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Structure Based Classification and Kinematic Analysis of Six-Joint Industrial Robotic Manipulators

Tuna Balkan, M. Kemal Özgören and M. A. Sahir Arıkan

1. Introduction

In this chapter, a complete set of compact, structure based generalized kinematic equations for six-joint industrial robotic manipulators are presented together with their sample solutions. Industrial robots are classified according to their kinematic structures, and their forward kinematic equations are derived according to this classification. The purpose of this classification is to obtain simplified forward kinematic equations considering the specific features of the classified manipulators and thus facilitate their inverse kinematic solutions. For the classification, one hundred industrial robots are surveyed. The robots are first classified into kinematic main groups and then into subgroups under each main group. The main groups are based on the end-effector rotation matrices and characterized by the twist angles. On the other hand, the subgroups are based on the wrist point positions and characterized by the link lengths and offsets. The reason for preferring the wrist point rather than the tip point in this classification is that, the wrist point and rotation matrix combination contain the same amount of information as the tip point and rotation matrix combination about the kinematic features of a manipulator, and the wrist point coordinates are simpler to express in terms of the joint variables. After obtaining the forward kinematic equations (i.e. the main group rotation matrix equations and the subgroup wrist point equations), they are simplified in order to obtain compact kinematic equations using the numerous properties of the exponential rotation matrices (Özgören, 1987-2002). The usage of the exponential rotation matrices provided important advantages so that simplifications are carried out in a systematic manner with a small number of symbolic matrix manipulations. Subsequently, an inverse kinematic solution approach applicable to the six-joint industrial robotic manipulators is introduced. The approach is based on the kinematic classification of the industrial robotic manipulators as explained above. In the inverse kinematic solutions of the surveyed industrial robots, most of the simplified compact equations can be solved analytically and the remaining few of them can be solved semi-analytically through a numerical solution of a single univariate equation. The semi-analytical method

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is named as the *Parametrized Joint Variable (PJV)* method. In these solutions, the singularities and the multiple configurations of the manipulators indicated by sign options can be determined easily. Using these solutions, the inverse kinematics can also be computerized by means of short and fast algorithms. Owing to the properties of the exponential rotation matrices, the derived simple and compact equations are easy to implement for computer programming of the inverse kinematic solutions. Besides, the singularities and the multiple configurations together with the working space limitations of the manipulator can be detected readily before the programming stage, which enables the programmer to take the necessary actions while developing the program. Thus, during the inverse kinematic solution, it becomes possible to control the motion of the manipulator in the desired configuration by selecting the sign options properly. In this approach, although the derived equations are manipulator dependent, for a newly encountered manipulator or for a manipulator to be newly designed, there will be no need to follow the complete derivation procedure starting from the beginning for most of the cases; only a few modifications will be sufficient. These modifications can be addition or deletion of a term, or just changing simply a subscript of a link length or offset. Even if the manipulator under consideration happens to generate a new main group, the equations can still be derived without much difficulty by using the procedure described here, since the approach is systematic and its starting point is the application of the Denavit-Hartenberg convention by identifying the twist angles and the other kinematic parameters. In this context, see (Özgören, 2002) for an exhaustive study that covers all kinds of six-joint serial manipulators. The presented method is applicable not only for the serial manipulators but also for the hybrid manipulators with closed chains. This is demonstrated by applying the method to an ABB IRB2000 industrial robot, which has a four-bar mechanism for the actuation of its third link. Thus, alongside with the serial manipulators, this particular hybrid manipulator also appears in this chapter with its compact forward kinematic equations and their inversion for the joint variables. Finally, the chapter is closed by giving the solutions to some typical trigonometric equations encountered during the inverse kinematic solutions.
For the solution of inverse kinematics problem, forward kinematic equations are required. There are three methods for inverse kinematic solution; namely, analytical, semi-analytical, and fully numerical. Presently, analytical methods can be used only for certain manipulators with specific kinematic parameter combinations such as PUMA 560. For a general case where the manipulator does not have specific kinematic parameter combinations, it becomes impossible to obtain analytical solutions. So, either semi-analytical or fully numerical methods have been developed. Since the present general semi-analytical methods are rather cumbersome to use (Raghavan & Roth, 1993; Manseur & Doty, 1996), fully numerical methods are mostly preferred. However, if the forward kinematic equations can be simplified, it becomes feasible to use semi-

analytical and even analytical methods for a large number of present industrial robot types. On the other hand, although the fully numerical methods can detect the singularities by checking the determinant of the Jacobian matrix, they have to do this continuously during the solution, which slows down the process. However, the type of the singularity may not be distinguished. Also, in case of multiple solutions, the desired configurations of the manipulator can not be specified during the solution. Thus, in order to clarify the singularities and the multiple configurations, it becomes necessary to make use of semi-analytical or analytical methods. Furthermore, the analytical or semi-analytical methods would be of practical use if they lead to compact and simple equations to facilitate the detection of singularities and multiple configurations. The methodology presented in this chapter provides such simple and compact equations by making use of various properties of the exponential rotation matrices, and the simplification tools derived by using these properties (Özgören, 1987-2002). Since different manipulator types with different kinematic parameters lead to different sets of simplified equations, it becomes necessary to classify the industrial robotic manipulators for a systematic treatment. For such a classification, one hundred currently used industrial robots are surveyed (Balkan et al., 1999, 2001).

The kinematics of robotic manipulators can be dealt with more effectively and faster by perceiving their particular properties rather than resorting to generality (Hunt, 1986). After the classification, it is found that most of the recent, well-known robotic manipulators are within a specific main group, which means that, instead of general solutions and approaches, manipulator dependent solutions and approaches that will lead to easy specific solutions are more reasonable. The usage of exponential rotation matrices provide important advantages so that simplifications can be carried out in a systematic manner with a small number of symbolic matrix manipulations and the resulting kinematic equations become much simpler especially when the twist angles are either 0° or ± 90°, which is the case with the common industrial robots.

For serial manipulators, the forward kinematics problem, that is, determination of the end-effector position and orientation in the Cartesian space for given joint variables, can easily be solved in closed-form. Unfortunately, the inverse kinematics problem of determining each joint variable by using the Cartesian space data does not guarantee a closed-form solution. If a closed-form solution can not be obtained, then there are different types of approaches for the solution of this problem. The most common one is to use a completely numerical solution technique such as the Newton-Raphson algorithm. Another frequently used numerical method is the "resolved motion rate control" which uses the inverse of the Jacobian matrix to determine the rates of the joint variables and then integrates them numerically with a suitable method (Wu & Paul, 1982). Runge-Kutta of order four is a common approach used for this purpose. As an analytical approach, it is possible to convert the forward kine-

matic equations into a set of polynomial equations. Then, they can be reduced to a high-order single polynomial equation through some complicated algebraic manipulations. Finally, the resulting high-order equation is solved numerically. However, requiring a lot of polynomial manipulations, this approach is quite cumbersome (Wampler & Morgan, 1991; Raghavan & Roth, 1993).
On the other hand, the approach presented in this chapter aims at obtaining the inverse kinematic solutions analytically by manipulating the trigonometric equations directly without converting them necessarily into polynomial equations. In a case, where an analytical solution cannot be obtained this way, then a semi-analytical solution is aimed at by using the method described below.
As explained before, the PJV method is a semi-analytical inverse kinematics solution method which can be applied to different kinematic classes of six-joint manipulators which have no closed-form solutions. In most of the cases, it is based on choosing one of the joint variables as a parameter and determining the remaining joint variables analytically in terms of this parametrized joint variable. Parametrizing a suitable joint variable leads to a single univariate equation in terms of the parametrized joint variable only. Then, this equation is solved using a simple numerical technique and as the final step remaining five joint variables are easily computed by substituting the parametrized joint variable in their analytical expressions. However, for certain kinematic structures and kinematic parameters two and even three equations in three unknowns may arise (Özgören, 2002). Any initial value is suitable for the solution and computational time is very small even for an initial condition far from the solution. The PJV method can also handle the singular configurations and multiple solutions. However, it is manipulator dependent and equations are different for different classes of manipulators. PJV works well also for non-spherical wrists with any structural kinematic parameter combination.
In this chapter, four different subgroups are selected for the demonstration of the inverse kinematic solution method. Two of these subgroups are examples to closed-form and semi-analytic inverse kinematic solutions for the most frequently seen kinematic structures among the industrial robots surveyed in (Balkan et al., 1999, 2001). Since the manipulators in these two subgroups have revolute joints only, the inverse kinematic solution of subgroup 4.4 which includes Unimate 4000 industrial robot is also given to demonstrate the method on manipulators with prismatic joints. The inverse kinematic solution for this class of manipulators happens to be either closed-form or needs the PJV method depending on the selection of one of its parameters. In addition, the inverse kinematic solution for ABB IRB2000 industrial robot, which has a closed chain, is obtained to show the applicability of the method to such manipulators.

2. Kinematic Equations for Six-Joint Robots

In the derivation of the kinematic equations for six-joint manipulators, Denavit-Hartenberg (D-H) convention is used as shown in Figure 1 (Denavit & Hartenberg, 1955), with notation adopted from (Özgören, 2002).

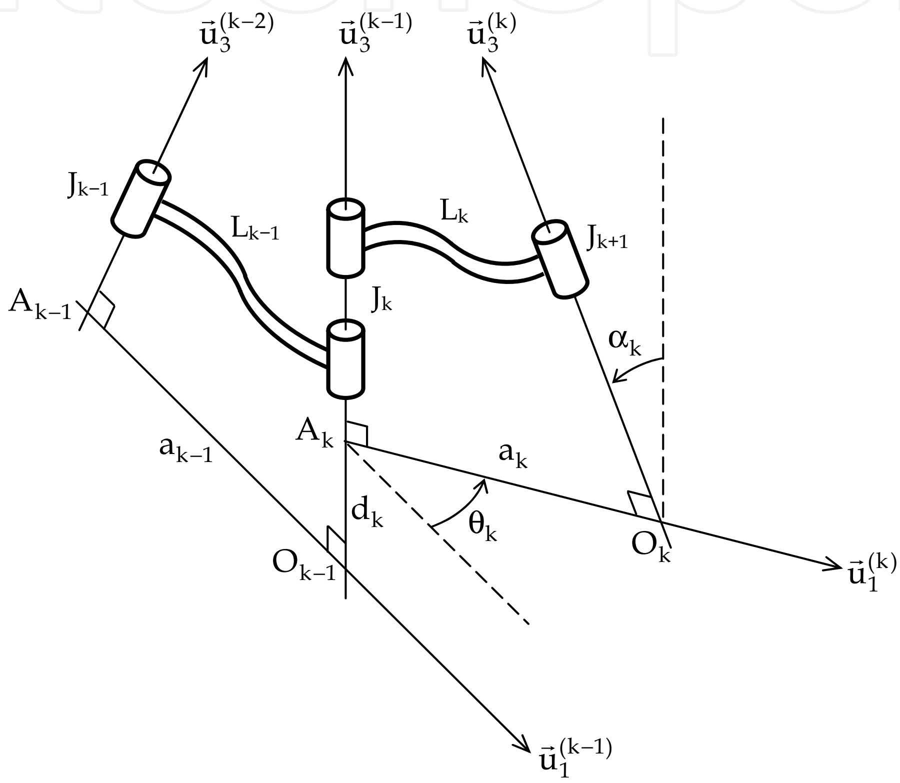

Figure 1. D-H Convention and Related Notation

The symbols in Fig. 1 are explained below.

J_k: Joint k.
L_k: Link k.
O_k: Origin of the reference frame F_k attached to L_k.
A_k: Auxiliary point between L_{k-1} and L_k.
$\vec{u}_i^{(k)}$: i^{th} unit basis vector of F_k ; $i = 1, 2, 3$.
a_k: Effective length A_kO_k of L_k along $\vec{u}_1^{\,k}$.
d_k: Distance $O_{k-1}A_k$ of L_k from L_{k-1} along $\vec{u}_3^{(k-1)}$. It is a constant parameter, called offset, if J_k is revolute. It is the k^{th} joint variable if J_k is prismatic. It is then denoted as s_k.

θ_k: Rotation angle of L_k with respect to L_{k-1} about $\vec{u}_3^{(k-1)}$. It is the k^{th} joint variable if J_k is revolute. If J_k is prismatic, it is a constant parameter which is either 0° or ±90° for common industrial robot manipulators.

