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Sub-Manifolds of a Riemannian Manifold

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Abstract

In this chapter, we introduce the theory of sub-manifolds of a Riemannian manifold. The fundamental notations are given. The theory of sub-manifolds of an almost Riemannian product manifold is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a sub-manifold, with respect to the action of almost Riemannian product structure of the ambient manifolds, we have three typical classes of sub-manifolds such as invariant sub-manifolds, anti-invariant sub-manifolds and semi-invariant sub-manifolds. In addition, slant, semi-slant and pseudo-slant sub-manifolds are introduced by many geometers.

Keywords: Riemannian product manifold, Riemannian product structure, integral manifold, a distribution on a manifold, real product space forms, a slant distribution

1. Introduction

Let $i : M \rightarrow \tilde{M}$ be an immersion of an n -dimensional manifold M into an m -dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Denote by $g = i^* \tilde{g}$ the induced Riemannian metric on M . Thus, i become an isometric immersion and M is also a Riemannian manifold with the Riemannian metric $g(X, Y) = \tilde{g}(X, Y)$ for any vector fields X, Y in M . The Riemannian metric g on M is called the induced metric on M . In local components, $g_{ij} = g_{AB} B_j^B B_i^A$ with $g = g_{ji} dx^j dx^i$ and $\tilde{g} = g_{BA} dU^B dU^A$.

If a vector field ξ_p of $\tilde{M}$ at a point $p \in M$ satisfies

$$\tilde{g}(X_p, \xi_p) = 0 \quad (1)$$

for any vector X_p of M at p , then ξ_p is called a normal vector of M in $\tilde{M}$ at p . A unit normal vector field of M in $\tilde{M}$ is called a normal section on M [3].

By $T^\perp M$, we denote the vector bundle of all normal vectors of M in $\tilde{M}$. Then, the tangent bundle of $\tilde{M}$ is the direct sum of the tangent bundle TM of M and the normal bundle $T^\perp M$ of M in $\tilde{M}$, i.e.,

$$T\tilde{M} = TM \oplus T^\perp M. \quad (2)$$

We note that if the sub-manifold M is of codimension one in $\tilde{M}$ and they are both orientable, we can always choose a normal section ξ on M , i.e.,

$$g(X, \xi) = 0, \quad g(\xi, \xi) = 1, \quad (3)$$

where X is any arbitrary vector field on M .

By $\tilde{\nabla}$, denote the Riemannian connection on $\tilde{M}$ and we put

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (4)$$

for any vector fields X, Y tangent to M , where $\nabla_X Y$ and $h(X, Y)$ are tangential and the normal components of $\tilde{\nabla}_X Y$, respectively. Formula (4) is called the Gauss formula for the sub-manifold M of a Riemannian manifold $(\tilde{M}, \tilde{g})$.

Proposition 1.1. ∇ is the Riemannian connection of the induced metric $g = i^* \tilde{g}$ on M and $h(X, Y)$ is a normal vector field over M , which is symmetric and bilinear in X and Y .

Proof: Let α and β be differentiable functions on M . Then, we have

$$\begin{aligned} \tilde{\nabla}_{\alpha X}(\beta Y) &= \alpha \{X(\beta)Y + \beta \tilde{\nabla}_X Y\} \\ &= \alpha \{X(\beta)Y + \beta \nabla_X Y + \beta h(X, Y)\} \\ \nabla_{\alpha X} \beta Y + h(\alpha X, \beta Y) &= \alpha \beta \nabla_X Y + \alpha X(\beta)Y + \alpha \beta h(X, Y) \end{aligned} \quad (5)$$

This implies that

$$\nabla_{\alpha X}(\beta Y) = \alpha X(\beta)Y + \alpha \beta \nabla_X Y \quad (6)$$

and

$$h(\alpha X, \beta Y) = \alpha \beta h(X, Y). \quad (7)$$

Eq. (6) shows that ∇ defines an affine connection on M and Eq. (4) shows that h is bilinear in X and Y since additivity is trivial [1].

Since the Riemannian connection $\tilde{\nabla}$ has no torsion, we have

$$0 = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \nabla_X Y + h(X, Y) - \nabla_Y X - h(Y, X) - [X, Y]. \quad (8)$$

By comparing the tangential and normal parts of the last equality, we obtain

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad (9)$$

and

$$h(X, Y) = h(Y, X). \quad (10)$$

These equations show that ∇ has no torsion and h is a symmetric bilinear map. Since the metric $\tilde{g}$ is parallel, we can easily see that

$$\begin{aligned} (\nabla_X g)(Y, Z) &= (\tilde{\nabla}_X \tilde{g})(Y, Z) \\ &= \tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(Y, \tilde{\nabla}_X Z) \\ &= \tilde{g}(\nabla_X Y + h(X, Y), Z) + \tilde{g}(Y, \nabla_X Z + h(X, Z)) \\ &= \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \end{aligned} \quad (11)$$

for any vector fields X, Y, Z tangent to M , that is, ∇ is also the Riemannian connection of the induced metric g on M .

We recall h the second fundamental form of the sub-manifold M (or immersion i), which is defined by

$$h : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(T^\perp M). \quad (12)$$

If $h = 0$ identically, then sub-manifold M is said to be totally geodesic, where $\Gamma(T^\perp M)$ is the set of the differentiable vector fields on normal bundle of M .

Totally geodesic sub-manifolds are simplest sub-manifolds.

Definition 1.1. Let M be an n -dimensional sub-manifold of an m -dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. By h , we denote the second fundamental form of M in $\tilde{M}$.

