$

Differentiate with respect to the total energy at a constant volume

$$\left(\frac{\partial S}{\partial E}\right)_V = k[-\mu] \quad (74)$$

By comparing Eqs. (70) and (74)

$$\frac{1}{T} = k[-\mu] \quad (75)$$

$$\mu = -\frac{1}{kT} \quad (76)$$

By adding value of “ μ ” in Eq. (64)

$$N_j = C e^{-\frac{E_j}{kT}} \quad (77)$$

To calculate the value of “ C ”, let us change the subscript “ j ” to “ i ” for Eq. (77) and take summation from “1” to “ m ”.

$$N = C \sum_{i=1}^m e^{-\frac{E_i}{kT}} \quad (78)$$

and

$$C = \frac{N}{\sum_{i=1}^m e^{-\frac{E_i}{kT}}} \quad (79)$$

Thus, Eq. (64) can be written as

$$N_j = \frac{N e^{-\frac{E_j}{kT}}}{\sum_{i=1}^m e^{-\frac{E_i}{kT}}} \quad (80)$$

$$\frac{N_j}{N} = \frac{e^{-\frac{E_j}{kT}}}{\sum_{i=1}^m e^{-\frac{E_i}{kT}}} \quad (81)$$

In a system, most of the energy levels in an atom are degenerated, that is, atoms have several states with the same energy. To find out the population of an atom at a particular level, the population of each constituent state is required to be added together. Thus, each term in Eq. (81) must be multiplied by the statistical weight (degeneracy) “ ϖ ” of the level

$$\frac{N_j}{N} = \frac{\varpi_j e^{-\frac{E_j}{kT}}}{\sum_{i=1}^m \varpi_i e^{-\frac{E_i}{kT}}} \quad (82)$$

The term “ $\sum_{i=1}^m \varpi_i e^{-\frac{E_i}{kT}}$ ” is called the partition function. The Eq. (82) gives the relative number of atoms in state “ j ” with respect to the total number of atoms in the system. The number of atoms in level “ j ” can also be compared with the number of atoms at the ground level

$$\frac{N_j}{N_0} = \frac{\varpi_j e^{-\frac{E_j}{kT}}}{\varpi_0 e^{-\frac{E_0}{kT}}} \quad (83)$$

$$N_j = \frac{N_0 \varpi_j e^{-\frac{E_j}{kT}}}{U} \quad (84)$$

and the number of atoms in level 2 relative to level 1, where level 2 is higher than level 1

$$\frac{N_2}{N_1} = \frac{\varpi_2 e^{-\frac{E_2}{kT}}}{\varpi_1 e^{-\frac{E_1}{kT}}} \quad (85)$$

$$\frac{N_2}{N_1} = \frac{\varpi_2}{\varpi_1} e^{-\frac{E_2-E_1}{kT}} \quad (86)$$

The Einstein A coefficient gives the probability of spontaneous emission. A quantum of radiation is emitted by an atom when it de-excites from an excited level to a lower level, which is given as

$$h\nu = \Delta E \quad (87)$$

where “ ν ” is the frequency of emitted radiation and “ $\Delta E = E_2 - E_1$ ” is the energy difference between two atomic states (or level) “ E_2 ” higher level and “ E_1 ” lower level.

Suppose the number of downward transitions per unit time is merely proportional to the number of atoms “ N_2 ” at a higher state, then, the number of transition per unit time is given as

$$-\dot{N}_2 = A_{21}N_2 \quad (88)$$

where “ A_{21} ” is proportionality constant known as Einstein coefficient for spontaneous emission for transition from level “ E_2 ” to level “ E_1 ”.

As “ $A_{21}N_2$ ” is the downward transition per unit time from “ E_2 ” to “ E_1 ”, thus the rate of emission of energy from these “ N_2 ”, that is, radiant power or flux, is given as

$$\Phi = A_{21}N_2h\nu_2 \quad (89)$$

Here, “ ν_{21} ” represents the frequency of radiation due to transition from level “ E_2 ” to level “ E_1 ”. As the radiation is emitted isotropically, thus the intensity is

$$I_{21} = \frac{A_{21}N_2h\nu_{21}}{4\pi} \quad (90)$$

From Eqs. (84) and (90)

$$I_{21} = \frac{N_o A_{21} \omega_2 h \nu_{21}}{4\pi U} e^{-\frac{E_2}{kT}} \quad (91)$$

$$I_{21} = \frac{N_o A_{21} \omega_2 h c}{4\pi \lambda_{21} U} e^{-\frac{E_2}{kT}} \quad (92)$$

