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Some Recent Advances in Nonlinear Inverse Scattering in 2D: Theory and Numerics

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Abstract

We survey our recently published results concerning scattering problems for the nonlinear Schrödinger equation

$$-\Delta u + h(x, |u|)u = k^2 u, \quad x \in \mathbb{R}^2,$$

where h is a quite general nonlinear analogue of the index of refraction. We will investigate direct scattering problem as well as several inverse scattering problems for this equation. We start by establishing sufficient conditions for the unique solvability of the direct scattering problem. Asymptotic behaviour of the scattering solutions is shown to give rise to scattering amplitude. Inverse problems are formulated as follows: extract information about the nonlinearity h from the knowledge of the scattering amplitude. At this point, one must specify more carefully the scattering data sets. We concentrate our attention to full data, backscattering data, fixed angle data and fixed energy data. The latter three data sets are collectively called limited data. For each data set, we define the inverse Born approximation and state theorems, which provide the recovery of main singularities of the function $h_0(x) = h(x, 1)$. The key idea here is to show that the difference between the Born approximation and $h_0(x)$ is less singular than $h_0(x)$. For practical applications, one must be able to compute the inverse Born approximations numerically. To this end, we proceed as follows. We formulate the computation as a solution of an underdetermined linear system. Due to the ill-posed nature of the system, regularization methods are used. Examples are given to illustrate the effectiveness of the method.

Keywords: Inverse scattering, Schrödinger equation, Born approximation, numerical solution, linear inverse problem

1. Introduction

We deal with the generalized nonlinear Schrödinger equation:

$$i \frac{\partial}{\partial t} E(x, t) = -\Delta E(x, t) + h(x, |E|) E(x, t),$$

where E denotes the electromagnetic field in two-dimensional case, Δ is the two-dimensional Laplacian and h describes in a general form the nonlinear contribution to the index of refraction. Considering harmonic time dependence $E(x, t) = e^{-i\omega t} u(x)$ with frequency $\omega > 0$, we obtain the steady-state nonlinear equation with fixed energy:

$$-\Delta u(x) + h(x, |u|)u(x) = k^2 u(x), \quad (1)$$

where $k^2 = \omega$ and fixed, and u denotes the complex amplitude of the field. Concerning the nonlinearity $h(x, s)$, we pose the basic assumptions.

1. $|h(x, s)| \leq c_\rho \alpha(x)$, $\alpha \in L^\sigma_\sigma(\mathbb{R}^2)$, $0 \leq s \leq \rho$,
2. $|h(x, s_1) - h(x, s_2)| \leq c'_\rho \beta(x) |s_1 - s_2|$, $\beta \in L^\sigma_\sigma(\mathbb{R}^2)$, $0 \leq s_1, s_2 \leq \rho$,

where c_ρ and c'_ρ are constants and

$$\sigma > 2 - 2/p, \quad 1 < p \leq \infty. \quad (2)$$

Here, $L^\sigma_\sigma(\mathbb{R}^2)$ denotes the weighted Lebesgue space with the norm.

$$\|f\|_{p,\sigma} = \left(\int_{\mathbb{R}^2} (1 + |x|)^{p\sigma} |f(x)|^p dx \right)^{1/p}, \quad \|f\|_p = \|f\|_{p,0}.$$

The main practical example (it can be considered as the motivation of this research) of such type equations (1) is the equation of the form:

$$-\Delta u(x) + q_1(x)u(x) + q_2(x) \frac{|u(x)|^2}{1+r|u(x)|^2} u(x) = k^2 u(x), \quad (3)$$

with real number k , complex valued function $q_1(x) \in L^2$ and real-valued function $q_2(x) \in L^2$, and parameter $r \geq 0$. A particular nonlinearity in (3) of cubic type ($r = 0$) can be met in the context of a Kerr-like nonlinear dielectric film, while the case when $r > 0$ corresponds to the saturation model (see [1–4]).

We consider the inverse scattering problems for (1). For these purposes, we are interested of the scattering solutions to (1), i.e. solutions of the form

$$u(x, k, \theta) = u_0(x, k, \theta) + u_{sc}(x, k, \theta), \quad (4)$$

where $\theta \in \mathbb{S}^1$, the unit sphere in $\mathbb{R}^2$, $u_0(x, k, \theta) = e^{ik(x, \theta)}$ is the incident wave and $u_{sc}(x, k, \theta)$ is the scattered wave. The scattered wave must satisfy the Sommerfeld radiation condition at infinity:

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u_{sc}(x, k, \theta)}{\partial r} - iku_{sc}(x, k, \theta) \right) = 0, \quad r = |x|,$$

for fixed $k > 0$ and uniformly in $\theta \in \mathbb{S}^1$. In that case, these solutions are the unique solutions of the Lippmann-Schwinger equation.

$$u(x, k, \theta) = u_0(x, k, \theta) - \int_{\mathbb{R}^2} G_k^+(|x - y|) h(y, |u(y, k, \theta)|) u(y, k, \theta) dy, \quad (5)$$

where G_k^+ is the outgoing fundamental solution of the operator $-\Delta - k^2$, i.e. the kernel of the integral operator $(-\Delta - k^2 - i0)^{-1}$. It is equal to

$$G_k^+(|x|) = \frac{i}{4} H_0^{(1)}(|kx|),$$

where $H_0^{(1)}$ denotes the Hankel function of order zero and first kind.