α_k: Twist angle of J_{k+1} with respect to J_k about $\vec{u}_1^{(k)}$. For common industrial robot manipulators, it is either 0° or ±90°.

Among the industrial robots surveyed in this chapter, there is no industrial robot whose last joint is prismatic. Thus, the wrist point, which is defined as the origin of F_6 is chosen to be coincident with the origin of F_5. That is, $O_5 = O_6$. The other features of the hand frame F_6 are defined as described below.

$$\vec{u}_3^{(6)} = \vec{u}_3^{(5)} \tag{1}$$

$$a_6 = 0,\ d_6 = 0,\ \alpha_6 = 0 \tag{2}$$

The end-effector is fixed in F_6 and assuming that its tip point P is on the axis along the approach vector $\vec{u}_3^{(6)}$, its location can be described as $d_p = O_6P$.
The relationship between the representations of the same vector in two different frames can be written as shown below.

$$\overline{n}^{(a)} = \hat{C}^{(a,b)}\ \overline{n}^{(b)} \tag{3}$$

Here, $\overline{n}^{(a)}$, $\overline{n}^{(b)}$ are the column representations of the vector $\vec{n}$ in the frames F_a and F_b while $\hat{C}^{(a,b)}$ is the transformation matrix between these two frames.
In order to make the kinematic features of the manipulators directly visible and to make the due simplifications easily, the hand-to-base transformation matrix $\hat{C}^{(0,6)}$ and the wrist point position vector $\overline{r}^{(0)}$, or the tip point position vector $\overline{p}^{(0)}$ are expressed separately, rather than concealing the kinematic features into the overcompact homogeneous transformation matrices, which are also unsuitable for symbolic manipulations. The wrist and tip point position vectors are related as follows:

$$\overline{p}^{(0)} = \overline{r}^{(0)} + d_p\hat{C}^{(0,6)}\overline{u}_3 \tag{4}$$

Here, $\overline{r}^{(0)}$ and $\overline{p}^{(0)}$ are the column matrix representations of the position vectors in the base frame F_0 whereas $\overline{u}_3$ is the column matrix representation of the approach vector in the hand frame F_6.
The overall relative displacement from F_{k-1} to F_k consists of two rotations and two translations, which are sequenced as a translation of s_k along $\vec{u}_3^{(k-1)}$, a rotation of θ_k about $\vec{u}_3^{(k-1)}$, a translation of a_k along $\vec{u}_1^{(k)}$, and a rotation of α_k about $\vec{u}_1^{(k)}$.

Using the link-to-link rotational transformation matrices, $\hat{C}^{(0,6)}$ can be formulated as follows:

$$\hat{C}^{(0,6)} = \hat{C}^{(0,1)}\ \hat{C}^{(1,2)}\ \hat{C}^{(2,3)}\ \hat{C}^{(3,4)}\ \hat{C}^{(4,5)}\ \hat{C}^{(5,6)} \tag{5}$$

According to the D-H convention, the transformation matrix between two successive link frames can be expressed using exponential rotation matrices (Özgören, 1987-2002). That is,

$$\hat{C}^{(k-1,k)} = e^{\tilde{u}_3\theta_k}\ e^{\tilde{u}_1\alpha_k} \tag{6}$$

On the other hand, assuming that frame F_b is obtained by rotating frame F_a about an axis described by a unit vector $\vec{n}$ through an angle θ, the matrix $\hat{C}^{(a,b)}$ is given as an exponential rotation matrix by the following equation (Özgören, 1987-2002):

$$\hat{C}^{(a,b)} = e^{\tilde{n}\theta} = \hat{I}\cos\theta + \tilde{n}\sin\theta + \bar{n}\,\bar{n}^T\ (1-\cos\theta) \tag{7}$$

Here, $\hat{I}$ is the identity matrix and $\tilde{n}$ is the skew symmetric matrix generated from the column matrix $\bar{n} = \bar{n}^{(a)}$. This generation can be described as follows.

$$\bar{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \rightarrow \quad \tilde{n} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \tag{8}$$

Furthermore, if $\vec{n} = \vec{u}_k^{(a)}$ where $\vec{u}_k^{(a)}$ is the k^{th} basis vector of the frame F_a, then $\bar{n} = \bar{u}_k$ and

$$\hat{C}^{(a,b)} = e^{\tilde{u}_k\theta} \tag{9}$$

Here,

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

Using Equation (6), Equation (5) can be written as

$$\hat{C} = \hat{C}^{(0,6)} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_1\alpha_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\alpha_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_1\alpha_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_1\alpha_5} e^{\tilde{u}_3\theta_6} \tag{11}$$

On the other hand, the wrist point position vector can be expressed as

$$\vec{r} = \vec{r}_{01} + \vec{r}_{12} + \vec{r}_{23} + \vec{r}_{34} + \vec{r}_{45} + \vec{r}_{56} \tag{12}$$

Here, $\vec{r}_{ij}$ is the vector from the origin O_i to the origin O_j.
Using the column matrix representations of the vectors in the base frame F_0, Equation (12) can be written as

$$\begin{aligned}\overline{r} = \overline{r}^{(0)} &= d_1\overline{u}_3 + a_1\hat{C}^{(0,1)}\overline{u}_1 + d_2\hat{C}^{(0,1)}\overline{u}_3 + a_2\hat{C}^{(0,2)}\overline{u}_1 + d_3\hat{C}^{(0,2)}\overline{u}_3 + a_3\hat{C}^{(0,3)}\overline{u}_1 \\ &+ d_4\hat{C}^{(0,3)}\overline{u}_3 + a_4\hat{C}^{(0,4)}\overline{u}_1 + d_5\hat{C}^{(0,4)}\overline{u}_3 + a_5\hat{C}^{(0,5)}\overline{u}_1\end{aligned} \tag{13}$$

Substitution of the rotational transformation matrices and manipulations using the exponential rotation matrix simplification tool E.2 (Appendix A) result in the following simplified wrist point equation in its most general form.

$$\begin{aligned}\overline{r} &= d_1\overline{u}_3 + a_1e^{\tilde{u}_3\theta_1}\overline{u}_1 + d_2e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}\overline{u}_3 + a_2e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}\overline{u}_1 \\ &+ d_3e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2}\overline{u}_3 + a_3e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2}e^{\tilde{u}_3\theta_3}\overline{u}_1 \\ &+ d_4e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2}e^{\tilde{u}_3\theta_3}e^{\tilde{u}_1\alpha_3}\overline{u}_3 \\ &+ a_4e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2}e^{\tilde{u}_3\theta_3}e^{\tilde{u}_1\alpha_3}e^{\tilde{u}_3\theta_4}\overline{u}_1 \\ &+ d_5e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2}e^{\tilde{u}_3\theta_3}e^{\tilde{u}_1\alpha_3}e^{\tilde{u}_3\theta_4}e^{\tilde{u}_1\alpha_4}\overline{u}_3 \\ &+ a_5e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\alpha_1}e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\alpha_2}e^{\tilde{u}_3\theta_3}e^{\tilde{u}_1\alpha_3}e^{\tilde{u}_3\theta_4}e^{\tilde{u}_1\alpha_4}e^{\tilde{u}_3\theta_5}\overline{u}_1\end{aligned} \tag{14}$$

3. Classification of Six-Joint Industrial Robotic Manipulators

As noticed in Equations (11) and (14), the general $\overline{r}$ expression contains five joint variables and the general $\hat{C}$ expression includes all of the angular joint variables. On the other hand, it is an observed fact that in the six-joint industrial robots, many of the structural length parameters (a_k and d_k) are zero (Balkan et al., 1999, 2001). Due to this reason, there is no need to handle the inverse kinematics problem in a general manner. Instead, the zero values of a_k and d_k of these robots can be used to achieve further simplifications in Equations (11) and (14). In order to categorize and handle the simplified equations in a systematic manner, the industrial robots are grouped using a two step classification scheme according to their structural parameters a_k, α_k, and d_k for revolute joints or θ_k for prismatic joints. The primary classification is based on the twist angles (α_k) and it gives the *main groups*. Whereas, the secondary classification is based on the other structural parameters (a_k and d_k or θ_k) and it gives the *subgroups* under each main group.

In the main groups, the simplified $\overline{r}$ and $\hat{C}$ expressions are obtained using the fact that the twist angles are either 0° or $\pm 90^\circ$. The $\hat{C}$ expression for each main group is the same, because the rotation angles (θ_k) are not yet distinguished at this level whether they are constant or not. At the level of the subgroups, the values of the twist and constant rotation angles are substituted into the $\overline{r}$ and $\hat{C}$ expressions, together with the other parameters. Then, the properties of the exponential rotation matrices are used in order to obtain simplified equations with reduced number of terms, which can be used with convenience for the inverse kinematic solutions. The main groups with their twist angles and the number of robots in each main group are given in Table 1 considering the industrial robots surveyed here. The subgroups are used for finer classification using the other structural parameters. For the manipulators in this classification, the $\overline{r}$ expressions are simplified to a large extent especially when zeros are substituted for the vanishing structural parameters.

Main Group	α_1	α_2	α_3	α_4	α_5	α_6	Number of Robots
1	− 90°	0°	90°	− 90°	90°	0°	73
2	− 90°	0°	0°	90°	− 90°	0°	12
3	− 90°	90°	0°	− 90°	90°	0°	5
4	− 90°	90°	− 90°	90°	− 90°	0°	4
5	0°	0°	0°	− 90°	90°	0°	1
6	− 90°	90°	0°	0°	− 90°	0°	1
7	0°	− 90°	0°	90°	− 90°	0°	1
8	0°	− 90°	90°	− 90°	90°	0°	2
9	− 90°	0°	90°	0°	− 90°	0°	1

Table 1. Main Groups of Surveyed Six-Joint Industrial Robots

3.1 Main Group Equations

Substituting all the nine sets of the twist angle values given in Table 1 into Equations (11) and (14), the main group equations are obtained. The terms of $\overline{r}$ involving a_k and d_k are denoted as $T(a_k)$ and $T(d_k)$ as described below.

$$T(a_k) = a_k\, e\,(\theta_1, \dots, \theta_k, \alpha_1, \dots, \alpha_k)\,\overline{u}_1 \tag{15}$$

$$T(d_k) = d_k\, e\,(\theta_1, \dots, \theta_{k-1}, \alpha_1, \dots, \alpha_{k-1})\,\overline{u}_3 \tag{16}$$

Here, e stands for a product of exponential rotation matrices associated with the indicated angular arguments as exemplified by the following terms.

$$T(a_2) = a_2 e(\theta_1, \theta_2, \alpha_1)\overline{u}_1 = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1} e^{\tilde{u}_3\theta_2}\overline{u}_1 \tag{17}$$

$$T(d_2) = d_2 e(\theta_1, \alpha_1)\bar{u}_3 = d_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\alpha_1}\bar{u}_3 \tag{18}$$

Here, the derivation of equations is given only for the main group 1, but the equations of the other groups can be obtained in a similar systematic manner by applying the *exponential rotation matrix simplification tools* given in Appendix A. The numbers (E.#) of the employed tools of Appendix A during the derivation of the $\hat{C}$ matrices and the terms (a_k) and (d_k) are shown in Table 2.

Main Group	$\hat{C}$	d_2	a_2	d_3	a_3	d_4	a_4	d_5	a_5
1	4, 6	8	10, 2, 6	2, 8	4, 6	4, 6	4, 6	4, 6, 8	4, 10, 2, 6
2	4, 6	8	10, 2, 6	2, 8	4, 10, 2, 6	4, 2, 8	4, 10, 2 , 6	4, 6	4, 6
3	4, 6	8	10, 2, 6	6	6	2, 6	4, 6	4, 6, 8	4, 10, 2, 6
4	4, 6	8	10, 2, 6	6	6	6, 8	10, 2, 6	6	6
5	4, 6	2	4	4, 2	4	4, 2	4	4, 8	4, 10, 2, 6
6	4, 6	8	10, 2, 6	6	6	2, 6	4, 6	4, 2, 6	4, 6
7	4, 6	2	4	4, 8	4, 10, 2, 6	2, 4, 8	4, 10, 2, 6	4, 6	4, 6
8	4, 6	2	4	4, 8	4, 10, 2, 6	4, 6	4, 6	4, 6, 8	4, 10, 2, 6
9	4, 6	8	10, 2, 6	2, 8	4, 10, 2, 6	4, 6	4, 6	2, 4, 6	4, 6

Table 2. Exponential Rotation Matrix Simplification Tool Numbers (E.#) Applied for Derivation of $\hat{C}$ Matrices and Terms (a_k) and (d_k) in Main Group Equations

Equations of Main Group 1

Let $\bar{\alpha}$ denote the set of twist angles. For the main group 1, $\bar{\alpha}$ is

$$\bar{\alpha} = [-90°,\ 0°,\ 90°,\ -90°,\ 90°\ ,0°]^T . \tag{19}$$

Substituting $\bar{\alpha}$ into the general $\hat{C}$ equation results in the following equation.

