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# Modeling and Control of Mechanical Systems in Terms of Quasi-Velocities

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## 1. Introduction

Multi-body systems' (MBS) dynamics are often described by the second-order nonlinear equations parameterized by a configuration-dependent inertia matrix and the nonlinear vector containing the Coriolis and centrifugal terms. Since these equations are the cornerstone for simulation and control of robotic manipulators, many researchers have attempted to develop efficient modelling techniques to derive the equations of motion of multi-body systems in novel forms. A unifying idea for most modeling techniques is to describe the equations of motion in terms of general coordinates and their time-derivatives. In classical mechanics of constrained systems, a generalized velocity is taken to be an element of tangential space of configuration manifold, and a generalized force is taken to be the cotangent space. However, neither does space possess a natural metric as the generalized coordinates or the constraints may have a combination of rotational and translational components. As a result, the corresponding dynamic formulation is not invariant and a solution depends on measure units or a weighting matrix selected Aghili (2005); Angeles (2003); Lipkin and Duffy (1988); Luca and Manes (1994); Manes (1992). There also exist other techniques to describe the equations of motion in terms of *quasi-velocities*, i.e., a vector whose Euclidean norm is proportional to the square root of the system's kinetic energy, which can lead to simplification of these equations Aghili (2008; 2007); Bedrossian (1992); Gu (2000); Gu and Loh (1987); Herman (2005); Herman and Kozlowski (2006); Jain and Rodriguez (1995); Junkins and Schaub (1997); Kodischeck (1985); Kozlowski (1998); Loduha and Ravani (1995); Papastavridis (1998); Rodriguez and Kertutz-Delgado (1992); Sinclair *et al.* (2006); Spong (1992). A recent survey on some of these techniques can be found in Herman and Kozlowski (2006). In short, the square-root factorization of mass matrix is used as a transformation to obtain the quasi-velocities, which are a linear combination of the velocity and the generalized coordinates Herman and Kozlowski (2006); Papastavridis (1998).

It was shown by Kodistchek Kodischeck (1985) that if the square-root factorization of the inertia matrix is integrable, then the robot dynamics can be significantly simplified. In such a case, transforming the generalized coordinates to quasi-coordinates by making use of the integrable factorization modifies the robot dynamics to a system of double integrator. Then, the cumbersome derivation of the Coriolis and centrifugal terms is not required. It was later realized by Gu *et al.* Gu and Loh (1987) that such a transformation is a canonical transformation because it satisfies Hamilton's equations. Rather than deriving the mass matrix of MBS first and then obtaining its factorization, Rodriguez *et al.* Rodriguez and Kertutz-Delgado (1992)

derived the closed-form expressions of the mass matrix factorization of an MBS and its inverse directly from the link geometric and inertial parameters. This eliminates the need for the matrix inversion required to compute the forward dynamics.

The interesting question of when the factorization of the inertia matrix is integrable, i.e., the factorization being the Jacobian of some quasi-coordinates, was addressed independently in Spong (1992) and Bedrossian (1992). Using the notion that the inertia matrix defines a metric tensor on the configuration manifold, Spong (1992) showed that the necessary and sufficient condition for the existence of an integrable factorization of the inertia matrix is that the metric tensor is a Euclidean metric tensor.<sup>1</sup> It turned out that for most of the practical robot systems, the condition is not satisfied meaning that integration of the quasi-velocities does not produce any quasi-coordinates. Jain and Rodriguez (1995) described quasi-velocities are obtained as a result of diagonalizing the inertia matrix. Instead of diagonalizing globally in configuration space, they look at a diagonalizing transformation in the velocity space. The transformation replaces generalized velocity with the quasi-velocities, without replacing the configuration variables. The concept of quasi-velocities has also been used for the set-point control of manipulators Herman (2005); Herman and Kozlowski (2001); Jain and Rodriguez (1995); Kozlowski (1998); Kozlowski and Herman (2000). However, the problem of the tracking control of manipulators using quasi-velocities feedback still remains unsolved owing to unintegrability of the quasi-velocities.

The goal of this chapter is to extend the concept of quasi-velocities for an gauge-invariant formulation of constrained MBS that can be used for simulation, analysis, and control purposes Aghili (2009). The main focus of previous works on modeling of constrained or unconstrained mechanical systems using the notion of quasi-velocities, e.g., Junkins and Schaub (1997); Loduha and Ravani (1995), has been decoupling of the equations of motion, which yields a dynamical system with an identity mass matrix. Analysis of constraint force has not been considered in the previous works. In this paper, we took advantages of the fact that quasi-velocities are not unique but they are related by unitary transformations and found a particular transformation which allows to decouple the equations of motions and the equation of constraints in such a way that separate control inputs are associated to each set of equations. This facilitates motion/force control of constrained systems such as robotic manipulators. Moreover, unlike other approaches Aghili (2005); Doty *et al.* (1993a); Luca and Manes (1994); Schutter and Bruyincx (1996), this formulation does not require any weighting matrix to achieve gauge-invariance when both translational and rotational components are involved in the generalized coordinates or in the constraint equations. Some properties of the quasi-velocities dynamic formulation are presented that could be useful for control purposes. Finally, the dynamic model is used for developing tracking control of constrained MBS based on a combination of feedbacks on the vector of reduced quasi-velocities and the vectors of configuration-variables Aghili (2009).

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<sup>1</sup> A manifold with a Euclidean metric is said to be "flat" and the curvature associated with it is identically zero Jain and Rodriguez (1995).

## 2. Quasi-Variables Transformation

### 2.1 Square-Root Factorization of the Mass Matrix

Dynamics of MBS with kinetic energy,  $T$ , and potential energy,  $P$ , obey the standard *Euler–Lagrange* (EL) equations, which are given as

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{f}, \quad (1)$$

where  $\mathbf{q} \in \mathbb{R}^n$  is the vector of configuration-variables<sup>2</sup> used to define the configuration of the system, and  $\mathbf{f}$  is the generalized forces acting on the system. The generalized forces  $\mathbf{f} = \mathbf{f}_p + \mathbf{f}_a$  contain all possible external forces including the conservative forces  $\mathbf{f}_p = -\partial P / \partial \mathbf{q}$  owing to gravitational energy plus all active and dissipative forces represented by  $\mathbf{f}_a$ . The system kinetic energy is in the following quadratic form:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}, \quad (2)$$

where the generalized inertia matrix  $\mathbf{M}(\mathbf{q})$  is *symmetric* and *positive definite* for all  $\mathbf{q}$ . It is well known that any symmetric positive-definite matrix  $\mathbf{M}$  can be decomposed as

$$\mathbf{M} = \mathbf{W} \mathbf{W}^T, \quad (3)$$

where  $\mathbf{W}$  is the *square root factorization* of  $\mathbf{M}$ , e.g. the Cholesky decomposition; see Appendix A.