$H = \frac{1}{n} \text{trace}(h)$ is called the mean curvature vector of M in $\tilde{M}$. If $H = 0$, the sub-manifold is called minimal.

On the other hand, M is called pseudo-umbilical if there exists a function λ on M , such that

$$\tilde{g}(h(X, Y), H) = \lambda g(X, Y) \quad (13)$$

for any vector fields X, Y on M and M is called totally umbilical sub-manifold if

$$h(X, Y) = g(X, Y)H. \quad (14)$$

It is clear that every minimal sub-manifold is pseudo-umbilical with $\lambda = 0$. On the other hand, by a direct calculation, we can find $\lambda = \tilde{g}(H, H)$ for a pseudo-umbilical sub-manifold. So, every

totally umbilical sub-manifold is a pseudo-umbilical and a totally umbilical sub-manifold is totally geodesic if and only if it is minimal [2].

Now, let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$ and V be a normal vector field on M , X be a vector field on M . Then, we decompose

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (15)$$

where $A_V X$ and $\nabla_X^\perp V$ denote the tangential and the normal components of $\nabla_X^\perp V$, respectively. We can easily see that $A_V X$ and $\nabla_X^\perp V$ are both differentiable vector fields on M and normal bundle of M , respectively. Moreover, Eq. (15) is also called Weingarten formula.

Proposition 1.2. Let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then

(a) $A_V X$ is bilinear in vector fields V and X . Hence, $A_V X$ at point $p \in M$ depends only on vector fields V_p and X_p .

(b) For any normal vector field V on M , we have

$$g(A_V X, Y) = g(h(X, Y), V). \quad (16)$$

Proof: Let α and β be any two functions on M . Then, we have

$$\begin{aligned} \tilde{\nabla}_{\alpha X}(\beta V) &= \alpha \tilde{\nabla}_X(\beta V) \\ &= \alpha \{X(\beta)V + \beta \tilde{\nabla}_X V\} \\ -A_{\beta V} \alpha X + \nabla_{\alpha X}^\perp \beta V &= \alpha X(\beta)V - \alpha \beta A_V X + \alpha \beta \nabla_X^\perp V. \end{aligned} \quad (17)$$

This implies that

$$A_{\beta V} \alpha X = \alpha \beta A_V X \quad (18)$$

and

$$\nabla_{\alpha X}^\perp \beta V = \alpha X(\beta)V + \alpha \beta \nabla_X^\perp V. \quad (19)$$

Thus, $A_V X$ is bilinear in V and X . Additivity is trivial. On the other hand, since g is a Riemannian metric,

$$X\tilde{g}(Y, V) = 0, \quad (20)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Eq. (12) implies that

$$\tilde{g}(\tilde{\nabla}_X Y, V) + \tilde{g}(Y, \tilde{\nabla}_X V) = 0. \quad (21)$$

By means of Eqs. (4) and (15), we obtain

$$\tilde{g}(h(X, Y), V) - g(A_V X, Y) = 0. \quad (22)$$

The proof is completed [3].

Let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$, and h and A_V denote the second fundamental form and shape operator of M , respectively.

The covariant derivative of h and A_V is, respectively, defined by

$$(\tilde{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \quad (23)$$

and

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{\nabla_X^\perp V} Y - A_V \nabla_X Y \quad (24)$$

for any vector fields X, Y tangent to M and any vector field V normal to M . If $\nabla_X h = 0$ for all X , then the second fundamental form of M is said to be parallel, which is equivalent to $\nabla_X A = 0$. By direct calculations, we get the relation

$$g((\nabla_X h)(Y, Z), V) = g((\nabla_X A)_V Y, Z). \quad (25)$$

Example 1.1. We consider the isometric immersion

$$\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^4, \quad (26)$$

$$\phi(x_1, x_2) = (x_1, \sqrt{x_1^2 - 1}, x_2, \sqrt{x_2^2 - 1}) \quad (27)$$

we note that $M = \phi(\mathbb{R}^2) \subset \mathbb{R}^4$ is a two-dimensional sub-manifold of $\mathbb{R}^4$ and the tangent bundle is spanned by the vectors

$$TM = S_p \left\{ e_1 = \left(\sqrt{x_1^2 - 1}, x_1, 0, 0 \right), e_2 = \left(0, 0, \sqrt{x_2^2 - 1}, x_2 \right) \right\} \text{ and the normal vector fields} \\ T^\perp M = sp \left\{ w_1 = \left(-x_1, \sqrt{x_1^2 - 1}, 0, 0 \right), w_2 = \left(0, 0, -x_1, \sqrt{x_2^2 - 1} \right) \right\}. \quad (28)$$

By $\tilde{\nabla}$, we denote the Levi-Civita connection of $\mathbb{R}^4$, the coefficients of connection, are given by

$$\tilde{\nabla}_{e_1} e_1 = \frac{2x_1 \sqrt{x_1^2 - 1}}{2x_1^2 - 1} e_1 - \frac{1}{2x_1^2 - 1} w_1, \quad (29)$$

$$\tilde{\nabla}_{e_2} e_2 = \frac{2x_2 \sqrt{x_2^2 - 1}}{2x_2^2 - 1} e_2 - \frac{1}{2x_2^2 - 1} w_2 \quad (30)$$

and

$$\nabla_{e_2} e_1 = 0. \quad (31)$$

Thus, we have $h(e_1, e_1) = -\frac{1}{2x_1^2-1}w_1$, $h(e_2, e_2) = -\frac{1}{2x_2^2-1}w_2$ and $h(e_2, e_1) = 0$. The mean curvature vector of $M = \phi(\mathbb{R}^2)$ is given by

