Chapter 11

Bifurcation Theory of Dynamical Chaos

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/intechopen.70987

Abstract

The purpose of the present chapter is once again to show on concrete new examples that chaos in one-dimensional unimodal mappings, dynamical chaos in systems of ordinary differential equations, diffusion chaos in systems of the equations with partial derivatives and chaos in Hamiltonian and conservative systems are generated by cascades of bifurcations under universal bifurcation Feigenbaum-Sharkovsky-Magnitskii (FShM) scenario. And all irregular attractors of all such dissipative systems born during realization of such scenario are exclusively singular attractors that are the nonperiodic limited trajectories in finite dimensional or infinitely dimensional phase space any neighborhood of which contains the infinite number of unstable periodic trajectories.

Keywords: nonlinear systems, dynamical chaos, bifurcations, singular attractors FShM theory

1. Introduction

Well-known, that chaotic dynamics is inherent practically in all nonlinear mappings and systems of differential equations having irregular attractors, distinct from stable fixed and singular points, limit cycles and tori. However, many years there was no clear understanding of that from itself represent irregular attractors and how they are formed. In this connection it was possible to find in the literature more than 20 various definitions of irregular attractors: stochastic, chaotic, strange, hyperbolic, quasiattractors, attractors of Lorenz, Ressler, Chua, Shilnikov, Chen, Sprott, Magnitskii and many others. It was considered that there are differences between attractors of autonomous and nonautonomous nonlinear systems, systems of ordinary differential equations and the equations with partial derivatives, and that the chaos in dissipative systems essentially differs from chaos in conservative and Hamiltonian systems. There was also an opinion which many outstanding scientists adhered, including Nobel prize winner I.R. Prigogine, that irregular attractors of complex nonlinear systems cannot be described by trajectory approach, that are systems of differential equations. And only in twenty-first century it has been proved and on numerous examples it was convincingly shown, that there is one universal bifurcation scenario of transition to chaos in nonlinear systems of mappings and differential equations: autonomous and nonautonomous, dissipative and conservative, ordinary, with private derivatives and with delay argument (see, for example, [1–9]). It is bifurcation Feigenbaum-Sharkovsky-Magnitskii (FShM) scenario, beginning with the Feigenbaum cascade of period-doubling bifurcations of stable cycles or tori and continuing from the Sharkovskii subharmonic cascade of bifurcations of stable cycles or tori of an arbitrary period up to the cycle or torus of the period three, and then proceeding to the Magnitskii homoclinic or heteroclinic cascade of bifurcations of stable cycles or tori. All irregular attractors born during realization of such scenario are exclusively singular attractors that are the nonperiodic limited trajectories in finite dimensional or infinitely dimensional phase space any neighborhood of which contains the infinite number of unstable periodic trajectories.

However, in the scientific literature many papers continue to appear in which authors, not understanding an essence of occurring processes, write about opened by them new attractors in nonlinear systems of differential equations. Such papers are, for example, papers [10, 11] which authors with surprise ascertain an existence of chaotic dynamics in nonlinear system of ordinary differential equations with one stable singular point and try to explain this phenomenon by presence in the system of Smale's horseshoe. Numerous papers continue to be published also in which presence of chaotic attractor in the system of ordinary differential equations is connected with Lyapunov's positive exponent found numerically, diffusion chaos in nonlinear system of equations with partial derivatives is explained by the Ruelle-Takens (RT) theory and is connected with birth of mythical strange attractor at destruction of three-dimensional torus, and presence of chaotic dynamics in Hamiltonian or conservative system is explained by the Kolmogorov-Arnold-Mozer (KAM) theory and is connected with consecutive destruction in the system of rational and mostly irrational tori of nonperturbed system.

The purpose of the present paper is once again to show on concrete new, not entered in [1–9], examples, that chaos in the system considered in Refs. [10, 11], and also chaos in one-dimensional unimodal mappings, dynamical chaos in systems of ordinary differential equations, diffusion chaos in systems of the equations with partial derivatives and chaos in Hamiltonian and conservative systems are generated by cascades of bifurcations under the FShM scenario. Thus, in any nonlinear system there can be an infinite number of various singular attractors, becoming complicated at change of bifurcation parameter in a direction of the cascade of bifurcations. Presence or absence in system of stable or unstable singular points, presence or absence of saddle-nodes or saddle-focuses, homoclinic or heteroclinic separatrix contours and Smale's horseshoes and also positivity of the calculated senior Lyapunov's exponent are not criteria of occurrence in system of chaotic dynamics. And the birth in the system of three-dimensional and even multi-dimensional stable torus leads not only to its destruction with birth of mythical strange attractor, but also to cascade of its period-doubling bifurcations along one of its frequencies or several frequencies simultaneously. Chaotic dynamics in Hamiltonian and conservative systems also is consequence of cascades of bifurcations of birth of new tori, instead of consequence of destruction of some already ostensibly existing mythical tori of nonperturbed system. Thus, for the analysis of chaotic dynamics of any nonlinear system, attempts of calculation of a positive Lyapunov's exponent, application of KAM and RT theories and the proof of existence of Smale's horseshoes are absolutely senseless. Let us notice, that the results of Feigenbaum and Sharkovsky are received only for one-dimensional unimodal maps and then were transferred by Magnitskii at first on two-dimensional systems of differential equations with periodic coefficients, then on three-dimensional, multi-dimensional and infinitely dimensional dissipative and conservative autonomous systems of ordinary differential equations and then on systems of the equations with partial derivatives. Besides this, it is proved by Magnitskii, that the subharmonic cascade of Sharkovsky bifurcations can be continued by homoclinic or heteroclinic bifurcations cascade both in the differential equations, and in continuous one-dimensional unimodal mappings.

1.1. FShM-cascades of bifurcations of stable cycles and a birth of singular attractors in one-dimensional unimodal mappings

Let us give a summary of bifurcation FShM theory of chaos in one-dimensional continuous unimodal mappings. Detailed proof of statements of the present section can be found in Ref. [1].

1.1.1. FShM-cascade of bifurcations in logistic mapping

Studying the properties of logistic mapping

$$f(x, \mu) = \mu x(1 - x), \quad x \in [0, 1], \quad \mu \in [1, 4] \tag{1}$$

Feigenbaum proved that in this equation there is a cascade of period-doubling bifurcations of its cycles and found a sequence of values of the parameter μ at which these bifurcations occur. Further studies have shown that the complex chaotic dynamics of the logistic mapping is also characteristic of any continuous difference equation of a kind $x_{n+1}=f(x_n, \mu)$ in which one-dimensional mapping $f: I \to I$ is unimodal at corresponding choice of scale, that is, it has the only extremum on an interval I. Return mapping f^{-1} has in this case two branches on I.