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_6} \tag{20}$$

Using the exponential rotation matrix simplification tools E.4 and E.6, the rotation matrix for the main group 1, i.e. $\hat{C}_1$, can be obtained as follows.

$$\hat{C}_1 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{21}$$

Here, $\theta_{jk} = \theta_j + \theta_k$ is used as a general way to denote joint angle combinations.

Substituting $\bar{\alpha}$ into the general $\bar{r}$ expression results in the following equation.

$$\begin{aligned}\bar{r} &= d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_2 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2}\bar{u}_3 + a_2 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2}\bar{u}_1 \\ &+ d_3 e^{\tilde{u}_3\theta_1}\, e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2}\bar{u}_3 + a_3 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_1 \\ &+ d_4 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2}\bar{u}_3 + a_4 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4}\bar{u}_1 \\ &+ d_5 e^{\tilde{u}_3\theta_1}\, e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{-\tilde{u}_1\pi/2}\bar{u}_3 \\ &+ a_5 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_4} e^{-\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_5}\bar{u}_1\end{aligned} \tag{22}$$

The simplifications can be made for the terms $T(a_k)$ and $T(d_k)$ of Equation (22) as shown in Table 3 using the indicated simplification tools given in Appendix A.

E.8	$T(d_2) = d_2 e^{\tilde{u}_3\theta_1}\ \bar{u}_2$
E.10, E.6 and E.2	$T(a_2) = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1$
E.2 and E.8	$T(d_3) = d_3 e^{\tilde{u}_3\theta_1}\bar{u}_2$
E.4 and E.6	$T(a_3) = a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_1$
E.4 and E.6	$T(d_4) = d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_3$
E.4 and E.6	$T(a_4) = a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\bar{u}_1$
E.4, E.6 and E.8	$T(d_5) = d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\bar{u}_2$
E.4, E.6 and E.10	$T(a_5) = a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5}\bar{u}_1$

Table 3. Simplifications of the terms $T(a_k)$ and $T(d_k)$ in Equation (22)

Replacing the terms $T(a_k)$ and $T(d_k)$ in Equation (22) with their simplified forms given in Table 3, the wrist point location for the main group 1, i.e. $\bar{r}_1$, can be obtained as follows:

$$\begin{aligned}\bar{r}_1 &= d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + d_3 e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_1 \\ &+ d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_3 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\bar{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\bar{u}_2 \\ &+ a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5}\bar{u}_1\end{aligned} \tag{23}$$

The simplified equation pairs for $\hat{C}$ and $\bar{r}$ pertaining to the other main groups can be obtained as shown below by using the procedure applied to the main group 1 and the appropriate simplification tools given in Appendix A. The subscripts indicate the main groups in the following equations. In these equations, d_{ij} denotes d_i+d_j. Note that, if J_k is prismatic, then the offset d_k is to be replaced with the joint variable s_k as done in obtaining the subgroup equations in Subsection 3.2.

$$\hat{C}_2 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{24}$$

$$\bar{r}_2 = d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_{234} e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_1$$
$$+ a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}}\bar{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}}\bar{u}_3 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} e^{\tilde{u}_3\theta_5}\bar{u}_1 \tag{25}$$

$$\hat{C}_3 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{26}$$

$$\bar{r}_3 = d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + d_{34} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3$$
$$+ a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\bar{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\bar{u}_2$$
$$+ a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}} e^{\tilde{u}_2\theta_5}\bar{u}_1 \tag{27}$$

$$\hat{C}_4 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{28}$$

$$\bar{r}_4 = d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + d_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3$$
$$+ a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_2 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\bar{u}_1$$
$$+ d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\bar{u}_3 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5}\bar{u}_1 \tag{29}$$

$$\hat{C}_5 = e^{\tilde{u}_3\theta_{1234}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{30}$$

$$\bar{r}_5 = d_{1234}\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\bar{u}_1 + a_3 e^{\tilde{u}_3\theta_{123}}\bar{u}_1 + a_4 e^{\tilde{u}_3\theta_{1234}}\bar{u}_1 + d_5 e^{\tilde{u}_3\theta_{1234}}\bar{u}_2$$
$$+ a_5 e^{\tilde{u}_3\theta_{1234}} e^{\tilde{u}_2\theta_5}\bar{u}_1 \tag{31}$$

$$\hat{C}_6 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{345}} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{32}$$

$$\bar{r}_6 = d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + d_{345} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_3$$
$$+ a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\bar{u}_1 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{345}}\bar{u}_1 \tag{33}$$

$$\hat{C}_7 = e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{34}$$

$$\bar{r}_7 = d_{12}\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\bar{u}_1 + d_{34} e^{\tilde{u}_3\theta_{12}}\bar{u}_2 + a_3 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\bar{u}_1$$
$$+ a_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}}\bar{u}_1 + d_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}}\bar{u}_3 + a_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_{34}} e^{\tilde{u}_3\theta_5}\bar{u}_1 \tag{35}$$

$$\hat{C}_8 = e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{36}$$

$$\bar{r}_8 = d_{12}\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\bar{u}_1 + d_3 e^{\tilde{u}_3\theta_{12}}\bar{u}_2 + a_3 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\bar{u}_1 + d_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\bar{u}_3$$
$$+ a_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4}\bar{u}_1 + d_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4}\bar{u}_2 + a_5 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5}\bar{u}_1 \tag{37}$$

$$\hat{C}_9 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_{45}} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} \tag{38}$$

$$\bar{r}_9 = d_1\bar{u}_3 + a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_{23} e^{\tilde{u}_3\theta_1}\bar{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_1$$
$$+ d_{45} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_3 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\bar{u}_1 + a_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_{45}}\bar{u}_1 \tag{39}$$

3.2 Subgroups and Subgroup Equations

The list of the subgroups of the nine main groups is given in Table 4 with the non-zero link parameters and the number of industrial robots surveyed in this study. In the table, the first digit of the subgroup designation indicates the underlying main group and the second non-zero digit indicates the subgroup of that main group (e.g., subgroup 2.6 indicates the subgroup 6 of the main group 2). The second zero digit indicates the main group itself. The brand names and the models of the surveyed industrial robots are given in Appendix B with their subgroups and non-zero link parameters. If the joint J_k of a manipulator happens to be prismatic, the offset d_k becomes a joint variable, which is then denoted by s_k. In the column titled "Solution Type", CF denotes that a *closed-form* inverse kinematic solution can be obtained analytically and PJV denotes that the inverse kinematic solution can only be obtained semi-analytically using the so called *parametrized joint variable* method. The details of these two types of inverse kinematic solutions can be seen in Section 4.

Sub-Group	Twist Angles or Nonzero Link Parameters	Nr of Robots	Solution Type	Sub-Group	Twist Angles or Nonzero Link Parameters	Nr of Robots	Solution Type
1.0	-90°, 0°, 90°,-90°, 90°, 0°	73	-	3.2	$a_1, a_3, s_1, s_2, s_3, d_4$	1	CF
1.1	a_2, d_4	25	CF	3.3	d_2, s_3	1	CF
1.2	a_2, d_2, d_4	4	CF	3.4	a_2, s_3, d_5	1	PJV
1.3	a_1, a_2, d_4	3	CF	4.0	-90°, 90°, -90°, 90°,-90°, 0°	4	-
1.4	a_2, a_3, d_4	9	CF	4.1	$a_1, a_2, s_1, s_2, s_3, d_4$	1	CF
1.5	a_2, d_2, d_3, d_4	2	CF	4.2	s_2, d_4	1	CF
1.6	a_1, a_2, a_3, d_4	21	CF	4.3	s_3, d_5	1	CF/PJV
1.7	a_2, d_4, d_5	6	PJV	4.4	a_2, s_3, d_5	1	CF/PJV
1.8	a_2, d_2, d_4, d_5	1	PJV	5.0	0°, 0°, 0°, - 90°, 90°, 0°	1	-
1.9	a_1, a_2, a_3, d_4, d_5	1	PJV	5.1	a_1, a_2, s_3	1	CF
1.10	a_1, a_2, a_3, d_4, d_5	1	PJV	6.0	- 90°, 90°, 0°, 0°, - 90°, 0°	1	-
2.0	-90°, 0°, 0°, 90°, -90°, 0°	12	-	6.1	a_3, a_4	1	CF
2.1	a_2, a_3, d_4	1	CF	7.0	0°, - 90°, 0°, 90°, - 90°, 0°	1	-
2.2	a_2, a_3, a_4	6	CF	7.1	a_2, s_2, s_3	1	CF
2.3	a_2, a_3, d_5	2	CF	8.0	0°, -90°, 90°, -90°, 90°, 0°	2	-
2.4	a_2, a_3, d_2, d_5	1	CF	8.1	a_1, d_3, d_4	1	CF
2.5	a_1, a_2, a_3, d_2, d_5	1	CF	8.2	a_1, a_2, d_3, d_4	1	CF
2.6	a_1, a_2, a_3, d_4, d_5	1	CF	9.0	- 90°, 0°, 90°, 0°, - 90°, 0°	1	-
3.0	-90°, 90°, 0°,-90°, 90°, 0°	5	-	9.1	$a_1, a_2, a_3, a_4, d_4, d_5$	1	PJV
3.1	s_1, s_2, s_3	1	CF				

Table 4. Subgroups of Six-Joint Robots

Using the information about the link lengths and the offsets, the simplified subgroup equations are obtained for the wrist locations as shown below by using again the exponential rotation matrix simplification tools given in Appendix A. In these equations, the first and second subscripts associated with the wrist locations indicate the related main groups and subgroups. For all subgroups of the main groups 1 and 2, the rotation matrix is as given in the main group equations.

$$\overline{r}_{11} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \tag{40}$$

$$\overline{r}_{12} = d_2 e^{\tilde{u}_3\theta_1} \overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \tag{41}$$

$$\overline{r}_{13} = a_1 e^{\tilde{u}_3\theta_1} \overline{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \tag{42}$$

$$\overline{r}_{14} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \tag{43}$$

$$\overline{r}_{15} = d_{23} e^{\tilde{u}_3\theta_1} \overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \tag{44}$$

$$\overline{r}_{16} = a_1 e^{\tilde{u}_3\theta_1} \overline{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \tag{45}$$

$$\overline{r}_{17} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \overline{u}_2 \tag{46}$$

$$\overline{r}_{18} = d_2 e^{\tilde{u}_3\theta_1} \overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \overline{u}_2 \tag{47}$$

$$\begin{aligned}\overline{r}_{19} = {} & a_1 e^{\tilde{u}_3\theta_1} \overline{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \\ & + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \overline{u}_2\end{aligned} \tag{48}$$

$$\begin{aligned}\overline{r}_{110} = {} & a_1 e^{\tilde{u}_3\theta_1} \overline{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_3 e^{\tilde{u}_3\theta_1} \overline{u}_2 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_3 \\ & + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} \overline{u}_2\end{aligned} \tag{49}$$

$$\overline{r}_{21} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} \overline{u}_2 \tag{50}$$

$$\overline{r}_{22} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \overline{u}_1 \tag{51}$$

$$\overline{r}_{23} = a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \overline{u}_3 \tag{52}$$

$$\overline{r}_{24} = d_2 e^{\tilde{u}_3\theta_1} \overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \overline{u}_3 \tag{53}$$

$$\overline{r}_{25} = a_1 e^{\tilde{u}_3\theta_1} \overline{u}_1 + d_2 e^{\tilde{u}_3\theta_1} \overline{u}_2 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \overline{u}_3 \tag{54}$$

$$\overline{r}_{26} = a_1 e^{\tilde{u}_3\theta_1} \overline{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} \overline{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} \overline{u}_1 + d_4 e^{\tilde{u}_3\theta_1} \overline{u}_2 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{234}} \overline{u}_3 \tag{55}$$

