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Robust Design, Sensitivity Analysis, and Tolerance Setting

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Abstract

This chapter describes various methods for reduction of uncertainties in the determination of characteristic values of random quantities (quantiles of normal and Weibull distribution, tolerance limits, linearly correlated data, interference method, Monte Carlo method, bootstrap method).

Keywords: Random quantity, uncertainty, normal distribution, Weibull distribution, tolerance limits, correlation, interference method, Monte Carlo method, bootstrap method

The reliability and safety of engineering objects are mostly formed during the design. Every design process has three stages:

1. Proposal of conception,
2. Determination of parameters,
3. Setting the tolerances.

Here, stages 2 and 3 will be explained in more detail, as they are very important for reliability.

1. Determination of optimum parameters — Robust design

After the concept of the construction (an engine, a bridge, a transmitter, etc.) has been proposed, it is necessary to determine all important parameters. However, input quantities often vary or can attain values different from those assumed in design. Good design ensures that the important output quantities will always lie within the allowable limits. This can be

achieved by a suitable choice of nominal values of input quantities and by setting their tolerances.

The nominal values of input quantities form together the **design point**. Its position should ensure the low sensitivity of the output parameters to the deviations of input quantities from nominal values. This is called **robust design** [1]. Figure 1 illustrates its principle on an example with one input variable x : the design point 1 is with high sensitivity, whereas point 2 is with low sensitivity. One can see that the changes of the output quantity y around point 2 are much smaller than around point 1, in both cases for the same changes of x . This also means that acceptable scatter of y can sometimes be achieved with lower demands on the accuracy of input parameters. The reliability is influenced not only by the scatter of input quantities, but also by the position of design point. The ideal position, with the lowest sensitivity to the parameter variations, corresponds to an extreme of the response function $y = f(x_1, x_2, \dots, x_n)$. Various optimization methods exist for finding this position, analytical or with computer modeling. Universal is the "simplex method", where the input variables approach the optimum step-by-step according to a simple algorithm [2, 3]. The graphical representation of the response is very informative. Also, the procedures of **design of experiments** (DOE) are suitable; see books by G. Taguchi and other authors [4 - 7]. The determination of optimum parameters should go hand in hand with the sensitivity analysis.

Figure 1. Principle of robust design. Note the influence of the position of the design point on the sensitivity of the output (Δy) to variations of input variable x .

2. Sensitivity analysis

After the design point has been found, the sensitivity analysis could be made to show the influence of the variations of input variables on the variability of the output [8]. The results

may be used for setting the tolerances of input quantities to keep the output in the allowable range. The sensitivity analysis can be done using analytical expressions or simulation methods. The analytical expression for the output variable y ,

$$y = f(x_1, x_2, \dots, x_n), \quad (1)$$

is known exactly only in simple cases (e.g. resonant frequency of an oscillator or deflection of a beam). Often, the **response function** must be found by numerical solution (e.g. using the finite element method). Then, an approximate expression is obtained by regression fitting the response computed for several combinations of input variables (Fig. 3 in Chapter 15).

The sensitivity analysis is usually done in two steps. First, the influence of individual variables is investigated. Several groups of computations are carried out, and in each group, only one variable (x_i) is changed, whereas the others keep their nominal values $x_{1,0}, x_{2,0}, \dots, x_{n,0}$, corresponding to the design point. Then, the y values for the individual groups are fitted by a suitable regression function (e.g. a polynomial),

$$y_i = a_0 + a_{i1}x_i + a_{i2}x_i^2 + a_{i3}x_i^3 + \dots, \quad (2)$$

or

$$y_i = y_0 + a_i(x_i - x_{i,0}) + b_i(x_i - x_{i,0})^2 + \dots, \quad (3)$$

the latter expression characterizes the changes of y as a function of deviations of the i -th input variable from the design point. These regression functions correspond to the cuts through the response surface (Fig. 3 in Chapter 15). The sensitivity analysis will depend on whether the deviations are small or large.