$$H = -\frac{1}{2}(w_1 + w_2). \quad (32)$$

Furthermore, by using Eq. (16), we obtain

$$\begin{aligned} g(A_{w_1} e_1, e_1) &= g(h(e_1, e_1), w_1) = -\frac{1}{2x_1^2-1}(x_1^2 + x_1^2 - 1) = -1, \\ g(A_{w_1} e_2, e_2) &= g(h(e_2, e_2), w_1) = -\frac{1}{2x_2^2-1}g(w_1, w_2) = 0, \\ g(A_{w_1} e_1, e_2) &= 0, \end{aligned} \quad (33)$$

and

$$\begin{aligned} g(A_{w_2} e_1, e_1) &= g(h(e_1, e_1), w_2) = 0, \\ g(A_{w_2} e_1, e_2) &= 0, g(A_{w_2} e_2, e_2) = 1. \end{aligned} \quad (34)$$

Thus, we have

$$A_{w_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_{w_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \quad (35)$$

Now, let M be a sub-manifold of a Riemannian manifold $(\tilde{M}, g)$, $\tilde{R}$ and R be the Riemannian curvature tensors of $\tilde{M}$ and M , respectively. From then the Gauss and Weingarten formulas, we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z \\ &= \tilde{\nabla}_X (\nabla_Y Z + h(Y, Z)) - \tilde{\nabla}_Y (\nabla_X Z + h(X, Z)) - \nabla_{[X, Y]} Z - h([X, Y], Z) \\ &= \tilde{\nabla}_X \nabla_Y Z + \tilde{\nabla}_X h(Y, Z) - \tilde{\nabla}_Y \nabla_X Z - \tilde{\nabla}_Y h(X, Z) - \nabla_{[X, Y]} Z - h(\nabla_X Y, Z) + h(\nabla_Y X, Z) \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + h(X, \nabla_Y Z) - h(\nabla_X Z, Y) + \nabla_X^\perp h(Y, Z) \\ &\quad - A_{h(Y, Z)} X - \nabla_Y^\perp h(X, Z) + A_{h(X, Z)} Y - \nabla_{[X, Y]} Z - h(\nabla_X Y, Z) + h(\nabla_Y X, Z) \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z + \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) \\ &\quad - h(Y, \nabla_X Z) - \nabla_Y^\perp h(X, Z) + h(\nabla_Y X, Z) + h(\nabla_Y Z, X) \\ &\quad + A_{h(X, Z)} Y - A_{h(Y, Z)} X \\ &= R(X, Y)Z + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + A_{h(X, Z)} Y - A_{h(Y, Z)} X \end{aligned} \quad (36)$$

from which

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \quad (37)$$

for any vector fields X, Y and Z tangent to M . For any vector field W tangent to M , Eq. (37) gives the Gauss equation

$$g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(h(Y, W), h(X, Z)) - g(h(Y, Z), h(X, W)). \quad (38)$$

On the other hand, the normal component of Eq. (37) is called equation of Codazzi, which is given by

$$(\tilde{R}(X, Y)Z)^\perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z). \quad (39)$$

If the Codazzi equation vanishes identically, then sub-manifold M is said to be curvature-invariant sub-manifold [4].

In particular, if $\tilde{M}$ is of constant curvature, $\tilde{R}(X, Y)Z$ is tangent to M , that is, sub-manifold is curvature-invariant. Whereas, in Kenmotsu space forms, and Sasakian space forms, this not true.

Next, we will define the curvature tensor $R^\perp$ of the normal bundle of the sub-manifold M by

$$R^\perp(X, Y)V = \nabla_X^\perp \nabla_Y^\perp V - \nabla_Y^\perp \nabla_X^\perp V - \nabla_{[X, Y]}^\perp V \quad (40)$$

for any vector fields X, Y tangent to sub-manifold M , and any vector field V normal to M . From the Gauss and Weingarten formulas, we have

$$\begin{aligned} \tilde{R}(X, Y)V &= \tilde{\nabla}_X \tilde{\nabla}_Y V - \tilde{\nabla}_Y \tilde{\nabla}_X V - \tilde{\nabla}_{[X, Y]} V \\ &= \tilde{\nabla}_X (-A_V Y + \nabla_Y^\perp V) - \tilde{\nabla}_Y (-A_V X + \nabla_X^\perp V) + A_V [X, Y] - \nabla_{[X, Y]}^\perp V \\ &= -\tilde{\nabla}_X A_V Y + \tilde{\nabla}_Y A_V X + \tilde{\nabla}_X \nabla_Y^\perp V - \tilde{\nabla}_Y \nabla_X^\perp V + A_V [X, Y] - \nabla_{[X, Y]}^\perp V \\ &= -\nabla_X A_V Y - h(X, A_V Y) + \nabla_Y A_V X + h(Y, A_V X) \\ &\quad + \nabla_X^\perp \nabla_Y^\perp V - \nabla_Y^\perp \nabla_X^\perp V - A_{\nabla_Y^\perp V} X + A_{\nabla_X^\perp V} Y + A_V [X, Y] - \nabla_{[X, Y]}^\perp V \\ &= \nabla_X^\perp \nabla_Y^\perp V - \nabla_Y^\perp \nabla_X^\perp V - \nabla_{[X, Y]}^\perp V - A_{\nabla_Y^\perp V} X + A_{\nabla_X^\perp V} Y + A_V [X, Y] \\ &\quad - \nabla_X A_V Y + \nabla_Y A_V X - h(X, A_V Y) + h(Y, A_V X) \\ &= R^\perp(X, Y)V + h(A_V X, Y) - h(X, A_V Y) - (\nabla_X A)_V Y + (\nabla_Y A)_V X. \end{aligned} \quad (41)$$