Considering the transformation

$$\bar{\mathbf{W}} = \mathbf{W} \mathbf{V},$$

where  $\mathbf{V}$  is an orthogonal matrix, i.e.,  $\mathbf{V} \mathbf{V}^T = \mathbf{V}^T \mathbf{V} = \mathbf{I}$ , one can trivially verify that  $\bar{\mathbf{W}} \bar{\mathbf{W}}^T = \mathbf{M}$ . Thus, we get the following remark

**Remark 1.** The square-root factorization (3) is not unique, rather they are related by unitary transformations.

Now, substituting (3) into (2) and then applying the EL formulation yields

$$\begin{aligned} \mathbf{f} &= \frac{d}{dt} \left( \mathbf{W} \mathbf{W}^T \dot{\mathbf{q}} \right) - \frac{1}{2} \left( \frac{\partial}{\partial \mathbf{q}} \|\mathbf{W}^T(\mathbf{q}) \dot{\mathbf{q}}\|^2 \right)^T \\ &= \mathbf{W} \frac{d}{dt} (\mathbf{W}^T \dot{\mathbf{q}}) + \left( \dot{\mathbf{W}} - \frac{\partial (\mathbf{W}^T(\mathbf{q}) \dot{\mathbf{q}})^T}{\partial \mathbf{q}} \right) \mathbf{W}^T \dot{\mathbf{q}} \end{aligned} \quad (4)$$

Note that (4) is obtained using the property that for any vector field  $\mathbf{a}(\mathbf{q})$ , we have

$$\frac{\partial}{\partial \mathbf{q}} \|\mathbf{a}(\mathbf{q})\|^2 = 2 \mathbf{a}^T \frac{\partial \mathbf{a}}{\partial \mathbf{q}}. \quad (5)$$

Define

$$\mathbf{v} \triangleq \mathbf{W}^T(\mathbf{q}) \dot{\mathbf{q}} \quad (6a)$$

$$\mathbf{u} \triangleq \mathbf{W}^{-1}(\mathbf{q}) \mathbf{f}, \quad (6b)$$

<sup>2</sup> also known as generalized coordinates

which are called here as the vectors of *quasi-velocities* and *quasi-forces*, respectively. It should be pointed out that in analytical dynamics, quasi-velocities are broadly defined as any linear combination of velocities Baruh (1999); Corben and Stehle (1960); Meirovitch (1970). Since  $\det \mathbf{W} = \sqrt{\det \mathbf{M}} \neq 0$ ,  $\mathbf{W}^{-1}$  is well-defined and hence the reciprocals of relations (6) always exist. Pre-multiplying (4) by  $\mathbf{W}^{-1}$  and the substituting (6) into the resultant equation, we arrive at the equations of mechanical systems expressed by the quasi-variables:

$$\dot{\mathbf{v}} + \mathbf{\Gamma} \mathbf{v} = \mathbf{u}, \quad (7a)$$

where

$$\mathbf{\Gamma} \triangleq \mathbf{W}^{-1} \left( \dot{\mathbf{W}} - \frac{\partial \mathbf{v}^T}{\partial \mathbf{q}} \right) \quad (7b)$$

is the Coriolis term associated with the quasi-velocities. Note that the quasi-velocities is factored out in the derivation of the Coriolis term (7b) that is different from the previous derivation Jain and Rodriguez (1995). As will be seen in the following section, this representation is useful when the formulation is extended for constrained MBS.

## 2.2 Changing Coordinates by Unitary Transformations

Remark 1 states that the quasi-velocities (and also quasi-forces) can not be uniquely determined. Rather, the following variables:

$$\bar{\mathbf{v}} = \mathbf{V}^T \mathbf{v} \quad \text{and} \quad \bar{\mathbf{u}} = \mathbf{V}^T \mathbf{u}, \quad (8)$$

obtained by any unitary transformation  $\mathbf{V}$ , are also valid choices for the new quasi-velocities and quasi-forces. Now we are interested to derive the equations of motion expressed by the new quasi-variables  $\bar{\mathbf{v}}$ . To this end, using the reciprocal of (8), i.e.,  $\mathbf{v} = \mathbf{V} \bar{\mathbf{v}}$  and  $\mathbf{f} = \mathbf{V} \bar{\mathbf{f}}$ , into (7a) and then multiplying the resultant equation by  $\mathbf{V}$ , we arrive at

$$\dot{\bar{\mathbf{v}}} + \mathbf{V}^T \dot{\mathbf{V}} \bar{\mathbf{v}} + \mathbf{V}^T \mathbf{\Gamma} \mathbf{V} \bar{\mathbf{v}} = \bar{\mathbf{u}} \quad (9)$$

Analogous to the rotation transformation in the three-dimensional Euclidean space, consider matrix  $\mathbf{V}$  as a transformation in the  $n$ -dimensional space. Then, it is known that the time-derivative of a differentiable orthogonal matrix  $\mathbf{V}$  satisfies a differential equation of this form Schaub *et al.* (1995)

$$\dot{\mathbf{V}} = -\mathbf{\Omega} \mathbf{V}, \quad (10)$$

where  $\mathbf{\Omega}$  is a *skew symmetric matrix* representing the angular rates in  $n - D$  space Bar-Itzhack (1989). The elements of  $\mathbf{\Omega}$  can be interpreted as a generalized eigenvector axis angular velocity Junkins and Schaub (1997). It is worth noting that in the three-dimensional space, the angular rate matrix can be obtained from the vector of angular velocity by  $\mathbf{\Omega} = [\boldsymbol{\omega} \times]$ . For the  $n$ -dimensional case, the method for computing the elements of matrix  $\mathbf{\Omega}$  can be found in Junkins and Schaub (1997); Oshman and Bar-Itzhack (1985); Schaub *et al.* (1995). Finally, by replacing (10) in (9), we can show that the latter equation is equivalent to

$$\dot{\bar{\mathbf{v}}} + \bar{\mathbf{\Gamma}} \bar{\mathbf{v}} = \bar{\mathbf{u}},$$

where

$$\bar{\mathbf{\Gamma}} = \mathbf{V}^T (\mathbf{\Gamma} - \mathbf{\Omega}) \mathbf{V}. \quad (11)$$

### 2.3 Conservation of Kinetic Energy

The kinetic energy expressed by the quasi-velocities is trivially

$$T = \frac{1}{2} \|v\|^2. \quad (12)$$

Note that in the following,  $\|\cdot\|$  denotes either the matrix 2-norm or the Euclidean vector norm. In the absence of any external force, the principle of conservation of kinetic energy dictates that the kinetic energy of mechanical system is bound to be constant, i.e.,  $u = 0 \implies \dot{T} = 0$ . On the other hand, the zero-input response of a mechanical system is  $\dot{v} = -\Gamma v$ . Substituting the latter equation in the time-derivative of (12) gives

$$v^T \Gamma v = 0, \quad (13)$$

which is consistent with the earlier result reported by Jain *et al.* Jain and Rodriguez (1995) that the Coriolis term associated with quasi-velocities does no mechanical work. Note that (13) is a necessary but not a sufficient condition for  $\Gamma$  to be a *skew-symmetric* matrix.

