Modelling and Simulation of the Shape Optimization Problems

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1. Introduction

Optimum shape design is an interesting and important field both mathematically and for industrial applications. Uniqueness, stability and existence of solution are important theoretical issues for scientists. Practical implementation issues are critical for realization for engineers and designers. As examples of industrial applications we can consider weight reduction in car engine, aircraft structures, electromagnetically optimum shapes, such as in stealth airplanes. There is also a great interest in shape optimization for fluid flow systems. The engineers and designers are interested in reducing the drag force on the wing of a plane or on a vehicle, or in reducing the viscous dissipation in hydraulic valves, pipes and tanks. The computation of optimal profiles that minimize the aerodynamic drag, the viscous energy which is dissipated in the fluid, the volume and weight of building structures plays nowadays a very important role.

In this paper, we investigate a methodology for the shape optimization problem. The problem consists in finding a shape (in two or three dimensions), which is optimal in a certain sense and satisfies certain requirements. In other words, we would like to find a bounded set D, which minimizes a functional $J(D)$ and satisfies constraints $B(D)=0$. The problem typically involves the solution of a system of nonlinear partial differential equations, which depend on parameters that define a geometrical domain. The solution is usually obtained numerically, by using iterative methods, for example by the finite element method (Strang & Fix, 1973). The continuum description of the geometrical domain is discretized with different meshing strategies. Some of them are fixed grid strategies (Xie & Steven, 1993; Li et al., 1999; Garcia & Gonzales, 2004), design element concepts (Imam, 1982), adaptive mesh strategies (Belegundu & Rajan, 1988) and remeshing strategies. The shape optimization problems are discussed by many scientists, and to mention only a few we note the works (Belegundu & Chandrupatla, 1999; Atanackovic, 2001; Delfour & Zolesio, 2001; Mohammadi & Pironneau, 2001; Skruch, 2001; Allaire, 2002; Bendsøe & Sigmund, 2003; Haslinger & Mäkinen, 2003; Allaire et al., 2004).

In recent years, some attempts have been made to use optimal control theory for the shape optimization problems (Szefer & Mikulski, 1978, 1984; Mitkowski & Skruch, 2001; Skruch, 2001; Laskowski, 2006; Skruch & Mitkowski, 2009). In this paper, we continue investigations of this topic and we show that the method based on the Pontryagin maximum principle

(Pontryagin et. al., 1962; Boltyanskii, 1971; Mitkowski, 1991) can be used for solving the formulated task of optimization. Of course, a general solution and proof are very often impossible. Therefore the main focus in this paper will be put on numerical solutions and simulations. The computer program has been designed in the MATLAB/Simulink environment. It uses an iterative method, that is, we start with an initial guess for a shape, and then gradually evolve it, until it falls into the optimum shape. Using the program we show how to find optimum shapes for different types of beam design. The approach can be also successfully applied to shape optimization of many mechanical systems.
The paper is organized as follows. In section 2 we formulate the problem. General solution of the problem is presented in section 3. To illustrate our approach, we consider a single span beam with rectangular cross-section (section 4), I cross-section (section 5) and a clamped beam with rectangular cross-section (section 6). We also show how to implement numerical solution of the problem (sections 4.2, 5.2 and 6.2). Numerical simulation results are presented in sections 4.3, 5.3 and 6.3. The study of the existence of local minimum for the optimization problem is given in section 4.4.
The following notation is used throughout this chapter:

$\lVert\cdot\rVert_\infty$	norm in the space $\mathrm{L}^\infty(T,X)$
γ	volume mass density of the beam material
b	width of the beam's cross-section
E	Young's modulus
h	height of the beam's cross-section
I	moment of inertia
l	length of the beam
$\mathrm{L}^\infty(T,X)$	Banach space with the norm $\lVert f\rVert_\infty = \operatorname{ess\,sup}_{t\in T}\lvert f(t)\rvert$, $f:T\to X$
$\mathrm{PC}(T,X)$	space of piecewise continuous functions $f:T\to X$
R	set of real numbers
R^n	real $n-$dimensional vector space over R
$\mathrm{W}^{1,\infty}(T,X)$	Banach space with the norm $\lVert f\rVert_{1,\infty} = \max\left(\lVert f\rVert_\infty, \left\lVert \frac{\mathrm{d}f}{\mathrm{d}t}\right\rVert_\infty\right)$
w^{T}	transpose of the vector w

