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Gabor-Fourier Analysis

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1. Introduction

The notion of Gabor transform, named after Dennis Gabor [1], is a special case of the short-time Fourier transform. The Gabor analysis, as it stands now, is a rather new field, but the idea goes back quite some while. Dennis Gabor investigated in [1] the representation of a one dimensional signal in two dimensions, time and frequency. He suggested to represent a function by a linear combination of translated and modulated Gaussians. Interestingly, there is a tight connection between this approach and quantum mechanics (c.f. [2]). On the mathematical side, the representation of functions by other functions was further investigated, leading to the theory of atomic decompositions, developed by Feichtinger and Gröchenig [3].

Gabor transform and Gabor expansion have long been recognized as very useful tools for the signal processing, and it is because of this reason over the recent years, an increasing attention has been given to the study of them in engineering and applied Mathematics, see for instance [4, 5]. Borichev et al. [6] studied the stability problem for the Gabor expansions generated by a Gaussian function. In [7], Ascensi and Bruna proved that the unique Gabor atom with analytical Gabor space, the image of $L^2(\mathbb{R})$ under the Gabor transform, is the Gaussian function. The structure of Gabor and super Gabor spaces inside $L^2(\mathbb{R}^{2d})$ is studied by Abreu [8]. Christensen [9] has done a comprehensive study of the Gabor system and has asked for the necessary and sufficient conditions to get a frame for $L^2(\mathbb{R})$.

Today Gabor analysis and the closely related wavelet analysis are considered topics in harmonic analysis. The basic idea behind wavelet analysis is that the notion of an orthonormal basis is not always useful. Sometimes it is more important for a decomposing set to have special properties, like good time frequency localization, than to have unique coefficients. This led to the concept of frames, which was introduced by Duffin and Schaefer in [10] and was made popular by Daubechies, and today is one of the most important foundations of Gabor theory and a fundamental subject in harmonic analysis.

Most examples of Gabor frames correspond to regular nets of points. That is, sets of the type $\{e^{2\pi i b n t} h(t - a m)\}_{n,m \in \mathbb{Z}^d}$. One can usually find sufficient and necessary conditions for the existence of such kind of frames, with a variety of applications. For technical reasons, however, one needs to work with frames coming from irregular grids. One of the main purposes of this chapter is to study perturbations of irregular Gabor frames and the problem of stability.

On the other hand, the theory of nonharmonic Fourier series is concerned with the completeness and expansion properties of sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^p[-\pi, \pi]$. In 1952, Duffin and Schaeffer [10] used frames to study this theory, and later Young put together many results in his book [11]. Reid [12] proved that if $\{\lambda_n\}$ is a sequence of real numbers whose differences are nondecreasing, then the set of complex exponentials $\{e^{i\lambda_n t}\}$ is a Riesz-Fischer sequence in $L^2[-A, A]$ for every $A > 0$. Jaffard [13] investigated how the regularity of nonharmonic Fourier series is related to the spacing of their frequencies, and this is obtained by using a transformation which simultaneously captures the advantages of the Gabor and wavelet transforms.

In this chapter, we restate and prove some classical results of (nonharmonic) Fourier expansions for Gabor systems instead of sets of complex exponentials. Some of the results may be known or obtainable via Hilbert space methods, but the main advantage of this work is that it uses analytic methods and can be fully understood with elementary knowledge of functional and complex analysis in several variables [14, 15].

2. Preliminaries

Let us introduce the notions and basic results, needed later in the chapter.

Definition 2.1 We say that $\Lambda = \{z_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^d$ is a *separated set* if there exists $\varepsilon > 0$ such that $|z_i - z_j| \geq \varepsilon$, $i \neq j$. The largest of such ε is called the *separation constant* of Λ . A finite union of separated sets is called a *relatively separated set*.

Definition 2.2 A sequence of vectors $\{x_n\}$ in a normed space $\mathcal{X}$ is said to be *complete* if its linear span is dense in $\mathcal{X}$, that is, if for each vector x and each $\varepsilon > 0$ there is a finite linear combination $c_1 x_1 + \dots + c_n x_n$ such that

$$\|x - (c_1 x_1 + \dots + c_n x_n)\| < \varepsilon.$$

Definition 2.3 A sequence $\{f_n\}$ in a Hilbert space H is said to be a *Bessel sequence* if

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 < \infty$$

for every element $f \in H$. It is called a *Riesz-Fischer sequence* if the moment problem

$$\langle f, f_n \rangle = c_n, \quad n \geq 1$$

admits at least one solution $f \in H$ whenever $\{c_n\} \in l^2$.

Proposition 2.4 Let $\{f_n\}$ be a sequence in a Hilbert space H . Then

(i) $\{f_n\}$ is a Bessel sequence with bound M if and only if the inequality

$$\left\| \sum c_n f_n \right\|^2 \leq M \sum |c_n|^2$$

holds for every finite sequence of scalars $\{c_n\}$;

(ii) $\{f_n\}$ is a Riesz-Fischer sequence with bound m if and only if the inequality

$$m \sum |c_n|^2 \leq \left\| \sum c_n f_n \right\|^2$$

holds for every finite sequence of scalars $\{c_n\}$.

Remark 2.5 For a sequence $\{f_n\}$ in a Hilbert space H , the moment problem

$$\langle f, f_n \rangle = c_n, \quad n \geq 1$$

has at most one solution for every choice of the scalars $\{c_n\}$ if and only if $\{f_n\}$ is complete.

Definition 2.6 A countable family $\{f_k\}_{k \in I}$ in a separable Hilbert space H is a *frame* for H if there exist constants A and B such that $0 < A \leq B < \infty$ and

$$A\|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad f \in H.$$

A, B are called the *lower* and *upper frame bounds* respectively. They are not unique: the biggest lower bound and the smallest upper bound are called the *optimal frame bounds*. Every element in H has at least one representation as an infinite linear combination of the frame elements.

Definition 2.7 Let $c \in \mathbb{R}^d$, the unitary operators T_c and M_c on $L^2(\mathbb{R}^d)$ defined by $T_c f(t) = f(t - c)$ and $M_c f(t) = e^{2\pi i c t} f(t)$ are called the *Translation* and *Modulation* operator, respectively. For a discrete set $\Lambda = \{z_j\}_{j \in \mathbb{Z}}$ in $\mathbb{C}^d$ and a fixed nonzero window function $h \in L^2(\mathbb{R}^d)$, we define the *Gabor system* $G(h, \Lambda)$ as:

$$G(h, \Lambda) = \{M_y T_x h(t) = e^{2\pi i y t} h(t - x); x + iy \in \Lambda\}.$$

For simplicity we denote $e^{2\pi i y t} h(t - x)$ by $h_z(t)$, where $z = x + iy$. Gabor systems were first introduced by Gabor [1] in 1946 for signal processing, and is still widely used. A Gabor

system is said to be *exact* in $L^2(\mathbb{R}^d)$ if it is complete, but fails to be complete on the removal of any one term.

If $G(h, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, it is called a *Gabor frame* or *Weyl-Heisenberg frame*.

Definition 2.8 Let f be an entire function. For $r > 0$, the *maximum modulus function* is $M(r) = \max\{|f(z)| : |z| = r\}$. Unless f is a constant of modulus less than or equal to 1, its *order*, which is denoted by ρ , is defined by

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}.$$

Simple examples of functions of finite order include e^z , $\sin z$, and $\cos z$, all of which are of order 1, and $\cos \sqrt{z}$, which is of order $\frac{1}{2}$. Every polynomial is of order 0; the order of a constant function is of course 0 and the function e^{e^z} is of infinite order.

Remark 2.9 An entire function has an order of growth $\leq \rho$ if $|f(z)| \leq A e^{B|z|^\rho}$.

The following is the fundamental factorization theorem for entire functions of finite order. It is due to Hadamard who used the result in his celebrated proof of the Prime Number Theorem. It is one of the classical theorems in function theory.