### 2.4 State-Space Model

It should be pointed out that despite of the one-to-one correspondence between velocity coordinate  $\dot{q}$  and the quasi-velocity  $v$ , they are not synonymous. This is because the integration of the former variable leads to the generalized coordinate, while that of the latter variable does not always lead to a meaningful vector describing the configuration of the mechanical system. Defining a matrix  $R = W\Gamma$ , we can calculate its  $ij$ th element from (7b) through the following equations

$$R_{ij} = \sum_k \left( \frac{\partial W_{ij}}{\partial q_k} - \frac{\partial W_{kj}}{\partial q_i} \right) \dot{q}_k. \quad (14)$$

Here  $W_{ij}$  and  $\dot{q}_k$  are the  $(i, j)$ th entry of matrix  $W$  and the  $k$ th element of vector  $\dot{q}$ , respectively. Now let us assume that  $\dot{\xi} = v$ , where  $\xi$  is called *quasi-coordinates*. For  $\xi$  to be an explicit function of  $q$ , i.e.,  $\xi = \xi(q)$ , it must be the gradient of a scalar function meaning that  $\xi$  is a *conservative vector field*. In that case, (6a) implies that  $W^T(q)$  is actually a Jacobian as  $W_{ij} = \partial \xi_j / \partial q_i$ . Since the Jacobian is an invertible matrix,  $\xi(q)$  must be an invertible function meaning that there is a one-to-one correspondence between  $\xi$  and  $q$ . Under this circumstance,  $\xi$  and  $v$  are indeed alternative possibilities for generalized coordinates and generalized velocities and that can fundamentally simplify the equations of motion Bedrossian (1992); Gu and Loh (1987); Kodischek (1985); Spong (1992). It can be also seen from (7b) that if  $\xi(q)$  exists and it is a smooth function, then the expression in the parenthesis of the right-hand side of (14) vanishes, i.e.,

$$\frac{\partial W_{ij}}{\partial q_k} - \frac{\partial W_{kj}}{\partial q_i} = \frac{\partial^2 \xi_j}{\partial q_i \partial q_k} - \frac{\partial^2 \xi_j}{\partial q_k \partial q_i} = 0,$$

because of the equality of mixed partials. Thus,  $\Gamma \equiv 0$  and the equations of motion become a simple integrator system.

Technically speaking, a necessary and sufficient condition for the existence of the quasi-coordinates,  $\xi$ , is that the Riemannian manifold defined by the robot inertia matrix  $M(q)$  be locally flat<sup>3</sup>. However, that has been proved to be a very stringent condition Bedrossian

<sup>3</sup> By definition, a Riemannian manifold that is locally isometric to Euclidean manifold is called a locally flat manifold Spong (1992).



(1992). Nevertheless, vector  $\mathbf{x}^T = [\mathbf{q}^T \ \mathbf{v}^T]$  is sufficient to describe completely the states of MBS. Hence, similar to Jain and Rodriguez (1995), we look at the transformation only in the velocity space. That is, only the velocity coordinate is replaced with the quasi-velocity whereas the generalized coordinate remains. Setting (6a) and (7a) in state space form gives

$$\frac{d}{dt} \begin{bmatrix} \mathbf{q} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{W}^{-T} \\ -\mathbf{\Gamma} \end{bmatrix} \mathbf{v} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}. \quad (15)$$

It is interesting to note that dynamics system (15) is in the form of the so-called *second-order kinematic model* of constrained mechanism, which appears in kinematics of nonholonomic systems. This is the manifestation of the fact that the integration of quasi-velocities, in general, does not lead to quasi-coordinates.

### 3. Constrained Mechanical Systems

#### 3.1 Equations of Motion

In this section, we extend the notion of the quasi-velocity for modeling of constrained mechanical systems where the coordinates are related by a set of  $m$  algebraic equations  $\Phi(\mathbf{q}) = \mathbf{0}$ . The constraints can be written in the Pfaffian form as

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0} \quad (16)$$

where Jacobian  $\mathbf{A} = \partial\Phi/\partial\mathbf{q} \in \mathbb{R}^{m \times n}$  is not necessarily a full-rank matrix because of the possible redundant constraints. The EL equations of the constrained MBS with kinetic energy  $T$  are

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{f} - \mathbf{A}^T \boldsymbol{\lambda}, \quad (17)$$

where  $\boldsymbol{\lambda} \in \mathbb{R}^m$  are the *generalized Lagrangian multipliers*.

Using any form of the square-root factorizations in a development similar to (6)-(7), we can show that (17) is equivalent to

$$\dot{\mathbf{v}} + \mathbf{\Gamma} \mathbf{v} = \mathbf{u} - \mathbf{\Lambda}^T \boldsymbol{\lambda}, \quad (18)$$

where

$$\mathbf{\Lambda} \triangleq \mathbf{A} \mathbf{W}^{-T}. \quad (19)$$

It can be verified that the quasi-velocities satisfy the following Paraffin constraint equation:

$$\mathbf{\Lambda} \mathbf{v} = \mathbf{0}. \quad (20)$$

Also, (20) may suggest that  $\mathbf{\Lambda}$  be taken as the Jacobian of the constraint with respect to the quasi-coordinates. However, this is true only if the quasi-coordinates ever exist. This means that, in general, system (18) together with (20) most likely constitutes a non-holonomic system even though the configuration-variables  $\mathbf{q}$  satisfies a holonomic constraint equation.