The constant joint angles associated with the prismatic joints are as follows for the subgroups of the main group 3: For the subgroups 3.1 and 3.2 having s_1, s_2, and s_3 as the variable offsets, the joint angles are $\theta_1 = 0^\circ$, $\theta_2 = 90^\circ$, $\theta_3 = 0^\circ$ or 90°. For the subgroups 3.3 and 3.4, having s_3 as the only variable offset, θ_3 is either 0° or 90°. This leads to the following equations:

$$\hat{C}_{31} = \hat{C}_{32} = e^{\tilde{u}_1\theta_{34}} e^{\tilde{u}_2\theta_5'} e^{\tilde{u}_3\theta_6} = \begin{cases} e^{\tilde{u}_1\theta_4} e^{\tilde{u}_2\theta_5'} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 0^\circ \\ e^{\tilde{u}_1\theta_4'} e^{\tilde{u}_2\theta_5'} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 90^\circ \end{cases} \tag{56}$$

Here, $\theta_4' = \theta_4 + 90^\circ$ and $\theta_5' = \theta_5 + 90^\circ$.

$$\overline{r}_{31} = s_3\overline{u}_1 + s_2\overline{u}_2 + s_1\overline{u}_3 \tag{57}$$

$$\overline{r}_{32} = s_3'\overline{u}_1 + s_2\overline{u}_2 + s_1\overline{u}_3 - a_3 e^{\tilde{u}_1\theta_3}\overline{u}_3 = \begin{cases} s_3'\overline{u}_1 + s_2\overline{u}_2 + s_1'\overline{u}_3 & \text{for } \theta_3 = 0^\circ \\ s_3'\overline{u}_1 + s_2'\overline{u}_2 + s_1\overline{u}_3 & \text{for } \theta_3 = 90^\circ \end{cases} \tag{58}$$

Here, $s_1' = s_1 - a_3$, $s_2' = s_2 + a_3$, and $s_3' = s_3 + a_1 + d_4$.

$$\hat{C}_{33} = \hat{C}_{34} = \begin{cases} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 0^\circ \\ e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_4'} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 90^\circ \end{cases} \tag{59}$$

$$\overline{r}_{33} = d_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 \tag{60}$$

$$\begin{aligned} \overline{r}_{34} &= a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\overline{u}_2 \\ &= \begin{cases} a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_4}\overline{u}_2 & \text{for } \theta_3 = 0^\circ \\ a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_4'}\overline{u}_2 & \text{for } \theta_3 = 90^\circ \end{cases} \end{aligned} \tag{61}$$

Here, $\theta_4' = \theta_4 + 90^\circ$.

The constant joint angles associated with the prismatic joints, for the subgroups of the main group 4 are as follows: For the subgroup 4.1 having s_1, s_2, and s_3 as the variable offsets, the joint angles are $\theta_1 = 0^\circ$, $\theta_2 = 90^\circ$, $\theta_3 = 0^\circ$ or 90°. For the subgroup 4.2 having s_2 as the only variable offset, θ_2 is either 0° or 90°. For the subgroups 4.3 and 4.4 having s_3 as the only variable offset, θ_3 is either 0° or 90°. This leads to the following equations:

$$\hat{C}_{41} = e^{\tilde{u}_1\theta_3} e^{\tilde{u}_2\theta_4'} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} = \begin{cases} e^{\tilde{u}_2\theta_4'} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_3 = 0° \\ e^{\tilde{u}_3\theta_4'} e^{-\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 90° \end{cases} \tag{62}$$

$$\overline{r}_{41} = s_3'\overline{u}_1 + s_2\overline{u}_2 + s_1'\overline{u}_3 + d_4 e^{\tilde{u}_1\theta_3}\overline{u}_2 = \begin{cases} s_3'\overline{u}_1 + s_2'\overline{u}_2 + s_1'\overline{u}_3 & \text{for } \theta_3 = 0° \\ s_3'\overline{u}_1 + s_2\overline{u}_2 + s_1''\overline{u}_3 & \text{for } \theta_3 = 90° \end{cases} \tag{63}$$

Here, $s_1' = s_1 - a_2$, $s_1'' = s_1' + d_4$, $s_2' = s_2 + d_4$, and $s_3' = s_3 + a_1$.

$$\hat{C}_{42} = \begin{cases} e^{\tilde{u}_3\theta_{13}} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_2 = 0° \\ e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\theta_3} e^{\tilde{u}_2\theta_4'} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_2 = 90° \end{cases} \tag{64}$$

Here, $\theta_4' = \theta_4 + 90°$.

$$\overline{r}_{42} = s_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\overline{u}_2 = \begin{cases} s_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + d_4 e^{\tilde{u}_3\theta_{13}}\overline{u}_2 & \text{for } \theta_2 = 0° \\ s_2 e^{\tilde{u}_3\theta_1}\overline{u}_2 + d_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\theta_3}\overline{u}_2 & \text{for } \theta_2 = 90° \end{cases} \tag{65}$$

$$\hat{C}_{43} = \hat{C}_{44} = \begin{cases} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{24}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_3 = 0° \\ e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\theta_4} e^{\tilde{u}_3\theta_5'} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for } \theta_3 = 90° \end{cases} \tag{66}$$

Here, $\theta_5' = \theta_5 + 90°$.

$$\begin{aligned} \overline{r}_{43} &= s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\overline{u}_3 \\ &= \begin{cases} s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{24}}\overline{u}_3 & \text{for } \theta_3 = 0° \\ s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\theta_4}\overline{u}_3 & \text{for } \theta_3 = 90° \end{cases} \end{aligned} \tag{67}$$

$$\begin{aligned} \overline{r}_{44} &= a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_2\theta_4}\overline{u}_3 \\ &= \begin{cases} a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{24}}\overline{u}_3 & \text{for } \theta_3 = 0° \\ a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_1 + s_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\overline{u}_3 + d_5 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_1\theta_4}\overline{u}_3 & \text{for } \theta_3 = 90° \end{cases} \end{aligned} \tag{68}$$

The constant joint angle θ_3 associated with the prismatic joint J_3 for the subgroup of the main group 5 is either $0°$ or $90°$. This leads to the following equations:

$$\hat{C}_{51} = e^{\tilde{u}_3\theta_{1234}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} = \begin{cases} e^{\tilde{u}_3\theta_{124}} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 0° \\ e^{\tilde{u}_3\theta_{124}'} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} & \text{for } \theta_3 = 90° \end{cases} \tag{69}$$

Here, $\theta'_{124} = \theta_{124} + 90°$.

$$\bar{r}_{51} = a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\bar{u}_1 + s_3\bar{u}_3 \tag{70}$$

For the subgroup of the main group 6, the rotation matrix is as given in the main group equations and the wrist point location is expressed as

$$\bar{r}_{61} = a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3}\bar{u}_1 + a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_{34}}\bar{u}_1 \tag{71}$$

The constant joint angles associated with the prismatic joints for the subgroup of the main group 7 are as follows: The joint angle θ_2 is $90°$ for the prismatic joint J_2 and the joint angle θ_3 is either $0°$ or $90°$ for the prismatic joint J_3. This leads to the following equations:

$$\hat{C}_{71} = e^{\tilde{u}_3\theta'_1} e^{\tilde{u}_2\theta_{34}} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} = \begin{cases} e^{\tilde{u}_3\theta'_1} e^{\tilde{u}_2\theta_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for} \quad \theta_3 = 0° \\ e^{\tilde{u}_3\theta'_1} e^{\tilde{u}_2\theta'_4} e^{\tilde{u}_3\theta_5} e^{\tilde{u}_2\theta_6} e^{-\tilde{u}_1\pi/2} & \text{for} \quad \theta_3 = 90° \end{cases} \tag{72}$$

Here, $\theta'_1 = \theta_1 + 90°$ and $\theta'_4 = \theta_4 + 90°$.

$$\bar{r}_{71} = s_2\bar{u}_3 + a_2 e^{\tilde{u}_3\theta_1}\bar{u}_2 - s_3 e^{\tilde{u}_3\theta_1}\bar{u}_1 \tag{73}$$

For the subgroups of the main group 8, the rotation matrix is as given in the main group equations and the wrist point locations are expressed as

$$\bar{r}_{81} = a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + d_3 e^{\tilde{u}_3\theta_{12}}\bar{u}_2 + d_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\bar{u}_3 \tag{74}$$

$$\bar{r}_{82} = a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + a_2 e^{\tilde{u}_3\theta_{12}}\bar{u}_1 + d_3 e^{\tilde{u}_3\theta_{12}}\bar{u}_2 + d_4 e^{\tilde{u}_3\theta_{12}} e^{\tilde{u}_2\theta_3}\bar{u}_3 \tag{75}$$

For the subgroup of the main group 9, the rotation matrix is as given in the main group equations and the wrist point location is expressed as

$$\begin{aligned} \bar{r}_{91} &= a_1 e^{\tilde{u}_3\theta_1}\bar{u}_1 + a_2 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_2}\bar{u}_1 + a_3 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_1 + d_{45} e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}}\bar{u}_3 \\ &+ a_4 e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4}\bar{u}_1 \end{aligned} \tag{76}$$

4. Classification Based Inverse Kinematics

In the inverse kinematics problem, the elements of $\hat{C}$ and $\bar{r}$ are available and it is desired to obtain the six joint variables. For this purpose, the required elements of the $\bar{r}$ and $\hat{C}$ matrices can be extracted as follows:

$$r_i = \bar{u}_i^T \bar{r} \quad \text{and} \quad c_{ij} = \bar{u}_i^T \hat{C} \, \bar{u}_j \tag{77}$$

For most of the manipulators, which are called *separable*, the wrist point position vector contains only three joint variables. The most typical samples of such manipulators are those with spherical wrists (Pieper & Roth, 1969). Therefore, for this large class of manipulators, Equation (14) is first used to obtain the *arm joint variables*, and then Equation (11) is used to determine the remaining three of the *wrist joint variables* contained in the $\hat{C}$ matrix. After obtaining the arm joint variables, $\hat{C}$ equation is arranged in such a way that the arm joint variables are collected at one side of the equation leaving the remaining joint variables to be found at the other side within a new matrix $\hat{M}$, which is called *modified orientation matrix*. The three arguments of $\hat{M}$ happen to be the wrist joint variables and they appear similarly as an Euler Angle sequence of three successive rotations. After this preparation, the solution of the modified orientation equation directly gives the wrist joint variables. The most commonly encountered sequences are given in Table 5 with their solutions and singularities. In the table, $\sigma = \pm 1$ indicates the alternative solutions. When the sequence becomes singular, the angles ϕ_1 and ϕ_3 can not be determined and the mobility of the wrist becomes restricted. For a more detailed singularity analysis, see (Özgören, 1999 and 2002).

However, there may also be other kinds of separable manipulators for which it is the $\hat{C}$ matrix that contains only three joint variables. In such a case, the solution is started naturally with the $\hat{C}$ equation and then the remaining three joint variables are found from the $\bar{r}$ equation. Besides, there are inseparable manipulators as well for which both of the $\hat{C}$ and $\bar{r}$ equations contain more than three joint variables. The most typical sample of this group is the Cincinnati Milacron-T3 (CM-T3 566 or 856) robot. It has four unknown variables in each of its $\hat{C}$ and $\bar{r}$ equations. For such manipulators, since the $\hat{C}$ and $\bar{r}$ equations are not separable, they have to be solved jointly and therefore a closed-form inverse kinematic solution cannot be obtained in general. Nevertheless, for some special forms of such manipulators, Cincinnati Milacron-T3 being one of them, it becomes possible to obtain a closed-form inverse kinematic solution. For a more detailed analysis and discussion of inverse kinematics covering all possible six-joint serial manipulators, see (Özgören, 2002).