Theorem 2.10 (Hadamard Factorization Theorem) Let f be an entire function of finite order ρ , $\{z_n\}$ be the zeros of f different from 0, k be the order of zero of f at the origin, and

$$f(z) = z^k e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

be its canonical Factorization, then $g(z)$ is a polynomial of degree no longer than ρ .

Definition 2.11 The (Bargmann-)Fock space, $\mathcal{F}(\mathbb{C}^d)$, is the Hilbert space of all entire functions f on $\mathbb{C}^d$ for which

$$\|f\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |f(z)|^2 e^{-\pi|z|^2} dz,$$

is finite. The natural inner product on $\mathcal{F}(\mathbb{C}^d)$ is defined by

$$\langle f, g \rangle_{\mathcal{F}} = \int_{\mathbb{C}^d} f(z) \overline{g(z)} e^{-\pi|z|^2} dz; \quad f, g \in \mathcal{F}(\mathbb{C}^d).$$

The *Bargmann transform* of a function $f \in L^2(\mathbb{R}^d)$ is the function Bf on $\mathbb{C}^d$ defined by

$$Bf(z) = 2^{\frac{d}{4}} e^{-\frac{\pi}{2} z^2} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{2\pi t z} dt.$$

Definition 2.12 Fix a function $h \in L^2(\mathbb{R}^d)$ (called the window function). The *Gabor transform* with respect to the window h is the isomorphic inclusion

$$V_h : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{C}^d),$$

defined by

$$V_h f(z) = \langle f(t), h_z(t) \rangle = 2^{d/4} \int_{\mathbb{R}^d} f(t) \overline{h(t-x)} e^{-2\pi i t y} dt$$

for every $f \in L^2(\mathbb{R}^d)$ and $z = x + iy \in \mathbb{C}^d$. The following subspace of $L^2(\mathbb{C}^d)$ which is the image of $L^2(\mathbb{R}^d)$ under the Gabor transform with the window h ,

$$\mathcal{G}_h = \{V_h f : f \in L^2(\mathbb{R}^d)\},$$

is called *Gabor space* or *model space*. A simple calculation shows that the Bargmann transform is related to the Gabor transform with the Gaussian window $g(t) = 2^{d/4} e^{-\pi t^2}$ in $L^2(\mathbb{R}^d)$ by the formula

$$V_g f(x - iy) = e^{i\pi xy} e^{-\pi \frac{|x+iy|^2}{2}} (Bf)(x + iy). \quad (1)$$

For more details we refer the reader to [2, 7, 8, 17].

3. Gabor Expansion

Here we discuss the fundamental completeness properties of the Gabor systems. The most extensive results in the case of the sets of complex exponentials $\{e^{i\lambda_n t}\}$ over a finite interval of the real axis were obtained by Paley and Wiener [16]. At the same time, we will be laying the groundwork for a more penetrating investigation of nonharmonic Gabor expansions in $L^2(\mathbb{R}^2)$.

Let $\{(\lambda_n, \mu_n)\}_{n \in \mathbb{Z}}$ be an arbitrary countable subset of $\mathbb{R}^2$ and

$$\{\varphi_n(\xi)\}_{n \in \mathbb{Z}} = \left\{ M_{\mu_n} T_{\lambda_n} g(\xi) \right\}_{n \in \mathbb{Z}} = \left\{ \sqrt{2} e^{2\pi i \mu_n \xi - \pi(\xi - \lambda_n)^2} \right\}_{n \in \mathbb{Z}}; \quad (2)$$

where $\xi \in \mathbb{R}^2$ or $\mathbb{C}$, be the corresponding Gabor system with respect to the Gaussian window g in $L^2(\mathbb{R}^2)$. If $\{\varphi_n\}_{n \in \mathbb{Z}}$ is incomplete in $L^2(\mathbb{R}^2)$ then the closed linear span $\mathcal{M}$ of $\{\varphi_n\}_{n \in \mathbb{Z}}$ is a proper subspace of $L^2(\mathbb{R}^2)$. By Hahn-Banach Theorem there exists a function F in $L^2(\mathbb{R}^2)$

such that $F|_{\mathcal{M}} = 0$ and $F \neq 0$. Riesz Representation Theorem implies that $F = F_\varphi$ for some φ in $L^2(\mathbb{R}^2)$ and

$$F(h) = F_\varphi(h) = \int_{\mathbb{R}^2} h \varphi d\xi; h \in L^2(\mathbb{R}^2).$$

For $(z, w) \in \mathbb{C}^2$ take

$$f(z, w) = \sqrt{2} \int_{\mathbb{R}^2} e^{2\pi i w \xi - \pi(\xi - z)^2} \varphi(\xi) d\xi, \quad (3)$$

then $f(\lambda_n, \mu_n) = F(\varphi_n) = 0$ (since $F|_{\mathcal{M}} = 0$).

Remark 3.1 The system (2) is incomplete in $L^2(\mathbb{R}^2)$ if and only if there exists a nontrivial entire function of the form (3) in the Gabor space $\mathcal{G}_g$, which is zero for every (λ_n, μ_n) . Furthermore, since

$$f(z, w) = V_g \varphi(z, -w) = e^{i\pi zw} e^{-\pi \frac{|z|^2 + |w|^2}{2}} (B\varphi)(z, w),$$

we have

$$|f(z, w)| \leq \|\varphi\|_2 e^{\frac{\pi}{2}|(z, w)|^2}.$$

Theorem 3.2 Let $\{\lambda_n\}_{n \in \mathbb{Z}}$ be a symmetric sequence of real numbers ($\lambda_{-n} = -\lambda_n$). If the Gabor system

$$\left\{ \sqrt[4]{2} e^{2\pi i \lambda_n t - \pi(t - \lambda_n)^2} \right\}_{n \in \mathbb{Z}} \quad (4)$$

is exact in $L^2(\mathbb{R})$, then the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2} \right) e^{\frac{z^2}{\lambda_n^2}}$$

converges to an entire function which belongs to the Gabor space with Gaussian window in $L^2(\mathbb{R})$.

Proof. By Remark 3.1, if the system (4) is exact, then there exists an entire function $f(z)$ in the Gabor space $\mathcal{G}_g$ such that $f(\lambda_n) = 0$ for $n \neq 0$, and

$$f(z) = \sqrt[4]{2} \int_{\mathbb{R}} e^{2\pi i z t - \pi(t - z)^2} \varphi(t) dt; \quad \varphi \in L^2(\mathbb{R}).$$

Since $f(\lambda_n) = 0$ for $n \neq 0$ and the sequence $\{\lambda_n\}$ is symmetric, $\varphi(-t)$ has the same orthogonality properties as $\varphi(t)$. But by Remark 2.5, $\varphi(t)$ is unique, so $\varphi(t)$ must be even.

Hence $f(z)$ is even. Now $f(z)$ vanishes only at the λ_n with $n \neq 0$. Indeed, if $f(z)$ vanishes at $z = \gamma$, then the function

$$\tilde{f}(z) = \frac{zf(z)}{z - \gamma}$$

would also belong to $\mathcal{G}_g$ and would vanish at every λ_n . The system (4) would then be incomplete in $L^2(\mathbb{R})$, contrary to hypothesis.