Since  $\mathbf{W}$  is a full-rank matrix, we can say  $\text{rank}(\mathbf{\Lambda}) = \text{rank}(\mathbf{A}) = r$ , where  $r \leq m$  is the number of independent constraints. Then, according to the *singular value decomposition* (SVD) there exist unitary (orthogonal) matrices  $\mathbf{U} = [\mathbf{U}_1 \ \mathbf{U}_2] \in \mathbb{R}^{m \times m}$  and  $\mathbf{V} = [\mathbf{V}_1 \ \mathbf{V}_2] \in \mathbb{R}^{n \times n}$  (i.e.,  $\mathbf{U}^T \mathbf{U} = \mathbf{I}_m$  and  $\mathbf{V}^T \mathbf{V} = \mathbf{I}_n$ ) such that

$$\mathbf{\Lambda} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \quad \text{where} \quad \mathbf{\Sigma} = \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (21)$$

and  $\mathbf{S} = \text{diag}(\sigma_1, \dots, \sigma_r)$  with  $\sigma_1 \geq \dots \geq \sigma_r > 0$  being the singular values Klema and Laub (1980); Press *et al.* (1988). The unitary matrices are partitioned so that the dimensions of the submatrices  $\mathbf{U}_1$  and  $\mathbf{V}_1$  are consistent with those of  $\mathbf{S}$ . That is the columns of  $\mathbf{U}_1$  and  $\mathbf{V}_2$  are the corresponding sets of orthonormal eigenvalues which span the range space and the null space of  $\mathbf{\Lambda}$ , respectively Golub and Loan (1996). As will be seen in the following, derivation of the equations of motion hinges on computing a basis for the kernel of matrix  $\mathbf{\Lambda}$ , which constitute the columns of  $\mathbf{V}_2$ . Fortunately, there are many powerful symbolic algorithms and even commercial softwares to do that Anton (2003).

Now, we take advantage of the arbitration in choosing the square-root factorization to find a particular one that leads to decoupling of the equations of motion and those of constrained force. Consider the unitary transformation (8) where the orthogonal matrix  $\mathbf{V}$  corresponds to decomposition (21). Then, the equations of motion expressed in terms of the new quasi-variables become

$$\dot{\bar{\mathbf{v}}} + \bar{\mathbf{\Gamma}}\bar{\mathbf{v}} = \bar{\mathbf{u}} - \bar{\mathbf{\Lambda}}^T \boldsymbol{\lambda}, \quad (22)$$

where  $\bar{\mathbf{\Lambda}} \triangleq \mathbf{\Lambda}\mathbf{V}$  and  $\bar{\mathbf{\Gamma}}$  has been already defined in (11). Again, it can be easily verified that the new quasi-velocities satisfy the following Pfaffian constraints:

$$\bar{\mathbf{\Lambda}}\bar{\mathbf{v}} = \mathbf{0}. \quad (23)$$

At the first glance, the transformed system (22)–(23) reassembles (18)–(20) without gaining any simplification. However, it is the structure of  $\bar{\mathbf{\Lambda}}$  that will result in further simplification. Using (21) in the definition of  $\bar{\mathbf{\Lambda}}$  gives

$$\bar{\mathbf{\Lambda}} = [\mathbf{\Lambda}_r \quad \mathbf{0}_{m \times (n-r)}] \quad \text{where} \quad \mathbf{\Lambda}_r \triangleq \mathbf{U}_1 \mathbf{S}. \quad (24)$$

Since  $\mathbf{\Lambda}_r \in \mathbb{R}^{m \times r}$  is a full-rank matrix, it can be inferred from (23) that the first  $r$ th elements of the transformed quasi-velocity  $\bar{\mathbf{v}}$  must be zero. That is,

$$\bar{\mathbf{v}} = \begin{bmatrix} \mathbf{0}_{r \times 1} \\ \mathbf{v}_r \end{bmatrix}, \quad (25)$$

where  $\mathbf{v}_r \in \mathbb{R}^{n-r}$  represents a set of *reduced quasi-velocities*—in the following, the subscript  $r$  denotes variables associated with the reduced-order variables. Clearly, the zero components of the transformed quasi-velocities are due to the  $r$ -independent constraints. It can be verified that (25) is equivalent to

$$\mathbf{V}_2^T \bar{\mathbf{v}} = \mathbf{v}_r. \quad (26)$$

Now, by using (26) in the reciprocal of relation (6a), we can show that there is a one-to-one correspondence between  $\mathbf{v}$  and  $\dot{\mathbf{q}}$  as

$$\dot{\mathbf{q}} = \mathbf{W}^{-T} \mathbf{V}_2 \mathbf{v}_r, \quad \text{and} \quad \mathbf{v}_r = \mathbf{V}_2^T \mathbf{W}^T \dot{\mathbf{q}}. \quad (27)$$

Moreover, by virtue of (25), we partition the quasi-forces accordingly as

$$\bar{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_o \\ \mathbf{u}_r \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} \mathbf{u}_o \triangleq \mathbf{V}_1^T \mathbf{W}^{-1} \mathbf{f} \\ \mathbf{u}_r \triangleq \mathbf{V}_2^T \mathbf{W}^{-1} \mathbf{f} \end{array}. \quad (28)$$

In addition, we assume that matrix  $\bar{\mathbf{\Gamma}}$  is divided into four block matrices

$$\bar{\mathbf{\Gamma}}_{ij} = \mathbf{V}_i^T (\mathbf{\Gamma} - \mathbf{\Omega}) \mathbf{V}_j, \quad i, j = 1, 2, \quad (29)$$



and then define

$$\Gamma_r \triangleq \bar{\Gamma}_{22}, \quad \text{and} \quad \Gamma_o \triangleq \bar{\Gamma}_{12}. \quad (30)$$

Now, substituting (25) into (22) and then using definitions (28) and (29), we arrive at

$$\dot{\mathbf{v}}_r + \Gamma_r \mathbf{v}_r = \mathbf{u}_r, \quad (31a)$$

and

$$\Lambda_r^T \boldsymbol{\lambda} + \Gamma_o \mathbf{v}_r = \mathbf{u}_o \quad (31b)$$

Apparently, (31a) and (31b) represent the equations of motion and those of constraint force which are completely decoupled from each other. Note that the partitioned components of the quasi-forces, i.e.,  $\bar{\mathbf{u}}_r$  and  $\bar{\mathbf{u}}_o$ , contribute exclusively to the motion system and the constraint force system, respectively. Now, we are ready to combine (31a) and (27) into the state-space form:

$$\frac{d}{dt} \begin{bmatrix} \mathbf{q} \\ \mathbf{v}_r \end{bmatrix} = \begin{bmatrix} \mathbf{W}^{-T} \mathbf{V}_2 \\ -\Gamma_r \end{bmatrix} \mathbf{v}_r + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_r. \quad (32)$$

The above equation can be viewed as the special case of Kane's equations Kane (1961); Kane and Levinson (1985) where all particles have unit mass.