Considering the map (1) on an interval $x \in [0, 1]$, Feigenbaum has established, that there is the infinite sequence μ_n of parameter values μ converging with a speed of the geometrical progression with a denominator $1/\delta \approx 1/4.67$ to value $\mu_\infty \approx 3.57$ in which period-doubling bifurcations of the cycles of logistic map occur. That is at all parameter values $\mu_n < \mu < \mu_{n+1}$ Eq. (1) has unique regular attractor—a stable cycle of the period 2^n and a set of unstable cycles of all periods $2^k, k=0, \dots, n-1$. Thus, the first most simple and low-power singular attractor, born in unimodal one-dimensional continuous mapping at the end of the Feigenbaum period-doubling bifurcation cascade, is a nonperiodic trajectory consisting of points, any neighborhood of each contains points belonging to some unstable cycles of the periods $2^n, n > 0$. This attractor is called Feigenbaum attractor. It is, obviously, everywhere not dense set of points on an interval. In the case of logistic mapping (1), Feigenbaum attractor exists at the parameter value $\mu_\infty \approx 3.57$. However, logistic mapping is defined on the interval $x \in [0, 1]$ at all parameter values $\mu \leq 4$. The answer to a question, that occurs with trajectories of logistic mapping and with any other unimodal continuous mapping at parameter values $\mu > \mu_\infty$, gives Sharkovsky theorem. It follows from this theorem that complication of structure of cycles of iterations of one-dimensional unimodal mappings, as a rule, does not come to the end with the cascade of

Feigenbaum bifurcations and Feigenbaum attractor, and it is continued by more complex cascade of bifurcations according to the order established by Sharkovsky in his theorem.

Definition**.** Ordering in set of the natural numbers, looking like

$$1 \lhd 2 \lhd 2^2 \lhd \ldots 2^n \lhd \ldots \lhd 2^2 \cdot 7 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 3 \lhd \ldots 2 \cdot 7 \lhd 2 \cdot 5 \lhd 2 \cdot 3 \lhd \ldots \lhd 7 \lhd 5 \lhd 3. \qquad (2)$$

is called as Sharkovsky's order. Theorem of Sharkovsky approves, that if continuous unimodal map $f: I \to I$ has a cycle of the period n then it has also all cycles of each period k, such that $k \lhd n$ in the sense of the order (2). Consequence of the theorem is the statement, that if map f has a cycle of the period 3, then it has cycles of all periods.

It also follows from the Sharkovsky theorem, that at change of values of bifurcation parameter, stable cycles in one-dimensional unimodal continuous mappings are obliged to be born according to the order (2). And their births occur in pairs together with unstable cycles as a result of saddle-node (tangent) bifurcations. Each stable cycle of Sharkovsky cascade, which has born thus, undergoes then the cascade of period-doubling bifurcations, generating its own window of periodicity (**Figure 1**). A limit of such cascade is more complex singular attractor—nonperiodic almost stable trajectory any neighborhood of which contains the infinite number of unstable periodic trajectories. Hence, the cascade of Feigenbaum bifurcations is an initial stage of the full subharmonic cascade of bifurcations, described by Sharkovsky order. In the case of logistic mapping (1) cycle of the period three is born at value $\mu \approx 3.828$ (**Figure 1**). Hence, the subharmonic cascade of Sharkovsky bifurcations does not cover all area of change of values of bifurcation parameter $\mu \leq 4$.

Behind subharmonic Sharkovsky cascade, homoclinic (heteroclinic) cascade of bifurcations lays, opened by Magnitskii at first in nonlinear systems of ordinary differential equations, and then found out in logistic and other unimodal continuous mappings. Homoclinic (heteroclinic) cascade of bifurcations consists of a consecutive birth of stable homoclinic (heteroclinic) cycles of the period n converging to a homoclinic (heteroclinic) contour. As a rule, it is a separatrix loop of a saddle-focus (heteroclinic separatrix contour) in nonlinear system of ordinary differential equations and a separatrix loop of a fixed point (heteroclinic separatrix contour) in one-dimensional unimodal mapping. Born before, unstable cycles and

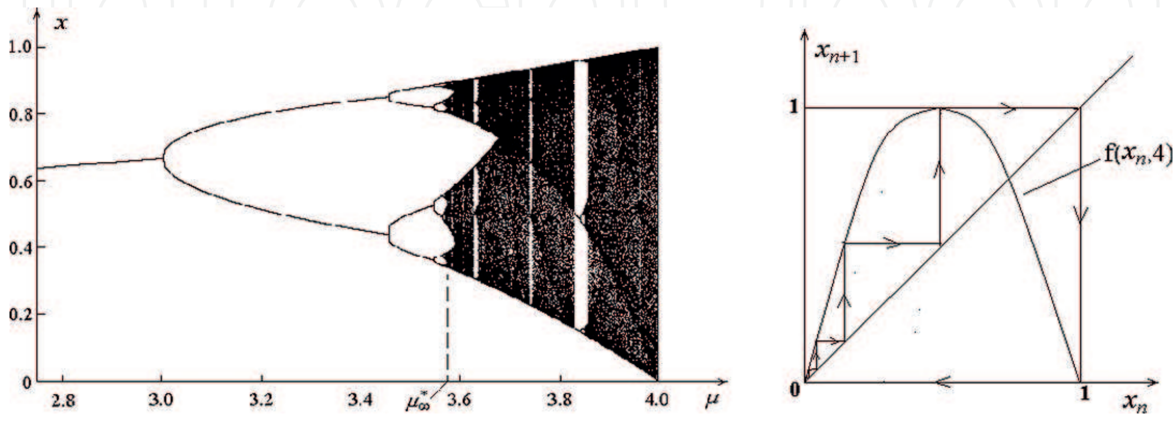

Figure 1. Full bifurcation diagram of logistic mapping at $\mu \leq 4$ and the separatrix loop of the zero fixed point at $\mu = 4$.

nonperiodic trajectories (singular attractors) remain in system, therefore dynamics of unimodal mapping in a neighborhood of a homoclinic (heteroclinic) contour is the most complex. The first cycles of homoclinic cascade are the most simple cycle of the period two of the Feigenbaum cascade and the most complex cycle of the period three of Sharkovsky cascade. In logistic mapping stable homoclinic cycle of the period four exists at $\mu = 3.9603$, and the separatrix loop of the fixed point $x = 0$ exists at $\mu = 4$, that completely covers all area of change of values of bifurcation parameter (**Figure 1**). So, in one-dimensional unimodal mappings at various parameter values stable periodic (regular) attractors and nonperiodic singular attractors can exist together with finite or infinite number of unstable periodic trajectories, and all such attractors are born as a result of cascades of soft bifurcations (saddle-node and period-doubling) in full accordance with the Feigenbaum-Sharkovsky-Magnitskii (FShM) theory.