For any normal vector U to M , we obtain

$$\begin{aligned}
g(\tilde{R}(X, Y)V, U) &= g(R^\perp(X, Y)V, U) + g(h(A_V X, Y), U) - g(h(X, A_V Y), U) \\
&= g(R^\perp(X, Y)V, U) + g(A_U Y, A_V X) - g(A_V Y, A_U X) \\
&= g(R^\perp(X, Y)V, U) + g(A_V A_U Y, X) - g(A_U A_V Y, X)
\end{aligned} \tag{42}$$

Since $[A_U, A_V] = A_U A_V - A_V A_U$, Eq. (42) implies

$$g(\tilde{R}(X, Y)V, U) = g(R^\perp(X, Y)V, U) + g([A_U, A_V]Y, X). \tag{43}$$

Eq. (43) is also called the Ricci equation.

If $R^\perp = 0$, then the normal connection of M is said to be flat [2].

When $(\tilde{R}(X, Y)V)^\perp = 0$, the normal connection of the sub-manifold M is flat if and only if the second fundamental form M is commutative, i.e. $[A_U, A_V] = 0$ for all U, V . If the ambient space $\tilde{M}$ is real space form, then $(\tilde{R}(X, Y)V)^\perp = 0$ and hence the normal connection of M is flat if and only if the second fundamental form is commutative. If $\tilde{R}(X, Y)Z$ tangent to M , then equation of codazzi Eq. (37) reduces to

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \tag{44}$$

which is equivalent to

$$(\nabla_X A)_V Y = (\nabla_Y A)_V X. \tag{45}$$

On the other hand, if the ambient space $\tilde{M}$ is a space of constant curvature c , then we have

$$\tilde{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} \tag{46}$$

for any vector fields X, Y and Z on $\tilde{M}$.

Since $\tilde{R}(X, Y)Z$ is tangent to M , the equation of Gauss and the equation of Ricci reduce to

$$\begin{aligned}
g(R(X, Y)Z, W) &= c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
&\quad + g(h(Y, Z), h(X, W)) - g(h(Y, W), h(X, Z))
\end{aligned} \tag{47}$$

and

$$g(R^\perp(X, Y)V, U) = g([A_U, A_V]X, Y), \tag{48}$$

respectively.

Proposition 1.3. A totally umbilical sub-manifold M in a real space form $\tilde{M}$ of constant curvature c is also of constant curvature.

Proof: Since M is a totally umbilical sub-manifold of $\tilde{M}$ of constant curvature c , by using Eqs. (14) and (46), we have

$$\begin{aligned} g(R(X, Y)Z, W) &= c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &\quad + g(H, H)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &= \{c + g(H, H)\}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}. \end{aligned} \quad (49)$$

This shows that the sub-manifold M is of constant curvature $c + \|H^2\|$ for $n > 2$. If $n = 2$, $\|H\| = \text{constant}$ follows from the equation of Codazzi [3].

This proves the proposition.

On the other hand, for any orthonormal basis $\{e_a\}$ of normal space, we have

$$\begin{aligned} g(Y, Z)g(X, W) - g(X, Z)g(Y, W) &= \sum_a \left[g(h(Y, Z), e_a)g(h(X, W), e_a) \right. \\ &\quad \left. - g(h(X, Z), e_a)g(h(Y, W), e_a) \right] \\ &= \sum_a g(A_{e_a}Y, Z)g(A_{e_a}X, W) - g(A_{e_a}X, Z)g(A_{e_a}Y, W) \end{aligned} \quad (50)$$

Thus, Eq. (45) can be rewritten as

$$\begin{aligned} g(R(X, Y)Z, W) &= c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &\quad + \sum_a [g(A_{e_a}Y, Z)g(A_{e_a}X, W) - g(A_{e_a}X, Z)g(A_{e_a}Y, W)] \end{aligned} \quad (51)$$

By using A_{e_a} , we can construct a similar equation to Eq. (47) for Eq. (23).

Now, let S - be the Ricci tensor of M . Then, Eq. (47) gives us

$$S(X, Y) = c\{ng(X, Y) - g(e_i, X)g(e_i, Y)\} \quad (52)$$

$$\begin{aligned} &+ \sum_{e_a} [g(A_{e_a}e_i, e_i)g(A_{e_a}X, Y) - g(A_{e_a}X, e_i)g(A_{e_a}e_i, Y)] \\ &= c(n-1)g(X, Y) + \sum_{e_a} [Tr(A_{e_a})g(A_{e_a}X, Y) - g(A_{e_a}X, A_{e_a}Y)], \end{aligned} \quad (53)$$

where $\{e_1, e_2, \dots, e_n\}$ are orthonormal basis of M .

Therefore, the scalar curvature r of sub-manifold M is given by

$$r = cn(n-1) \sum_{e_a} Tr^2(A_{e_a}) - \sum_{e_a} Tr(A_{e_a})^2 \quad (54)$$

$\sum_{e_a} Tr(A_{e_a})^2$ is the square of the length of the second fundamental form of M , which is denoted by $|A_{e_a}|^2$. Thus, we also have

$$\|h^2\| = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)) = \|A^2\|. \quad (55)$$

2. Distribution on a manifold

An m -dimensional distribution on a manifold $\tilde{M}$ is a mapping $\mathcal{D}$ defined on $\tilde{M}$, which assigns to each point p of $\tilde{M}$ an m -dimensional linear subspace $\mathcal{D}_p$ of $T_{\tilde{M}}(p)$. A vector field X on $\tilde{M}$ belongs to $\mathcal{D}$ if we have $X_p \in \mathcal{D}_p$ for each $p \in \tilde{M}$. When this happens, we write $X \in \Gamma(\mathcal{D})$. The distribution $\mathcal{D}$ is said to be differentiable if for any $p \in \tilde{M}$, there exist m -differentiable linearly independent vector fields $X_j \in \Gamma(\mathcal{D})$ in a neighborhood of p .