Let us observe that the function $\tilde{f}$ belongs to $\mathcal{G}_g$. Since the Bargmann transform is related to the Gabor transform by the formula (1), it is sufficient to show that the function $e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)} \tilde{f}(z)$; $z = x + iy$, belongs to the Fock space $\mathcal{F}(\mathbb{C})$. In other words, we must show that the integral

$$\int_{\mathbb{C}} \frac{|z|^2}{|z - \gamma|^2} |f(z) e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)}|^2 e^{-\pi(|x|^2 + |y|^2)} dx dy; \quad z = x + iy,$$

is finite. Since $\lim_{z \rightarrow \infty} \left| \frac{z}{z - \gamma} \right| = 1$, we have $\left| \frac{z}{z - \gamma} \right| \leq 3/2$ outside a square T with complement T^c . Thus the above integral is no larger than

$$\begin{aligned} & \int_T \frac{|z|^2}{|z - \gamma|^2} |f(z) e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)}|^2 e^{-\pi(|x|^2 + |y|^2)} dx dy \\ & + 9/4 \int_{T^c} |f(z) e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)}|^2 e^{-\pi(|x|^2 + |y|^2)} dx dy \\ & \leq \int_T \frac{|z|^2}{|z - \gamma|^2} |f(z) e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)}|^2 e^{-\pi(|x|^2 + |y|^2)} dx dy \\ & + 9/4 \int_{\mathbb{C}} |f(z) e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)}|^2 e^{-\pi(|x|^2 + |y|^2)} dx dy. \end{aligned}$$

In the last expression, since T is compact the first integral is finite, and since the function $f(z) e^{i\pi xy} e^{\frac{\pi}{2}(|x|^2 + |y|^2)}$ is in the Fock space $\mathcal{F}(\mathbb{C})$, so is the second integral. Next since

$$|f(z)| \leq \|\varphi\|_2 e^{\frac{\pi}{2}|z|^2},$$

the order of growth of f is at most 2, and by Hadamard's factorization theorem,

$$f(z) = e^{Az} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2} \right) e^{\frac{z^2}{\lambda_n^2}}; \quad A \in \mathbb{R}.$$

Since $f(z)$ and the canonical product are both even, $A = 0$ and

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2} \right) e^{\frac{z^2}{\lambda_n^2}}.$$

We have the following version of Plancherel-Pólya theorem. We give the proof which is similar to [11, Th. 2.16] for the sake of completeness.

Theorem 3.3 (Plancherel-Pólya). If $f(z)$ is an entire function of order of growth $\leq \tau$ and if for some positive number p ,

$$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty,$$

then

$$\int_{-\infty}^{\infty} |f(x + iy)|^p dx \leq e^{p\tau|y|} \int_{-\infty}^{\infty} |f(x)|^p dx.$$

The proof will require two preliminary lemmas. Suppose that $q(z)$ is a non constant continuous function in the closed upper half-plane, $\text{Im}z \geq 0$, and analytic in its interior. Let a and p be positive real numbers and put

$$Q(z) = \int_{-a}^a |q(z+t)|^p dt.$$

It is clear that $Q(z)$ is continuous for $\text{Im}z \geq 0$. Since $|q(z)|^p$ is subharmonic for $\text{Im}z > 0$ (see [12, p.83]), so is $Q(z)$.

Lemma 3.4 Let $q(z)$ be a function of order of growth $\leq \tau$ in the half-plane $\text{Im}z \geq 0$ and suppose that the following quantities are both finite:

$$M = \sup_{-\infty < x < \infty} Q(x) \text{ and } N = \sup_{y > 0} Q(iy).$$

Then on this half-plane,

$$Q(z) \leq \max(M, N).$$

Proof. Since $q(z)$ is of order of growth $\leq \tau$, then there exist positive numbers A and B such that

$$|q(z)| \leq Ae^{B|z|^\tau} \quad (\text{Im}z \geq 0). \quad (5)$$

For each positive number ε , define the auxiliary function

$$q_\varepsilon(z) = q(z)e^{-\varepsilon(\lambda(z+a))^{3/2}}, \quad (6)$$

where $\lambda = e^{-i\pi/4}$. The exponent of e in (6) has two possible determinations in the half-plane $\text{Im}z > 0$; we choose the one whose real part is negative in the quarter-plane $x > -a, y \geq 0$. Put

$$Q_\varepsilon(z) = \int_{-a}^a |q_\varepsilon(z+t)|^p dt,$$

which is then defined and continuous in the upper half-plane $\text{Im}z \geq 0$, and subharmonic in its interior. A simple calculation involving (5) and (6) shows that in the quarter plane $x > -a, y \geq 0$,

$$|q_\varepsilon(z)| \leq Ae^{B|z|^\tau - \varepsilon\gamma|z+a|^{3/2}}, \quad (7)$$

where $\gamma = \cos 3\pi/8$, and $|q_\varepsilon(z)| \leq |q(z)|$. Hence

$$Q_\varepsilon(z) \leq Q(z) \quad (x \geq 0, y \geq 0),$$

and in particular

$$Q_\varepsilon(x) \leq M \quad \text{for } x \geq 0 \quad \text{and} \quad Q_\varepsilon(iy) \leq N \quad \text{for } y \geq 0.$$

Let z_0 be a fixed but arbitrary point in the first quadrant. We shall apply the maximum principle to $Q_\varepsilon(z)$ in the region $\Omega = \{z : \text{Re}z \geq 0, \text{Im}z \geq 0, |z| \leq R\}$, choosing R large enough so that (i) $z_0 \in \Omega$, and (ii) the maximum value of $Q_\varepsilon(z)$ on Ω is not attained on the circular $\text{arc}|z| = R$ (this is possible by virtue of (7)). Since $Q_\varepsilon(z)$ does not reduce to a constant, the maximum value of $Q_\varepsilon(z)$ on Ω must be attained on one of the coordinate axes, and in particular,

$$Q_\varepsilon(z_0) \leq \max(M, N).$$

Now let $\varepsilon \rightarrow 0$. This establishes the result for the first quadrant; the proof for the second quadrant is similar.

Lemma 3.5 In addition to the hypotheses of Lemma 3.4, suppose that

$$\lim_{y \rightarrow \infty} q(x+iy) = 0 \quad (8)$$

uniformly in x , for $-a \leq x \leq a$. Then

$$Q(z) \leq M, \quad \text{Im}z \geq 0.$$

Proof. It is sufficient to show that $N \leq M$. By virtue of (8), we see that the function $Q(iy)$ tends to zero as $y \rightarrow \infty$, and so must attain its least upper bound N for some finite value of y , say $y = y_0$. If $y_0 = 0$, then

$$N = Q(iy_0) = Q(0) \leq M.$$

If $y_0 > 0$, then the maximum principle shows that the least upper bound of $Q(z)$ in the half-plane $\text{Im}z \geq 0$ cannot be attained at the interior point $z = iy_0$. Therefore, by Lemma 3.4,

$$N = Q(iy_0) < \max(M, N),$$

and again $N < M$.

Theorem 3.3 now follows.

Proof of Theorem 3.3. It is sufficient to prove the theorem when $y > 0$ and $f(z)$ is not identically zero. Let ε be a fixed positive number and consider the function

$$q(z) = f(z)e^{i(\tau+\varepsilon)z}.$$

It is easy to see that, for each positive number a , the functions $q(z)$ and $Q(z)$ satisfy the conditions Lemmas 3.4 and 3.5. Consequently, for $y > 0$,

$$Q(iy) \leq M < \int_{-\infty}^{\infty} |q(x)|^p dx.$$

This together with the definitions of $q(z)$ and $Q(z)$ implies

$$e^{-p(\tau+\varepsilon)y} \int_{-a}^a |f(x+iy)|^p dx < \int_{-\infty}^{\infty} |f(x)|^p dx.$$

To get the result, first let $a \rightarrow \infty$, then let $\varepsilon \rightarrow 0$.