The following main results concerning the direct scattering problem were proved in [5].

Under the basic assumptions and (2) for h , it is proved that for any $\rho > 1$ there is $k_0 > 0$ such that for any $k \geq k_0$ in the ball $B_\rho = \{u \in L^\infty(\mathbb{R}^2) : \|u\|_\infty \leq \rho\}$, there is a unique scattering solution (or the solutions of the form in (4) to (5) which satisfies the condition:

$$\|u_{sc}\|_\infty \rightarrow 0, k \rightarrow \infty \quad (6)$$

uniformly in $\theta \in \mathbb{S}^1$. What is more, the solution is obtained as the limit

$$u(x, k, \theta) = \lim_{j \rightarrow \infty} u_j(x, k, \theta),$$

where

$$u_{j+1}(x, k, \theta) = u_0(x, k, \theta) - \int_{\mathbb{R}^2} G_k^+(|x - y|) h(y, |u_j(y, k, \theta)|) u_j(y, k, \theta) dy,$$

for $j = 0, 1, \dots$ with u_0 as above. Let the function h have the same properties as above, but now with

$$\sigma > \begin{cases} 2 - 2/p, & 4/3 \leq p \leq \infty \\ 1/2, & 1 < p < 4/3. \end{cases} \quad (7)$$

Then for fixed $k \geq k_0$, the solution $u(x, k, \theta)$ admits the representation

$$u(x, k, \theta) = e^{ik(x, \theta)} - \frac{1+i}{4\sqrt{\pi k|x|}} e^{ik|x|} A(k, \theta', \theta) + o(|x|^{-1/2}), \quad |x| \rightarrow \infty.$$

The function $A(k, \theta', \theta)$ is called the scattering amplitude and it is defined as

$$A(k, \theta', \theta) = \int_{\mathbb{R}^2} e^{-ik(\theta', y)} h(y, |u(y, k, \theta)|) u(y, k, \theta) dy,$$

where $\theta' = \frac{x}{|x|} \in \mathbb{S}^1$ is the direction of observation. This function A gives us the scattering data for inverse problem. More precisely, the inverse problem that is considered here is to extract some information about the function h by the knowledge of the scattering amplitude A for different sets of scattering data. There are four well-known inverse scattering data sets: (i) the full (scattering) data:

$$D = \{A(k, \theta', \theta): k > 0, \theta', \theta \in \mathbb{S}^1\},$$

(ii) the backscattering data:

$$D_B = \{A(k, \theta', \theta): k > 0, \theta' = -\theta \in \mathbb{S}^1\},$$

(iii) the fixed angle data:

$$D_A = \{A(k, \theta', \theta): k > 0, \theta = \theta_0 \text{ fixed } \theta' \in \mathbb{S}^1\}$$

and (iv) the fixed energy data:

$$D_E = \{A(k, \theta', \theta): k = k_0 > 0 \text{ fixed}, \theta', \theta \in \mathbb{S}^1\}.$$

We use the following notations for the direct and inverse Fourier transforms:

$$F(f)(\xi) = \int_{\mathbb{R}^2} e^{i(\xi, y)} f(y) dy \quad (8)$$

$$F^{-1}(f)(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(\xi, x)} f(\xi) d\xi, \quad (9)$$

where (ξ, x) denotes the inner product in $\mathbb{R}^2$, i.e. $(\xi, x) = \xi_1 x_1 + \xi_2 x_2$. By $C > 0$, we denote a generic constant that may change from one step to another. By $H^t(\mathbb{R}^2) = W_2^t(\mathbb{R}^2)$, $t \in \mathbb{R}$ we denote the standard L^2 based Sobolev space. A weighted Sobolev space $W_{p,\sigma}^1(\mathbb{R}^2)$ is defined here by

$$W_{p,\sigma}^1(\mathbb{R}^2) = \{f \in L_\sigma^p(\mathbb{R}^2) : \nabla f \in L_\sigma^p(\mathbb{R}^2)\}.$$

The following notation for the characteristic function is used:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

2. Inverse scattering problems

The direct scattering theory described above can also be reversed. The inverse scattering theory treats the function h as unknown and attempts to recover it from the knowledge of the scattering amplitude A for different data. Usually, the model in (1) is probed with one or more incident plane waves u_0 and the resulting scattered waves are measured at a distance. This gives rise to several different scattering data sets which can be used to recover the unknowns.

The inverse backscattering problem for (1) was treated in [6]. Also for (1), the recovery of unknown function h is possible from the full scattering data. In addition to the two-dimensional studies mentioned above, certain particular nonlinear cases of (1) have been investigated in other dimensions too. In one-space dimension, we refer to [7] and the references therein. In higher dimensions $n \geq 3$ we are only aware of [8,9]. Similar problems with formally more general equation but with bounded h are considered in [10] and [11].