2. Dynamical chaos in nonlinear dissipative systems of ordinary differential equations

Bases of the FShM theory with reference to nonlinear dissipative systems of ordinary differential equations are stated in Refs. [1–3, 7]. Thus in systems with strong dissipation it is realized both the full subharmonic cascade of Sharkovsky bifurcations, and full (or incomplete) homoclinic (or heteroclinic) cascade of Magnitskii bifurcations depending on, whether exists homoclinic (or heteroclinic) separatrix contour in the system. In systems with weak dissipation the FShM-order of bifurcations can be broken in its right part. Hence, attractors of such systems are regular attractors (stable singular points, stable cycles and stable tori of any dimension), or singular cyclic or toroidal attractors—limited nonperiodic almost stable trajectories or the toroidal manifolds, being limits of cascades of the period-doubling bifurcations of regular attractors (cycles, tori). In Refs. [1–3, 7] it is proved, that the FShM scenario of transition to chaos takes place in such classical two-dimensional dissipative systems with periodic coefficients, as systems of Duffing-Holmes, Mathieu, Croquette and Krasnoschekov; in three-dimensional autonomous dissipative systems, as systems of Lorenz, Ressler, Chua, Magnitskii, Vallis, Anishchenko-Astakhov, Rabinovich-Fabricant, Pikovskii-Rabinovich-Trakhtengertz, Sviregev, Volterra-Gause, Sprott, Chen, Rucklidge, Genezio-Tesi, Wiedlich-Trubetskov and many others; in multi-dimensional and infinitely dimensional autonomous dissipative systems, as systems of Rikitaki, Lorenz complex system, Mackey-Glass equation and many others. These systems describe processes and the phenomena in all areas of scientific researches. Lorenz system is a hydrodynamic system, Ressler system is a chemical system, Chua system describes the electro technical processes, Magnitskii system is a macroeconomic system, Widlich-Trubetskov system describes the social processes and phenomena, Mackey-Glass equation describes the processes of hematopoiesis.

2.1. Transition to chaos in the system with one stable singular point

In this chapter, let us consider the three-dimensional system of ordinary differential equations with one stable singular point which has been proposed in Ref. [10]

$$\dot{x} = yz + 0.006, \quad \dot{y} = x^2 - y, \quad \dot{z} = 1 - 4x \tag{3}$$

This system has the only stable singular point $(0.25, 0.0625, -0.096)$ of stable focus type as Jacobian matrix in the singular point has eigenvalues $(-0.96069, -0.01966 \pm 0.50975\,i)$, where $i^2 = -1$. The system (3) has no neither saddle-focuses, nor a saddle-nodes and, hence, it has no homoclinic or heteroclinic contours, but it has strongly expressed chaotic dynamics (see in Ref. [10] and below in **Figure 2**). In Ref. [11] attempt is undertaken to explain chaos in system (3) by presence in it of Smale's horseshoe. We shall show now, that transition to chaos in system (3) actually occurs in full accordance with universal bifurcation scenario of Feigenbaum-Sharkovsky-Magnitskii. For this purpose, it is necessary only to define correctly bifurcation parameter at which change the cascade of bifurcations under FShM scenario is realized in the system.

As bifurcation parameter we choose the parameter b and consider the system

$$\dot{x} = yz + 0.006, \quad \dot{y} = x^2 - by, \quad \dot{z} = 1 - 4x \tag{4}$$

At $b = 1$, the system (4) obviously passes into system (3). We shall search stable cycles of the system (4) by numerical modeling of the system by the Runge-Kutta method of the fourth order. The system (4) remains dissipative at all parameter values $b > 0$. At values $b < 0.39$ there are no attractors in the system, except for a singular point of a stable focus type. At value $b \approx 0.39$ there is a stable cycle in the system as a result a saddle-node bifurcation of births of stable and unstable cycles. This cycle exists up to the value $b \approx 0.8$, when the stable cycle of the double period is born in the system . Further the cascade of Feigenbaum period-doubling bifurcations follows: the cycle of period 2 is observed up to value $b \approx 0.9$, the cycle of the period 4—up to value $b \approx 0.926$, generating a stable cycle of the period 8, etc. At the further increase in parameter values b, the next cycles have been found: of the period 7 at $b \approx 0.956$, of the period 5 at $b \approx 0.965$ and of the period 3 at $b \approx 0.982$. This indicates the realization of full subharmonic cascade of Sharkovsky bifurcations in the system (4) (**Figure 2**). At $b = 1$ there exists a chaos in the system (4) and, hence, in the system (3), corresponding to an area of FShM scenario, which lies behind the Sharkovsky cascade. Homoclinic cascade in the system (4) is not found out, in view of absence in it of unstable singular points and homoclinic separatrix contours.

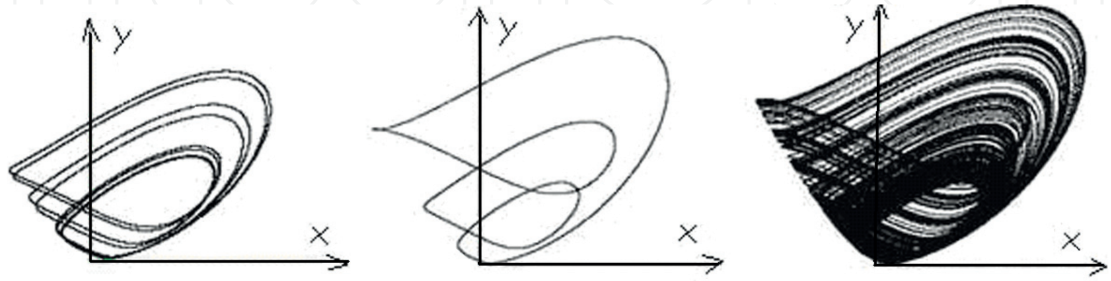

Figure 2. Projections to a plane (x, y) of cycles of periods 8 ($b = 0.927$), 3 ($b = 0.982$) and singular attractor ($b = 1$) of the system (4).

3. Dynamical chaos in Hamiltonian and conservative systems

Conservative system saves its volume at movement along the trajectories and, hence, cannot have attractors. Therefore studying of dynamical chaos in Hamiltonian and, especially, simply conservative systems is more a difficult task in comparison with the analysis of chaotic dynamics in dissipative systems which can be described by universal bifurcation FShM theory. The main problem solved by the modern classical theory of Hamiltonian systems (the Kolmogorov-Arnold-Moser theory) is the problem of integrability of such a system, that is, the problem of its reduction to the "action-angle" variables by constructing some canonical transformation. It is assumed that in such variables the motion in a Hamiltonian system is periodic or quasiperiodic and occurs on the surface of an n-dimensional torus. In this formulation, any non-integrable Hamiltonian system is considered as a perturbation of the integrable system, and the analysis of the dynamics of the perturbed system reduces to studying the problem of the destruction of the tori of an unperturbed system with increasing values of the perturbation parameter. But numerous examples of Hamiltonian and simply conservative systems, considered by the author in [4–7], deny existence such classical KAM-scenario of transition to chaos.