Small changes of the input and output quantities

In this case, linear approximation of the response function may be used, which yields simple expressions. The **sensitivity of the response** to the variations of individual variables is obtained from partial derivatives at the pertinent point,

$$c_i = \partial y / \partial x_i \approx \Delta y / \Delta x_i. \quad (4)$$

For linear approximation, the sensitivity coefficients c_i correspond to the constants $a_{i,1}$ in (2) and a_i in (3). Further information is obtained from **relative sensitivities**,

$$c_{ri} = \frac{\partial y}{\partial x_i} \frac{x_{i,0}}{y_0} \approx \frac{\Delta y}{y_0} \bigg/ \frac{\Delta x_i}{x_{i,0}}, \quad (5)$$

where y_0 and $x_{i,0}$ are the values corresponding to the design point. Coefficient c_{ri} expresses the change of y (in %, for example) caused by 1% deviation of x_i from the nominal value $x_{i,0}$. For linear approximation, $c_{ri} = a_i(x_{i,0}/y_0)$.

Generally, two kinds of sensitivity analysis can be made: (1) deterministic, which assumes that the deviations of individual quantities from nominal values have constant magnitude, and (2) stochastic, which assumes the random scatter of individual input quantities around their nominal values.

Both approaches will be illustrated on an example [9]. A cantilever flat spring of rectangular cross-section (Fig. 2) should be used in a precise measuring device. It is necessary to get an idea how the deviations of its individual dimensions and material properties from the nominal values will influence its compliance. The spring compliance C is given by the formula:

$$C = y / F = 4L^3 / (Ewt^3) ; \quad (6)$$

y is deflection, F is load, L is length, E is elastic modulus, w is spring width, and t is spring thickness.

Figure 2. Spring for a measuring device (a schematic).

Deterministic analysis for small deviations

The increments of y are calculated via the first derivatives. The response surface is replaced by a tangent plane at the investigated point. For $y = f(x_1, x_2, \dots, x_n)$, the infinitesimal increment of y can be expressed generally as

$$dy = (\partial y / \partial x_1)dx_1 + (\partial y / \partial x_2)dx_2 + \dots + (\partial y / \partial x_n)dx_n, \quad (7)$$

where $\partial y / \partial x_1$ expresses partial derivatives. For practical reasons, the differentials are replaced by small finite increments Δ ,

$$\Delta y = (\partial y / \partial x_1)\Delta x_1 + (\partial y / \partial x_2)\Delta x_2 + \dots + (\partial y / \partial x_n)\Delta x_n. \quad (8)$$

In our example with the spring, the partial derivative of Equation (6) with respect to the first variable ($x_1 = L$) is

$$\partial C / \partial L = 3L^2 \times 4 / (Ewt^3) = \left[4L^3 / (Ewt^3) \right] \times 3 / L = (3 / L) \times C, \quad (9)$$

and the increment of compliance due to a small increment of the beam length ΔL is thus

$$\Delta C = 3C (\Delta L / L). \quad (10)$$

The formulas for other variables are obtained in a similar way. The resultant expression, involving the changes of all variables, is

$$\Delta C = C(3\Delta L / L - \Delta E / E - \Delta w / w - 3\Delta t / t), \quad (11)$$

and the relative sensitivity of the stiffness is

$$\Delta C / C = 3\Delta L / L - \Delta E / E - \Delta w / w - 3\Delta t / t. \quad (12)$$

This formula shows the influence of individual quantities. If the spring will be longer by 1% than the nominal value, the compliance will be higher by 3%; if the elastic modulus E will be higher by 1%, the compliance will be lower by 1%, etc. The constants at individual terms correspond to their exponents in Equation (6), and the signs depend on whether the quantity was in the numerator or denominator.

This preliminary analysis reveals which input quantities have very small influence on the variability of the output quantity y and may thus be considered as constants in the following analysis of simultaneous random variance of the input quantities. However, one must always keep in mind that the variance of the output depends on both the sensitivity c_i and the variance of the pertinent input quantity x_i .