Our point of view is that the nonlinearity may contain local singularities in the space coordinate x , and therefore we work in the frame of weighted Lebesgue spaces. These local singularities can be recovered from the scattering amplitude using the method of Born approximation. As a unifying result, we obtain mathematically more general results that have far wider applicability in physical experiments.

Let us set

$$h_0(x) = h(x, 1).$$

In the subsections that follow we consider the inverse problems of recovering information about h_0 from the knowledge of full data D , backscattering data D_B , fixed angle data D_A and fixed energy data D_E .

2.1. Full scattering data

The inverse problem with full data D was investigated in [5]. Here we summarize the main results without proofs.

Theorem 1 (Saito's formula). Under the basic assumptions and (7) for the function h we have,

$$\lim_{k \rightarrow \infty} k \int_{\mathbb{S}^1 \times \mathbb{S}^1} e^{-ik(\theta - \theta', x)} A(k, \theta', \theta) d\theta d\theta' = 4\pi \int_{\mathbb{R}^2} \frac{h_0(y)}{|x - y|} dy, \quad x \in \mathbb{R}^2,$$

where the limit is valid in the sense of distributions for $4/3 < p \leq 2$ and pointwise (even uniformly) for $2 < p \leq \infty$.

Corollary 1 (Uniqueness). Let σ be as in (7). Consider the scattering problems for two sets of potentials h and $\tilde{h}$. If the scattering amplitudes coincide for some sequence $k_j \rightarrow \infty$ and for all $\theta', \theta \in \mathbb{S}^1$, then

$$h_0(x) = \tilde{h}_0(x)$$

holds in the sense of distributions for $4/3 < p \leq \infty$.

Corollary 2 (Representation formula). Let σ be as in (7). Then the representation

$$h_0(x) = \lim_{k \rightarrow \infty} \frac{k^2}{8\pi^2} \int_{\mathbb{S}^1 \times \mathbb{S}^1} e^{-ik(\theta - \theta', x)} A(k, \theta', \theta) |\theta - \theta'| d\theta d\theta'$$

holds in the sense of distributions for $4/3 < p \leq \infty$.

Remark 1. In addition to providing the above results, Saito's formula might be applied numerically too. It can be written as a convolution equation:

$$4\pi h_0 * |x|^{-1} = f,$$

where the function f can be computed from the full scattering data in principle. A numerical inversion of this equation would yield a full recovery of h_0 but this is an open problem as far as we know. What is more, unlike the representation formula above it holds pointwise in the important case of bounded ($p = \infty$) nonlinearities.

We assume that the function $x \mapsto h(x, s)$ is real-valued and recall that

$$A(k, \theta', \theta) = \int_{\mathbb{R}^2} e^{-ik(\theta', y)} h(y, |u|) u(y, k, \theta) dy, \quad k \geq k_0.$$

For reasons of purely technical nature we define the scattering solutions $u(x, k, \theta)$ for negative k as

$$u(x, k, \theta) = \overline{u(x, -k, \theta)}, \quad k < 0.$$

These are the unique solutions of the integral equation:

$$u(x, k, \theta) = e^{ik(x, \theta)} - \int_{\mathbb{R}^2} \overline{G_k^+(|x - y|)} h(y, |u|) u(y, k, \theta) dy,$$

provided that h is real-valued. This allows us to extend A to negative $k \leq -k_0$ by

$$A(k, \theta', \theta) = \overline{A(-k, \theta', \theta)}.$$

We also put $A(k, \theta', \theta) = 0$ for $|k| \leq k_0$. Splitting

$$h(y, |u|)u = h_0(x)u_0 + h(y, |u|)u - h_0(x)u_0$$

we have that

$$A(k, \theta', \theta) = \int_{\mathbb{R}^2} e^{ik(\theta - \theta', y)} h_0(y) dy + \int_{\mathbb{R}^2} e^{-ik(\theta', y)} (h(y, |u|)u(y, k, \theta) - h_0(y)u_0) dy$$

or

$$A(k, \theta', \theta) = F(h_0)(k(\theta - \theta')) + A_R(k, \theta', \theta),$$

where F denotes the Fourier transform (8). Using the basic assumptions for the function h and (6), we can easily obtain that

$$\begin{aligned}
|A_R(k, \theta', \theta)| &\leq \int_{\mathbb{R}^2} |h(y, |u|)u(y, k, \theta) - h_0(y)u_0| dy \leq C \int_{\mathbb{R}^2} (\alpha(y) + \beta(y)) |u_{sc}(y)| dy \\
&\leq C \|u_{sc}\|_{\infty} (\|\alpha\|_1 + \|\beta\|_1) \rightarrow 0, \quad |k| \rightarrow \infty.
\end{aligned}$$

We have used here the fact that the basic assumptions for the function h guarantee that the functions α and β both belong to $L^1(\mathbb{R}^2)$.