The distribution $\mathcal{D}$ is said to be involutive if for all vector fields $X, Y \in \Gamma(\mathcal{D})$ we have $[X, Y] \in \Gamma(\mathcal{D})$. A sub-manifold M of $\tilde{M}$ is said to be an integral manifold of $\mathcal{D}$ if for every point $p \in M$, $\mathcal{D}_p$ coincides with the tangent space to M at p . If there exists no integral manifold of $\mathcal{D}$ which contains M , then M is called a maximal integral manifold or a leaf of $\mathcal{D}$. The distribution $\mathcal{D}$ is said to be integrable if for every $p \in \tilde{M}$, there exists an integral manifold of $\mathcal{D}$ containing p [2].

Let $\tilde{\nabla}$ and distribution be a linear connection on $\tilde{M}$, respectively. The distribution $\mathcal{D}$ is said to be parallel with respect to $\tilde{M}$, if we have

$$\tilde{\nabla}_X Y \in \Gamma(\mathcal{D}) \text{ for all } X \in \Gamma(T\tilde{M}) \text{ and } Y \in \Gamma(\mathcal{D}) \quad (56)$$

Now, let $(\tilde{M}, \tilde{g})$ be Riemannian manifold and $\mathcal{D}$ be a distribution on $\tilde{M}$. We suppose $\tilde{M}$ is endowed with two complementary distribution $\mathcal{D}$ and $\mathcal{D}^\perp$, i.e., we have $T\tilde{M} = \mathcal{D} \oplus \mathcal{D}^\perp$. Denoted by P and Q the projections of $T\tilde{M}$ to $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively.

Theorem 2.1. All the linear connections with respect to which both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel, are given by

$$\nabla_X Y = P \nabla'_X P Y + Q \nabla'_X Q Y + P S(X, P Y) + Q S(X, Q Y) \quad (57)$$

for any $X, Y \in \Gamma(T\tilde{M})$, where ∇' and S are, respectively, an arbitrary linear connection and arbitrary tensor field of type $(1, 2)$ on $\tilde{M}$.

Proof: Suppose ∇' is an arbitrary linear connection on $\tilde{M}$. Then, any linear connection ∇ on $\tilde{M}$ is given by

$$\nabla_X Y = \nabla'_X Y + S(X, Y) \quad (58)$$

for any $X, Y \in \Gamma(T\tilde{M})$. We can put

$$X = PX + QX \quad (59)$$

for any $X \in \Gamma(T\tilde{M})$. Then, we have

$$\begin{aligned} \nabla_X Y &= \nabla_X (PY + QY) = \nabla_X PY + \nabla_X QY = \nabla'_X PY + S(X, PY) \\ &\quad + \nabla'_X QY + S(X, QY) = P\nabla'_X PY + Q\nabla'_X PY + PS(X, PY) + QS(X, PY) \\ &\quad + P\nabla'_X QY + Q\nabla'_X QY + PS(X, QY) + QS(X, QY) \end{aligned} \quad (60)$$

for any $X, Y \in \Gamma(T\tilde{M})$.

The distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are both parallel with respect to ∇ if and only if we have

$$\phi(\nabla_X PY) = 0 \text{ and } P(\nabla_X QY) = 0. \quad (61)$$

From Eqs. (58) and (61), it follows that $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to ∇ if and only if

$$Q\nabla'_X PY + QS(X, PY) = 0 \text{ and } P\nabla'_X QY + PS(X, QY) = 0. \quad (62)$$

Thus, Eqs. (58) and (62) give us Eq. (57).

Next, by means of the projections P and Q , we define a tensor field F of type $(1, 1)$ on $\tilde{M}$ by

$$FX = PX - QX \quad (63)$$

for any $X \in \Gamma(T\tilde{M})$. By a direct calculation, it follows that $F^2 = I$. Thus, we say that F defines an almost product structure on $\tilde{M}$. The covariant derivative of F is defined by

$$(\nabla_X F)Y = \nabla_X FY - F\nabla_X Y \quad (64)$$

for all $X, Y \in \Gamma(T\tilde{M})$. We say that the almost product structure F is parallel with respect to the connection ∇ , if we have $\nabla_X F = 0$. In this case, F is called the Riemannian product structure [2].

Theorem 2.2. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold and $\mathcal{D}, \mathcal{D}^\perp$ be orthogonal distributions on $\tilde{M}$ such that $T\tilde{M} = \mathcal{D} \oplus \mathcal{D}^\perp$. Both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to ∇ if and only if F is a Riemannian product structure.