2.7.1. Boltzmann plot

Taking log and solving Eq. (92),

$$\ln \left(\frac{I_{21} \lambda_{21}}{A_{21} \omega_2} \right) = -\frac{E_2}{kT} + \ln \left(\frac{N_o h c}{4\pi U} \right) \quad (93)$$

2.7.2. Intensity ratio method

Consider two different emission lines from level $i \rightarrow j$ and $m \rightarrow n$, where i and m are higher energy levels, j and n are lower energy levels. By using Eq. (92),

$$I_{ij} = \frac{N_o A_{ij} \omega_i h c}{4\pi \lambda_{ij} U} e^{-\frac{E_i}{kT}} \quad (94)$$

$$I_{mn} = \frac{N_0 A_{mn} \varpi_m h c}{4\pi \lambda_{mn} U} e^{-\frac{E_m}{kT}} \quad (95)$$

Taking ratio of Eqs. (94) and (95) and solving for temperature “T”,

$$\ln \left(\frac{I_{ij} \lambda_{ij} A_{mn} \varpi_m}{I_{mn} \lambda_{mn} A_{ij} \varpi_i} \right) = -\frac{E_i - E_m}{kT} \quad (96)$$

$$T = -\frac{E_i - E_m}{k \ln \left(\frac{I_{ij} \lambda_{ij} A_{mn} \varpi_m}{I_{mn} \lambda_{mn} A_{ij} \varpi_i} \right)} \quad (97)$$

2.8. Saha equation

The Boltzmann equation gives only the relative number of atoms or ions in a particular excitation state with respect to the ground state and it does not provide the total number of atoms that have been ionized. In order to determine the total abundance of a given element, it is necessary to know how the atoms are distributed among their several ionization stages. To quantify the number of atoms/ions in different ionization states, Saha’s equation is used which gives an expression for the total number of ions in an ionization state relative to lower ionization state.

For a system at local thermal equilibrium with a constant volume, the Boltzmann equation for the number of ions “ N^i ” relative to atoms/ions “ N_1^i ” in the ground state of ionization state “ i ” can be written as (by using Eq. (85))

$$\frac{N^i}{N_1^i} = \frac{\varpi^i e^{-\frac{E_i}{kT}}}{\varpi_1^i e^{-\frac{E_1^i}{kT}}} \quad (98)$$

Taking the sum of all the excited states “ j ” of ionization state “ i ”, Eq. (98) can be written as

$$\frac{N_j^i}{N_1^i} = \frac{\sum_{j=1}^{\infty} \varpi_j^i e^{-\frac{E_j^i}{kT}}}{\varpi_1^i e^{-\frac{E_1^i}{kT}}} \quad (99)$$

$$\frac{N_j^i}{N_1^i} = \frac{\sum_{j=1}^{\infty} \varpi_j^i e^{-\frac{E_j^i}{kT}}}{Z_1} \quad (100)$$

where $Z_1 = \varpi_1^i e^{-\frac{E_1^i}{kT}}$, which represents the number of atoms at the ground level. The same expression for the ionization state “ $i + 1$ ” will include not only the excitation states of ion “ $i + 1$ ” but also the free electrons. Let us consider the energy of the free electron “ E_e ” and momentum “ P_e ” with “ $E_e = P_e^2/2m_e$ ”. Then, at any state of the system of ion “ $i + 1$ ” and an electron is characterized by the total energy “ $E_j^{i+1} - E_e$ ” and statistical weight “ $\varpi_j^{i+1} \varpi_e(P_e)$ ”. As

the energies of free electrons are continuous, the Boltzmann expression analogous to Eq. (100) over all possible momenta of free electron can be written as

$$\frac{N_j^{i+1}}{N_1^i} = \frac{1}{\omega_1^i} \sum_{j=1}^{\infty} \omega_j^{i+1} e^{-\frac{(E_j^{i+1} - E_1^i)}{kT}} \int_0^{\infty} \varpi_e(P_e) e^{-\frac{E_e}{kT}} dP_e \quad (101)$$