Deterministic analysis for large deviations

The above approach is acceptable if the response function is linear or if the errors due to approximation by linear function are small. If the response function is nonlinear and the investigated ranges of input quantities are not small, the errors will not be negligible (Fig. 3). In such case, it is better to study the influence of deviations of input quantities by modeling the response without simplifications. For example, the influence of j -th variable can be studied from Equation (1), in which only x_j varies, whereas the others keep their values corresponding to the design point.

Influence of random variability – small scatter

The influence of random variability of input quantities can be investigated using the formula for the scatter of a function of several random variables. For small scatter,

Figure 3. Error caused by linear approximation of response function $y = f(x)$.

$$s_y^2 = \left(\frac{\partial y}{\partial x_1} \right)^2 s_{x_1}^2 + \left(\frac{\partial y}{\partial x_2} \right)^2 s_{x_2}^2 + \dots + 2 \left(\frac{\partial y}{\partial x_1} \right) \left(\frac{\partial y}{\partial x_2} \right) \text{cov}(x_1, x_2) + \dots, \quad (13)$$

where $s_{x_j}^2$ is the scatter of the j -th variable (quadrate of standard deviation). The far right-hand term is nonzero if the variables are correlated; often, it can be omitted. For linear approximation of y ,

$$y = a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n, \quad (14)$$

the scatter is

$$s_y^2 = a_1^2 s_{x_1}^2 + a_2^2 s_{x_2}^2 + \dots + a_n^2 s_{x_n}^2 + \dots \quad (15)$$

The individual components, $s_{y_j}^2 = a_j^2 s_{x_j}^2$, give the scatter of y caused by random variations of j -th variable. Similarly to deterministic analysis, the contribution of a certain variable x_j to the total scatter is larger for large scatter of this variable ($s_{x_j}^2$) and for large sensitivity (a_j) of the output y to its changes.

The expression obtained by dividing Equation (10) or (12) by the total scatter s_y^2 gives the relative proportions of individual factors in the total scatter,

$$1 = a_1^2 \frac{s_{x_1}^2}{s_y^2} + a_2^2 \frac{s_{x_2}^2}{s_y^2} + \dots + a_n^2 \frac{s_{x_n}^2}{s_y^2} + \dots \quad (16)$$

The square root of scatter (10) is the standard deviation s_y . If the input quantities have normal distribution, the confidence interval for the output quantity y can be calculated as

$$y_{\text{lower,upper}} = y_0 \pm u_\alpha s_y; \quad (17)$$

the + or – sign corresponds to the upper (or lower) confidence limit and u_α is the α -critical value of standard normal distribution. The probability that y will lie out of these limits is 2α .

If Formula (12) is applied on the above example with a spring, one obtains the following expression for the standard deviation of the compliance caused, for example, by random variability of the length L :

$$s_{CL} = 3C(s_L / L); \quad (18)$$

cf. Equation (11). Similar expressions can be written for other variables. The random variability of all input quantities causes the following variability of the spring compliance:

$$s_C = [(3C / L)^2 s_L^2 + (C / E)^2 s_E^2 + (C / w)^2 s_w^2 + (3C / t)^2 s_t^2]^{1/2}. \quad (19)$$

The ratio of the standard deviation of a quantity and its mean is the variation coefficient,

$$v = \sigma / \mu, \quad (20)$$

so that the combination of Equations (16) and (17) gives the variation coefficient of the compliance,

$$v_C = s_C / C = [9v_L^2 + v_E^2 + v_w^2 + 9v_t^2]^{1/2}. \quad (21)$$

Stochastic analysis for large scatter

The above approach, based on the linearization of the response function, is suitable for small values of variance coefficients of input quantities, say $v_j \leq 10\%$. If their scatter is large, it is better to study the influence of variability or deviations of input quantities by the Monte Carlo simulation method. A preliminary assessment consists of making m simulation experiments with random variable only x_j , for $j = 1, 2, \dots, n$, and then calculating partial scatter $s_{y_j}^2$ of the obtained values y . Using the characteristics s_{x_j} , $x_{i,0}$, and y_0 , one can determine the variation coefficients v_j or the sensitivity coefficients $a_j (= s_y / s_{x_j})$.