Hence, for k large, we have approximately that

$$A(k, \theta', \theta) \approx F(h_0)(k(\theta - \theta')). \quad (10)$$

These considerations and real valuedness of h suggest and justify the following definition:

We define the inverse Born approximation q_B via the equality

$$A(k, \theta', \theta) = F(q_B)(k(\theta' - \theta)) \quad (11)$$

which is understood in the sense of tempered distributions. In order to recover main singularities of h_0 from q_B , we must study their difference and show that it is locally less singular than h_0 . To this end, we have the following main result from [5].

Theorem 2. Let σ be as in (7). Then

$$q_B - h_0 \in H_{loc}^t(\mathbb{R}^2),$$

where $t < 3 - 4/p$ if $4/3 < p \leq 3/2$ and $t < 1 - 1/p$ if $3/2 < p \leq \infty$.

Remark 2. Theorem 2 means that, for $4/3 < p < \infty$, the main singularities of h_0 can be recovered from the inverse scattering Born approximation q_B with full data D . In the case of $p = \infty$, we have no singularities but may have finite jumps. Under such circumstances, we record the following special case.

Corollary 3. If a piecewise smooth compactly supported function h_0 contains a jump over a smooth curve, then the curve and the height function of the jump are uniquely determined by the full scattering data. Especially, for the function h_0 being the characteristic function of a smooth bounded domain, this domain is uniquely determined by the full scattering data.

Concluding, in this part of the work, the uniqueness of the direct problem for the nonlinearities h satisfying the appropriate properties was proved. These properties allow local singularities and do not require compact support, but rather some sufficient decay at infinity. Under similar properties, we were also able to establish the asymptotic behaviour of scattering solutions, which gives us the scattering data so we can investigate the inverse scattering problems. Note that both results were proved without assuming smallness of the norm of the nonlinearity as is necessary in dimensions three and higher.

What can we regard as the main result of this section is the Saito's formula since it implies a uniqueness result and a representation formula for our unknown function h_0 . In addition, we managed to extract more information about the nonlinearity by applying the method of Born approximation. More precisely, the main singularities (or jumps over smooth curve) of h_0 can be recovered from the Born approximation q_B which corresponds to the full scattering data D .

2.2. Backscattering and fixed angle data

In this section, we consider backscattering data D_B and fixed angle scattering data D_A following [12]. Using (10) we introduce the inverse backscattering and inverse fixed angle scattering Born approximations q_B^b and $q_B^{\theta_0}$ as follows:

$$A(k, -\theta, \theta) = F(q_B^b)(2k\theta) \quad (12)$$

and

$$A(k, \theta', \theta_0) = F(q_B^{\theta_0})(k(\theta' - \theta_0)), \quad (13)$$

where θ_0 is fixed.

Furthermore, we assume in addition that the nonlinearity h possesses the Taylor expansion:

$$h(x, 1+s) = h(x, 1) + \partial_s h(x, 1)s + \frac{1}{2} \partial_s^2 h(x, s^*)s^2, \quad 1 < s^* < 1+s$$

where

$$|\partial_s h(x, 1)| \leq \eta_1(x), \quad |\partial_s^2 h(x, s^*)| \leq \eta_2(x)$$

uniformly in $s \in (0, s_0)$, $s_0 > 0$ and with $\eta_1, \eta_2 \in L^\sigma_\sigma(\mathbb{R}^2)$, where σ as in (7). From this we obtain the expansion:

$$h(x, |u|)u = h_0(x)u_0 + g_1(x)u_{sc} + g_2(x)u_0^2 \overline{u_{sc}} + \eta(x)O(|u_{sc}|^2),$$

where $g_2(x) = \frac{1}{2} \partial_s^2 h(x, 1)$, $g_1(x) = h_0(x) + g_2(x)$ and $\eta \in L^\sigma_\sigma(\mathbb{R}^2)$ with the same σ as above.

Again, the main result for the recovery of main singularities is formulated as the following theorem.

Theorem 3. Let σ be as in (7) with $2 < p \leq \infty$. Suppose in addition

$$\widehat{h_0}, \widehat{g_2} \in W_s^1(\mathbb{R}^2), \quad 1 < s < 2,$$

and $2 < s' < p \leq \infty$, where s' is the Hölder conjugate of s . Then

$$q_B^{\theta_0} - h_0, q_B^b - h_0 \in H_{loc}^t(\mathbb{R}^2)$$

for any $t < 1 - 1/p$ if $1 < s \leq 4/3$ and for any $t < \min\{1 - 1/p, 4/s - 2\}$ if $4/3 < s < 2$.