Proof: For any $X, Y \in \Gamma(T\tilde{M})$, we can write

$$\tilde{\nabla}_Y PX = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX \quad (65)$$

and

$$\tilde{\nabla}_Y X = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{PY} QX + \tilde{\nabla}_{QY} PX + \tilde{\nabla}_{QY} QX, \quad (66)$$

from which

$$g(\tilde{\nabla}_{QY} PX, QZ) = QYg(PX, QZ) - g(\nabla_{QY} QZ, PX) = 0 - g(\tilde{\nabla}_{QY} QZ, PX) = 0, \quad (67)$$

that is, $\nabla_{QY} PX \in \Gamma(\mathcal{D})$ and so $P\tilde{\nabla}_{QY} PX = \tilde{\nabla}_{QY} PX$,

$$Q\tilde{\nabla}_{QY} PX = 0. \quad (68)$$

In the same way, we obtain

$$g(\tilde{\nabla}_{PY} QX, PZ) = PYg(QX, PZ) - g(QX, \tilde{\nabla}_{PY} PZ) = 0, \quad (69)$$

which implies that

$$P\tilde{\nabla}_{PY} QX = 0 \text{ and } Q\tilde{\nabla}_{PY} QX = \tilde{\nabla}_{PY} QX. \quad (70)$$

From Eqs. (66), (68) and (70), it follows that

$$P\tilde{\nabla}_Y X = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX. \quad (71)$$

By using Eqs. (64) and (71), we obtain

$$(\tilde{\nabla}_Y P)X = \tilde{\nabla}_Y PX - P\tilde{\nabla}_Y X = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX - \tilde{\nabla}_{PY} PX - \tilde{\nabla}_{QY} PX = 0. \quad (72)$$

In the same way, we can find $\tilde{\nabla}Q = 0$. Thus, we obtain

$$\tilde{\nabla}F = \tilde{\nabla}(P - Q) = 0. \quad (73)$$

This proves our assertion [2].

Theorem 2.3. Both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to Levi-Civita connection ∇ if and only if they are integrable and their leaves are totally geodesic in $\tilde{M}$.

Proof: Let us assume both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel. Since ∇ is a torsion free linear connection, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(\mathcal{D}), \text{ for any } X, Y \in \Gamma(\mathcal{D}) \quad (74)$$

and

$$[U, V] = \nabla_U V - \nabla_V U \in \Gamma(\mathcal{D}^\perp), \text{ for any } U, V \in \Gamma(\mathcal{D}^\perp) \quad (75)$$

Thus, $\mathcal{D}$ and $\mathcal{D}^\perp$ are integrable distributions. Now, let M be a leaf of $\mathcal{D}$ and denote by h the second fundamental form of the immersion of M in $\tilde{M}$. Then by the Gauss formula, we have

$$\nabla_X Y = \nabla'_X Y + h(X, Y) \quad (76)$$

for any $X, Y \in \Gamma(\mathcal{D})$, where ∇' denote the Levi-Civita connection on M . Since $\mathcal{D}$ is parallel from Eq. (76) we conclude $h = 0$, that is, M is totally in $\tilde{M}$. In the same way, it follows that each leaf of $\mathcal{D}^\perp$ is totally geodesic in $\tilde{M}$.

Conversely, suppose $\mathcal{D}$ and $\mathcal{D}^\perp$ be integrable and their leaves are totally geodesic in $\tilde{M}$. Then by using Eq. (4), we have

$$\nabla_X Y \in \Gamma(\mathcal{D}) \text{ for any } X, Y \in \Gamma(\mathcal{D}) \quad (77)$$

and

$$\nabla_U V \in \Gamma(\mathcal{D}^\perp) \text{ for any } U, V \in \Gamma(\mathcal{D}^\perp). \quad (78)$$

Since g is a Riemannian metric tensor, we obtain

$$g(\nabla_U Y, V) = -g(Y, \nabla_U V) = 0 \quad (79)$$

and

$$g(\nabla_X V, Y) = -g(V, \nabla_X Y) = 0 \quad (80)$$

for any $X, Y \in \Gamma(\mathcal{D})$ and $U, V \in \Gamma(\mathcal{D}^\perp)$. Thus, both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel on $\tilde{M}$.

3. Locally decomposable Riemannian manifolds

Let $(\tilde{M}, \tilde{g})$ be n -dimensional Riemannian manifold and F be a tensor $(1,1)$ -type on $\tilde{M}$ such that $F^2 = I$, $F \neq \mp I$.

If the Riemannian metric tensor $\tilde{g}$ satisfying

$$\tilde{g}(X, Y) = \tilde{g}(FX, FY) \quad (81)$$

for any $X, Y \in \Gamma(T\tilde{M})$ then $\tilde{M}$ is called almost Riemannian product manifold and F is said to be almost Riemannian product structure. If F is parallel, that is, $(\tilde{\nabla}_X F)Y = 0$, then $\tilde{M}$ is said to be locally decomposable Riemannian manifold.

Now, let $\tilde{M}$ be an almost Riemannian product manifold. We put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F). \quad (82)$$

Then, we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \quad \text{and} \quad F = P - Q. \quad (83)$$

Thus, P and Q define two complementary distributions P and Q globally. Since $F^2 = I$, we easily see that the eigenvalues of F are 1 and -1 . An eigenvector corresponding to the eigenvalue 1 is in P and an eigenvector corresponding to -1 is in Q . If F has eigenvalue 1 of multiplicity p and eigenvalue -1 of multiplicity q , then the dimension of P is p and that of Q is q . Conversely, if there exist in $\tilde{M}$ two globally complementary distributions P and Q of dimension p and q , respectively. Then, we can define an almost Riemannian product structure F on $\tilde{M}$ by $\tilde{M}$ by $F = P - Q$ [7].

Let $(\tilde{M}, \tilde{g}, F)$ be a locally decomposable Riemannian manifold and we denote the integral manifolds of the distributions P and Q by M^p and M^q , respectively. Then we can write $\tilde{M} = M^p \times M^q$, $(p, q > 2)$. Also, we denote the components of the Riemannian curvature R of $\tilde{M}$ by R_{dcba} , $1 \leq a, b, c, d \leq n = p + q$.