The approximate value of the total scatter is obtained by summing up the partial scatters,

$$s_y^2 = s_{y1}^2 + s_{y2}^2 + \dots + s_{yn}^2 + \dots \quad (22)$$

More accurate value is obtained if all input variables, $x_1, x_2, \dots, x_n$ are considered as random in the Monte Carlo simulations, and the scatter is calculated from all values y_j . Dividing Equation (19) by the total scatter s_y^2 gives the relative influence of individual factors, like in Equation (13).

3. Determination of tolerances of input quantities

If the variability or deviation of the output quantity y from the nominal value is larger than allowed, it must be reduced. The procedure depends on whether the variability is random or deterministic.

Deterministic deviations

If the deviation of y is caused by the deviation of one or more input quantities, Equation (12) or (12), showing the contribution of individual factors to the total deviation of y , can be used to decide which factor should be aimed at. Let us assume that the deviation of y in Equation (12) is caused only by the deviation of x_j . The allowable magnitude of Δx_j , ensuring that the deviation of y does not exceed Δy , is

$$\Delta x_j \leq \Delta y / \left(\partial y / \partial x_j \right). \quad (23)$$

For example, the allowable length tolerance of the above spring, ensuring the compliance tolerance ΔC , is

$$\Delta L = \Delta C (L / 3C), \text{ or } \Delta L / L = (1 / 3) (\Delta C / C). \quad (24)$$

The tolerances of other quantities can be determined in similar way. One must respect that the deviations of some input quantities influence the output in one direction, whereas the deviations of other quantities can have the opposite influence. Generally, the deviations of y depend on the deviations of input quantities and also on the sensitivity of y to the changes of x_j . The reduction of the tolerance of y can thus be accomplished by tightening the tolerances of individual input quantities or by changing the position of the design point towards lower sensitivity. The decision will also depend on the costs related to the individual adjustments.

Random variability of input quantities

The following analysis assumes that the range of probable occurrence of y (i.e. the half-width Δy_α of the α -confidence interval for y) is directly proportional to the standard deviation s_y , equal to the square root of the scatter. In production, the allowable limits of a quantity x are

also often determined as $x_{nom} \pm ks_{x'}$ where k is a constant (e.g. a suitable quantile of standard normal distribution). With this assumption, the tolerance of y can be reduced from Δy_α to $\Delta y_{\alpha'}$ by reducing the standard deviation of y from the original value s_y to $s_{y'}$. This may be accomplished by the reduction of the variance or influence of input factors.

Often, the influence of one factor prevails (e.g. x_k). In such case, most of the scatter of y can be reduced by reducing its component due to this factor. As it follows from Equation (12), the scatter of y can be reduced by reducing the standard deviation $s_{x,k}$ or the sensitivity of y to the changes of x_k (coefficient a_k). The reduction of variance of y_k can be attained by more accurate manufacturing or by better control and sorting out the components that are out of the tolerance limits. The reduction of sensitivity of y to the changes of x_k can be accomplished by changing the parameters of the design point (Fig. 1). An example is a prestressed flange connection in steam turbines: the use of long bolts increases the compliance of the joint and reduces the sensitivity of the prestress to the variations of pressure in the pipe and thermal dilatations of the flanges. Sometimes, both ways, the reduction of $s_{x,k}$ and a_k is combined.

If several input variables vary, one must decide, which of them should be reduced. As the standard deviation equals the square root of the scatter, it is obvious that the reduction of scatter of a quantity, contributing to the total scatter by only 5% to 10%, will have negligible effect. Also, the costs of the pertinent improving operation must be considered, as they usually increase with tightening the tolerances.

After having obtained the corrected standard deviation $s_{xi'}$, the lower (L) and upper (U) allowable limit for the input quantity x_j can be determined as

$$x_{i,L,U} = x_{i,0} \pm k_{L,U} s_{xi'} \quad (25)$$

$x_{j,0}$ is the nominal (design) value and k is a constant (e.g. 5% quantile of standard normal distribution). k_L corresponds to the lower limit, whereas k_U corresponds to the upper limit.