Sequence	3-2-3	2-3-2	1-2-3
$\hat{C}$	$e^{\tilde{u}_3\phi_1}e^{\tilde{u}_2\phi_2}e^{\tilde{u}_3\phi_3}$	$e^{\tilde{u}_2\phi_1}e^{\tilde{u}_3\phi_2}e^{\tilde{u}_2\phi_3}$	$e^{\tilde{u}_1\phi_1}e^{\tilde{u}_2\phi_2}e^{\tilde{u}_3\phi_3}$
ϕ_1	$\text{atan2}(\sigma c_{23}, \sigma c_{13})$	$\text{atan2}(\sigma c_{32}, -\sigma c_{12})$	$\text{atan2}(\sigma c_{23}, \sigma c_{33})$
ϕ_2	$\text{atan2}(\sigma\sqrt{1-c_{33}^2}, c_{33})$	$\text{atan2}(\sigma\sqrt{1-c_{22}^2}, c_{22})$	$\text{atan2}(-c_{13}, \sigma\sqrt{1-c_{13}^2})$
ϕ_3	$\text{atan2}(\sigma c_{32}, -\sigma c_{31})$	$\text{atan2}(\sigma c_{23}, \sigma c_{21})$	$\text{atan2}(-\sigma c_{12}, \sigma c_{11})$
Singularity	$\phi_2 = 0°$ or $\phi_2 = \pm 180°$	$\phi_2 = 0°$ or $\phi_2 = \pm 180°$	$\phi_2 = \pm 90°$

Table 5. Wrist Joint Variables in the Most Commonly Encountered Sequences

For all the groups of six-joint manipulators considered in this chapter, there are two types of inverse kinematic solution, namely the *closed-form* (CF) solution and the *parametrized joint variable* (PJV) solution where one of the joint variables is temporarily treated as if it is a known parameter. For many of the six-joint manipulators, a closed-form solution can be obtained if the wrist point location equation and the end-effector orientation equation are separable, i.e. if it is possible to write the wrist point equation as

$$\bar{r} = \bar{r}(q_1, q_2, q_3) \tag{78}$$

where q_k denotes the k^{th} joint variable, which is either θ_k or s_k. Since there are three unknowns in the three scalar equations contained in Equation (78), the unknowns q_1, q_2, q_3 can be directly obtained from those equations. The end-effector orientation variables q_4, q_5, q_6 are then obtained by using the equation for $\hat{C}$.

In general, the necessity for a PJV solution arises when a six-joint manipulator has a non-zero value for one or more of the structural length parameters a_4, a_5, and d_5. However, in all of the manipulators that are considered here, only d_5 exists as an offset at the wrist. In this case, $\bar{r}$ will be a function of four variables as

$$\bar{r} = \bar{r}(q_1, q_2, q_3, q_4) \tag{79}$$

Since, there are more unknowns now than the three scalar equations contained in Equation (79), the variables q_1, q_2, q_3, and q_4 can not be obtained directly. So, one of the joint variables is parametrized and the remaining five joint variables are obtained as functions of this parametrized variable from five of the six scalar equations contained in the $\hat{C}$ and $\bar{r}$ equations. Then, the remaining sixth scalar equation is solved for the parametrized variable using a suitable numerical method. Finally, by substituting the numerically found value of this joint variable into the previously obtained expressions of the remaining joint variables, the complete solution is obtained.

There may also be a situation that a six-joint manipulator can have non-zero values for the structural parameters d_5 and a_5 so that

$$\overline{r} = \overline{r}(q_1, q_2, q_3, q_4, q_5) \quad (80)$$

In this case, two joint variables can be chosen as the parametrized joint variables and the remaining four joint variables are obtained as functions of these two parametrized variables. Then, using a suitable numerical method, the remaining two equations are solved for the parametrized joint variables. Afterwards, the inverse kinematic solution is completed similarly as described above. However, if desired, it may also be possible to reduce the remaining two equations to a single but rather complicated univariate polynomial equation by using methods similar to those in (Raghavan & Roth, 1993; Manseur & Doty, 1996; Lee et al, 1991).
Although the analytical or semi-analytical solution methods are necessarily dependent on the specific features of the manipulator of concern, the procedure outlined below can be used as a general course to follow for most of the manipulators considered here.

1. The wrist location equation is manipulated symbolically to obtain three scalar equations using the simplification tools given in Appendix A.

2. The three scalar equations are worked on in order to cast them into the forms of the trigonometric equations considered in Appendix C, if they are not so already.

3. As a sufficient condition for a CF solution, if there exists a scalar equation containing only one joint variable, or if such an equation can be generated by combining the other available equations, it can be solved for that joint variable to start the solution.

4. If such an equation does not exist or cannot be generated, then the PJV method is used. Thus, except the parametrized joint variable, there will again be a single equation with a single unknown to start the solution.

5. The two remaining scalar equations pertaining to the wrist location are then used to determine the remaining two of the arm joint variables again by using the appropriate ones of the trigonometric equation solutions given in Appendix C.

6. Once the arm joint variables are obtained, the solution of the orientation equation for the three wrist joint variables is straightforward since it will be the same as the solution pattern of one of the commonly encountered rotation sequences, such as 1-2-3, 3-2-3, etc, which are shown in Table 5.

When the manipulators in Table 4 are considered from the viewpoint of the solution procedure described above, they are accompanied by the designations CF (having inverse kinematic solution in closed-form) and PJV (having inverse kinematic solution using a parametrized joint variable). As noted, almost all the manipulators in Table 4 are designated exclusively either with CF or PJV. Exceptionally, however, the subgroups 4.3 and 4.4 have both of the designations. This is because the solution type can be either CF or PJV depending on whether $\theta_3 = 0°$ or $\theta_3 = 90°$, respectively.

In this section, the inverse kinematic solutions for the subgroups 1.1 (e.g. KUKA IR 662/10), 1.7 (e.g. GMF S-3 L or R) and 4.4 (e.g. Unimate 4000) are given in order to demonstrate the solution procedure described above. As indicated in Table 4, the subgroup 1.1 can have the inverse kinematic solution in closed-form, whereas the subgroup 1.7 necessitates a PJV solution. On the other hand, for the subgroup 4.4, which has a prismatic joint, the inverse kinematic solution can be obtained either in closed-form or by using the PJV method depending on whether the structural parameter θ_3 associated with the prismatic joint is 0° or 90°. Although $\theta_3 = 0°$ for the enlisted industrial robot of this subgroup, the solution for $\theta_3 = 90°$ is also considered here for sake of demonstrating the application of the PJV method to a robot with a prismatic joint as well. It should be noted that, the subgroups 1.1 and 1.7 are examples to robots with only revolute joints and the subgroup 4.4 is an example to robots with revolute and prismatic joints. The subgroups 1.1 and 1.7 are considered particularly because the number of industrial robots is high within these categories. As an additional example, the ABB IRB2000 industrial robot is also considered to demonstrate the applicability of the method to manipulators containing closed kinematic chains. However, the solutions for the other subgroups or a new manipulator with a different kinematic structure can be obtained easily by using the same systematic approach.

4.1 Inverse Kinematics of Subgroups 1.1 and 1.7

For all the subgroups of the main group 1, the orientation matrix is

$$\hat{C}_1 = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_2\theta_{23}} e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} \tag{81}$$

Since all the subgroups have the same $\hat{C}_1$ matrix, they will have identical equations for θ_4, θ_5 and θ_6. In other words, these variables can always be determined from the following equation, after finding the other variables somehow from the wrist location equations of the subgroups:

$$e^{\tilde{u}_3\theta_4} e^{\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6} = \hat{M}_1 \tag{82}$$

Here, $\hat{M}_1 = e^{-\tilde{u}_2\theta_{23}} e^{-\tilde{u}_3\theta_1} \hat{C}_1$ and $\theta_{23} = \theta_2 + \theta_3$. Since the sequence in Equation (82) is 3-2-3, using Table 5, the angles θ_4, θ_5 and θ_6 are obtained as follows, assuming that θ_1 and θ_{23} have already been determined as explained in the next subsection:

$$\theta_4 = \operatorname{atan2}(\sigma_5 m_{23},\ \sigma_5 m_{13}) \tag{83}$$

$$\theta_5 = \operatorname{atan2}(\sigma_5 \sqrt{1 - m_{33}^2},\ m_{33}) \tag{84}$$

$$\theta_6 = \operatorname{atan2}(\sigma_5 m_{32},\ -\sigma_5 m_{13}) \tag{85}$$

Here, $\sigma_5 = \pm 1$ and $m_{ij} = \bar{u}_i^T \hat{M}_1 \bar{u}_j$.

Note that this 3-2-3 sequence becomes singular if $\theta_5 = 0$ or $\theta_5 = \pm 180^\circ$, but the latter case is not physically possible. This is the first kind of singularity of the manipulator, which is called *wrist singularity*. In this singularity with $\theta_5 = 0$, the axes of the fourth and sixth joints become aligned and Equation (82) degenerates into

$$e^{\tilde{u}_3\theta_4} e^{\tilde{u}_3\theta_6} = e^{\tilde{u}_3(\theta_4 + \theta_6)} = e^{\tilde{u}_3\theta_{46}} = \hat{M}_1 \tag{86}$$

This equation implies that, in the singularity, θ_4 and θ_6 become arbitrary and they cannot be determined separately although their combination $\theta_{46} = \theta_4 + \theta_6$ can still be determined as $\theta_{46} = \operatorname{atan2}(m_{21},\ m_{11})$. This means that one of the fourth and sixth joints becomes redundant in orienting the end-effector, which in turn becomes underivable about the axis normal to the axes of the fifth and sixth joints.

4.1.1 Inverse Kinematics of Subgroup 1.1

The wrist point position vector of this subgroup given in Equation (40) can be written again as follows by transposing the leading exponential matrix on the right hand side to the left hand side:

$$e^{-\tilde{u}_3\theta_1} \bar{r}_{11} = a_2 e^{\tilde{u}_2\theta_2} \bar{u}_1 + d_4 e^{\tilde{u}_2\theta_{23}} \bar{u}_3 \tag{87}$$

Premultiplying both sides of Equation (87) by $\bar{u}_1^T$, $\bar{u}_2^T$, $\bar{u}_3^T$ and using the simplification tool E.8 in Appendix A, the following equations can be obtained.

$$r_1 \cos\theta_1 + r_2 \sin\theta_1 = a_2 \cos\theta_2 + d_4 \sin\theta_{23} \tag{88}$$

$$r_2 \cos\theta_1 - r_1 \sin\theta_1 = 0 \tag{89}$$

$$r_3 = -a_2 \sin\theta_2 + d_4 \cos\theta_{23} \tag{90}$$

Here, r_1, r_2 and r_3 are the base frame components of the wrist position vector, $\bar{r}_{11}$.
From Equation (89), θ_1 can be obtained as follows by using the trigonometric equation T1 in Appendix C, provided that $r_2^2 + r_1^2 \neq 0$:

$$\theta_1 = \operatorname{atan2}(\sigma_1 r_2, \sigma_1 r_1) \quad \text{and} \quad \sigma_1 = \pm 1 \tag{91}$$

If $r_2^2 + r_1^2 = 0$, i.e. if $r_2 = r_1 = 0$, i.e. if the wrist point is located on the axis of the first joint, the second kind of singularity occurs, which is called *shoulder singularity*. In this singularity, Equation (89) degenerates into 0 = 0 and therefore θ_1 cannot be determined. In other words, the first joint becomes ineffective in positioning the wrist point, which in turn becomes underivable in the direction normal to the arm plane (i.e. the plane formed by the links 2 and 3).

To continue with the solution, let

$$\rho_1 = r_1 \cos\theta_1 + r_2 \sin\theta_1 \tag{92}$$

Thus, Equation (88) becomes

$$\rho_1 = a_2 \cos\theta_2 + d_4 \sin\theta_{23} \tag{93}$$

Using Equations (90) and (93) in accordance with T6 in Appendix C, θ_3 can be obtained as follows, provided that $-1 \leq \rho_2 \leq 1$:

$$\theta_3 = \operatorname{atan2}(\rho_2, \sigma_3\sqrt{1-\rho_2^2}) \quad \text{and} \quad \sigma_3 = \pm 1 \tag{94}$$

Here,

$$\rho_2 = \frac{(\rho_1^2 + r_3^2) - (a_2^2 + d_4^2)}{2a_2 d_4} \tag{95}$$

Note that the constraint on ρ_2 implies a working space limitation on the manipulator, which can be expressed more explicitly as

$$(a_2 - d_4)^2 \leq \rho_1^2 + r_3^2 \leq (a_2 + d_4)^2 \tag{96}$$

Expanding sin θ_{23} and cos θ_{23} in Equation (90) and (93) and rearranging the terms as coefficients of sin θ_2 and cos θ_2, the following equations can be obtained.

$$\rho_1 = \rho_3 \cos\theta_2 + \rho_4 \sin\theta_2 \tag{97}$$

$$r_3 = \rho_4 \cos\theta_2 - \rho_3 \sin\theta_2 \tag{98}$$

Here,

$$\rho_3 = a_2 + d_4 \sin\theta_3 \tag{99}$$

$$\rho_4 = d_4 \cos\theta_3 \tag{100}$$

According to T4 in Appendix C, Equations (97) and (98) give θ_2 as follows, provided that $\rho_3^2 + \rho_4^2 \neq 0$:

$$\theta_2 = \mathrm{atan2}(\rho_4\rho_1 - \rho_3 r_3 \ , \ \rho_4 r_3 - \rho_3\rho_1) \tag{101}$$

If $\rho_3^2 + \rho_4^2 = 0$, i.e. if $\rho_3 = \rho_4 = 0$, the third kind of singularity occurs, which is called *elbow singularity*. In this singularity, both of Equations (97) and (98) degenerate into $0 = 0$. Therefore, θ_2 cannot be determined. Note that, according to Equations (99) and (100), it is possible to have $\rho_3 = \rho_4 = 0$ only if $a_2 = d_4$ and $\theta_3 = \pm 180^{\circ}$. This means that the elbow singularity occurs if the upper and front arms (i.e. the links 2 and 3) have equal lengths and the front arm is folded back onto the upper arm so that the wrist point coincides with the shoulder point. In this configuration, the second joint becomes ineffective in positioning the wrist point, which in turn becomes underivable neither along the axis of the second joint nor in a direction parallel to the upper arm.
As seen above, the closed-form inverse kinematic solution is obtained for the subgroup 1.1 as expressed by Equations (83)-(85) and (91)-(101). The completely analytical nature of the solution provided all the multiplicities (indicated by the sign variables σ_1, σ_2, etc), the singularities, and the working space limitations alongside with the solution.