Let us sketch the main ideas in the proof of Theorem 3. Using the definition, we may expand the difference in several terms as

$$q_B^{\theta_0} - h_0 = q_1^{\theta_0} + q_\infty^{\theta_0} + q_2^{\theta_0} + q_R^{\theta_0}.$$

In straightforward manner one sees that $q_\infty^{\theta_0} \in C^\infty(\mathbb{R}^2)$ and $q_2^{\theta_0}, q_R^{\theta_0} \in H^t(\mathbb{R}^2)$ with $t < 3 - 4/p$ if $4/3 < p \leq 3/2$ and $t < 1 - 1/p$ if $3/2 < p \leq \infty$. The first nonlinear term $q_1^{\theta_0}$ cannot be analyzed directly from its definition. Instead, we first proved the representation

$$\begin{aligned} q_1^{\theta_0}(x) = & -F_4^{-1} \left(p v \frac{\widehat{g_1}(\xi) \widehat{h_0}(\eta)(\xi + \eta, \theta_0)}{(|\eta|^2(\xi + \eta) - |\xi + \eta|^2 \eta, \theta_0)} \right) (x, x) \\ & - F_4^{-1} \left(p v \frac{\widehat{g_2}(\xi) \widehat{h_0}(\eta)(\xi + \eta, \theta_0)}{(|\eta|^2(\xi + \eta) + |\xi + \eta|^2 \eta, \theta_0)} \right) (x, x), \end{aligned}$$

where F_4^{-1} denotes the four-dimensional inverse Fourier transform. This formula might have independent interest too, but primarily it allows us to prove the following regularity: the term $q_1^{\theta_0}$ belongs to the space

1. $C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ if $1 < s < 4/3$;
2. $H^1(\mathbb{R}^2)$ if $s = 4/3$;
3. $H^t(\mathbb{R}^2)$, $t < 4/s - 2$ if $4/3 < s < 2$.

If we combine all these steps, we obtain Theorem 3 for fixed angle scattering.

The inverse backscattering Born approximation is treated similarly. Namely, we write

$$q_B^b - h_0 = q_1^b + q_\infty^b + q_2^b + q_R^b.$$

The latter four terms have exactly the same regularity results as their counterparts in fixed angle scattering. For the first nonlinear term, the representation is now

$$q_1^b(x) = 4F_4^{-1} \left(p v \frac{\widehat{g}_1(\xi) \widehat{h}_0(\eta)}{(\xi, \eta)} \right) (x, x) - 4F_4^{-1} \left(p v \frac{\widehat{g}_2(\xi) \widehat{h}_0(\eta)}{(\xi + 2\eta, \eta)} \right) (x, x).$$

The additional assumption $\widehat{h}_0 \in W_s^1(\mathbb{R}^2)$ in Theorem 3 implies that $h_0 \in L_1^{s'}(\mathbb{R}^2)$. This explains why we restrict $s' < p$. By Sobolev embedding, we obtain

$$q_B^{\theta_0} - h_0, q_B^b - h_0 \in W_p^\epsilon(\mathbb{R}^2)$$

with some positive $\epsilon < \min\{1/p, 1 - 2/p\}$. Hence, h_0 is locally more singular than either of these differences. That's why both Born approximations recover the main singularities of h_0 . On the other hand, we may perform the comparison also in the scale of Sobolev spaces. Indeed, if $h_0 \in H_{comp}^r(\mathbb{R}^2)$ with some $0 < r < 1$, then

$$q_B^{\theta_0} - h_0, q_B^b - h_0 \in H_{loc}^t(\mathbb{R}^2)$$

for any $t < 2r$ if $0 < r \leq 1/3$ and for any $t < (1+r)/2$ if $1/3 < r < 1$. In both cases this t can be made bigger than r . It means that we can reconstruct all singularities from Sobolev space H^r , $0 < r < 1$ from data D_B and D_A by the method of Born approximation.

Corollary 4. If a piecewise smooth compactly supported function h_0 contains a jump over a smooth curve, then the curve and the height function of the jump are uniquely determined by backscattering and fixed angle scattering data. Especially, for the function h_0 being the characteristic function of a smooth bounded domain, this domain is uniquely determined by backscattering and fixed angle scattering data.

Concluding, in this section we proved that all singularities and jumps (in the absence of a uniqueness theorem) of the nonlinearity h can be recovered from the inverse scattering Born approximation corresponding to fixed angle scattering and backscattering data D_A and D_B , respectively. No assumptions about the smallness of the norm of nonlinearity h were used as it were in previous publications even in the linear case.

2.3. Fixed energy data

The two-dimensional fixed energy problem was a long-standing open problem. In the case of linear Schrödinger operator, the first uniqueness and reconstruction algorithm was proved by Nachman [13] via $\bar{\partial}$ -methods for potentials of conductivity type. Sun and Uhlmann [14] proved uniqueness for potentials satisfying nearness conditions to each other. The question of global uniqueness for the linear Schrödinger equation with fixed energy was settled only in 2008 by Bukhgeim [15] for compactly supported potentials from L^p , $p > 2$. This result has recently been improved and extended to related inverse problems (see for example [16] and [17]). Note that Grinevich and Novikov [18] proved that in two dimensions there are nonzero real potentials of the Schwartz class with zero scattering amplitude at fixed positive energy.

Thus, the compactness of the supports of the potentials is very natural condition in our considerations.

The results of this section are proved in [19] and they slightly generalize the linear case to a special type of nonlinearity. It turned out that (as we can see in this section) inverse fixed energy scattering problem is much more difficult than the others.