One of the most effective approaches to the decision of a problem of the analysis of chaotic dynamics in conservative systems is offered by the author in Ref. [4] (see also [5–7]). The approach assumes consideration of conservative system in the form of limiting transition from corresponding extended dissipative system (in which the dissipative member is added) to initial conservative system. This approach can be evidently shown by means of construction of two-parametrical bifurcation diagram which corresponds to transition from dissipative state to conservative state. Attractors (stable cycles, tori and singular attractors) of extended dissipative system can be found numerically with use of results of universal bifurcation FShM theory. Further transition to chaos in conservative (Hamiltonian) system is carried out through cascades of bifurcations of attractors of extended dissipative system when dissipation parameter tends to zero. Areas of stability of stable cycles of the extended system at zero dissipation turn into tori of conservative (Hamiltonian) system around of its elliptic cycles into which stable cycles transform. Thus tori of conservative (Hamiltonian) system touch through hyperbolic cycles into which saddle cycles of extended dissipative system transform. In [4–7] the considered above approach is described in detail with reference to Hamiltonian systems with one and a half, two, two and a half and three degrees of freedom, and also to simply conservative systems of differential equations, including the conservative Croquette equation, the equation of a pendulum with oscillating point of fixing, the conservative generalized Mathieu equation, well-known Hamiltonian system of Henon-Heiles equations. In Refs. [12, 13] the given approach has been applied and strictly proved by continuation along parameter of solutions from dissipative into conservative areas by means of the Magnitskii method of stabilization of unstable periodic orbits [1] at research bifurcations and chaos in the Duffing-Holmes equation

$$\ddot{x} + \mu \dot{x} - \delta x + x^3 - \varepsilon \cos(\omega t) = 0, \tag{5}$$

and in the model of a space pendulum

$$\ddot{x} + \mu\dot{x} + kx + \varepsilon\sin(2\pi x) = h\cos\omega t. \tag{6}$$

Corresponding bifurcation diagrams in a plane (ε, μ) of existence of cycles of various periods down to a conservative case at $\mu = 0$ are shown in [12–14]. Application of Magnitskii approach has revealed the essence of dynamical chaos in Hamiltonian and simply conservative systems. It became clear, that the chaos in such systems is not a result of destruction or non-destruction of some mythical tori of nonperturbed systems, as it follows from the KAM theory, but absolutely on the contrary, it is a consequence of limit transition of infinite number of cycles, tori and singular attractors, born according to the FShM theory as a result of cascades of bifurcations in extended dissipative system when dissipation parameter tends to zero.

3.1. Hamiltonian Yang-Mills-Higgs system with two degrees of freedom

In this chapter, let us illustrate Magnitskii approach by the example of Yang-Mills-Higgs system with two degrees of freedom and with Hamiltonian

$$H = \left(\dot{x}^2 + \dot{z}^2\right)/2 + x^2z^2/2 + \nu\left(x^2 + z^2\right)/2, \tag{7}$$

passing into classical system of the Yang-Mills equations at $\nu = 0$. We shall consider four-dimensional phase space of the system with coordinates $x, y = \dot{x},\ z, r = \dot{z}$:

$$\dot{x} = y,\ \ \dot{y} = -x\left(\nu + z^2\right),\ \ \ \dot{z} = r,\ \ \ \dot{r} = -z\left(\nu + x^2\right). \tag{8}$$

The system (8) has four sets of periodic solutions to which there correspond four basic cycles in phase space

$$C_x : z = r = 0,\ \ y^2 + \nu x^2 = 2;\ \ C_z : x = y = 0, r^2 + \nu z^2 = 2;\ \ C^{\pm} : z = \pm x, y^2 + \nu x^2 + x^4/2 = 1. \tag{9}$$

Assuming $H = 1$, we shall consider four-dimensional extended two-parametrical dissipative system of differential equations of a kind

$$\dot{x} = y,\ \ \dot{y} = -x\left(\nu + z^2\right) - \mu y,\ \ \ \dot{z} = r + (1 - H(x, y, z, r))z,\ \ \ \dot{r} = -z\left(\nu + x^2\right), \tag{10}$$

where $r = \dot{z}$. Complication of solutions of Hamiltonian system (8) of the Yang-Mills-Higgs equations down to full chaotic dynamics occurs at $\nu \to\ 0$. In turn for each value $\nu > 0$ the structure of solutions of Hamiltonian system (8) is completely determined by cascades of bifurcations of cycles of extended dissipative system (10) when dissipation parameter $\mu \to 0$. Stable cycles of dissipative system (10), born as a result of cascades of bifurcations, pass into elliptic cycles of Hamiltonian system (8), and their areas of stability—into tori around of these elliptic cycles. The contact of born tori of conservative system occurs on hyperbolic cycles in which corresponding unstable cycles of extended dissipative system transform. These unstable cycles are born in the dissipative system together with stable cycles as a result a saddle-node bifurcations, or at loss of stability of a cycle as a result of pitchfork

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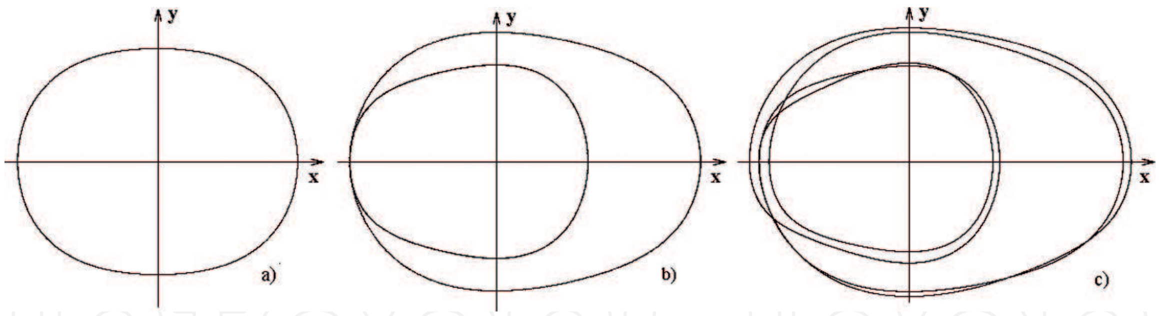

Figure 3. Projections of cycles of Hamiltonian system (8) on a plane (x, y): an initial cycle C^+ at $\nu = 0.73$ (a), cycles of the double and quadruple periods at $\nu = 0.65$ (b) and at $\nu = 0.534$ (c).

bifurcation or period-doubling bifurcation. In neighborhoods of separatrix surfaces of hyperbolic cycles there is a formation of new more complex hyperbolic and elliptic cycles according to nonlocal effect of multiplication of cycles and tori in conservative systems (see [4–7]). Last effect plays a key role in the system of Yang-Mills-Higgs equations at an initial stage of transition from regular dynamics to chaotic dynamics. At the same time, as numerical calculations show, the further complication of dynamics of solutions of system (10) at reduction of parameter value ν occurs not only by means of multiplication of elliptic and hyperbolic cycles and tori, but also by means of the cascade of period-doubling bifurcations of the basic cycles and by means of the subharmonic cascade of bifurcations. Initial cycles of the cascade of period-doubling bifurcations of the cycle C^+ are presented in **Figure 3**. In Ref. [14] stabilization of unstable cycles of system (8) by modified Magnitskii method [1] is carried out.