4.1.2 Inverse Kinematics of Subgroup 1.7

The wrist point position vector of this subgroup is given in Equation (46). From that equation, the following scalar equations can be obtained as done previously for the subgroup 1.1:

$$r_1 \cos\theta_1 + r_2 \sin\theta_1 = a_2 \cos\theta_2 + d_4 \sin\theta_{23} - d_5 \cos\theta_{23} \sin\theta_4 \tag{102}$$

$$r_2 \cos\theta_1 - r_1 \sin\theta_1 = d_5 \cos\theta_4 \tag{103}$$

$$r_3 = -a_2 \sin\theta_2 + d_4 \cos\theta_{23} + d_5 \sin\theta_{23} \sin\theta_4 \tag{104}$$

Here, r_1, r_2 and r_3 are the components of the wrist position vector, $\overline{r}_{17}$. Note that Equations (102)-(104) contain four unknowns (θ_1, θ_2, θ_3, θ_4). Therefore, it now becomes necessary to use the PJV method. That is, one of these four unknowns must be parametrized. On the other hand, Equation (103) is the simplest one of the three equations. Therefore, it will be reasonable to parametrize either θ_1 or θ_4. As it is shown in (Balkan et al., 1997, 2000), the solutions obtained by parametrizing θ_1 and θ_4 expose different amounts of explicit information about the multiple and singular configurations of the manipulators belonging to this subgroup. The rest of the information is concealed within the equation to be solved numerically. It happens that the solution obtained by parametrizing θ_4 reveals more information so that the critical shoulder singularity of the manipulator can be seen *explicitly* in the relevant equations; whereas the solution obtained by parametrizing θ_1 conceals it. Therefore, θ_4 is chosen as the parametrized joint variable in the solution presented below. As the starting step, θ_1 can be obtained from Equation (103) as follows by using T3 in Appendix C, provided that $r_2^2 + r_1^2 > 0$ and $r_2^2 + r_1^2 \geq \rho_5^2$:

$$\theta_1 = \text{atan2}\,(-r_2, r_1) + \sigma_1 \text{atan2}\,(\sqrt{r_1^2 + r_2^2 - \rho_5^2}, \rho_5) \quad \text{and} \quad \sigma_1 = \pm 1 \tag{105}$$

Here,

$$\rho_5 = d_5 \cos\theta_4 \tag{106}$$

If $r_2^2 + r_1^2 = 0$, which necessitates that $\rho_5 = 0$ or $\theta_4 = \pm 90^\circ$, the shoulder singularity occurs. In that case, Equation (103) degenerates into 0 = 0 and therefore θ_1 becomes arbitrary. The consequences are the same as those of the subgroup 1.1.

On the other hand, the inequality constraint $r_2^2 + r_1^2 \geq \rho_5^2$ indicates a working space limitation on the manipulator.

Equations (102) and (104) can be arranged as shown below:

$$x_1 = a_2 \cos\theta_2 + d_4 \sin\theta_{23} - \rho_6 \cos\theta_{23} \tag{107}$$

$$r_3 = -a_2 \sin\theta_2 + d_4 \cos\theta_{23} + \rho_6 \sin\theta_{23} \tag{108}$$

Here,

$$\rho_6 = d_5 \sin\theta_4 \tag{109}$$

According to T9 in Appendix C, Equations (107) and (108) give θ_3 as follows, provided that $\rho_1^2 + d_4^2 \geq \rho_7^2$:

$$\theta_3 = \operatorname{atan2}(d_4, -\rho_6) + \sigma_3 \operatorname{atan2}\left(\sqrt{\rho_1^2 + d_4^2 - \rho_7^2}, \rho_7\right) \quad \text{and} \quad \sigma_3 = \pm 1 \tag{110}$$

Here,

$$\rho_7 = \frac{(\rho_1^2 + r_3^2) - (a_2^2 + d_4^2 + \rho_6^2)}{2a_2} \tag{111}$$

As noted, the inequality constraint $\rho_1^2 + d_4^2 \geq \rho_7^2$ constitutes another limitation on the working space of the manipulator.

Expanding sin θ_{23} and cos θ_{23} in Equation (107) and (108) and collecting the relevant terms as coefficients of sin θ_2 and cos θ_2, the following equations can be obtained.

$$\rho_1 = \rho_8 \cos\theta_2 + \rho_9 \sin\theta_2 \tag{112}$$

$$r_3 = \rho_9 \cos\theta_2 - \rho_8 \sin\theta_2 \tag{113}$$

Here,

$$\rho_8 = a_2 + d_4 \sin\theta_3 - \rho_6 \cos\theta_3 \tag{114}$$

$$\rho_9 = d_4 \cos\theta_3 + \rho_6 \sin\theta_3 \tag{115}$$

According to T4 in Appendix C, Equation (112) and (113) give θ_2 as follows, provided that $\rho_8^2 + \rho_9^2 \neq 0$:

$$\theta_2 = \operatorname{atan2}(\rho_9\rho_1 - \rho_3 r_3, \rho_9 r_3 - \rho_8\rho_1) \tag{116}$$

If $\rho_8^2 + \rho_9^2 = 0$, the elbow singularity occurs. Then, θ_2 becomes arbitrary with the same consequences as those of the subgroup 1.1.

Note that the matrix $\hat{M}_1 = e^{-\tilde{u}_2\theta_{23}} e^{-\tilde{u}_3\theta_1} \hat{C}_1$ of this subgroup comes out to be a function of θ_4 because the angles θ_1 and $\theta_{23} = \theta_2 + \theta_3$ are determined above as functions of θ_4. Therefore, the equation for the parametrized joint variable θ_4 is nothing but Equation (83), which is written here again as

$$\theta_4 = f_4(\theta_4) = \operatorname{atan2}\,[\sigma_5 m_{23}(\theta_4), \sigma_5 m_{13}(\theta_4)] \quad \text{and} \quad \sigma_5 = \pm 1 \tag{117}$$

As noticed, Equation (117) is a highly complicated equation for the unknown θ_4 and it can be solved only with a suitable numerical method. However, after it is solved for θ_4, by substituting θ_4 into the previously derived equations for the other joint variables, the complete solution is obtained. Here, it is worth to mention that, although this solution is not completely analytical, it is still capable of giving the multiple and singular configurations as well as the working space limitations.

Although the PJV method is demonstrated above as applied to the subgroup 1.7, it can be applied similarly to the other subgroups that require it. For example, as a detailed case study, its quantitatively verified application to the FANUC ArcMate Sr. robot of the subgroup 1.9 can be seen in (Balkan et al. 1997 and 2000).

4.2 Inverse Kinematics of Subgroup 4.4

The inverse kinematic solution for the subgroup 4.4 is obtained in a similar manner and the related equations are given in Table 6 indicating the multiple solutions by $\sigma_i = \pm 1$. The orientation matrix $\hat{C}_4$ is simplified using the kinematic properties of this subgroup and denoted as $\hat{C}_{44}$. Actually, the Unimate 4000 manipulator of this subgroup does not have two versions with $\theta_3 = 0°$ and $\theta_3 = 90°$ as given below. It has simply $\theta_3 = 0°$ and the other configuration is a fictitious one. However, aside from constituting an additional example for the PJV method, this fictitious manipulator also gives a design hint for choosing the structural parameters so that the advantage of having a closed-form inverse kinematic solution is not lost.

Orientation Matrix, $\hat{C}_{44}$	$\hat{C}_{44}=e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_{24}}e^{\tilde{u}_3\theta_5}e^{\tilde{u}_2\theta_6}e^{-\tilde{u}_1\pi/2}$ for $\theta_3=0°$ $\hat{C}_{44}=e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}e^{-\tilde{u}_1\theta'_4}e^{-\tilde{u}_2\theta'_5}e^{\tilde{u}_3\theta_6}$ for $\theta_3=90°$ Here, $\theta'_4=\theta_4+90°$, $\theta'_5=\theta_5+90°$	
Modified Orientation Matrix, $\hat{M}_{44}$	$e^{-\tilde{u}_3\theta_1}\hat{C}_{44}e^{\tilde{u}_1\pi/2}=\hat{M}_{44}=e^{\tilde{u}_2\theta_{24}}e^{\tilde{u}_3\theta_5}e^{\tilde{u}_2\theta_6}$ for $\theta_3=0°$ $e^{-\tilde{u}_2\theta_2}e^{-\tilde{u}_3\theta_1}\hat{C}_{44}=\hat{M}_{44}=e^{-\tilde{u}_1\theta'_4}e^{-\tilde{u}_2\theta'_5}e^{\tilde{u}_3\theta_6}$ for $\theta_3=90°$	
θ_3 Selection	$\theta_3=0°$	$\theta_3=90°$
Solution Method	CF	PJV with θ_4
Wrist Orientation Sequence	2-3-2	1-2-3
Solved Joint Variables from $\hat{M}_{44}$	$\theta_{24}=\text{atan2}(\sigma_5 m_{32},-\sigma_5 m_{12})$ $\theta_5=\text{atan2}(\sigma_5\sqrt{1-m_{22}^2}, m_{22})$ $\theta_6=\text{atan2}(\sigma_5 m_{23}, \sigma_5 m_{21})$	$\theta'_4=\text{atan2}(\sigma_5 m_{23}, \sigma_5 m_{33})$ $\theta'_5=\text{atan2}(-m_{13}, \sigma_5\sqrt{1-m_{13}^2})$ $\theta_6=\text{atan2}(-\sigma_5 m_{12}, \sigma_5 m_{11})$
Wrist Singularity	$\theta_5=0$	$\theta_5=\pm 90°$
Wrist Point Position Vector $\overline{r}_{44}$	$\overline{r}_{44}=a_2e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}\overline{u}_1+s_3e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}\overline{u}_3+d_5e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}e^{\tilde{u}_3\theta_3}e^{\tilde{u}_2\theta_4}\overline{u}_3$ $\overline{r}_{44}=a_2e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}\overline{u}_1+s_3e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}\overline{u}_3+d_5e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_{24}}\overline{u}_3$ for $\theta_3=0°$ $\overline{r}_{44}=a_2e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}\overline{u}_1+s_3e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}\overline{u}_3-d_5e^{\tilde{u}_3\theta_1}e^{\tilde{u}_2\theta_2}e^{-\tilde{u}_1\theta'_4}\overline{u}_2$ for $\theta_3=90°$	
Scalar Wrist Point Equations	$r_1\cos\theta_1+r_2\sin\theta_1=\begin{cases}a_2\cos\theta_2+s_3\sin\theta_2+d_5\sin\theta_{24} & \text{for } \theta_3=0° \\ a_2\cos\theta_2+(s_3+d_5\cos\theta_4)\sin\theta_2 & \text{for } \theta_3=90°\end{cases}$ $r_2\cos\theta_1-r_1\sin\theta_1=\begin{cases}0 & \text{for } \theta_3=0° \\ d_5\sin\theta_4 & \text{for } \theta_3=90°\end{cases}$ $r_3=\begin{cases}-a_2\sin\theta_2+s_3\cos\theta_2+d_5\cos\theta_{24} & \text{for } \theta_3=0° \\ -a_2\sin\theta_2+(s_3+d_5\cos\theta_4)\cos\theta_2 & \text{for } \theta_3=90°\end{cases}$	
θ_3 Selection	$\theta_3=0°$	$\theta_3=90°$
Used T# Equations of Appendix C	T1 for θ_1, T2 for s_3, and T4 for θ_2	T3 for θ_1, T2 for s_3, and T4 for θ_2
Solved Joint Variables	$\theta_1=\text{atan2}(\sigma_1 r_2,\sigma_1 r_1)$ $s_3=\sqrt{\rho_{10}^2+\rho_{11}^2-a_2^2}$ $(s_3>0)$ $\theta_2=\text{atan2}(\rho'_2, \rho_2)$ $\rho_2=s_3\rho_{11}+a_2\rho_{10})$ $\rho'_2=s_3\rho_{10}-a_2\rho_{11}$ $\theta_4=\theta_{24}-\theta_2$ $\rho_{10}=\rho_1-d_5\sin\theta_{24}$ $\rho_{11}=r_3-d_5\cos\theta_{24}$ $\rho_1=r_1\cos\theta_1+r_2\sin\theta_1$	$\theta_1=\text{atan2}(-r_1, r_2)$ $+\sigma_1\text{atan2}(\sqrt{r_1^2+r_2^2-\rho_{12}^2}, \rho_{12})$ $s_3=\sigma_3\sqrt{\rho_1^2+r_3^2-a_2^2}-\rho_{13}$ $(s_3>0)$ $\theta_2=\text{atan2}(\rho'_2, \rho_2)$ $\rho_2=(s_3+\rho_{13})r_3+a_2\rho_1$ $\rho'_2=(s_3+\rho_{13})\rho_1-a_2r_3$ Here, $\rho_{12}=d_5\sin\theta_4$ $\rho_{13}=d_5\cos\theta_4$
Shoulder Singularity	$r_1=r_2=0$	$r_1=r_2=0$ together with $\theta_4=0$ or $\theta_4=\pm 180°$