The Lagrangian multipliers can be uniquely obtained from (31b) through matrix inversion only if  $r = m$ , i.e., in the presence of no redundant constraints. Otherwise, there are fewer equations than unknowns, and hence there is no unique solution to (31b). Nevertheless, the minimum norm solution can be found by

$$\min \|\boldsymbol{\lambda}\| \quad \leftarrow \quad \boldsymbol{\lambda} = \mathbf{U}_1 \mathbf{S}^{-1} (\mathbf{u}_o - \Gamma_o \mathbf{v}_r). \quad (33)$$

### 3.2 Calculating the Coriolis Term

The Coriolis force term  $\Gamma_r$  itself characterized completely the motion dynamics of a constrained mechanical system expressed by reduced quasi-velocities. In this section, we describe  $\Gamma_r$  expressed in terms of  $\mathbf{v}_r$  that appears to be simpler than (30). First, in view of (5) and the facts that  $\mathbf{v} = \mathbf{V}_2 \mathbf{v}_r$  and  $\|\mathbf{v}_r\| = \|\mathbf{v}\|$ , one can verify that

$$\frac{\partial \mathbf{v}_r}{\partial \mathbf{q}} = \frac{\partial \mathbf{v}}{\partial \mathbf{q}} \mathbf{V}_2. \quad (34)$$

Now, consider the relation between  $\mathbf{v}_r$  and  $\dot{\mathbf{q}}$  as

$$\mathbf{v}_r = \mathbf{W}_r^T(\mathbf{q}) \dot{\mathbf{q}},$$

where  $\mathbf{W}_r = \mathbf{W} \mathbf{V}_2$ . Then, from (7b), (10), (30), and (35) we obtain

$$\begin{aligned} \Gamma_r &= \mathbf{V}_2^T \mathbf{W}^{-1} \left( \dot{\mathbf{W}} - \frac{\partial \mathbf{v}^T}{\partial \mathbf{q}} \right) \mathbf{V}_2 + \mathbf{V}_2^T \dot{\mathbf{V}}_2 \\ &= \mathbf{V}_2^T \mathbf{W}^{-1} (\dot{\mathbf{W}}_r - \mathbf{W} \dot{\mathbf{V}}_2 - \frac{\partial \mathbf{v}^T}{\partial \mathbf{q}} \mathbf{V}_2) + \mathbf{V}_2^T \dot{\mathbf{V}}_2 \\ &= \mathbf{V}_2^T \mathbf{W}^{-1} (\dot{\mathbf{W}}_r - \frac{\partial \mathbf{v}_r^T}{\partial \mathbf{q}}). \end{aligned} \quad (35)$$

Finally, by noting that  $\mathbf{V}_2^T \mathbf{W}^{-1} = \mathbf{W}_r^+$  is a left inverse of  $\mathbf{W}_r$ , that is,  $\mathbf{W}_r^+ \mathbf{W}_r = \mathbf{I}$ , we can express (35) by

$$\Gamma_r = \mathbf{W}_r^+ (\dot{\mathbf{W}}_r - \frac{\partial \mathbf{v}_r^T}{\partial \mathbf{q}}), \quad (36)$$

which closely resembles the Coriolis term of unconstrained mechanical systems in (7b). It is interesting to note that  $\mathbf{W}_r \in \mathbb{R}^{n \times (n-r)}$  can be thought of as the factorization of the semi-positive "mass matrix"  $\mathbf{M}_r = \mathbf{W}_r \mathbf{W}_r^T = \mathbf{W} \mathbf{P} \mathbf{W}^T$ , where  $\mathbf{P} = \mathbf{V}_2 \mathbf{V}_2^T$  is a projection matrix which projects vectors from  $\mathbb{R}^n$  to the null space of system (20). A comparison between systems (31a)–(36) and (7) reveals that the formulation of constrained mechanical systems remains essentially similar to that of unconstrained mechanical systems if the quasi-velocity is simply replaced by a reduced quasi-velocity.

Finally, a development similar to (35) shows that

$$\Gamma_o = \mathbf{W}_r^- \left( \dot{\mathbf{W}}_r - \frac{\partial \mathbf{v}_r^T}{\partial \mathbf{q}} \right),$$

where  $\mathbf{W}_r^- = \mathbf{V}_1^T \mathbf{W}^{-1}$  is an annihilator for  $\mathbf{W}_r$ , i.e.,  $\mathbf{W}_r^- \mathbf{W}_r = \mathbf{0}$ .

#### 4. Force/Motion Control

In general, it should be always possible to choose a minimal set of independent velocity coordinate, equal in number of the degrees-of-freedom (DOF) exhibited by the mechanical system. However, a minimal set of independent generalized coordinates may not exist; a well-known example is the orientation configuration of a rigid-body that can not be expressed by a three-dimensional vector. However, the conventional control of constrained mechanical system relies on the existence of a minimal set of parameters defining the configuration of a constrained MBS. In this section, we provide velocity and position feedbacks from (reduced) quasi-velocities and (dependent) configuration variables, respectively, for tracking control and regulating a constrained MBS. Interestingly enough, the control challenge, then, becomes similar to that of non-holonomic systems, as the configuration of MBS can not be represented by any quasi-coordinates.

##### 4.1 Properties

First, we explore some properties of system (31) that will be useful in control design purposes.

**Remark 2.** Using (13) and the fact that  $\boldsymbol{\Omega}$  is a skew-symmetric matrix in definition (29), we can say

$$\mathbf{v}_r^T \Gamma_r \mathbf{v}_r = 0.$$

##### 4.2 Tracking Control

Due to presence of only  $r$  independent constraints, the actual number of degrees of freedom of the system is reduced to  $n - r$ . Thus, in principle, there must be  $n - r$  independent variables  $\boldsymbol{\theta}(\mathbf{q}) \in \mathbb{R}^{n-r}$ , which is also called a *minimal set of generalized coordinates*. In view of the time-derivative of the minimal set of generalized coordinates,  $\frac{d}{dt} \boldsymbol{\theta}(\mathbf{q})$ , and (27), we get

$$\dot{\boldsymbol{\theta}} = \mathbf{B}(\boldsymbol{\theta}) \mathbf{v}_r, \quad \text{where} \quad \mathbf{B} \triangleq \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{q}} \mathbf{W}^{-T}(\mathbf{q}) \mathbf{V}_2. \quad (37)$$

Since both velocities variables  $\mathbf{v}_r$  and  $\dot{\boldsymbol{\theta}}$  are with the same dimension, the reciprocal of mapping (37) must uniquely exist, i.e.,  $\mathbf{v}_r = \mathbf{B}^{-1}(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}}$ .