In fixed energy scattering problem, instead of the scattering solutions (4) we need the so-called complex geometrical optics solutions. Complex geometrical optics (CGO) solutions or exponentially growing solutions of the form:

$$u(x, z) = e^{i(x, z)}(1 + R(x, z)), \quad z \in \mathbb{C}^2, (z, z) = 0 \quad (14)$$

with R decaying at infinity for $|z|$ large for the homogeneous Schrödinger equation

$$-\Delta u(x) + h(x, |u(x)|)u(x) = 0 \quad (15)$$

are obtained as follows. Substituting (14) into (15) yields

$$-\Delta R - 2i(z, \nabla)R + h(x, |e^{i(x, z)}(1 + R)|)(1 + R) = 0.$$

It means that using the Faddeev Green's function

$$g_z(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{-i(x, \xi)}}{\xi^2 + 2(z, \xi)} d\xi$$

as the fundamental solution of the differential operator

$$-\Delta - 2i(z, \nabla)$$

we see that the function R solves the integral equation

$$R(x, z) = - \int_{\mathbb{R}^2} g_z(x - y) h(y, |e^{i(y, z)}(1 + R(y, z))|)(1 + R(y, z)) dy. \quad (16)$$

It remains to establish unique solvability of this equation. To this end we again use iterations in the sense of next theorem.

We assume that h is compactly supported in $\Omega \subset \mathbb{R}^2$ and

1. $|h(x, s)| \leq \alpha(x)$ with some $\alpha \in L^2(\Omega)$ and $s \geq 0$

2. $|h(x, |e^{i(x,z)}(1 + R_1)|) - h(x, |e^{i(x,z)}(1 + R_2)|)| \leq \beta(x)|R_1 - R_2|$ with some $\beta \in L^2(\Omega)$ and for any $R_1, R_2 \in L^\infty(\mathbb{R}^2)$ such that $\|R_j\|_\infty < 1 - \delta, j = 1, 2$ for some fixed $\delta \in (0, 1)$ and for any $z \in \mathbb{C}^2$

3. $\|\alpha\|_2, \|\beta\|_2 > 0$

Theorem 4. Under the above conditions for h , there exists a constant $C_0 > 0$ such that for all $|z| \geq C_0$ the equation (16) has a unique solution in $L^\infty(\mathbb{R}^2)$ and this solution can be obtained as $\lim_{j \rightarrow \infty} R_j$ in $L^\infty(\mathbb{R}^2)$, where

$$R_{j+1}(x, z) = - \int_{\mathbb{R}^2} g_z(x - y) h\left(y, \left|e^{i(y,z)}(1 + R_j(y, z))\right|\right) (1 + R_j(y, z)) dy, j = 0, 1, \dots$$

with $R_0 = 0$. Moreover, the following estimates hold

$$\|R\|_\infty \leq \frac{c}{|z|^\gamma}, \left\|R - R_j\right\|_\infty \leq \frac{c_j}{|z|^{\gamma(j+1)}}, j = 0, 1, \dots$$

The proof of Theorem 4 is based on the fact that for any $\gamma < 1$ there is constant $c_\gamma > 0$ such that

$$\|g_z * f\|_\infty \leq \frac{c_\gamma}{|z|^\gamma} \|f\|_2, |z| > 1,$$

for any $f \in L^2(\Omega)$, see [20].

Turning now to the inverse fixed energy scattering problem, we define the scattering transform by

$$T_h(\xi) = \int_{\mathbb{R}^2} e^{i(x,\xi)} h(x, e_0 |1 + R(x, z)|) (1 + R(x, z)) dx, \quad |\xi| \geq \sqrt{2} C_0$$

and $T_h(\xi) = 0$ for $|\xi| < \sqrt{2} C_0$. Here $z = \frac{1}{2}(\xi - iJ\xi)$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $e_0 = |e^{i(x,z)}| = e^{\frac{1}{2}(x_1\xi_2 - x_2\xi_1)}$. What is more, we have the uniform limit $T_h(\xi) = \lim_{j \rightarrow \infty} T_{h,j}(\xi)$, where

$$T_{h,j}(\xi) = \int_{\mathbb{R}^2} e^{i(x,\xi)} h(x, e_0 |1 + R_j(x, z)|) (1 + R_j(x, z)) dx.$$

We point out that the scattering transform is somehow an auxiliary object (see D_E in Introduction). But it is connected to the scattering amplitude as follows. It is well known that the scattering amplitude at fixed energy uniquely determines the Dirichlet-to-Neumann map $\Lambda_{h-k_0^2}$ which in turn uniquely determines the scattering transform (see the details, for example, in [19] and [20]). Recall that $\Lambda_h f = \partial_\nu u$, where u satisfies the Dirichlet problem:

$$\begin{cases} -\Delta u + h(x, |u|)u = 0, x \in \Omega \\ u(x) = f(x), x \in \partial\Omega. \end{cases}$$

Here, Ω is a domain with boundary $\partial\Omega$ and outward unit normal vector ν .

Next, we define the inverse fixed energy Born approximation by

$$q_B^f(x) = F^{-1}(T_h(\xi))(x). \quad (17)$$

In contrast to the preceding inverse problems, we now set the unknown function to be

$$h_0(x) = F^{-1}(T_{h,0}(\xi))(x). \quad (18)$$

In linear case h_0 is the actual potential appearing in the Schrödinger equation, but otherwise the connection to physical scatterers is not known to us.