Further, the process continues with the birth of infinitely folded heteroclinic separatrix manifold, stretched over separatrix Feigenbaum tree, both in extended dissipative system (10), and in Hamiltonian system (8) close to it. Accordions of corresponding heteroclinic separatrix zigzag fill the whole phase space of the system, however the limited accuracy of numerical methods does not allow to track this process up to the value $\nu = 0$, corresponding to the initial system of the Yang-Mills equations.

4. Spatio-temporal chaos in nonlinear partial differential equations

Bases of FShM theory with reference to a wide class of nonlinear systems of partial differential equations are stated in Refs. [6–9]. This class includes systems of the equations of reaction-diffusion type, describing various autowave oscillatory processes in chemical, biological, social and economic systems, including the well-known brusselator equations; the equations of FitzHugh-Nagumo type, describing processes of chemical and biological turbulence in excitable media; the equations of Kuramoto-Tsuzuki (or Time Dependent Ginzburg-Landau) type, describing complex autooscillating processes after loss of stability of a thermodynamic branch in reaction-diffusion systems; the systems of Navier-Stokes equations, describing laminar-turbulent transitions in hydrodynamical and gazodynamical mediums.

4.1. Diffusion chaos in reaction-diffusion systems

Wide class of physical, chemical, biological, ecological and economic processes and phenomena is described by reaction-diffusion systems of partial differential equations

$$u_t = D_1 u_{xx} + f(u, v, \mu), \quad v_t = D_2 v_{xx} + g(u, v, \mu), \; 0 \le x \le l, \tag{11}$$

depending on the scalar or vector parameter μ. The dynamics of the solutions of such a complex system of equations depends on the boundary conditions, the length of the spatial region, and the values of the diffusion coefficients. In many cases, there is a value of the system parameter μ_0, such that for $\mu < \mu_0$ the system (11) has a stable spatial homogeneous stationary solution (U, V), called the thermodynamic branch. In the case of loss of stability of the thermodynamic branch, when $\mu > \mu_0$, solutions of the system (11) can be various homogeneous and inhomogeneous periodic solutions, spiral waves, running impulses, stationary dissipative structures, as well as nonstationary nonperiodic inhomogeneous solutions, called space-time or diffusion chaos.

The nonlinear processes occurring in so-called excitable media, are described by a special case of systems of the reaction-diffusion equations—FitzHugh-Nagumo type systems

$$u_t = D u_{xx} + f(u, v, \mu), \; v_t = g(u, v, \mu). \tag{12}$$

Solutions of the system (12) are: switching waves, traveling waves and running impulses, dissipative stationary spatially inhomogeneous structures, and diffusion chaos—nonstationary nonperiodic inhomogeneous structures, sometimes called biological or chemical turbulence. All such solutions can be analyze on a line by replacement $\xi = x - ct$ and transition to three-dimensional system of ordinary differential equations

$$\dot{u} = y, \; \dot{y} = -(cy + f(u, v, \mu))/D, \;\; \dot{v} = -g(u, v, \mu)/c, \tag{13}$$

where the derivative is taken over the variable ξ. Therefore, the separatrix of the heteroclinic contour of system (13) describes the switching wave of the system (12), the limit cycle and the separatrix loop of the singular point of system (13) describe the traveling wave and the running impulse of system (12). And diffusion chaos in system (12) is described by singular attractors of the system of ordinary differential Eq. (13) in full accordance with the universal bifurcation Feigenbaum-Sharkovsky-Magnitskii theory. The greatest interest represents a case when c is a bifurcation parameter, describing a speed of wave distribution along an axis x, which is not included obviously into initial system (12). This case means, that system of a kind (12) with the fixed parameters can have infinitely number of various autowave solutions of any period running along a spatial axis with various speeds, and infinite number of modes of diffusion chaos. One of such system, describing chemical turbulence in autocatalytic chemical reactions, is studied in [6, 7, 15].

In this chapter, let us consider the system of a kind (12) describing distribution of nervous impulses in a cardiac muscle [16]:

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$$u_t = u_{xx} + \frac{1}{\varepsilon}u(1-u)\left(u - \frac{0.06+v}{0.75}\right), \quad v_t = u^3 - v. \tag{14}$$

where u is the activator, v—ingibitor, slowing down development of the activator, the parameter $1/\varepsilon$ defines time of restoration of the system after perturbation. Let us show, that transition to diffusion chaos in system (14), during complication of periodic fluctuations occurring in it, occurs according to universal bifurcation scenario of the FShM theory. We shall analyze solutions of system (14) by means of automodeling replacement of independent variables $\xi = x - c\,t$, having reduced the initial system of partial differential equations to three-dimensional system of ordinary differential equations

$$\dot{u} = w, \quad \dot{w} = -\left(cw + \frac{1}{\varepsilon}u(1-u)\left(u - \frac{0.06+v}{0.75}\right)\right), \quad \dot{v} = \left(v - u^3\right)/c, \tag{15}$$

where derivative is taken with respect to the variable ξ. If $(u(\xi), v(\xi), w(\xi))$ is the solution of system of ODE (15) then $(u(x-ct), v(x-ct), w(x-ct))$ will be the solution of system in private derivatives (14). Thus running waves in system (14) are described by limit cycles of system (15), and running impulses—by separatrix loops of saddle-focuses. Let us carry out numerical research of system (15) in the field of where one of singular points is a saddle-focus. The greatest interest represents a case when c is the bifurcation parameter which describes a speed of waves distribution along an axis x and which is not included obviously into initial system (14). This case means, that the system (14) with the fixed parameters can have infinitely number of various autowave solutions of any period running along a spatial axis with various speeds, and infinite number of modes of diffusion chaos. Let us fix a parameter value $\varepsilon : 1/\varepsilon = 17.4$, and take the parameter c as bifurcation parameter. At $c \in [1.6305, 1.6316]$ there is a stable cycle in the system (15). At $c \approx 1.6317$ the cascade of Feigenbaum period-doubling bifurcations of the initial cycle begins, and at $c \in [1.6317, 1.6331]$ the cycle of period 2 is observed, at $c \in [1.6332, 1.6335]$—the cycle of period 4, and at $c \approx 1.63375$ the first singular attractor—Feigenbaum attractor is found out (**Figure 4**).

At the further reduction of values of parameter c, cycles of period 5 and period 3 are found out at $c \approx 1.6344$ and at $c \approx 1.6347$ (**Figure 4**). Thus, it is established, that in system (15) at change of parameter c, Feigenbaum cascade of period-doubling bifurcations of stable limit cycles and the full subharmonic Sharkovsky cascade of bifurcations of stable cycles according to the Sharkovsky order are realized. To the found cycles of system (15) there correspond running waves of system (14), some of which are represented in **Figure 5**.