Now, we suppose that the two components are both of constant curvature λ and μ . Then, we have

$$R_{dcba} = \lambda \{g_{da}g_{cb} - g_{ca}g_{db}\} \quad (84)$$

and

$$R_{zyxw} = \mu \{g_{zw}g_{yx} - g_{yw}g_{zx}\}. \quad (85)$$

Then, the above equations may also be written in the form

$$\begin{aligned} R_{kjih} = & \frac{1}{4}(\lambda + \mu) \{ (g_{kh}g_{ji} - g_{jh}g_{ki}) + (F_{kh}F_{ji} - F_{jh}F_{ki}) \} \\ & + \frac{1}{4}(\lambda - \mu) \{ (F_{kh}g_{ji} - F_{jh}g_{ki}) + (g_{kh}F_{ji} - g_{jh}F_{ki}) \}. \end{aligned} \quad (86)$$

Conversely, suppose that the curvature tensor of a locally decomposable Riemannian manifold has the form

$$\begin{aligned} R_{kjih} = & a \{ (g_{kh}g_{ji} - g_{jh}g_{ki}) + (F_{kh}F_{ji} - F_{jh}F_{ki}) \} \\ & + b \{ (F_{kh}g_{ji} - F_{jh}g_{ki}) + (g_{kh}F_{ji} - g_{jh}F_{ki}) \}. \end{aligned} \quad (87)$$

Then, we have

$$R_{dcba} = 2(a + b) \{g_{da}g_{cb} - g_{ca}g_{db}\} \quad (88)$$

and

$$R_{zyxw} = 2(a - b) \{g_{zw}g_{yx} - g_{yw}g_{zx}\}. \quad (89)$$

Let $\tilde{M}$ be an m -dimensional almost Riemannian product manifold with the Riemannian structure $(F, \tilde{g})$ and M be an n -dimensional sub-manifold of $\tilde{M}$. For any vector field X tangent to M , we put

$$FX = fX + wX, \quad (90)$$

where fX and wX denote the tangential and normal components of FX , with respect to M , respectively. In the same way, for $V \in \Gamma(T^\perp M)$, we also put

$$FV = BV + CV, \quad (91)$$

where BV and CV denote the tangential and normal components of FV , respectively.

Then, we have

$$f^2 + Bw = I, Cw + wf = 0 \quad (92)$$

and

$$fB + BC = 0, wB + C^2 = I. \quad (93)$$

On the other hand, we can easily see that

$$g(X, fY) = g(fX, Y) \quad (94)$$

and

$$g(X, Y) = g(fX, fY) + g(wX, wY) \quad (95)$$

for any $X, Y \in \Gamma(TM)$ [6].

If $wX = 0$ for all $X \in \Gamma(TM)$, then M is said to be invariant sub-manifold in $\tilde{M}$, i.e., $F(T_M(p)) \subset T_M(p)$ for each $p \in M$. In this case, $f^2 = I$ and $g(fX, fY) = g(X, Y)$. Thus, (f, g) defines an almost product Riemannian on M .

Conversely, (f, g) is an almost product Riemannian structure on M , the $w = 0$ and hence M is an invariant sub-manifold in $\tilde{M}$.

Consequently, we can give the following theorem [7].

Theorem 3.1. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. The induced structure (f, g) on M is an almost Riemannian product structure if and only if M is an invariant sub-manifold of $\tilde{M}$.

Definition 3.1. Let M be a sub-manifold of an almost Riemannian product $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. For each non-zero vector $X_p \in T_M(p)$ at $p \in M$, we denote the slant angle between FX_p and $T_M(p)$ by $\theta(p)$. Then M said to be slant sub-manifold if the angle $\theta(p)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X_p \in T_M(p)$ [5].

Thus, invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A proper slant immersion is neither invariant nor anti-invariant.

Theorem 3.2. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. M is a slant sub-manifold if and only if there exists a constant $\lambda \in (0, 1)$, such that

$$f^2 = \lambda I. \quad (96)$$

Furthermore, if the slant angle is θ , then it satisfies $\lambda = \cos^2 \theta$ [9].

Definition 3.2. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. M is said to be semi-slant sub-manifold if there exist distributions $\mathcal{D}^\theta$ and $\mathcal{D}^T$ on M such that

- (i) TM has the orthogonal direct decomposition $TM = \mathcal{D} \oplus \mathcal{D}^T$.
- (ii) The distribution $\mathcal{D}^\theta$ is a slant distribution with slant angle θ .
- (iii) The distribution $\mathcal{D}^T$ is an invariant distribution, .e., $F(\mathcal{D}^T) \subseteq \mathcal{D}^T$.

In a semi-slant sub-manifold, if $\theta = \frac{\pi}{2}$, then semi-slant sub-manifold is called semi-invariant sub-manifold [8].

Example 3.1. Now, let us consider an immersed sub-manifold M in $\mathbb{R}^7$ given by the equations

$$x_1^2 + x_2^2 = x_5^2 + x_6^2, x_3 + x_4 = 0. \quad (97)$$

By direct calculations, it is easy to check that the tangent bundle of M is spanned by the vectors

$$\begin{aligned} z_1 &= \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \beta \frac{\partial}{\partial x_5} + \sin \beta \frac{\partial}{\partial x_6} \\ z_2 &= -u \sin \theta \frac{\partial}{\partial x_1} + u \cos \theta \frac{\partial}{\partial x_2}, z_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \\ z_4 &= -u \sin \beta \frac{\partial}{\partial x_5} + u \cos \beta \frac{\partial}{\partial x_6}, z_5 = \frac{\partial}{\partial x_7}, \end{aligned} \quad (98)$$

where θ, β and u denote arbitrary parameters.