Proposition 3.6 Let $f(z, w)$ be an entire function of order of growth $\leq \tau$ and suppose that $\{\lambda_n\}, \{\mu_n\}$ are increasing sequences of real numbers such that

$$\lambda_{n+1} - \lambda_n \geq \varepsilon_1 > 0 \quad \text{and} \quad \mu_{n+1} - \mu_n \geq \varepsilon_2 > 0.$$

If for some positive number p ,

$$\sup_n \int_{-\infty}^{\infty} |f(x_z, \mu_n)|^p dx_z < \infty \quad \text{and} \quad \sup_n \int_{-\infty}^{\infty} |f(\lambda_n, x_w)|^p dx_w < \infty, \quad (9)$$

then

$$\sum_n |f(\lambda_n, \mu_n)|^p < \infty.$$

Proof. First, using the Plancherel-Pólya Theorem, observe that conditions (9) imply that

$$\sup_n \int_{-\infty}^{\infty} |f(z, \mu_n)|^p dx_z \leq e^{p\tau|y_z|} \int_{-\infty}^{\infty} \sup_n |f(x_z, \mu_n)|^p dx_z,$$

and

$$\sup_n \int_{-\infty}^{\infty} |f(\lambda_n, w)|^p dx_w \leq e^{p\tau|y_w|} \int_{-\infty}^{\infty} \sup_n |f(\lambda_n, x_w)|^p dx_w.$$

Now since $|f|^p$ is plurisubharmonic, the inequality

$$|f(z_0, w_0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f((z_0, w_0) + (\zeta, \eta)e^{i\theta})|^p d\theta \quad (10)$$

holds for all values of (ζ, η) . Fix $\eta = 0$, multiply both sides of (10) by ζ and integrate between 0 and δ_1 ,

$$\int_0^{\delta_1} |f(z_0, w_0)|^p \zeta d\zeta \leq \frac{1}{2\pi} \int_0^{\delta_1} \int_0^{2\pi} |f(z_0 + \zeta e^{i\theta}, w_0)|^p d\theta \zeta d\zeta.$$

Then

$$|f(z_0, w_0)|^p \leq \frac{1}{\pi\delta_1^2} \iint_{\Omega_1} |f(z, w_0)|^p dx_z dy_z,$$

where $\Omega_1 = \{(z, w_0) : |z - z_0| \leq \delta_1\}$. Similarly fix $\zeta = 0$, multiply both sides of (10) by η and integrate between 0 and δ_2 ,

$$|f(z_0, w_0)|^p \leq \frac{1}{\pi\delta_2^2} \iint_{\Omega_2} |f(z_0, w)|^p dx_w dy_w,$$

where $\Omega_2 = \{(z_0, w) : |w - w_0| \leq \delta_2\}$. Then

$$\begin{aligned} 2|f(z_0, w_0)|^p &\leq \frac{1}{\pi\delta_1^2} \iint_{\Omega_1} |f(z, w_0)|^p dx_z dy_z \\ &\quad + \frac{1}{\pi\delta_2^2} \iint_{\Omega_2} |f(z_0, w)|^p dx_w dy_w. \end{aligned}$$

Now let $\Omega_1^n = \{(\lambda_n + z, \mu_n) : |z| \leq \delta_1\}$ and $\Omega_2^n = \{(\lambda_n, \mu_n + w) : |w| \leq \delta_2\}$, then

$$\begin{aligned}
 2 \sum_n |f(\lambda_n, \mu_n)|^p &\leq \sum_n \left(\frac{1}{\pi \delta_1^2} \iint_{\Omega_1^n} |f(\lambda_n + z, \mu_n)|^p dx_z dy_z \right. \\
 &\quad \left. + \frac{1}{\pi \delta_2^2} \iint_{\Omega_2^n} |f(\lambda_n, \mu_n + w)|^p dx_w dy_w \right) \\
 &\leq \sum_n \left(\frac{1}{\pi \delta_1^2} \int_{-\delta_1}^{\delta_1} \int_{-\delta_1}^{\delta_1} |f(\lambda_n + z, \mu_n)|^p dx_z dy_z \right. \\
 &\quad \left. + \frac{1}{\pi \delta_2^2} \int_{-\delta_2}^{\delta_2} \int_{-\delta_2}^{\delta_2} |f(\lambda_n, \mu_n + w)|^p dx_w dy_w \right) \\
 &= \sum_n \left(\frac{1}{\pi \delta_1^2} \int_{-\delta_1}^{\delta_1} \int_{\lambda_n - \delta_1}^{\lambda_n + \delta_1} |f(z, \mu_n)|^p dx_z dy_z \right. \\
 &\quad \left. + \frac{1}{\pi \delta_2^2} \int_{-\delta_2}^{\delta_2} \int_{\mu_n - \delta_2}^{\mu_n + \delta_2} |f(\lambda_n, w)|^p dx_w dy_w \right).
 \end{aligned}$$

It is clear that the last expression above is no larger than

$$\begin{aligned}
 &\sum_n \left(\frac{1}{\pi \delta_1^2} \int_{-\delta_1}^{\delta_1} \int_{\lambda_n - \delta_1}^{\lambda_n + \delta_1} \sup_n |f(z, \mu_n)|^p dx_z dy_z \right. \\
 &\quad \left. + \frac{1}{\pi \delta_2^2} \int_{-\delta_2}^{\delta_2} \int_{\mu_n - \delta_2}^{\mu_n + \delta_2} \sup_n |f(\lambda_n, w)|^p dx_w dy_w \right).
 \end{aligned}$$

Now for $\delta_1 = \frac{\varepsilon_1}{2}$ and $\delta_2 = \frac{\varepsilon_2}{2}$, the intervals $(\lambda_n - \delta_1, \lambda_n + \delta_1)$ are pairwise disjoint, and similarly for the intervals $(\mu_n - \delta_2, \mu_n + \delta_2)$, thus

$$\begin{aligned}
 2 \sum_n |f(\lambda_n, \mu_n)|^p &\leq \frac{1}{\pi \delta_1^2} \int_{-\delta_1}^{\delta_1} \int_{-\infty}^{\infty} \sup_n |f(z, \mu_n)|^p dx_z dy_z \\
 &\quad + \frac{1}{\pi \delta_2^2} \int_{-\delta_2}^{\delta_2} \int_{-\infty}^{\infty} \sup_n |f(\lambda_n, w)|^p dx_w dy_w.
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 2 \sum_n |f(\lambda_n, \mu_n)|^p &\leq \frac{1}{\pi \delta_1^2} \int_{-\delta_1}^{\delta_1} \left(e^{p\tau|y_z|} \int_{-\infty}^{\infty} \sup_n |f(x_z, \mu_n)|^p dx_z \right) dy_z \\
 &\quad + \frac{1}{\pi \delta_2^2} \int_{-\delta_2}^{\delta_2} \left(e^{p\tau|y_w|} \int_{-\infty}^{\infty} \sup_n |f(\lambda_n, x_w)|^p dx_w \right) dy_w \\
 &= B_1 \sup_n \int_{-\infty}^{\infty} |f(x_z, \mu_n)|^p dx_z \\
 &\quad + B_2 \sup_n \int_{-\infty}^{\infty} |f(\lambda_n, x_w)|^p dx_w < \infty,
 \end{aligned}$$

where $B_1 = B_1(p, \tau, \varepsilon_1)$ and $B_2 = B_2(p, \tau, \varepsilon_2)$.

Remark 3.7 In the above proposition, if we replace the conditions (9) by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x_z, x_w)|^p dx_z dx_w < \infty$$

the interior integral is finite everywhere except on a null set. If we use Fubini to change the order of integration, we get a null set for the integral against the second variable. If we know that none of λ_n 's and μ_n 's lie in these null sets, the conclusion still holds.

Theorem 3.8 If $\{\lambda_n\}_{n \in \mathbb{Z}}$ and $\{\mu_n\}_{n \in \mathbb{Z}}$ are separated sequences of real numbers such that $0 \leq \lambda_n \leq 1$ and $0 \leq \mu_n \leq 1$ for each n , then the Gabor system (2) forms a Bessel sequence in $L^2(\mathbb{R}^2)$. If $\sum_n |c_n|^2 < \infty$, then the Gabor expansion

$$\sum_n c_n e^{2\pi i \mu_n \xi - \pi(\xi - \lambda_n)^2}$$

converges in mean to an element of $L^2(\mathbb{R}^2)$.