We assume that the nonlinearity h admits the Taylor expansion

$$h(x, e_0(1+s)) = h(x, e_0) + \partial_s h(x, e_0(1+s))|_{s=0} s + O(\beta_1(x)s^2),$$

where $|\partial_s h(x, e_0(1+s))|_{s=0}| \leq \beta_1(x)$ and $O(\beta_1(x)s^2)$ with $\beta_1(x) \in L^2(\Omega)$ and with small s in the neighbourhood of zero and where O is uniform in $x \in \Omega$ and such s .

Suppose in addition that the nonlinearity h satisfies the asymptotic expansions

$$h(x, e_0) = \sum_{j=0}^{\infty} \frac{\alpha_j(x)}{|\xi|^j}, \quad h(x, e_0|1 + R_1|) = \sum_{j=0}^{\infty} \frac{\tilde{\alpha}_j(x)}{|\xi|^j}$$

where . Then we have the following main result concerning the recovery of singularities of $h_0(x)$ defined by (18).

Theorem 5. Under the foregoing conditions for the potential function h , it is true that

$$q_B^f - h_0 \in H^t(\mathbb{R}^2)$$

for any $t < 1$ modulo $C^\infty(\mathbb{R}^2)$ - functions.

Remark 3. The embedding theorem for Sobolev spaces says that the difference $q_B^f - h_0$ belongs to $L^q(\mathbb{R}^2)$ for any $q < \infty$ modulo $C^\infty(\mathbb{R}^2)$ functions. It means that all singularities from $L_{loc}^p(\mathbb{R}^2)$, $p < \infty$ of unknown function h_0 can be obtained exactly by the Born approximation which corresponds to the inverse scattering problem with fixed positive energy.

We note that under fixed energy data we have some additional assumptions on h . This limits the applicability of the main result to, for example, saturation type nonlinearities. In particular, cubic nonlinearity is excluded from these considerations. Moreover, the unknown function h_0 has no direct connection to original scatterers in nonlinear cases.

3. Numerical examples

Here we discuss the numerical computation of the Born approximation first proposed in [21] for backscattering and fixed angle data. We assume that the scatterer $h_0(x)$ is supported in the rectangle $R_0 = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We consider the following examples:

Example 1: $h(x, |u|) = \chi_\Omega(x) |u|$ (linear)

Example 2: $h(x, |u|) = \chi_\Omega(x) |u| \frac{|u|^2}{1 + |u|^2}$ (saturation)

Example 3: $h(x, |u|) = 0.75 \chi_{\Omega_1}(x) |u| + \chi_{\Omega_2}(x) \frac{|u|^2}{1 + |u|^2}$ (linear, saturation)

Example 4: $h(x, |u|) = 0.75 \chi_{\Omega_2}(x) \frac{|u|^2}{1 + |u|^2} + \chi_{\Omega_1}(x) |u|$ (saturation, linear)

Here $\Omega, \Omega_1, \Omega_2$ are the shifted ellipses rotated in angle Θ theta (counter-clockwise) detailed in **Table 1** and shown in **Figure 1**.

	Semi axis	Θ	Centre
Ω	1/2, 1/5	$17\pi/18$	$(-0.3, -0.4)$
Ω_1	1/2, 1/4	$\pi/3$	$(0.5, 0)$
Ω_2	1/3, 1/5	$2\pi/3$	$(-0.5, 0)$

Table 1. Geometries of the ellipses Ω , Ω_1 and Ω_2

Figure 1. Geometries of the ellipses Ω , Ω_1 and Ω_2 .

Consider the Born approximation for full data (11) in the form

$$\int_{\mathbb{R}^2} e^{ik(\theta' - \theta, y)} f(y) dy = A(k, \theta', \theta), \quad f = q_B. \quad (19)$$

To discretize the unknown function f we divide the rectangle R_0 into $N = n \times n$ disjoint subrectangles r_j of equal size, i.e.

$$R_0 = \bigcup_{j=1}^N r_j.$$

Then, we represent f on R_0 in piecewise constant form:

$$f(y) = \sum_{j=1}^N f_j \chi_{r_j}(y),$$

where f_j 's are the unknown values on r_j 's. Substitution into (19) yields

$$\sum_{j=1}^N f_j \int_{r_j} e^{ik(\theta' - \theta, y)} dy = A(k, \theta', \theta).$$

Evaluating this at some points $k = k_s$, $\theta' = \theta'_t$ and $\theta = \theta_p$ leads us to

$$\sum_{j=1}^N f_j \int_{r_j} e^{ik_s(\theta'_t - \theta_p, y)} dy = A(k_s, \theta'_t, \theta_p), \quad s = 1, \dots, N_1, t = 1, \dots, N_2, p = 1, \dots, N_3.$$

If we denote $M = N_1 N_2 N_3$, we may form one linearized index $l = l(s, t, p)$, $l = 1, \dots, M$ and write the latter equation as

$$\sum_{j=1}^N f_j E(j, l) = g_l, \quad l = 1, \dots, M, \quad (20)$$

where $E(j, l) = \int_{r_j} e^{ik_s(\theta'_t - \theta_p, y)} dy$ are easily evaluated and $g_l = A(k_s, \theta'_t, \theta_p)$ needs to be computed. The computation of g_l is carried out using numerical integration, see [21] for details. In matrix form (20) is clearly $Ef = g$.