Table 6. Inverse Kinematic Solution for Subgroup 4.4

4.3 Inverse Kinematics of Manipulators with Closed Kinematic Chains

The method of inverse kinematics presented here is not limited to the serial manipulators only. It can also be applied to robots with a main open kinematic chain supplemented with auxiliary closed kinematic chains for the purpose of motion transmission from the actuators kept close to the base. As a typical example, the ABB IRB2000 industrial robot is selected here in order to demonstrate the application of the method to such manipulators. The kinematic sketch of this manipulator with its four-link transmission mechanism is shown in Figure 2. It can be seen from the kinematic sketch that the four-link mechanism can be considered in a sense as a satellite of the manipulator's main open kinematic chain. In other words, its relative position with respect to the main chain is determined completely by the angle θ_3. Once θ_3 is found by the inverse kinematic solution, the angular position of the third joint actuator ϕ_3 can be determined in terms of θ_3 as follows by considering the kinematics of the four-link mechanism:

$$\phi_3 = \psi_2 + \theta_2 \tag{118}$$

Here,

$$\psi_2 = \mathrm{atan2}(b, a) + \sigma_3 \mathrm{atan2}(\sqrt{a^2 + b^2 + c^2}, c) \quad \text{and} \quad \sigma_3 = \pm 1 \tag{119}$$

$$a = a_2 + b_4 \sin\theta_3, \quad b = b_4 \cos\theta_3, \quad c = \frac{a_2^2 + b_2^2 + b_4^2 - b_3^2}{2b_2} - \frac{a_2 b_4}{b_2} \sin\theta_3 \tag{120}$$

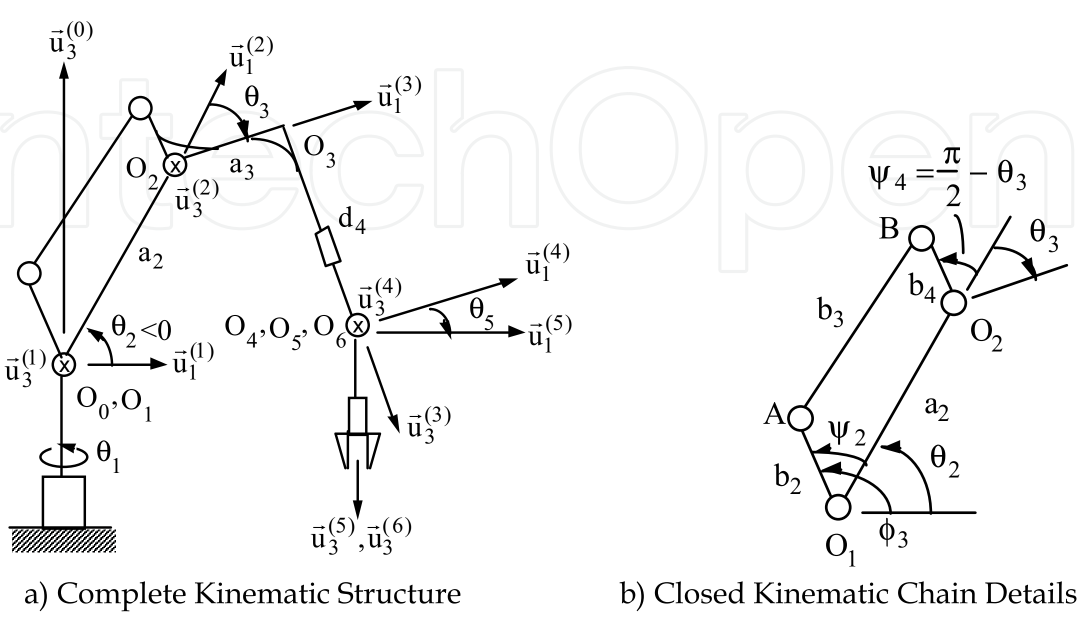

Figure 2. Kinematic Sketch of the ABB IRB2000 Manipulator

However, in this particular manipulator, the four-link mechanism happens to be a parallelogram mechanism so that $\psi_2 = \psi_4$ and $\phi_3 = \theta_2 + \frac{\pi}{2} - \theta_3$. Note that, if the auxiliary closed kinematic chain is separated from the main open kinematic chain, then this manipulator becomes a member of the subgroup 1.4 and the pertinent inverse kinematic solution can be obtained in closed-form similarly as done for the subgroup 1.1.

4.4 Comments on the Solutions

The inverse kinematic solutions of all the subgroups given in Table 4 are obtained. In main group 1, subgroup 1.2 has parameter d_2 in excess when compared to subgroup 1.1. This has an influence only in the solution of θ_1. The remaining joint variable solutions are the same. Similarly subgroup 1.3 has parameter a_1 and subgroup 1.5 has d_{23} (d_2+d_3) in excess when compared to subgroup 1.1. Considering subgroup 1.3, only the solutions of θ_2 and θ_3 are different than the solution of subgroup 1.1, whereas the solution of subgroup 1.5 is identical to the solution of subgroup 1.2 except that d_{23} is used in the formulas instead of d_2. Subgroup 1.6 has parameter a_1 in excess compared to subgroup 1.4. Thus, the solutions of θ_2 and θ_3 are different than the solution of subgroup 1.4. Subgroup 1.8 has parameter d_2 and subgroup 1.9 has a_1 and a_3 in excess when compared to subgroup 1.7. Considering subgroup 1.8, only the solution of θ_1 is different. For subgroup 1.9, a_1 and a_3 causes minor changes in the parameters defined in the solutions of θ_2 and θ_3. The last subgroup, that is subgroup 1.10 has the parameters a_1, a_3 and d_3 in excess when compared to subgroup 1.7. θ_1, θ_2 and θ_3 have the same form as they have in the solution of subgroup 1.7, except the minor changes in the parameters defined in the solutions. It can be concluded that d_2 affects the solution of θ_2 and a_1 affects the solutions of θ_2 and θ_3 through minor changes in the parameters defined in the solution. In main group 2, subgroup has parameter d_2 in excess when compared to subgroup 2.3 and thus the solution of θ_1 has minor changes. Subgroup 2.5 has parameter a_1 in excess when compared to subgroup 2.4 and the solutions of θ_2 and θ_3 have minor changes. Subgroup 2.6 has parameter d_4 in excess and the term including it is identical to the term including d_2 in subgroup 2.5 except θ_1 which includes d_4 instead of d_2. In main group 8, subgroup 8.2 has the parameter a_2 in excess compared to subgroup 8.1. This leads to a minor change in the solutions of θ_1 and θ_2 through the parameters defined in the solution.
For main group 1, if θ_1 is obtained analytically θ_4, θ_5 and θ_6 can be solved in closed-form. Any six-joint manipulator belonging to main group 2 can be solved in closed-form provided that θ_1 is obtained in closed-form. Using θ_1, θ_{234} can easily be determined using the orientation matrix equation. Since θ_4 appears in the terms including a_4, d_5 and a_5 as θ_{234}, this lead to a complete

closed-form solution. In main group 3, directly obtaining θ_1 and θ_2 analytically results in a complete closed-form solution provided that the kinematic parameter $a_5 = 0$. In main group 6, even there is the offset d_5, a complete closed-form solution can be obtained since the term of d_5 does not include θ_4. Moreover, if θ_1 can be obtained in closed-form and a_5 is a nonzero kinematic parameter, simply solving θ_{345} from $\hat{C}$ leads to a complete closed-form solution. Among all the main groups, main group 5 has the least complicated equations. Joint variable θ_{1234} is directly obtained from $\hat{C}$ and if d_5 or a_4 are kinematic parameters of a manipulator belonging to this main group, the whole solution will be in closed-form. The offset in main group 9 does not lead to a PJV solution since it does not include θ_4 in the term including it. Also θ_1 even if a_5 is a nonzero kinematic parameter, a closed-form solution can be obtained using θ_{45} provided that θ_1 is obtained analytically. Since α_4 of main groups 6 and 9 is 0°, θ_4 does not appear in the term including offset d_5. The kinematic parameter a_5 does not appear in any of the subgroups, so it can be concluded that this parameter might appear only in some very specific six-joint manipulators. On the other hand, the more the number of prismatic joints, the easier the solution is, since the joint angles become constant for prismatic joints.

5. Conclusion

In this chapter, a general approach is introduced for a classification of the six-joint industrial robotic manipulators based on their kinematic structures, and a complete set of compact kinematic equations is derived according to this classification. The algebraic tools based on the properties of the exponential rotation matrices have been very useful in simplifying the kinematic equations and obtaining them in compact forms. These compact kinematic equations can be used conveniently to obtain the inverse kinematic solutions either analytically in closed forms or semi-analytically using parametrized joint variables. In either case, the singular and multiple configurations together with the working space limitations are also determined easily along with the solutions.

Moreover, both types of these inverse kinematic solutions provide much easier programming facilities and much higher on-line application speeds compared to the general manipulator-independent but purely numerical solution methods.

On the other hand, although the classification based method presented here is naturally manipulator-dependent, it is still reasonable and practical to use for the inverse kinematic solutions, because a manipulator having all non-zero kinematic parameters does not exist and it is always possible to make a considerable amount of simplification on the kinematic equations before attempting to solve them.

6. References

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Appendix A

Exponential Rotation Matrix Simplification Tools
(Özgören, 1987-2002; Balkan et al., 2001)

E.1 : $(e^{\tilde{u}_i\theta_k})^{-1} = e^{-\tilde{u}_i\theta_k}$

E.2 : $e^{\tilde{u}_i\theta_k}\bar{u}_i = \bar{u}_i$

E.3 : $\bar{u}_i^T e^{\tilde{u}_i\theta_k} = \bar{u}_i^T$

E.4 : $e^{\tilde{u}_i\theta_j} e^{\tilde{u}_i\theta_k} = e^{\tilde{u}_i(\theta_j+\theta_k)} = e^{\tilde{u}_i\theta_{jk}}$

E.5 : $e^{\tilde{u}_i\pi} = e^{-\tilde{u}_i\pi}$

E.6 : $e^{\tilde{u}_i\pi/2} e^{\tilde{u}_j\theta} e^{-\tilde{u}_i\pi/2} = e^{\tilde{n}_{ij}\theta}$ where $\bar{n}_{ij} = \tilde{u}_i\bar{u}_j$.