The ionization potential of the ion “ i ” can be defined as

$$\chi_i = E_1^{i+1} - E_1^i \quad (102)$$

Thus, we can write

$$E_j^{i+1} - E_1^i = E_j^{i+1} - E_1^{i+1} + \chi_i \quad (103)$$

Eq. (101) will become

$$\frac{N_j^{i+1}}{N_1^i} = \frac{1}{\omega_1^i} \sum_{j=1}^{\infty} \omega_j^{i+1} e^{-\frac{(E_j^{i+1} - E_1^{i+1} + \chi_i)}{kT}} \int_0^{\infty} \varpi_e(P_e) e^{-\frac{E_e}{kT}} dP_e \quad (104)$$

$$\frac{N_j^{i+1}}{N_1^i} = \frac{1}{\omega_1^i e^{-\frac{E_1^{i+1}}{kT}}} \sum_{j=1}^{\infty} \omega_j^{i+1} e^{-\frac{(E_j^{i+1} + \chi_i)}{kT}} \int_0^{\infty} \varpi_e(P_e) e^{-\frac{E_e}{kT}} dP_e \quad (105)$$

$$\frac{N_j^{i+1}}{N_1^i} = \frac{1}{Z_1} \sum_{j=1}^{\infty} \omega_j^{i+1} e^{-\frac{(E_j^{i+1} + \chi_i)}{kT}} \int_0^{\infty} \varpi_e(P_e) e^{-\frac{E_e}{kT}} dP_e \quad (106)$$

where $Z_1 = \omega_1^i e^{-\frac{E_1^{i+1}}{kT}}$, which represents the number of atoms at the ground level. Although the momenta of the free electrons have a continuous distribution, but according to Heisenberg’s uncertainty principle the electrons within a phase-space volume ΔV are indistinguishable, unless they have an opposite spin orientation, that is,

$$\Delta V P_e^3 = h^3 \quad (107)$$

There are two possible distinguishable electron states within phase space of $\Delta V P_e^3 = h^3$. Thus, the statistical weight for the free electron can be written as

$$\varpi_e(P_e) dP_e = 2 \frac{\Delta V P_e^3}{h^3} \quad (108)$$

If the distribution of electron momenta is isotropic, then

$$\Delta P_e^3 = 4\pi P_e^2 dP_e \quad (109)$$

If “ n_e ” is the number density of electron, then the differential volume per electron will be

$$\Delta V = \frac{1}{n_e} \tag{110}$$

Thus, Eq. (108) can be written as

$$\varpi_e(P_e)dP_e = \frac{8\pi P_e^2}{n_e h^3} dP_e \tag{111}$$

We can write the integral in Eq. (106) as

$$\int_0^\infty \varpi_e(P_e)e^{-\frac{E_e}{kT}}dP_e = \frac{8\pi}{n_e h^3} \int_0^\infty P_e^2 e^{-\frac{E_e}{kT}}dP_e \tag{112}$$

The integral can be transformed into an integral over variable “ z ” such that

$$z = \frac{E_e}{kT} = \frac{P_e^2}{2m_eKT} \tag{113}$$

$$P_e = \sqrt{2m_eKTz} \tag{114}$$

$$dP_e = \frac{1}{2} \sqrt{\frac{2m_eKT}{z}} dz \tag{115}$$

By replacing values of P_e and dP_e in Eq. (112)

$$\int_0^\infty \varpi_e(P_e)e^{-\frac{E_e}{kT}}dP_e = \frac{4\pi}{n_e h^3} (2m_eKT)^{3/2} \int_0^\infty \sqrt{z}e^{-z} dz \tag{116}$$

The integral on the right-hand side of Eq. (116) is the Gamma function of the argument 3/2, which is $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$. By adding this value in Eq. (116)

$$\int_0^\infty \varpi_e(P_e)e^{-\frac{E_e}{kT}}dP_e = \frac{4\pi}{n_e h^3} (2m_eKT)^{3/2} (\sqrt{\pi}/2) \tag{117}$$

$$\int_0^\infty \varpi_e(P_e)e^{-\frac{E_e}{kT}}dP_e = \frac{2}{n_e h^3} (2\pi m_eKT)^{3/2} \tag{118}$$

From Eqs. (106) and (118),

$$\frac{N_j^{i+1}}{N_1^i} = \frac{1}{Z_1} \sum_{j=1}^\infty \varpi_j^{i+1} e^{-\frac{(E_j^{i+1} + X_j)}{kT}} \frac{2}{n_e h^3} (2\pi m_eKT)^{3/2} \tag{119}$$