For backscattering data and fixed angle data the system (20) is modified accordingly. We note that the system $Ef = g$ does not depend on the scatterer but only on data type and measurement setup.

The fixed energy case differs from the first three data sets. In fixed energy inversion we approximate the scattering transform as

$$T_h(\xi) \approx T_{h,1}(\xi).$$

We choose $M = m \times m$ points ξ uniformly from the rectangle $[-s, s] \times [-s, s]$. The function $T_{h,1}(\xi)$ is evaluated by numerical integration, see [19,21] for details. Then the inverse Born approximation (17) is computed similarly to (19).

We use the following parameter values:

$$k_s = s, \theta_p = \theta'_p = \left(\cos \frac{p2\pi(N_2 - 1)}{N_2}, \sin \frac{p2\pi(N_2 - 1)}{N_2} \right)$$

We use $N = 2500$ and $N_1 = 12$ for each data set. For full data we use $N_2 = N_3 = 6$. For backscattering and fixed angle data, we use $N_2 = 24$. For fixed energy scattering we use $m = 40$ and $s = 6$.

In all four cases we obtain the linear system $Ef = g$ whose coefficient matrix E is of size $M \times N$. The data g is corrupted with zero mean Gaussian noise with standard deviation $\sigma = 0.01 \max |g|$.

The size M as well as the ranks $r(E)$ and (approximate) condition numbers $\log_{10}\kappa(E)$ measuring the ill-posedness of the linear system $EF = g$ are shown in **Table 2**.

Data	M	$r(E)$	$\log_{10}\kappa(E)$
Full	1728	360	63
Backscattering	288	281	15
Fixed angle	288	241	17
Fixed energy	1600	268	15

Table 2. Matrix sizes, ranks and condition numbers

As the linear system is rank-deficient and ill-posed, we use regularization method to solve it. More precisely, we apply the total variation method (TV) which is defined as

$$f = \arg \min_z \{ \|Ez - g\|_2^2 + \delta \|Lz\|_1 \},$$

where the matrix L implements the differences between neighbouring elements in horizontal and vertical directions (for details, see [21]). As in [21] we formulate this minimization problem as a quadratic problem in standard form for more efficient solution. As the regularization parameter we use $\delta = 2 \cdot 10^{-3}$ for D_E and $\delta = 10^{-3}$ otherwise.

All computations are carried out in a UNIX system with 512 GB of memory and 40 logical CPU cores, each running at 2.8 GHz. The software platform is MATLAB R2015b. We have used 12 workers in parallel in computing the right-hand side g . We note that a desktop PC with 8 logical cores running at 3.4 GHz and 16 GB of memory is also capable of our computations, but with 7 workers it is considerably slower in computing g .

The computational times to both form and solve the linear system are shown in **Table 3**. We point out that the right-hand side g contains synthetic data and actual physical measurements may take longer or shorter time.

	Example 1			Example 3	
Data	E	g	TV	g	TV
Full	1.3s	47s	70s	3m20s	63s
Backscattering	0.4s	9s	59s	37s	45s
Fixed angle	0.3s	9s	66s	33s	58s
Fixed energy	0.1s	4m11s	78s	-	-

Table 3. Computational times

The contour plots of scatterers $h_0(x)$ and their TV reconstructions via Born approximation from full data, backscattering data and fixed angle data (with $\theta_0 = (1, 0)$) are shown in **Figures 2–5** for all examples. For fixed energy scattering we only consider the linear Example 1, since otherwise we do not have direct comparison to a scatterer. The TV reconstruction is shown in **Figure 6**. In each figure solid white line indicates the true geometry of the scatterer.

We see that the location of the scatterer is located quite nicely in all cases. The shape of the scatterer is best seen from full data and backscattering data. By computing the Born approximation from full angle data, we close an open problem from [5].

Figure 2. Scatterer $h_0(x)$ and its TV reconstruction via Born approximation, Example 1. (a) scatterer; (b) full data; (c) backscattering data; (d) fixed angle data.

Figure 3. Scatterer $h_0(x)$ and its TV reconstruction via Born approximation, Example 2. (a) scatterer; (b) full data; (c) backscattering data; (d) fixed angle data

Figure 4. Scatterer $h_0(x)$ and its TV reconstruction via Born approximation, Example 3. (a) scatterer; (b) full data; (c) backscattering data; (d) fixed angle data.

Figure 5. Scatterer $h_0(x)$ and its TV reconstruction via Born approximation, Example 4. (a) scatterer; (b) full data; (c) backscattering data; (d) fixed angle data.

Figure 6. Scatterer $h_0(x)$ and its TV reconstruction via Born approximation, Example 1. (a) scatterer; (b) fixed energy.

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