E.7 : $e^{\tilde{u}_i\theta_k} e^{\tilde{u}_j\pi} = e^{\tilde{u}_j\pi} e^{-\tilde{u}_i\theta_k}$

E.8 : $e^{\tilde{u}_i\theta_k}\bar{u}_j = \bar{u}_j\cos\theta + (\tilde{u}_i\bar{u}_j)\sin\theta$ for $i \neq j$

E.9 : $\bar{u}_j^T e^{\tilde{u}_i\theta_k} = \bar{u}_j^T\cos\theta + (\tilde{u}_j\bar{u}_i)^T\sin\theta$ for $i \neq j$

E.10 : $e^{-\tilde{u}_i\theta_k} e^{\tilde{u}_i\theta_k} = e^{\tilde{u}_i 0} = \hat{I}$

Appendix B

List of Six-joint Industrial Robots Surveyed (Balkan et al., 2001)

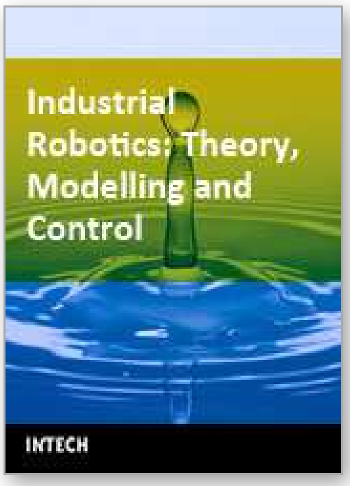
Robot Name	Non-Zero Link Parameters	Subgroup	Robot Name	Non-Zero Link Parameters	Subgroup
ABB IRB 1400	a_1, a_2, a_3, d_4	1.6	Kawasaki UT120	a_1, a_2, a_3, d_4	1.6
ABB IRB 1400H	a_1, a_2, a_3, d_4	1.6	Kawasaki UT150	a_1, a_2, a_3, d_4	1.6
ABB IRB 2400	a_1, a_2, a_3, d_4	1.6	Kawasaki UX100	a_2, a_3, d_4	1.4
ABB IRB 2400L	a_1, a_2, a_3, d_4	1.6	Kawasaki UX120	a_2, a_3, d_4	1.4
ABB IRB 4400	a_1, a_2, a_3, d_4	1.6	Kawasaki UX150	a_2, a_3, d_4	1.4
ABB IRB 4400L	a_1, a_2, a_3, d_4	1.6	Kawasaki UX70	a_2, a_3, d_4	1.4
ABB IRB 6400	a_1, a_2, a_3, d_4	1.6	KUKA IR 662/10	a_2, d_4	1.1
ABB IRB4400FS	a_1, a_2, a_3, d_4	1.6	Limanat RT-280	a_2, a_3, d_4	2.1
ABB IRB6400PE	a_1, a_2, a_3, d_4	1.6	Mack BASE Rob	$a_1, a_2, s_1, s_2, s_3, d_4$	4.1
Acrobe AG-4	a_2, a_3, a_4	2.2	Motoman J10	a_2, d_4	1.1
AKR 3000	a_2, d_4, d_5	1.7	Motoman L3	a_2, a_3, d_5	2.3
Altınay ASR-60	a_2, a_3, d_4	1.4	Motoman SK-10	a_1, a_2, a_3, d_4	1.6
Altınay EG-6	$a_1, a_2, a_3, a_4, d_4, d_5$	9.1	Motoman SK-16	a_1, a_2, a_3, d_4	1.6
Altınay EV-6	a_1, a_2, a_3, d_4	1.6	Motoman SK-30	a_1, a_2, a_3, d_4	1.6
Altınay HSR-6	a_2, d_4	1.1	Motoman SK-45	a_1, a_2, a_3, d_4	1.6
Anorad-Anorobot	$a_1, a_3, s_1, s_2, s_3, d_4$	3.2	Motoman SK-6	a_1, a_2, a_3, d_4	1.6
ATI Robo-Master	a_2, a_3, a_4	2.2	Motoman SK120	a_1, a_2, a_3, d_4	1.6
Bendix AA670	a_1, a_2, d_4	1.3	MotomanSK16-6	a_1, a_2, a_3, d_4	1.6
Binks 88-800	a_2, d_4	1.1	MTS 200A	a_2, a_3, d_4	1.4
CM T3-566	a_2, a_3, a_4	2.2	Nachi Robot8601	a_2, d_4, d_5	1.7
CM T3-856	a_2, a_3, a_4	2.2	Narthwest Tech.	a_2, a_3, a_4	2.2
CMV Inst. DDaII	a_1, a_2, d_3, d_4	8.2	Nordson Robot	a_2, d_4, d_5	1.7
Comau SmartS-2	$a_1, a_2, a_3, d_3, d_4, d_5$	1.10	Numan I&II	s_2, d_4	4.2
CRS A255	a_2, a_3, a_4	2.2	Pharemme BMR	a_1, a_2, d_4	1.3
CRS A465	a_2, d_4	1.1	Portque 80	s_1, s_2, s_3	3.1
Cyclomat Merlin	a_2, d_2, d_4	1.2	PUMA 200	a_2, d_2, d_3, d_4	1.5
Daihen LK	a_2, a_3, d_5	2.3	PUMA 260	a_2, d_2, d_3, d_4	1.5
FANUC S-420	a_1, a_2, a_3, d_4	1.6	PUMA 560	a_2, d_2, d_4	1.2
FanucArcMateSR	a_1, a_2, a_3, d_4, d_5	1.9	PUMA 562	a_2, d_2, d_4	1.2
GCA DKP 1000	a_2, d_4	1.1	PUMA 700	a_2, d_2, d_4	1.2
GCP 132	a_2, d_4	1.1	REIS RR625	a_1, a_2, d_3, d_4	8.2
GMF S-2L	a_2, d_4	1.1	Rhino Charger	a_2, d_4	1.1
GMF S-3L	a_2, d_4, d_5	1.7	SCARA Robot	a_1, a_2, s_3	5.1
GMF S-3R	a_2, d_4, d_5	1.7	Scheinman	a_2, s_3, d_5	3.4
GMF S-LR	a_2, d_4	1.1	Scorbot ERIX	a_1, a_2, a_3, d_2, d_5	2.5
Griffin	a_2, d_4	1.1	Scorbot PerfMK2	a_1, a_2, a_3, d_4, d_5	2.6
Intelledex 660	a_3, a_4	6.1	SEF-SR10	a_1, a_2, d_4	1.3
Kawasaki EE10	a_2, d_2, d_4, d_5	1.8	Stanford JPL	d_2, s_3	3.3
Kawasaki EH120	a_1, d_3, d_4	8.1	Staubli RX-line	a_2, d_4	1.1
Kawasaki JA-5	a_2, d_4	1.1	Staubli RX130	a_2, d_4	1.1
Kawasaki JC-5	a_2, d_4	1.1	Staubli RX60	a_2, d_4	1.1
Kawasaki JS-10	a_2, d_4, d_5	1.7	Staubli RX90	a_2, d_4	1.1
Kawasaki JS-30	a_2, d_4	1.1	TH8	a_2, s_2, s_3	7.1
Kawasaki JS-40	a_2, d_4	1.1	Thermwood 6	a_2, d_4	1.1
Kawasaki JS-5	a_2, d_4	1.1	Thermwood PM	a_2, d_4	1.1
Kawasaki JS-6	a_2, d_4	1.1	Toshiba TSR 500	a_2, a_3, d_2, d_5	2.4
Kawasaki U2100	a_2, a_3, d_4	1.4	Unimate 4000	a_2, s_3, d_5	4.4
Kawasaki U2120	a_2, a_3, d_4	1.4	Unimate Robot	s_3, d_5	4.3
Kawasaki U2150	a_2, a_3, d_4	1.4	V80	a_2, d_4	1.1
Kawasaki UT100	a_1, a_2, a_3, d_4	1.6	6P01	a_2, d_4	1.1

Appendix C

Solutions to Some Trigonometric Equations Encountered in Inverse Kinematic Solutions

T0a T0b T0c	$\sin\theta_i = a$ (alone) $\cos\theta_i = b$ (alone) $\sin\theta_i = a$, $\cos\theta_i = b$ (together)	$\theta_i = \text{atan2}\,(a, \sigma_i\sqrt{1-a^2})$; $\sigma_i = \pm 1$ $\theta_i = \text{atan2}\,(\sigma_i\sqrt{1-b^2}, b)$; $\sigma_i = \pm 1$ $\theta_i = \text{atan2}\,(a, b)$
T1	$a\cos\theta_i - b\sin\theta_i = 0$	$\theta_i = \text{atan2}\,(\sigma_i a, \sigma_i b)$; $\sigma_i = \pm 1$ if $a = b = 0$ singularity occurs
T2	$s_i\cos\theta_j = b$, $s_i\sin\theta_j = a$	$s_i = \sigma_i\sqrt{a^2+b^2}$; $\sigma_i = \pm 1$ $\theta_i = \text{atan2}\,(\sigma_i a, \sigma_i b)$ if $s_i = 0$ singularity occurs
T3	$a\cos\theta_i + b\sin\theta_i = c$	$\theta_i = \text{atan2}\,(b,a) + \sigma_i \text{atan2}\,(\sqrt{a^2+b^2-c^2}, c)$ $\sigma_i = \pm 1$; if $a = b = 0$ singularity occurs
T4	$a\cos\theta_i - b\sin\theta_i = c$ $a\sin\theta_i + b\cos\theta_i = d$	$\theta_i = \text{atan2}\,(ad - bc, ac + bd)$ provided that $a^2+b^2 = c^2+d^2$ and $a^2+b^2 \neq 0$ if $a^2+b^2 = 0$ singularity occurs
T5	$a\cos\theta_i + b\cos\theta_{ij} = c$ $a\sin\theta_i + b\sin\theta_{ij} = d$	$\theta_j = \text{atan2}\,(\sigma_j\sqrt{1-e^2}, e)$; $\sigma_j = \pm 1$ $e = \dfrac{(c^2+d^2)-(a^2+b^2)}{2ab}$ and $\theta_{ij} = \theta_i + \theta_j$
T6	$a\cos\theta_i + b\sin\theta_{ij} = c$ $-a\sin\theta_i + b\cos\theta_{ij} = d$	$\theta_j = \text{atan2}\,(e, \sigma_j\sqrt{1-e^2})$; $\sigma_j = \pm 1$ $e = \dfrac{(c^2+d^2)-(a^2+b^2)}{2ab}$ and $\theta_{ij} = \theta_i + \theta_j$
T7	$a\cos\theta_i + b\sin\theta_i = c + d\cos\theta_j$ $b\cos\theta_i - a\sin\theta_i = d\sin\theta_j$	$\theta_j = \text{atan2}\,(\sigma_j\sqrt{1-e^2}, e)$; $\sigma_j = \pm 1$ $e = \dfrac{(a^2+b^2)-(c^2+d^2)}{2cd}$
T8	$a\cos\theta_i + b\sin\theta_i = c - d\sin\theta_j + e\cos\theta_j$ $b\cos\theta_i - a\sin\theta_i = d\sin\theta_j + e\cos\theta_j$	$\theta_j = \text{atan2}\,(-d, e) + \sigma_j \text{atan2}\,(\sqrt{d^2+e^2-f^2}, f)$ $f = \dfrac{(a^2+b^2)-(c^2+d^2+e^2)}{2c}$; $\sigma_j = \pm 1$
T9	$a\sin\theta_{ij} + b\cos\theta_{ij} + d\cos\theta_i = e$ $-a\cos\theta_{ij} + b\sin\theta_{ij} + d\sin\theta_i = f$	$\theta_j = \text{atan2}\,(a,b) + \sigma_j \text{atan2}\,(\sqrt{a^2+b^2-g^2}, g)$ $g = \dfrac{(e^2+f^2)-(a^2+b^2+d^2)}{2d}$; $\sigma_j = \pm 1$

The two-unknown trigonometric equations T5-T8 and T9 become similar to T4 and T0c respectively once θ_j is determined from them as indicated above. Then, θ_i can be determined as described for T4 or T0c.



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This book covers a wide range of topics relating to advanced industrial robotics, sensors and automation technologies. Although being highly technical and complex in nature, the papers presented in this book represent some of the latest cutting edge technologies and advancements in industrial robotics technology. This book covers topics such as networking, properties of manipulators, forward and inverse robot arm kinematics, motion path-planning, machine vision and many other practical topics too numerous to list here. The authors and editor of this book wish to inspire people, especially young ones, to get involved with robotic and mechatronic engineering technology and to develop new and exciting practical applications, perhaps using the ideas and concepts presented herein.