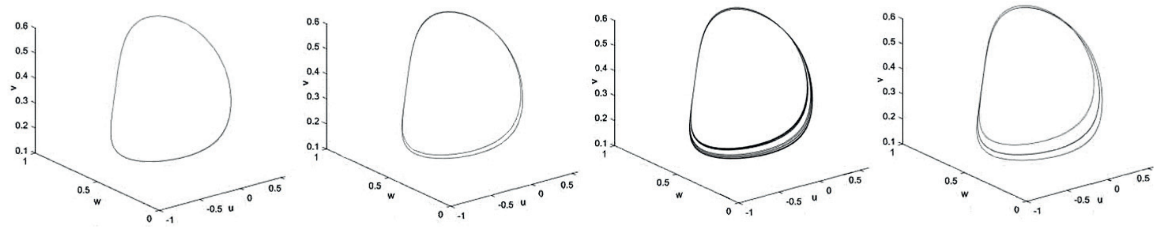

Figure 4. Cycles of periods 1, 2, 3 from Sharkovsky cascade and Feigenbaum attractor.

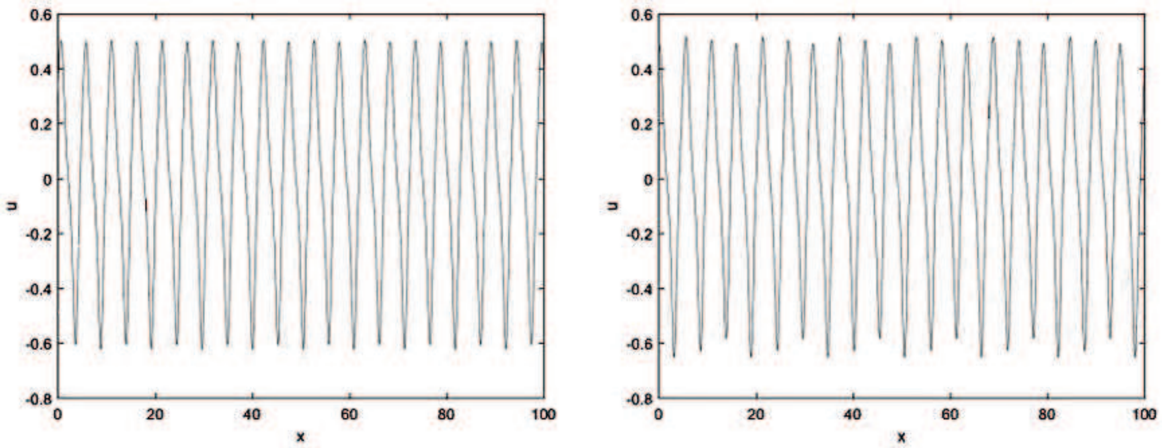

Figure 5. Running waves of system (14), corresponding to cycles of system (15) with periods 2, 3.

4.2. Spatio-temporal chaos in autooscillating mediums

It is well-known that any solution of the reaction-diffusion system (11) in a neighborhood $\mu > \mu_0$ of the thermodynamic branch can be approximated by some complex-valued solution $W(r, \tau) = u(r, \tau) + iv(r, \tau)$ of the Kuramoto-Tsuzuki (or Time Dependent Ginzburg-Landau) equation (see [1, 2, 6, 7]):

$$W_\tau = W + (1 + ic_1)W_{rr} - (1 + ic_2)|W|^2W, \tag{16}$$

where $r = \varepsilon x, \tau = \varepsilon^2 t,\ \varepsilon = \sqrt{\mu - \mu_0}, 0 \leq r \leq R,\ c_1, c_2$—two real constants. Obviously, Eq. (16) has a spatial homogeneous solution $W(\tau) = \exp(-i(c_2\tau + \varphi))$ for an arbitrary phase φ. Consequently, each element of the medium (16) oscillates with a frequency c_2. This solution is stable in a certain area of parameters c_1 and c_2. So, such media are called as autooscillating media. Research of solutions of the Kuramoto-Tsuzuki (Ginzburg-Landau) Eq. (16) directly in its phase space has shown, that actually in this equation there is subharmonic cascade of bifurcations of stable two-dimensional tori of any period according to the Sharkovsky order over each of frequencies and over two frequencies simultaneously. In [1, 2, 6, 7] solutions of the second boundary-value problem for the Eq. (16) on an interval are analyzed in detail. It has been constructed four-dimensional subspace $(u(0), v(0), u(l/2), v(l/2))$ of infinitely dimensional phase space of solutions of the problem, and its Poincare section by the plane $u(l/2) = 0$ for various values of bifurcation parameters c_1 and c_2 has been considered. Poincare's method of the analysis of phase space of solutions of the Eq. (16) has allowed to find all cascades of bifurcations of two-dimensional tori in full accordance with the FShM theory.

In this chapter, we consider the problem of research of nonlinear effects in model of surface plasmon-polyariton. The passage of an electromagnetic wave through a configuration from three various environments dielectric-metal-dielectric can be described by following system of the equations in partial derivatives in the complex variables, turning out of Maxwell equations (see [17]):

$$
\begin{aligned}
&i\frac{\partial\psi_p}{\partial z}+\frac{1}{2\beta}\frac{\partial^2\psi_p}{\partial y^2}+(il-\Delta\beta)\psi_p+\kappa\psi_a=0,\\
&i\frac{\partial\psi_a}{\partial z}+\frac{1}{2\beta}\frac{\partial^2\psi_a}{\partial y^2}+(i(l-g)+\Delta\beta)\psi_a+f\Upsilon|\psi_a|^2\psi_a\kappa\psi_p=0,
\end{aligned}
\tag{17}
$$

The system (17) represents two connected Ginzburg-Landau equations, ψ_p and ψ_a—complex-valued functions, y and z—independent variables. The role of time in Ginzburg-Landau equation in this case is played with spatial coordinate z. The equation for ψ_p corresponds to a wave on border of metal and passive dielectric, and for ψ_a—on border of metal and active nonlinear dielectric. Parameters l g, κ—accordingly dimensionless factors of losses, strengthening and connection between two borders. In Ref. [17] the following fixed values of parameters were considered: $l=0.0026$, $\kappa=0.0028$, $\Delta\beta=0$, $\beta=1.43$, $f\Upsilon=f(\Upsilon'+i\Upsilon'')=3.5\cdot10^{-3}(1+0.1i)$. We shall research dynamics of system (17) at various values of parameter g, and as boundary conditions on spatial variable y we shall consider periodic boundary conditions. In analysis of dynamics of the system (17) we use the real functions: u_1, v_1, u_2, v_2 instead of complex-valued functions ψ_p and ψ_a where $\psi_p=u_1+iv_1$, $\psi_a=u_2+iv_2$. And vector of independent variables is denoted $\vec{x}=(u_1,v_1,u_2,v_2)^T$.