We adopt a Lyapunov-based control scheme (Canudas de Wit *et al.*, 1996, p. 74) for designing a feedback control in terms of quasi-velocities. Define the composite error

$$\boldsymbol{\epsilon} \triangleq \tilde{\mathbf{v}}_r + \mathbf{B}^{-1}(\boldsymbol{\theta}) \mathbf{K}_p \tilde{\boldsymbol{\theta}}, \quad (38)$$

where  $K_p > 0$ ,  $\tilde{v}_r = v_r - v_{r_d}$ , and  $\tilde{\theta} = \theta - \theta_d$ . Also, define the new variable as  $s = v_{r_d} - B^{-1}K_p\tilde{\theta}$ , which is used in the following control law:

$$u_r = \dot{s} + \Gamma_r s - K_d \epsilon, \quad (39)$$

where  $K_d > 0$ . Applying control law (39) to system (31a) gives the dynamics of the error  $\epsilon$  in terms of the first-order differential equation:

$$\dot{\epsilon} = -(\Gamma_r + K_d)\epsilon. \quad (40)$$

As shown in Appendix B, the solution of (40) is bounded by

$$\|\epsilon\| \leq \|\epsilon(0)\|e^{-\eta_1 t}, \quad (41)$$

where  $\eta_1 = 2\lambda_{\min}(K_d)$ , and hence the composite error  $\epsilon$  is exponentially stable.

Pre-multiplying both sides of (38) by  $B(\theta)$ , the resultant equation can be rearranged to the following differential equation

$$\dot{\tilde{\theta}} = -K_p\tilde{\theta} + (B(\theta) - B(\theta_d))v_d + B(\theta)\epsilon. \quad (42)$$

Now, it remains to show that the solution of the above non-autonomous system converges to zero. We assume that the matrix function  $B(\theta)$  is bounded and sufficiently smooth so that it satisfies the Lipschitz condition, i.e., there exists a finite scalar  $c_l > 0$  such that

$$\|B(\theta) - B(\theta^*)\| \leq c_l \|\theta - \theta^*\| \quad \forall \theta, \theta^* \in \mathbb{R}^{n-r}. \quad (43)$$

Furthermore, there exists scalar  $c_b > 0$  such that

$$B(\theta) \leq c_b I. \quad (44)$$

Assuming that the command velocity is bounded, i.e.,  $\|v_{r_d}\| \leq c_v$ , we can show that the solution of the above differential equation satisfies

$$\|\tilde{\theta}\| \leq \|\tilde{\theta}(0)\|e^{-\eta_2 t} + \frac{c_b}{c_l c_v} \|\epsilon(0)\| (e^{-\eta_1 t} - e^{-\eta_2 t}), \quad (45)$$

where  $\eta_2 = \lambda_{\min}(K_p) - c_l c_v$ ; see Appendix C for details. Equation (45) implies exponential stability of error  $\|\tilde{\theta}\|$  provided that  $\eta_2 > 0$ , i.e.,

$$\lambda_{\min}(K_p) > c_l c_v. \quad (46)$$

The above development can be summarized in the following theorem.

**Theorem 1.** Assume that the mass matrix factorization is a smooth function satisfying the Lipschitz condition and that  $\|v_d\|$  is bounded. Then, for a sufficiently large position gain, i.e., (46) is satisfied, the error trajectories of the configuration-variables and quasi-velocities of a constrained MBS under control law (38)–(39) exponentially converge to zero.

Tracking of the desired constraint force  $\lambda_d$  can be achieved simply by compensating for the velocity perturbation term in (31b), i.e.,

$$u_o = \Lambda_r^T \lambda_d + \Gamma_o v_r. \quad (47)$$

It is worth noting that in view of the norm identity  $\|u\|^2 = \|u_r\|^2 + \|u_o\|^2$ , we can say that  $\|u\|$  is minimum if  $u_o \equiv 0$ . That is tantamount to minimization of weighted norm of the generalized forces where the weight matrix is the inertia matrix because  $\|u\|^2 = f^T M(q)f$ . Interestingly enough, no additional weighing matrix is required even if the elements of the generalized contains both force and torque components.

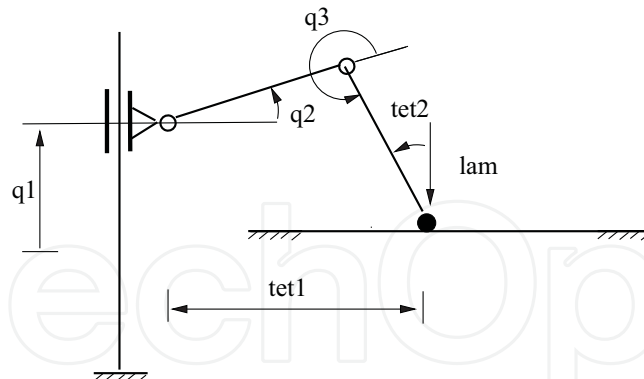


Fig. 1. A constrained 3-PRR manipulator.

### 4.3 Gauge Invariant

A problem that often arises in robotics, namely hybrid control or minimum solution to joint rate or force, is that generalized coordinate  $q$  may have a combination of rotational and translational components that can be even compounded by having combination of rotational and translational constraints Doty *et al.* (1993b). This may lead to inconsistent results, i.e., results that are invariant with respect to changes in dimensional units unless adequate weighting matrixes are used Aghili (2005); Doty *et al.* (1993b); Featherstone and Fijany (1999); Featherstone *et al.* (1999); Manes (1992). For example, the minimum joint rate rates,  $\min \|\dot{q}\|$ , or minimum norm force,  $\min \|f\|$ , are not meaningful quantities if the robot has both revolute and prismatic joints Doty *et al.* (1993b).