Proof. If $\phi \in L^2(\mathbb{R}^2)$ then the inner product

$$a_n = \langle \sqrt{2} e^{2\pi i \mu_n \xi - \pi(\xi - \lambda_n)^2}, \phi(\xi) \rangle;$$

is just the value $f(\lambda_n, \mu_n)$ of the entire function

$$f(z, w) = \sqrt{2} \int_{\mathbb{R}^2} \phi(\xi) e^{2\pi i w \xi - \pi(\xi - z)^2} d\xi; \phi(\xi) = \overline{\phi(\xi)},$$

in the Gabor space $\mathcal{G}_g$ and f is of order of growth 2. we have

$$\begin{aligned} & \sup_n \int_{-\infty}^{\infty} |f(x_z, \mu_n)|^p dx_z \\ & \leq 2^{p/2} M^p e^{\pi} \int_{-\infty}^{\infty} \left[\int_{\mathbb{R}} e^{-2\pi(x_{\xi} - x_z)^2} dx_{\xi} \int_{\mathbb{R}} e^{-2\pi(|y_{\xi}| - 1)^2} dy_{\xi} \right]^{p/2} dx_z < \infty, \end{aligned}$$

and similarly

$$\sup_n \int_{-\infty}^{\infty} |f(\lambda_n, x_w)|^p dx_w < \infty.$$

Therefore f satisfies conditions (9) and by Proposition 3.6 we have

$$\sum_n |\langle \sqrt{2} e^{2\pi i \mu_n \xi - \pi(\xi - \lambda_n)^2}, \phi(\xi) \rangle|^2 = \sum_n |a_n|^2 = \sum_n |f(\lambda_n, \mu_n)|^2 < \infty.$$

Thus the Gabor system (2) forms a Bessel sequence in $L^2(\mathbb{R}^2)$. The second part follows from the first by Proposition 2.4.

Paley and Wiener, showed in Theorem XLII of [16] that whenever

$$\lim_{n \rightarrow \pm\infty} (\lambda_{n+1} - \lambda_n) = \infty,$$

for a sequence of real numbers $\{\lambda_n\}$, then the exponentials are weakly independent over an arbitrarily short interval: $\sum a_n e^{i\lambda_n t} = 0$ only when all the a_n are zero. The next lemma states a similar statement for the set of complex exponentials replaced by the system (4). Here l.i.m. is used to show the limit in mean-square in L^2 . The proof is almost identical to that of Paley and Wiener.

Lemma 3.9 Let no a_n vanish, $\sum_{-\infty}^{\infty} |a_n|^2$ converge, and let

$$\cdots < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

such that

$$\lim_{n \rightarrow \pm\infty} (\lambda_{n+1} - \lambda_n) = \infty.$$

Let

$$f(t) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{-N}^N a_n e^{2\pi i \lambda_n t - \pi(t - \lambda_n)^2};$$

over every finite range. If $f(t)$ is equivalent to zero over any interval (a, b) then $f(t)$ is equivalent to zero over every interval, and all the a_n 's vanish.

Now we want to show that if the separation of the λ_n 's is great enough then system (4) is a Riesz-Fischer sequence.

Theorem 3.10 Let $\{\lambda_n\}$ be a sequence of real numbers whose differences are nondecreasing and satisfy

$$\sum \frac{1}{(\lambda_{k+1} - \lambda_k)^2} < \infty.$$

Then the Gabor system (4) is a Riesz-Fischer sequence in $L^2(\mathbb{R})$.

Proof. We adapt the proof of [16, Th. 1]. By the second part of the Proposition 2.4 we have to show that for all finite sequences of scalars $\{c_n\}$ and some constant $m > 0$,

$$m \sum |c_n|^2 \leq \left\| \sum c_n \sqrt{2} e^{2\pi i \lambda_n t - \pi(t - \lambda_n)^2} \right\|^2. \quad (11)$$

Using c to denote an l^2 sequence $\{c_1, c_2, \dots\}$, inequality (11) is the same as

$$\frac{\langle Gc, c \rangle_{l^2}}{\langle c, c \rangle_{l^2}} \geq m,$$

where the l^2 operator G is the Gram matrix of the members of the Gabor system (4). It is to be shown that the eigenvalues of finite subsections of G are bounded away from zero, which in turn follows from these two conditions:

- (1) $Gv = 0$ implies $v = 0$, for every l^2 sequence v .
- (2) $G = I + M$, where M is a compact operator.

The first condition is satisfied by Lemma 3.9. To verify condition (2), observe that the entries of $G = I + M$ are

$$g_{nm} = \sqrt{2} \int_{-\infty}^{\infty} e^{2\pi i(\lambda_n - \lambda_m)t - \pi(t - \lambda_n)^2 - \pi(t - \lambda_m)^2} dt.$$

Now M can be shown to be compact by showing that its Schmidt norm is finite. Since G is symmetric, it suffices to show that

$$\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}^2 < \infty.$$

The sum is bounded above,

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}^2 &= 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \left(\int_{-\infty}^{\infty} e^{2\pi i(\lambda_n - \lambda_m)t - \pi(t - \lambda_n)^2 - \pi(t - \lambda_m)^2} dt \right)^2 \\ &\leq 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \left(\int_{-\infty}^{\infty} e^{2\pi i(\lambda_n - \lambda_m)t} dt \right)^2 \\ &= 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \lim_{A \rightarrow \infty} \left(\int_{-A}^A e^{2\pi i(\lambda_n - \lambda_m)t} dt \right)^2 \\ &\leq \frac{2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{(\lambda_n - \lambda_m)^2} \\ &< \frac{2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{(\lambda_{n+1} - \lambda_n)^2 (m - n)^2}, \end{aligned}$$

where $(\lambda_m - \lambda_n) \leq (\lambda_{n+1} - \lambda_n)(m + n)$ follows from the assumption that differences are nondecreasing. Letting $k = m + n$, one concludes that

$$\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}^2 < C \sum_{n=1}^{\infty} \frac{1}{(\lambda_{n+1} - \lambda_n)^2} < \infty,$$

establishing the theorem.

Theorem 3.11 Let

$$f(z) = \int_{-\infty}^{\infty} \alpha(t) e^{2\pi i z t - \pi(t-z)^2} dt,$$

where $\alpha \in L^2(\mathbb{R})$. If $f(\mu) = 0$ and $g(z) = \frac{z - \lambda}{z - \mu} f(z)$, then there exists a function β in $L^2(\mathbb{R})$ such that

$$g(z) = \int_{-\infty}^{\infty} \beta(t) e^{2\pi i z t - \pi(t-z)^2} dt. \quad (12)$$

Moreover,

$$\beta(t) = \alpha(t) + 2\pi(i+1)(\lambda - \mu) e^{-2\pi i \mu t + \pi(t-\mu)^2} \int_{-\infty}^t \alpha(s) e^{2\pi i \mu s - \pi(s-\mu)^2} ds \quad (13)$$

almost everywhere on $\mathbb{R}$.

Proof. To motivate the proof, let us suppose that $g(z)$ is in fact representable in the form (12), and try to deduce (13). If (12) holds, then

$$\frac{1}{z - \mu} \int_{-\infty}^{\infty} \alpha(t) e^{2\pi i z t - \pi(t-z)^2} dt = \frac{1}{z - \lambda} \int_{-\infty}^{\infty} \beta(t) e^{2\pi i z t - \pi(t-z)^2} dt.$$

The trick in solving for $\beta(t)$ is to transform each of these integrals by first rewriting $e^{2\pi i z t - \pi(t-z)^2}$ as

$$e^{2\pi i z t - \pi(t-z)^2} = e^{2\pi i(z-\mu)t + 2\pi i \mu t - \pi(t-\mu)^2 + \pi(z-\mu)(2t-z-\mu)}.$$

and then integrating by parts. When this is done, the result is

$$\frac{1}{z - \mu} \int_{-\infty}^{\infty} \alpha(t) e^{2\pi i z t - \pi(t-z)^2} dt = \int_{-\infty}^{\infty} \alpha_1(t) e^{2\pi i z t - \pi(t-z)^2} dt,$$

with

$$\alpha_1(t) = -2(i+1)\pi e^{-2\pi i\mu t + \pi(t-\mu)^2} \int_{-\infty}^t \alpha(s) e^{2\pi i\mu s - \pi(s-\mu)^2} ds,$$

and

$$\frac{1}{z-\lambda} \int_{-\infty}^{\infty} \beta(t) e^{2\pi i z t - \pi(t-z)^2} dt = \int_{-\infty}^{\infty} \beta_1(t) e^{2\pi i z t - \pi(t-z)^2} dt,$$

with

$$\beta_1(t) = -2(i+1)\pi e^{-2\pi i\lambda t + \pi(t-\lambda)^2} \int_{-\infty}^t \beta(s) e^{2\pi i\lambda s - \pi(s-\lambda)^2} ds.$$