In the considered initial boundary-value problem it is possible to allocate a subclass of spatially homogeneous solutions, not dependent on a variable y. They can be found, solving the system of ordinary differential equations received from (17) by rejection of members, containing derivatives on y. The received system of ODE in coordinates $\vec{x}$ is

$$
\frac{d\vec{x}}{dz}=\begin{pmatrix}-l & \Delta\beta & 0 & -\kappa\\ -\Delta\beta & -l & \kappa & 0\\ 0 & -\kappa & -(l-g) & -\Delta\beta\\ \kappa & 0 & \Delta\beta & g-l\end{pmatrix}\times\vec{x}+\left(u_2^2+v_2^2\right)f\begin{pmatrix}0\\0\\ \Upsilon''\cdot u_2+\Upsilon'\cdot v_2\\ \Upsilon''\cdot v_2-\Upsilon'\cdot u_2\end{pmatrix}
\tag{18}
$$

Critical value of parameter is $g=0.0052$. At smaller parameter values the zero solution is stable. At great values the solution loses stability, and the signs on the real parts are changed at once with four roots of the characteristic equation. Approximately at parameter value $g=0.00357$ a pair of periodic solutions appears in system (18) as a result of the saddle-node bifurcation At parameter value $g=0.0052$, when zero singular point loses its stability, the unstable periodic solution disappears as a result of subcritical Andronov-Hopf bifurcation. Thus, at $g>0.0052$ there is a stable limit cycle in the system. Let us consider the scenario of complication of dynamics of solutions in system (17) at value L = 10, where L is the size of physical area on a variable y. Phase portraits of system we build in a point y = L/3: $u_1(L/3,z)$, $v_1(L/3,z)$, $u_2(L/3,z)$, $v_2(L/3,z)$. In case of periodic boundary conditions, the first stages of complication go according to the Landau-Hopf scenario, that is, occurrence of periodic and quasiperiodic solutions of the increasing phase dimension have been found out. At parameter values $g<0.0095$ the homogeneous cycle described above saves the stability, and at a parameter value $g=0.0096$ he becomes non-homogeneous. The further complication of dynamics of system occurs at parameter value $g\approx0.0105$. At this value a quasiperiodic solution—torus of dimension two is born in the system as a result Andronov-Hopf bifurcation. A kind of this solution in phase space and its section by a plane $u_1 = 0$ are represented in **Figure 6**. It is visible, that the section represents the closed curve.

Following bifurcation in the system (17) occurs in a range of parameter values g from 0.01385 till 0.01390. As a result of one more Andronov-Hopf bifurcation more complex quasiperiodic solution is formed in the system—it is torus of dimension three. A phase portrait of this torus at $g = 0.0139$ and its Poincare sections are represented in **Figure 7**. The first section $u_1 = 0$ represents two-dimensional torus which in turn in section $u_2 = 0$ gives two closed curves.

For the problem with Neumann's homogeneous boundary conditions also it was possible to observe a non-homogeneous stable cycle at $g = 0.0060$. At $g = 0.0095$ stable two-dimensional torus is born from this cycle, and at $g \approx 0.013$ stable three-dimensional torus is born from it as a result of the second Andronov-Hopf bifurcation. Thus, it is proved, that in complex nonlinear systems of partial differential equations stable three-dimensional tori can exist, that contradicts to the Ruelle-Takens theorem. The natural is not the decay of three-dimensional torus with forming uncertain mythical strange attractor, but further complication of dynamics of solutions as a result of following Andronov-Hopf bifurcation with forming four-dimensional torus, or as a result of period-doubling bifurcation of three-dimensional torus along one of its frequencies or along all frequencies simultaneously (that takes place in systems of Navier-Stokes equations).

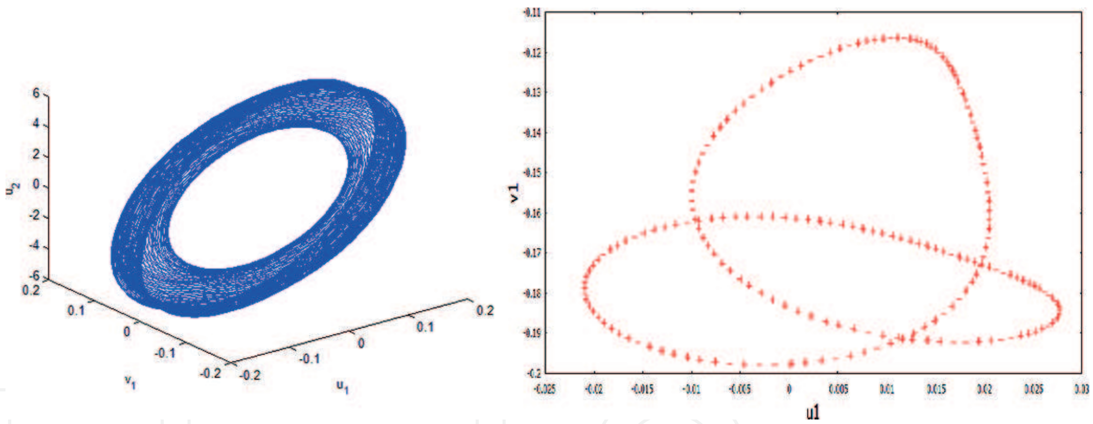

Figure 6. Phase portrait of system (17) and its section by the plane $u_1 = 0$, $g = 0.0105$.

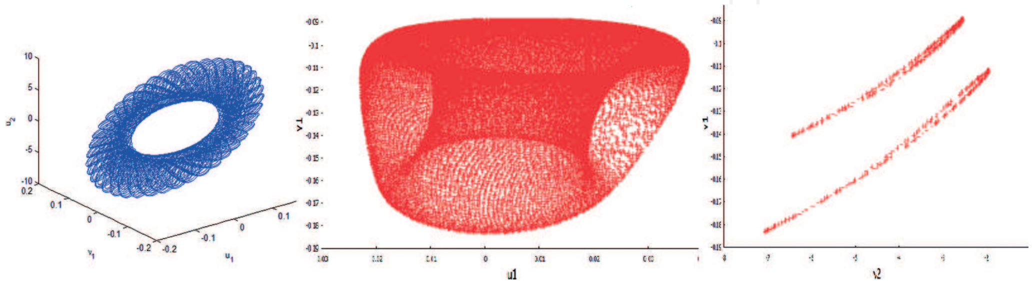

Figure 7. Phase portrait of system (17), its first section by the plane $u_1 = 0$ and second section by the plane $u_2 = 0$, $g = 0.0139$.