An important property of the reduced quasi-velocities and quasi-forces is that they always have homogenous units. As a matter of fact, since

$$\|v_r\| = \|v\| = \sqrt{2T},$$

we can say that all elements of the vector of quasi-velocity  $v$  or  $v_r$  must have a homogenous unit  $[\sqrt{\text{kgm/s}}]$ . This is true even if the vector of the generalized coordinate or the constraints have combinations of rotational and translational components. Similarly, one can argue that the elements of the quasi-forces have always identical unit  $[\sqrt{\text{kgm/s}^2}]$ , regardless of the units of the generalized force of the constraint wrench. Therefore, minimization of  $\|v\|$  or  $\min \|u\|$  is legitimate because the latter vectors have always homogeneous units. Moreover, the selection matrices which are often needed in hybrid position-force control of manipulators when both translational and rotational constraints are involved between its end effector and its environment Featherstone *et al.* (1999)—and that yields a problem with gauge invariance—becomes a non-issue here.

## 5. Analytical Example

Fig. 1 illustrates a PRR manipulator, with one prismatic and two revolute joints. The vertical motion of the manipulator tip-point is constrained by a solid surface. The prismatic joint provides the vertical motion of the robot base, which is with mass of  $m$ . Clearly, the vectors of generalized coordinates,  $q$ , and generalized force,  $f$ , have inhomogeneous components. We assume that each link is uniform with length of  $l$  and mass of  $m$ . Then, the constraint Jacobian can be expressed by

$$A(q) = [1 \quad l(c_{23} + c_2) \quad lc_{23}],$$

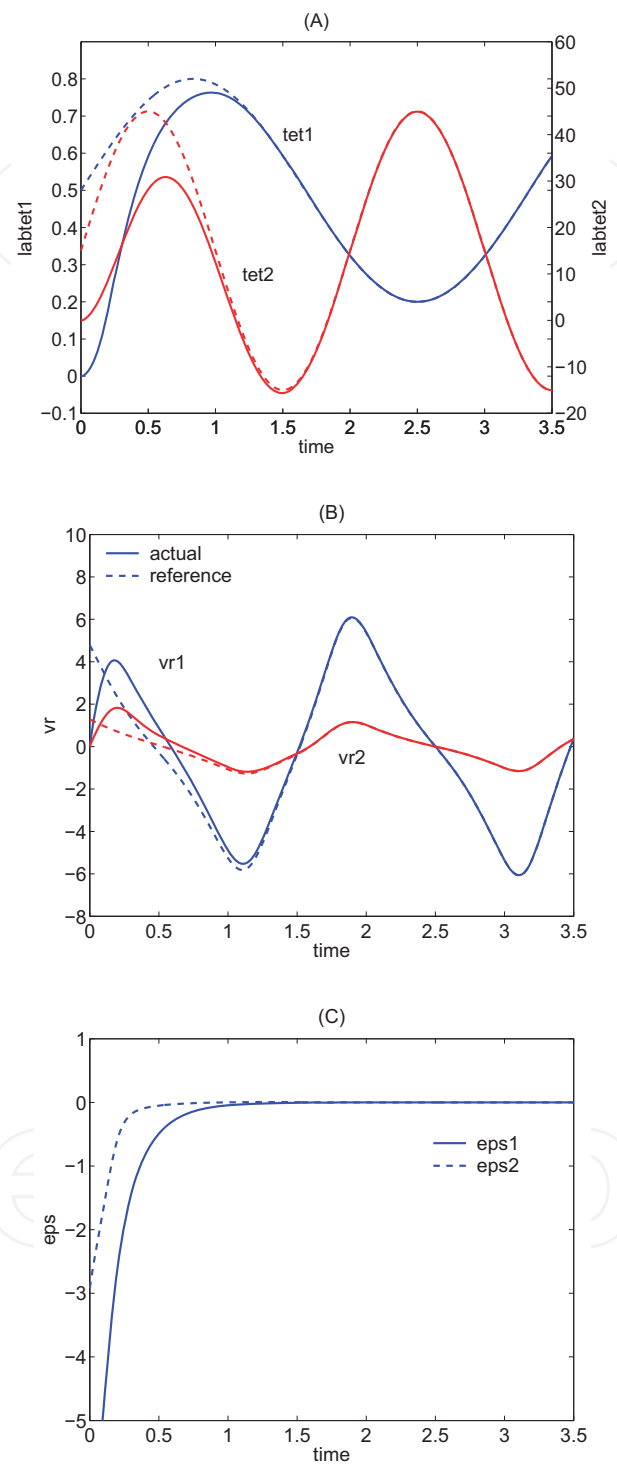


Fig. 2. Simulated motion tracking.

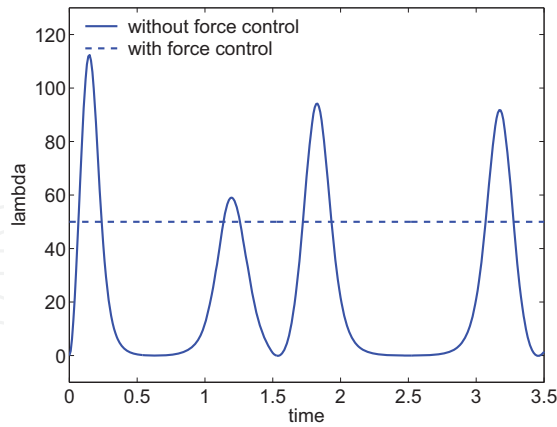


Fig. 3. Simulated constrained force.

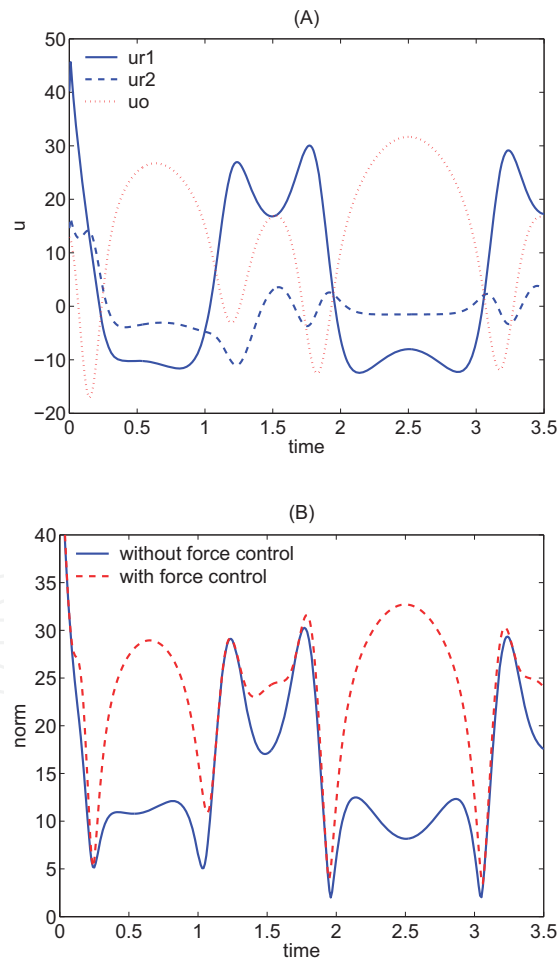


Fig. 4. Trajectories of the quasi-forces.



where  $c_{23} = \cos(q_2 + q_3)$  and  $c_2 = \cos(q_2)$ .