For the coordinate system of $\mathbb{R}^7 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) | x_i \in \mathbb{R}, 1 \leq i \leq 7\}$, we define the almost product Riemannian structure F as follows:

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}, 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 7. \quad (99)$$

Since Fz_1 and Fz_3 are orthogonal to M and Fz_2, Fz_4, Fz_5 are tangent to M , we can choose a $\mathcal{D} = S_p\{z_2, z_4, z_5\}$ and $\mathcal{D}^\perp = S_p\{z_1, z_3\}$. Thus, M is a 5-dimensional semi-invariant sub-manifold of $\mathbb{R}^7$ with usual almost Riemannian product structure $(F, \langle \cdot, \cdot \rangle)$.

Example 3.2. Let M be sub-manifold of $\mathbb{R}^8$ by given

$$(u + v, u - v, u \cos \alpha, u \sin \alpha, u + v, u - v, u \cos \beta, u \sin \beta) \quad (100)$$

where u, v and β are the arbitrary parameters. By direct calculations, we can easily see that the tangent bundle of M is spanned by

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos\alpha \frac{\partial}{\partial x_3} + \sin\alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} + \cos\beta \frac{\partial}{\partial x_7} + \sin\beta \frac{\partial}{\partial x_8} \\ e_2 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}, e_3 = -u \sin \frac{\partial}{\partial x_3} + u \cos \alpha \frac{\partial}{\partial x_4}, \\ e_4 &= -u \sin \beta \frac{\partial}{\partial x_7} + u \cos \beta \frac{\partial}{\partial x_8}. \end{aligned} \quad (101)$$

For the almost Riemannian product structure F of $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$, $F(TM)$ is spanned by vectors

$$\begin{aligned} Fe_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos\alpha \frac{\partial}{\partial x_3} + \sin\alpha \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} - \cos\beta \frac{\partial}{\partial x_7} - \sin\beta \frac{\partial}{\partial x_8}, \\ Fe_2 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}, Fe_3 = e_3 \text{ and } Fe_4 = -e_4. \end{aligned} \quad (102)$$

Since Fe_1 and Fe_2 are orthogonal to M and Fe_3 and Fe_4 are tangent to M , we can choose $\mathcal{D}^T = Sp\{e_3, e_4\}$ and $\mathcal{D}^\perp = Sp\{e_1, e_2\}$. Thus, M is a four-dimensional semi-invariant sub-manifold of $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ with usual Riemannian product structure F .

Definition 3.3. Let M be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. M is said to be pseudo-slant sub-manifold if there exist distributions $\mathcal{D}_\theta$ and $\mathcal{D}_\perp$ on M such that

- i. The tangent bundle $TM = \mathcal{D}_\theta \oplus \mathcal{D}^\perp$.
- ii. The distribution $\mathcal{D}_\theta$ is a slant distribution with slant angle θ .
- iii. The distribution $\mathcal{D}^\perp$ is an anti-invariant distribution, i.e., $F(\mathcal{D}^\perp) \subseteq T^\perp M$.

As a special case, if $\theta = 0$ and $\theta = \frac{\pi}{2}$, then pseudo-slant sub-manifold becomes semi-invariant and anti-invariant sub-manifolds, respectively.

Example 3.3. Let M be a sub-manifold of $\mathbb{R}^6$ by the given equation

$$(\sqrt{3}u, v, v \sin \theta, v \cos \theta, s \cos t, -s \cos t) \quad (103)$$

where u, v, s and t arbitrary parameters and θ is a constant.

We can check that the tangent bundle of M is spanned by the tangent vectors

$$\begin{aligned} e_1 &= \sqrt{3} \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial y_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \theta \frac{\partial}{\partial y_2}, \\ e_3 &= \cos t \frac{\partial}{\partial x_3} - \cos t \frac{\partial}{\partial y_3}, e_4 = -s \sin t \frac{\partial}{\partial x_3} + s \sin t \frac{\partial}{\partial y_3}. \end{aligned} \quad (104)$$

For the almost product Riemannian structure F of $\mathbb{R}^6$ whose coordinate systems $(x_1, y_1, x_2, y_2, x_3, y_3)$ choosing

$$\begin{aligned} F\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial y_i}, 1 \leq i \leq 3, \\ F\left(\frac{\partial}{\partial y_j}\right) &= \frac{\partial}{\partial x_j}, 1 \leq j \leq 3, \end{aligned} \quad (105)$$

Then, we have

$$\begin{aligned} Fe_1 &= \sqrt{3} \frac{\partial}{\partial y_1}, Fe_2 = -\frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial y_2} - \cos\theta \frac{\partial}{\partial x_2} \\ Fe_3 &= \cos\theta \frac{\partial}{\partial y_3} + \sin\theta \frac{\partial}{\partial x_3}, Fe_4 = -\sin\theta \frac{\partial}{\partial y_3} + \cos\theta \frac{\partial}{\partial x_3}. \end{aligned} \quad (106)$$

Thus, $\mathcal{D}_\theta = S_p\{e_1, e_2\}$ is a slant distribution with slant angle $\alpha = \frac{\pi}{4}$. Since Fe_3 and Fe_4 are orthogonal to M , $\mathcal{D}^\perp = S_p\{e_3, e_4\}$ is an anti-invariant distribution, that is, M is a 4-dimensional proper pseudo-slant sub-manifold of $\mathbb{R}^6$ with its almost Riemannian product structure $(F, \langle \cdot, \cdot \rangle)$.

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