The above optimization can be performed even if the scatter s_y^2 from the preliminary design is smaller than the allowable value. The optimization assigns such tolerances that the total costs are minimal. Sometimes, the tolerances may even be made wider, with lower costs.

Often, the scatter of some input quantities cannot be changed continuously. In such cases, the response must be evaluated for each possible value of every discontinuous quantity.

The determination of suitable tolerances will be illustrated on the following example, adapted from [9].

Example 1

A cantilever microbeam from Figure 2, with length $L = 10$ mm, width $w = 1.0$ mm and thickness $t = 50$ μ m, made of a material with elastic modulus $E = 200$ GPa, has compliance $C = 4L^3/(Ewt^3) = 0.16$ mm/mN. Each input quantity has coefficient of variation $v_l = v_w = v_t = v_E = v = 0.01 = 1\%$. The variation coefficient of the compliance, Equation (17), is $v_C = (9v_L^2 + v_E^2 + v_w^2 + 9v_t^2)^{1/2} =$

0.0447, and the standard deviation $s_c = Cv_c = 0.00716$ mm/mN. Such variation of compliance is unacceptably high and must be reduced to $s_c' = 0.004$ mm/mN.

Solution. The corresponding reduced variation coefficient is $v_c' = s_c'/C = 0.004/0.16 = 0.025$. It is possible to reduce the scatter of L , w , and t ; the material (E) remains unchanged. The easiest way is to reduce the scatter of L . However, even if this scatter were zero, the variation coefficient of compliance would be $v_c' = 0.033$, which is much more than demanded. Therefore, the variance of all three quantities (L , w , and t) must be reduced by more accurate manufacturing. If the new variation coefficients of L , w , and t would have the same value, $v_L' = v_w' = v_t' = v'$, this value v' can be calculated from the modified Equation (18):

$$v_c' = [9v'^2 + v_E'^2 + v'^2 + 9v'^2]^{1/2}. \quad (26)$$

With the variation coefficient of elastic modulus unchanged, $v_E = 0.01$, the new coefficients of variation of L , w , and t must be reduced to $v' \leq 0.005256$, which is approximately $v' = 0.005$. The corresponding allowable standard deviations, obtained by multiplying the variation coefficient v' by the nominal values of L , w , and t , are $s_L' \leq 0.05$ mm, $s_w' \leq 0.005$ mm, and $s_t' \leq 0.25$ μ m. In the limit case, $v_c' = 0.024$ and $s_c' = 0.0038$ m/N. However, the tolerances of individual dimensions could be adjusted with respect to the manufacturing possibilities, the principal condition being $s_c' \leq 0.004$ m/N.

4. Uncertainties in ensuring safety and lifetime using proof testing

If the high reliability of a certain object must be ensured, a proof-test is often used: the component is exposed to some overload, specified so that only sufficiently strong components survive it; the weaker ones are destroyed. In the same way, sufficient lifetime can be ensured for components made of brittle materials suffering by static fatigue. The minimum time to failure of a component that has passed a proof-test is [10 - 12]

$$t_{\min} = \frac{2\sigma_{pt}^{N-2}}{(N-2)AY^2K_{IC}^{N-2}\sigma_0^N}, \quad (27)$$

where K_{IC} is the fracture toughness of the material, N and A are the parameters of subcritical crack growth, Y is the geometrical factor of typical crack, responsible for fatigue failure, σ_0 is the characteristic operational stress (assumed constant), and σ_{pt} is the proof-test stress. A rearrangement of Equation (24) gives the formula for the proof stress needed to guarantee the minimum lifetime:

$$\sigma_{pt} = \sigma_0^{N/(N-2)} K_{IC} \left[\frac{N-2}{2} AY^2 t_{\min} \right]^{1/(N-2)}. \quad (28)$$

However, K_{IC} , N , and A were determined by measurement and are known only approximately and Y was estimated. Therefore, it is recommended to perform sensitivity analysis and correct

the proof stress appropriately. The pertinent theory, based on probabilistic analysis, is explained in [11, 12] or in [10]. For easier application, strength-probability-time diagrams were developed [13 – 15], in which the necessary proof stress can be found for the demanded time to failure and confidence level.

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