Let us define the minimal set generalized coordinates as  $\theta = [\theta_1 \ \theta_2]^T$  with  $\theta_1$  and  $\theta_2$  being the horizontal location of the tip and the angle of the last link with respect to the vertical line, respectively; see Fig. 1. Then from the kinematics, we get

$$\theta(q) = \begin{bmatrix} lc_{23} + lc_2 \\ q_2 + q_3 - \frac{3\pi}{2} \end{bmatrix}$$

Now assume that the control objective is to track the following desired trajectories

$$\begin{aligned} \theta_{1_d}(t) &= 0.3 \sin(0.6\pi t) + 0.5 \quad (\text{m}) \\ \theta_{2_d}(t) &= \frac{\pi}{6} \sin(\pi t) + \frac{\pi}{12} \quad (\text{rad}) \end{aligned}$$

Figs. 2A and 2B show the actual and desired trajectories of the position and quasi-velocities when the quasi-velocity feedback (38)-(39) is applied for the following parameters:  $m = 5 \text{ kg}$ ,  $l = 1 \text{ m}$ ,  $K_p = 3I$ , and  $K_d = 5I$ . The time-history of the composite error,  $\epsilon$ , shown in Fig. 2C demonstrates tracking of the reference motion trajectory. Fig.3 illustrates trajectories of the constraint force,  $\lambda$ , for two cases: i) no force control is applied, ii) the force control law (47) is applied to achieve

$$\lambda_d = 50 \quad (\text{N}).$$

Trajectories of the corresponding motion input,  $u_r$ , and force input,  $u_o$ , components of the quasi-forces are shown in Fig. 4A. Trajectories of the Euclidean norm the quasi-forces with and without force control are illustrated in the bottom of Fig. 4B. Clearly, the quasi-forces norm is automatically minimized norm if the force control input,  $u_o$  is set to zero. It is worth noting that the norm of quasi-forces is an invariant quantity even though the vector of generalized force has both force and torque components.

## Appendix A

According to the *Cholesky decomposition*, a symmetric and positive-definite matrix  $M$  can be decomposed efficiently into  $M = LL^T$ , where  $L$  is a lower-triangular matrix with strictly positive-diagonal elements;  $L$  is also called the *Cholesky triangle*. The Cholesky decomposition is a particular case of the well-known LU decomposition for symmetric matrices. Nevertheless, the Cholesky decomposition is twice as efficient as the LU decomposition. The following formula can be used to obtain the Cholesky triangle through some elementary operations

$$\begin{aligned} l_{ii} &= (m_{ii} - \sum_{k=1}^{i-1} l_{ik}^2)^{1/2} \quad \forall i = 1, \dots, n \\ l_{ji} &= (m_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik}) / l_{ii} \quad \forall j = i + 1, \dots, n \end{aligned} \quad (48)$$

Since  $L$  is a lower-triangular matrix, its inverse can be simply computed by the back substitution technique.

## Appendix B

Consider the following positive-definite function:

$$V = \frac{1}{2} \|\epsilon\|^2$$

In view of Remark 2, the time-derivative of the above function along the error trajectory (40) is obtained as

$$\begin{aligned}\dot{V} &= -\epsilon^T \Gamma_r \epsilon - \epsilon^T K_d \epsilon \\ &= -\epsilon^T K_d \epsilon\end{aligned}$$

which gives

$$\dot{V} \leq -2\lambda_{\min}(K_d)V.$$

Thus

$$V \leq V(0)e^{-2\lambda_{\min}(K_d)t},$$

which is equivalent to (41).

## Appendix C

Consider the following positive-definite function

$$V = \frac{1}{2} \|\tilde{\theta}\|^2, \quad (49)$$

whose time-derivative along (42) gives

$$\dot{V} = -\tilde{\theta}^T K_p \tilde{\theta} + \tilde{\theta}^T B(\theta) \epsilon + \tilde{\theta}^T (B(\theta) - B(\theta_d)) v_d.$$

From (44) and (43), we can find a bound on  $\dot{V}$  as

$$\begin{aligned}\dot{V} &\leq -\lambda_{\min}(K_p) \|\tilde{\theta}\|^2 + c_b \|\tilde{\theta}\| \|\epsilon\| + c_l c_v \|\tilde{\theta}\|^2 \\ &\leq -2\eta_2 V + c_b \sqrt{2V} \|\epsilon\|,\end{aligned} \quad (50)$$

which is in the form of a Bernoulli differential inequality. The above nonlinear inequality can be linearized by the following change of variable  $U = \sqrt{V}$ , i.e.,

$$\dot{U} \leq -\eta_2 U + \frac{c_b}{\sqrt{2}} \|\epsilon(0)\| e^{-\eta_1 t} \quad (51)$$

In view of the comparison lemma (Khalil, 1992, p. 222) and (41), one can show that the solution of (51) must satisfy

$$U \leq U(0)e^{-\eta_2 t} + \frac{c_b \|\epsilon(0)\|}{\sqrt{2}} \int_0^t e^{-\eta_2(t-\tau)-\eta_1 \tau} d\tau,$$

which is equivalent to (45).

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## **Robotics 2010 Current and Future Challenges**

Edited by Houssem Abdellatif

ISBN 978-953-7619-78-7

Hard cover, 494 pages

**Publisher** InTech

**Published online** 01, February, 2010

**Published in print edition** February, 2010

Without a doubt, robotics has made an incredible progress over the last decades. The vision of developing, designing and creating technical systems that help humans to achieve hard and complex tasks, has intelligently led to an incredible variety of solutions. There are barely technical fields that could exhibit more interdisciplinary interconnections like robotics. This fact is generated by highly complex challenges imposed by robotic systems, especially the requirement on intelligent and autonomous operation. This book tries to give an insight into the evolutionary process that takes place in robotics. It provides articles covering a wide range of this exciting area. The progress of technical challenges and concepts may illuminate the relationship between developments that seem to be completely different at first sight. The robotics remains an exciting scientific and engineering field. The community looks optimistically ahead and also looks forward for the future challenges and new development.

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