It follows that $\alpha_1(t) = \beta_1(t)$ almost everywhere on $\mathbb{R}$, and so

$$\begin{aligned} e^{2\pi i(\lambda-\mu)t + \pi(\lambda-\mu)(2t-(\lambda+\mu))} \int_{-\infty}^t \alpha(s) e^{2\pi i\mu s - \pi(s-\mu)^2} ds \\ = \int_{-\infty}^t \beta(s) e^{2\pi i\lambda s - \pi(s-\lambda)^2} ds. \end{aligned}$$

To obtain (13), differentiate both sides of this equation with respect to t . Now simply observe that all of the above steps are reversible, that is $\beta \in L^2(\mathbb{R})$.

Remark 3.12 A similar result holds when f is of the form

$$f(z) = \int_{-\infty}^{\infty} e^{2\pi i z t - \pi(t-z)^2} d\alpha(t),$$

and α is of bounded variation on $\mathbb{R}$, only now

$$g(z) = \int_{-\infty}^{\infty} e^{2\pi i z t - \pi(t-z)^2} d\beta(t),$$

with

$$d\beta(t) = d\alpha(t) + 2\pi i(\lambda - \mu) e^{-2\pi i\mu t + \pi(t-\mu)^2} \int_{-\infty}^t e^{2\pi i\mu s - \pi(s-\mu)^2} d\alpha(s).$$

Corollary 3.13 The completeness of system (4) is unaffected if one λ_n is replaced by another number.

Nowak [18] showed that the deficit of the regular Gabor system generated by $h \in L^2(\mathbb{R}^d)$ and $a, b > 0$ is either zero or infinite if the system is a Bessel sequence in $L^2(\mathbb{R}^d)$. The next result on the deficit of the irregular Gabor system (4) is proved as in [19, Th. 4.6]. Here we give the proof for the sake of completeness.

Theorem 3.14 If $\{\lambda_n\}$ is a separated sequence of real numbers such that

$$\lambda_{n+1} - \lambda_n > 1; (n = 0, \pm 1, \pm 2, \dots)$$

then the Gabor system (4) has infinite deficiency in $L^2(\mathbb{R})$.

Proof. Let N be a fixed but arbitrary positive integer. If K is large enough, then we can replace

$$\lambda_0, \lambda_1, \dots, \lambda_K$$

by

$$\mu_0, \mu_1, \dots, \mu_{K+N+1},$$

so that the resulting sequence, relabeled $\{\mu_n\}$, satisfies

$$\inf_n (\mu_{n+1} - \mu_n) > 1.$$

By Theorem 3.10 there is a function $\varphi \in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \varphi(t) \sqrt[4]{2} e^{-2\pi i \mu_n t - \pi(t - \mu_n)^2} dt = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Thus the system

$$\left\{ \sqrt[4]{2} e^{2\pi i \mu_n t - \pi(t - \mu_n)^2} : n \neq 0 \right\}$$

is incomplete in $L^2(\mathbb{R})$, and we conclude by the above corollary that the deficiency of the system in $L^2(\mathbb{R})$ is at least N .

4. Stability

In this section we study stability of sampling sets in Gabor spaces. Here we let $\mathcal{G}_h$ to be the Gabor space of a Gabor window $h \in L^2(\mathbb{R}^d)$, and Λ be a discrete set in $\mathbb{C}^d$.

Proposition 4.1 [17, Cor. 3.2.3] (Inversion formula for the Gabor transform) Let $h, \gamma \in L^2(\mathbb{R}^d)$ be such that $\langle h, \gamma \rangle \neq 0$, and we consider $V_h f(z) = \langle f, h_z \rangle$, for every $f \in L^2(\mathbb{R}^d)$. Then it is fulfilled that $V_h f \in L^2(\mathbb{C}^d)$. Moreover we have inversion formula given by:

$$f(t) = \frac{1}{\langle h, \gamma \rangle} \int_{\mathbb{C}^d} V_h f(z) M_y T_x \gamma(t) dy dx; \quad z = x + iy \in \mathbb{C}^d.$$

The image of $L^2(\mathbb{R}^d)$ under the Gabor transform with the window h , forms a reproducing kernel Hilbert space

$$\mathcal{G}_h = \{V_h f : f \in L^2(\mathbb{R}^d)\}$$

(a closed subspace of $L^2(\mathbb{C}^d)$) which is called *Gabor space* or *model space*.

The following result is proved for $d = 1$ in [19, Prop. 1.29], the proof given here is based on [17].

Proposition 4.2 The Gabor space $\mathcal{G}_h$ of a Gabor window $h \in L^2(\mathbb{R}^d)$ is a Hilbert subspace of $L^2(\mathbb{C}^d)$ that is characterized for the following reproducing kernel:

$$k_h(z, z_0) = k(z, z_0) = k_{z_0}(z) = \langle h_{z_0}, h_z \rangle.$$

That is, $F \in \mathcal{G}_h$ if and only if

$$F \in L^2(\mathbb{C}^d)$$

and

$$F(z_0) = \int_{\mathbb{C}^d} F(z) \overline{k(z, z_0)} dx dy. \quad (14)$$

Proof. We introduced the inversion formula for the Gabor transform in Proposition 4.1. Without loss of generality, we may assume that $\|h\| = 1$. Now for $z_0 = x_0 + iy_0 \in \mathbb{C}^d$ we have

$$\begin{aligned} V_h f(z_0) &= \int_{\mathbb{R}^d} f(t) \overline{M_{y_0} T_{x_0} h(t)} dt \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{C}^d} V_h f(z) M_y T_x h(t) dx dy \right) \overline{M_{y_0} T_{x_0} h(t)} dt \end{aligned}$$

where $z = x + iy$. Switching the integrals we have

$$\begin{aligned} V_h f(z_0) &= \int_{\mathbb{C}^d} V_h f(z) \left(\int_{\mathbb{R}^d} M_y T_x h(t) \overline{M_{y_0} T_{x_0} h(t)} dt \right) dx dy \\ &= \int_{\mathbb{C}^d} V_h f(z) \left(\int_{\mathbb{R}^d} h_z(t) \overline{h_{z_0}(t)} dt \right) dx dy, \end{aligned}$$

that is,

$$\begin{aligned} F(z_0) &= \int_{\mathbb{C}^d} F(z) \langle h_z, h_{z_0} \rangle dx dy \\ &= \int_{\mathbb{C}^d} F(z) \overline{k(z, z_0)} dx dy. \end{aligned}$$

Therefore k replays all the functions of the space, and as it belongs to this space, it is its reproducing kernel.