By dividing Eqs. (100) and (119),

$$\frac{N_j^{i+1}}{N_j^i} = \frac{\sum_{j=1}^{\infty} \varpi_j^{i+1} e^{-\frac{(E_j^{i+1} + \chi_i)}{kT}}}{\sum_{j=1}^{\infty} \varpi_j^i e^{-\frac{E_j^i}{kT}}} \frac{2}{n_e h^3} (2\pi m_e kT)^{3/2} \quad (120)$$

$$\frac{N_j^{i+1}}{N_j^i} = \frac{\sum_{j=1}^{\infty} \varpi_j^{i+1}}{\sum_{j=1}^{\infty} \varpi_j^i} \frac{2}{n_e h^3} (2\pi m_e kT)^{3/2} e^{-\frac{(E_j^{i+1} + \chi_i - E_j^i)}{kT}} \quad (121)$$

$$\frac{N_j^{i+1}}{N_j^i} = \frac{\sum_{j=1}^{\infty} \varpi_j^{i+1}}{\sum_{j=1}^{\infty} \varpi_j^i} \frac{2}{n_e h^3} (2\pi m_e kT)^{3/2} e^{-\frac{(E_j^i - E_j^i + \chi_i)}{kT}} \quad (122)$$

For a particular state,

$$\frac{N_j^{i+1}}{N_j^i} = \frac{2(2\pi m_e kT)^{3/2}}{n_e h^3} \frac{\varpi_j^{i+1}}{\varpi_j^i} e^{-\frac{(E_j^{i+1} - E_j^i + \chi_i)}{kT}} \quad (123)$$

$$n_e = \frac{2(2\pi m_e kT)^{3/2}}{h^3} \frac{N_j^i \varpi_j^{i+1}}{N_j^{i+1} \varpi_j^i} e^{-\frac{(E_j^{i+1} - E_j^i + \chi_i)}{kT}} \quad (124)$$

Eq. (90) can be written as

$$N^{i+1} = \frac{4\pi h I^{i+1} \lambda^{i+1}}{A^{i+1} c} \quad (125)$$

$$N^i = \frac{4\pi h I^i \lambda^i}{A^i c} \quad (126)$$

From Eqs. (124)–(126),

$$n_e = \frac{2(2\pi m_e kT)^{3/2}}{h^3} \frac{I^i \lambda^i A^{i+1} \varpi^{i+1}}{I^{i+1} \lambda^{i+1} A^i \varpi^i} e^{-\frac{(E^{i+1} - E^i + \chi_i)}{kT}} \quad (127)$$

3. Summary

Kinetic theory provides the essential material for an introductory course on plasma physics as well as the basis for advanced kinetic theory. It offers a wide-range coverage of the field. Plasma kinetics deals with the relationship between velocity and forces and the study of continua in velocity space. The understanding of the most important plasma parameters, that is, plasma oscillations, plasma frequency, Debye shielding, Debye length, plasma temperature, and electron density, is important for studying plasmas. Kinetic theory has a wide scope. Plasma kinetics mathematical equations that will aid the readers in understanding simple techniques on how to investigate plasma dynamics and kinetics are discussed in this chapter.

Kinetic theory provides the basics and essential introduction to plasma physics and subsequently advanced kinetic theory. Plasma understanding, waves, oscillations, frequencies, and applications are covered in kinetic theory. In fact, we have attempted to present a precise discussion of plasma kinetic theory which includes the basic plasma parameters mathematical formulation for an easy understanding to the reader.

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References

- [1] Bittencourt JA. Fundamentals of Plasma Physics. Munich, Germany: Springer Science & Business Media; 2013
- [2] Sitenko AG. Fluctuations and Non-Linear Wave Interactions in Plasmas: International Series in Natural Philosophy. Brazil: Elsevier; 2016. Vol. 107
- [3] Thompson WB. An Introduction to Plasma Physics. Brazil: Elsevier, 2013
- [4] Akhiezer AI, Akhiezer I, Polovin R. Collective Oscillations in a Plasma: International Series of Monographs in Natural Philosophy. Brazil: Elsevier; 2017. Vol. 7
- [5] Davidson R. Methods in Nonlinear Plasma Theory. Brazil: Elsevier; 2012
- [6] Klimontovich YL. The Statistical Theory of Non-Equilibrium Processes in a Plasma: International Series of Monographs in Natural Philosophy. Brazil: Elsevier; 2013. Vol. 9
- [7] Rice SA, Gray P. The Statistical Mechanics of Simple Liquids. Olney, Bucks, UK: Interscience Publ; 1965
- [8] Halliday D, Resnick R, Walker J. Principles of Physics Extended, International Student Version. India: Wiley India Pvt. Ltd.; 2016
- [9] Thompson CJ. Mathematical Statistical Mechanics. Princeton, New Jersey, USA: Princeton University Press; 2015