4.3. Laminar-turbulent transition in Navier-Stokes equations

The problem of turbulence consists in explaining the nature of the disordered chaotic motion of a nonlinear continuous medium and in finding ways and methods of its adequate mathematical description. Originating more than a 100 years ago, the problem of turbulence is still one of the most complicated and most interesting problems in mathematical physics. It is in the list of seven mathematical millennium problems, named so by the Clay Institute of Mathematics [18]. In addition, the turbulence problem is formulated in the list of S.Smale's 18 most important mathematical problems of the twenty-first century [19]. The most important and interesting in the problem of turbulence is to elucidate the causes and mechanisms of chaos generation in a nonlinear continuous medium when passing from the laminar to the turbulent state. Currently, there are several mathematical models that claim to explain the mechanisms of generation of chaos and turbulence in nonlinear continuous media. The most famous among these models are: the Landau-Hopf model explaining turbulence by motion along an infinite-dimensional torus generated by an infinite cascade of Andronov-Hopf bifurcations; and the Ruelle-Takens model, which explains turbulence by moving along a strange attractor generated by the destruction of a three-dimensional torus. In recent years, the author and his pupils have proved (see [8, 9, 20–22]) that the universal bifurcation FShM mechanism for the transition to space-time chaos in nonlinear systems of partial differential equations through subharmonic cascades of bifurcations of stable cycles or two-dimensional and multi-dimensional tori also takes place in problems of laminar-turbulent transitions for Navier-Stokes equations

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right) \vec{u} = -\nabla p + R^{-1} \Delta \vec{u} + \vec{f}, \\ &\nabla \vec{u} = 0, \end{aligned} \tag{19}$$

where R is the bifurcation parameter (Reynolds number). The existence of stable two-dimensional tori of doubled period and stable three-dimensional tori and their further bifurcations is established for the problem of fluid flow from the ledge [20]. The existence of subharmonic cascades of bifurcations of stable cycles and two-dimensional tori is established for Rayleigh-Benard convection in Ref. [21]. A numerically complete subharmonic cascade of bifurcations of stable two-dimensional tori is found up to a torus of period three in the famous Kolmogorov problem in two-dimensional and three-dimensional spatial cases [22]. The features of compressible flow and instabilities triggered by Kelvin-Helmholtz (KH) and Rayleigh-Taylor (RT) mechanisms are considered in Ref [9]. The Kelvin-Helmholtz instability is the instability of the shear layer, which is a tangential discontinuity for the inviscid liquid and which arises when there is a velocity difference at the interface of two liquids or when there is a velocity shift in one of the liquids. Rayleigh-Taylor instability is the instability of the boundary between two liquids, where a lighter liquid supports a heavier fluid in a gravitational or external potential field, the gradient vector of which is directed from the heavier liquid to the lighter one. Light fluid can also push heavier one. Those two instabilities are often considered together. Indeed, RT instability causes movement of adjusted fluids in different directions with the appearance of the shear layer that is subject to KH instability.

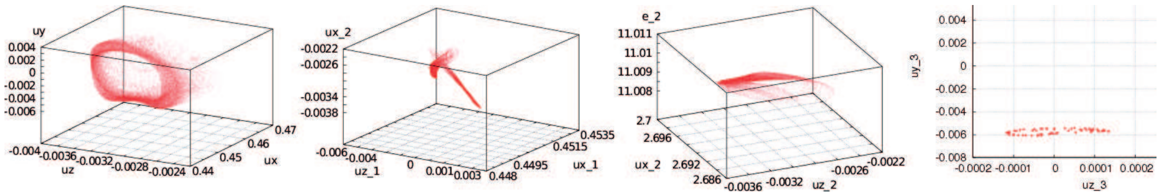

Figure 8. Projection of the invariant four-dimensional torus into three-dimensional phase subspace and sequential first, second and third sections in the phase space for $R = 520.5$ (left to right). Only parts of sections are shown.

In this chapter, we consider shortly the bifurcation scenario in coupled Kelvin-Helmholtz and Rayleigh-Taylor problem. This problem is solved in detail in Ref. [9]. We begin our consideration from the value of $R = 1$ for which the system has a stationary solution corresponding to a stable singular point in the phase space of solutions. Approximately for $R = 10.5$ the first bifurcation of the stationary solution occurs with the formation from the singular point of the stable limit cycle in the phase space of solutions. The next attractor that can be able to detect is the limited torus. Close resemblance to the cycle may indicate that this attractor was formed from the cycle as the result of Andronov-Hopf bifurcation. This indicates the existence of two irrational frequencies in the system. Further increase of the Reynolds number up to $R = 516$ resulted in the other Andronov-Hopf bifurcation with the formation of the three-dimensional invariant torus. This torus becomes singular (by period-doubling bifurcations along one of the frequencies). However this cascade of period-doubling bifurcations is reversed to the original 3D torus. The next bifurcation that could be traced at $R = 520.5$ is second Andronov-Hopf bifurcation leading to the formation of the four-dimensional invariant torus (**Figure 8**) . Further increase of the Reynolds number leads to the chaotic solution that corresponds to the dense field of points in phase subspace projections up to $R = 2100$. With the further increase of R, formation of inverse bifurcation cascades is observed. Thus, it seems reasonable, that there is no unified laminar-turbulent transition scenario in problems described by Navier-Stokes equations, it can be a cascade of stable limit cycles or cascade of stable two-dimensional or many dimensional tori, but all these scenarios lay in the frameworks of the FShM theory. However, the existence of computationally stable 4D invariant torus is a remarkable fact. It took $2.6 \cdot 10^9$ time samples to analyze and about 3.5 month to calculate this torus and its Poincare sections.

5. Conclusion

We make some general remarks on the chaotic dynamics of nonlinear systems of differential equations, since the very publication of papers [10, 11] and many similar papers, even in prestigious refereed journals, attests to a complete lack of understanding of the mechanism of transition to chaos in nonlinear systems of differential equations. In this chapter, on numerous examples, it is convincingly demonstrated that there exists one universal FShM bifurcation scenario of transition to chaos in all systems of nonlinear differential equations without exception: autonomous and nonautonomous, dissipative and conservative, ordinary, with partial derivatives and with delayed argument. All irregular attractors that arise during the implementation of this scenario are exclusively singular attractors. Each nonlinear system can have

http://dx.doi.org/10.5772/intechopen.70987

infinitely many different structurally unstable singular attractors for different values of the bifurcation parameter, which can enter implicitly into the equations of the system. Thus, neither the presence or absence of stable or unstable singular points in the system, nor the presence or absence of saddle-nodes or saddle-focuses, as well as homoclinic or heteroclinic separatrix contours, is not a criterion for the appearance of chaotic dynamics in the system. Also, neither the positivity of the senior Lyapunov exponent, nor the proof of existence of Smale's horseshoe, nor the KAM (Kolmogorov-Arnold-Moser) theory, nor the theory of RT (Ruelle-Takens), are such criteria either. The positivity of the Lyapunov exponent is purely a consequence of computational errors, because due to the presence of an everywhere dense set of nonperiodic trajectories, numerical motion is possible only over the whole region occupied by the trajectory of the singular attractor, and not along its trajectory itself. In addition, the Lyapunov exponent will also be positive when moving along a stable periodic trajectory of a large period in the vicinity of some singular attractor. The presence of Smale's horseshoe in the system testifies to the complex dynamics of the solutions, however, even in the neighborhood of the separatrix loop of saddle-focus, where by Shilnikov's theorem there exists an infinite number of Smale's horseshoes, the dynamics of solutions are determined not by horseshoes, but by a much more complex infinite set of unstable periodic solutions generated at all stages of all three cascades of bifurcations of the FShM scenario, whose homoclinic cascade of cycles ends in the limit precisely with the separatrix loop of saddle-focus. The only method that allows establishing reliably the presence of chaotic dynamics in the system is the numerical finding of stable cycles or tori of the FSM-cascades of bifurcations.

Acknowledgements

Paper is supported by Russian Foundation for Basic Research (grants 14-07-00116-a).

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