For $x, y \in \mathbb{R}^d$ we recall T_x describes a translation by x also called a time shift and M_y a modulation by y also called a frequency shift. So the operators of the form $M_y T_x$ or $T_x M_y$ are known as time-frequency shifts. They satisfy the commutation relations

$$\begin{aligned} T_x M_y f(t) &= (M_y f)(t - x) \\ &= e^{2\pi i y \cdot (t - x)} f(t - x) \\ &= e^{-2\pi i y \cdot x} M_y T_x f(t) \end{aligned}$$

Then we have

$$\begin{aligned} \langle h_z, h_{z_0} \rangle &= \langle M_y T_x h, M_{y_0} T_{x_0} h \rangle \\ &= \langle h, T_{-x} M_{y_0 - y} T_{x_0} h \rangle \\ &= \langle h, e^{2\pi i x \cdot (y_0 - y)} M_{y_0 - y} T_{x_0 - x} h \rangle \\ &= e^{2\pi i x \cdot (y - y_0)} \langle h, M_{y_0 - y} T_{x_0 - x} h \rangle \\ &= e^{2\pi i x \cdot (y - y_0)} \langle h, h_{z_0 - z} \rangle \end{aligned}$$

In terms of $k_h(z) = k_h(z, 0) = \langle h, h_z \rangle$ one has

$$\begin{aligned} k_h(z, z_0) &= \overline{\langle h_z, h_{z_0} \rangle} = \overline{e^{2\pi i x \cdot (y - y_0)} \langle h, h_{z_0 - z} \rangle} \\ &= e^{2\pi i x \cdot (y - y_0)} k_h(z_0 - z) \end{aligned}$$

and hence the reproduction formula (14) takes the form

$$F(z_0) = \int_{\mathbb{C}^d} F(z) e^{2\pi i x \cdot (y - y_0)} k_h(z_0 - z) dx dy$$

Using this notations we can deduce that

$$\begin{aligned} V_h f_{z_0}(z) &= \langle f_{z_0}, h_z \rangle = e^{2\pi i x_0 \cdot (y_0 - y)} \langle f, h_{z - z_0} \rangle \\ &= e^{2\pi i x_0 \cdot (y_0 - y)} V_h f(z - z_0). \end{aligned}$$

In this way, to be consistent with the notation and the definition of the transform, we have to define the translations in $\mathbb{C}^d$ of a function $F \in \mathcal{G}_h$ (or in $L^2(\mathbb{C}^d)$ in a general way) as:

$$F_{z_0}(z) = e^{2\pi i x_0 \cdot (y_0 - y)} F(z - z_0).$$

It is necessary to observe that these translations do not coincide in general with the usual translation of $\mathbb{C}^d$. But if we look at the function, then we have

$$|F_{z_0}(z)| = |F(z - z_0)|.$$

Since in general the function $F(z - z_0)$ can not belong to $\mathcal{G}_h$. Taking this into account we can write the reproduction formula in a bit more compact way

$$F(z_0) = \int_{\mathbb{C}^d} F(z) k_z(z_0) dx dy$$

The Gabor space has certain good continuity properties. More precisely, the functions of the space will be uniformly continuous. For $F \in \mathcal{G}_h$, since F is defined as a definite integral, it is uniformly continuous with respect to the free variable of the integrand, i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|z_1 - z_2| < \delta$ by using triangle inequality we have

$$||F(z_1)| - |F(z_2)|| \leq |F(z_1) - F(z_2)| < \varepsilon$$

Ascensi [19] formalized this idea in the case $d = 1$ with the following result.

Proposition 4.3 [19, Prop. 5.1] Let $\mathcal{G}_h$ be the Gabor space of normalized Gabor window $h \in L^2(\mathbb{R}^d)$. Then, given ε there exists δ such that if $|z_1 - z_2| < \delta$, then for every $F \in \mathcal{G}_h$

$$F(z) = \langle f, h_z \rangle$$

it is fulfilled that

$$||F(z_1)| - |F(z_2)|| < \|F\|\varepsilon = \|f\|\varepsilon.$$

A good description of the Gabor space is most convenient if there are some complete characterizations. The best situation occurs when for some analyzing function the Gabor space is a space of holomorphic functions. The most important example and the only possible one is the Gaussian function, for which the Gabor space can be identified with the Fock space, in which the sampling and interpolation sets are completely characterized [20]. The following assertion is proved for $d = 1$ in [7], we do not know if the same holds in higher dimensions.

Problem 4.4 Consider the Gabor space with a Gabor window $h \in L^2(\mathbb{R}^d)$

$$\mathcal{G}_h = \left\{ F(z) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i y \cdot t} \overline{h(t - x)} dt, f \in L^2(\mathbb{R}^d) \right\}.$$

Then this space is a space of antiholomorphic functions (i.e., $F(x, -y)$ is holomorphic), modulo a multiplication by a weight, if and only if h is a time-frequency translation of the Gaussian function.

As the space in that we will work is formed by continuous functions and it has reproducing kernel, we can sample the functions at any point. Given a set of points of $\mathbb{C}^d$ we can consider, for each $F \in \mathcal{G}_h$, the succession of values that F takes in this set.

Definition 4.5 A discrete set $\Lambda = \{z_j\}_{j \in \mathbb{Z}}$ in $\mathbb{C}^d$ is said to be a *sampling set* for $\mathcal{G}_h$ if there are constants $A, B > 0$ such that

$$A\|F\|^2 \leq \sum_{j \in \mathbb{Z}} |F(z_j)|^2 \leq B\|F\|^2 \quad F \in \mathcal{G}_h.$$

These sets are very important since they correspond with frames. We give some properties of Gabor space and sampling set in the case $d > 1$. The case $d = 1$ was considered by Ascensi and Bruna in [7]. The proofs are essentially the same (with little changes in certain cases). Recall that $\mathcal{G}_h = \{V_h f : f \in L^2(\mathbb{R}^d)\}$ is the Gabor space. Here we assume that $\|h\| = 1$.

Proposition 4.6 If $\Lambda = \{z_j\}_{j \in \mathbb{N}}$ is a sampling set for $\mathcal{G}_h$ then Λ is a relatively separated set.

Proof. The proof is exactly similar to the proof of [7, Prop. 3.1].

Definition 4.7 Given a continuous function F defined in $\mathbb{C}^d$ we define its *local maximal function* as:

$$MF(z) = \sup_{|w-z|<1} |F(w)|$$

Lemma 4.8 Let $\Lambda = \{\lambda_j\}_{j \in \mathbb{Z}}$ be a separated set with separation constant ε . Then

$$\sum_{\lambda \in \Lambda} |k(\lambda)| < \frac{1}{c\varepsilon^{2d}} \|Mk\|_1,$$

where $c = m(B(0, 1))$.

Proof. We suppose without loss of generality that $1 < \varepsilon < 2$. Then using sub-mean-value inequality

$$\begin{aligned} \sum_{\lambda \in \Lambda} |k(\lambda)| &\leq \sum_{\lambda \in \Lambda} \frac{1}{|B(\lambda, \varepsilon)|} \int_{B(\lambda, \varepsilon)} |k(z)| \, dm(z) \\ &\leq \sum_{\lambda \in \Lambda} \frac{1}{|B(\lambda, \varepsilon)|} \int_{B(\lambda, \varepsilon)} Mk(z) \, dm(z) \\ &\leq \sum_{\lambda \in \Lambda} \frac{1}{|B(\lambda, \varepsilon)|} \int_{\mathbb{C}^d} Mk(z) \, dm(z). \end{aligned}$$

Since by hypothesis those balls are disjoint, then

$$\sum_{\lambda \in \Lambda} |k(\lambda)| < \frac{1}{c\varepsilon^{2d}} \|Mk\|_1.$$

Proposition 4.9 If Λ is a separated set, there exists $B > 0$ such that

$$\sum_{\lambda \in \Lambda} |F(\lambda)|^2 \leq B \|F\|^2; \quad F \in \mathcal{G}_h.$$

Proof. Calculating directly we have that

$$\begin{aligned} \sum_{\lambda \in \Lambda} |F(\lambda)|^2 &= \sum_{\lambda \in \Lambda} \left| \int_{\mathbb{C}^d} F(z) \overline{k_\lambda(z)} dm(z) \right|^2 \\ &\leq \sum_{\lambda \in \Lambda} \left(\int_{\mathbb{C}^d} |F(z)|^2 |k_\lambda(z)| dm(z) \right) \times \left(\int_{\mathbb{C}^d} |k_\lambda(z)| dm(z) \right) \\ &= \int_{\mathbb{C}^d} |F(z)|^2 \sum_{\lambda \in \Lambda} |k(z - \lambda)| dm(z) \times \int_{\mathbb{C}^d} |k(z)| dm(z). \end{aligned}$$

Here $\int_{\mathbb{C}^d} |k(z)| dm(z) = m < \infty$ because the kernel is integrable and also

$$\sum_{\lambda \in \Lambda} |k(z - \lambda)| = \sum_{\gamma \in (z - \Lambda)} |k(\gamma)|$$

is bounded independently of z , and since $z - \Lambda$ has the same separation constant as Λ we can apply Lemma 4.8. Then

$$\begin{aligned} \sum_{\lambda \in \Lambda} |F(\lambda)|^2 &\leq \int_{\mathbb{C}^d} |F(z)|^2 \sum_{\gamma \in (z - \Lambda)} |k(\gamma)| dm(z) \int_{\mathbb{C}^d} |k(z)| dm(z) \\ &\leq B \|F\|^2, \end{aligned}$$

where $B = \frac{m}{c\varepsilon^{2d}} \|Mk\|_1$ and ε is the separation constant of Λ .

Next, we want to know when the Gabor system $G(h, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. First we observe that $V_h f(z) = \langle f, h_z \rangle$ and $\|V_h f\| = \|f\|$ (if $\|h\| = 1$). As we have bijective correspondence between $\mathcal{G}_h$ and $L^2(\mathbb{R}^d)$ by the Gabor transform, we can write

$$\sum_{\lambda \in \Lambda} |F(\lambda)|^2 = \sum_{\lambda \in \Lambda} |V_h f(\lambda)|^2 = \sum_{\lambda \in \Lambda} |\langle f, h_\lambda \rangle|^2$$

for every $f \in L^2(\mathbb{R}^d)$ or for every $V_h f \in \mathcal{G}_h$. Therefore the frame condition and that of sampling set are equivalent. We conclude that: given a discrete set $\Lambda \subset \mathbb{C}^d$ and a Gabor window $h \in L^2(\mathbb{R}^d)$, $G(h, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if Λ is a sampling set for $\mathcal{G}_h$.

Let $\Sigma_\alpha = \{h \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) : h' + zh \in L^2(\mathbb{R}) \text{ and } \|h' + zh\|_2 \leq \alpha \|h\|_2 \text{ for } z(t) = t, t \in \mathbb{R}\}$. It is clear that $C_c^\infty(\mathbb{R}) \subseteq \Sigma_\alpha$ and so Σ_α is a non empty set.

Lemma 4.10 Let $h \in \Sigma_\alpha$ and let $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ be sequences of scalars and suppose that there exist positive numbers B and L such that

$$\sum_n |F(\lambda_n)|^2 \leq B \|F\|^2; (F \in \mathcal{G}_h)$$

and

$$|\mu_n - \lambda_n| \leq L, (n = 1, 2, 3, \dots).$$

Then for every $F \in \mathcal{G}_h$

$$\sum_n |F(\lambda_n) - F(\mu_n)|^2 \leq B(e^{\alpha L} - 1)^2 \|F\|^2.$$

Proof. Let F be an element of $\mathcal{G}_h$. By expanding F in a Taylor series about λ_n , we find that

$$F(\mu_n) - F(\lambda_n) = \sum_{k=1}^{\infty} \frac{F^{(k)}(\lambda_n)}{k!} (\mu_n - \lambda_n)^k (n = 1, 2, 3, \dots).$$

If ρ is an arbitrary positive number, then by multiplying and dividing the summand by ρ^k we find also that

$$|F(\mu_n) - F(\lambda_n)|^2 \leq \sum_{k=1}^{\infty} \frac{|F^{(k)}(\lambda_n)|^2}{\rho^{2k} k!} \sum_{k=1}^{\infty} \frac{\rho^{2k} |\mu_n - \lambda_n|^{2k}}{k!}$$

Since $\mathcal{G}_h$ is closed under differentiation and $\|F'\|_2 \leq \alpha \|F\|_2$ it follows that

$$\begin{aligned} \sum_n |F^{(k)}(\lambda_n)|^2 &\leq B \|F^{(k)}\|^2 \\ &\leq B \alpha^{2k} \|F\|^2, \quad k = 1, 2, 3, \dots \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \sum_n |F(\lambda_n) - F(\mu_n)|^2 &\leq B \|F\|^2 \sum_{k=1}^{\infty} \frac{\alpha^{2k}}{\rho^{2k} k!} \sum_{k=1}^{\infty} \frac{(\rho L)^{2k}}{k!} \\ &= B \|F\|^2 (e^{\frac{\alpha^2}{\rho^2}} - 1) (e^{\rho^2 L^2} - 1) \end{aligned}$$

since $|\mu_n - \lambda_n| \leq L$.

Now by choosing $\rho = \sqrt{\frac{\alpha}{L}}$ we get

$$\sum_n |F(\lambda_n) - F(\mu_n)|^2 \leq B(e^{\alpha L} - 1)^2 \|F\|^2.$$

Theorem 4.11 If $\{\lambda_n\}_{n \in \mathbb{N}}$ is a sampling set for $\mathcal{G}_h$ ($h \in \Sigma_\alpha$) then there exists positive constant L such that if $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies $|\lambda_n - \mu_n| \leq L$ for all n , then $\{\mu_n\}_{n \in \mathbb{N}}$ is also sampling set.

Proof. Since $\{\lambda_n\}_{n \in \mathbb{N}}$ is a sampling set for $\mathcal{G}_h$, then there exist positive constants A and B such that

$$A\|F\|^2 \leq \sum_n |F(\lambda_n)|^2 \leq B\|F\|^2$$

for every function F belonging to the Gabor space $\mathcal{G}_h$. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be complex scalars for which $|\lambda_n - \mu_n| \leq L$ ($n = 1, 2, 3, \dots$). It is to be shown that if L is sufficiently small, then similar inequalities hold for the μ_n 's.

By virtue of the previous lemma, for every $F \in \mathcal{G}_h$,

$$\sum_n |F(\lambda_n) - F(\mu_n)|^2 \leq B(e^{\alpha L} - 1)^2 \|F\|^2$$

and since $\|F\|^2 \leq \frac{1}{A} \sum_n |F(\lambda_n)|^2$ to have

$$\sum_n |F(\lambda_n) - F(\mu_n)|^2 \leq C \sum_n |F(\lambda_n)|^2$$

where $C = \frac{B}{A}(e^{\alpha L} - 1)^2$. Applying Minkowski's inequality, we find that

$$\left| \sqrt{\sum |F(\lambda_n)|^2} - \sqrt{\sum |F(\mu_n)|^2} \right| \leq \sqrt{C \sum |F(\lambda_n)|^2},$$

and hence

$$\sqrt{A}(1 - \sqrt{C})\|F\| \leq \sqrt{\sum |F(\mu_n)|^2} \leq \sqrt{B}(1 + \sqrt{C})\|F\|$$

for every F . Since C is less than 1 if L is sufficiently small, $\{\mu_n\}_{n \in \mathbb{N}}$ is a sampling set for $\mathcal{G}_h$.

Problem 4.12 Let $h, k \in L^2(\mathbb{R})$. It would be desirable to show that there exists $\varepsilon > 0$ such that if $\|k - h\| < \varepsilon$ and $\{\lambda_n\}$ is a sampling set for $\mathcal{G}_h$ then $\{\lambda_n\}$ is a sampling set for $\mathcal{G}_k$. If one can show this and $h \in \Sigma_\alpha$, then for each $k \in L^2(\mathbb{R})$ with $\|k - h\| < \varepsilon$ the stability result of Theorem 4.11 holds for $\mathcal{G}_k$ as well. Now since Σ_α is norm dense in $L^2(\mathbb{R})$, one could conclude that Theorem 4.11 holds for each $h \in L^2(\mathbb{R})$. At present we are not able to prove that a "small" perturbation does not effect sampling set.

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