2. Formulation of the problem

Consider a physical system of which the statics can be described by the following equation

$$\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi} = \boldsymbol{f}(\boldsymbol{x}(\xi),\boldsymbol{u}(\xi),\xi), \tag{1}$$

where $\boldsymbol{x}\in \mathrm{W}^{1,\infty}([\xi_0,\xi_1],\mathrm{R}^n)$, $\xi_0,\xi_1\in\mathrm{R}$, $\xi_0<\xi_1$, $\xi\in[\xi_0,\xi_1]$, $\boldsymbol{u}\in U_{\mathrm{ad}}$, U_{ad} stands for the set of admissible controls

$$U_{\mathrm{ad}} = \left\{\boldsymbol{u}\in \mathrm{PC}([\xi_0,\xi_1],\mathrm{R}^m)\subset \mathrm{L}^\infty([\xi_0,\xi_1],\mathrm{R}^m) : \boldsymbol{u}(\xi)\in U\right\}, \tag{2}$$

$$U = \left\{ \boldsymbol{v} \in \mathrm{R}^m : \boldsymbol{u}_{\min} \le \boldsymbol{v} \le \boldsymbol{u}_{\max}, \boldsymbol{u}_{\min} < \boldsymbol{u}_{\max} \right\}, \ \boldsymbol{u}_{\min}, \boldsymbol{u}_{\max} \in \mathrm{R}^m , \tag{3}$$

$\boldsymbol{f} : \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R} \to \mathrm{R}^n$ is a vector function that is continuous with respect to each variable and whose partial derivative $\nabla_{\boldsymbol{\theta}} \boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{\eta}, \tau)$ exists and is continuous for all $(\boldsymbol{\theta}, \boldsymbol{\eta}, \tau) \in \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}$. Geometrical and strength constraints are imposed on the system in the form of equalities and weak inequalities

$$I_i(\boldsymbol{x}, \boldsymbol{u}, \xi_0, \xi_1) = \int_{\xi_0}^{\xi_1} G_i(\boldsymbol{x}(\xi), \boldsymbol{u}(\xi), \xi) \mathrm{d}\xi + \gamma_i(\xi_0, \boldsymbol{x}(\xi_0), \xi_1, \boldsymbol{x}(\xi_1)) \le 0, \quad i = 1,2,\ldots,p , \tag{4}$$

$$E_j(\boldsymbol{x}, \boldsymbol{u}, \xi_0, \xi_1) = \int_{\xi_0}^{\xi_1} B_j(\boldsymbol{x}(\xi), \boldsymbol{u}(\xi), \xi) \mathrm{d}\xi + \beta_j(\xi_0, \boldsymbol{x}(\xi_0), \xi_1, \boldsymbol{x}(\xi_1)) = 0, \quad j = 1,2,\ldots,q , \tag{5}$$

where $G_i : \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R} \to \mathrm{R}^n$, $\gamma_i : \mathrm{R} \times \mathrm{R}^n \times \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}$, $B_j : \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R} \to \mathrm{R}^n$, $\beta_j : \mathrm{R} \times \mathrm{R}^n \times \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}$, $i = 1,2,\ldots,p$, $j = 1,2,\ldots,q$. Existence of the constraints (3), (4) and (5) should secure the solutions the proper physical meaning. We assume that the functions G_i, B_j, γ_i, β_j are continuous with respect to each variable, the partial derivatives $\nabla_{\boldsymbol{\theta}} G_i(\boldsymbol{\theta}, \boldsymbol{\eta}, \tau)$, $\nabla_{\boldsymbol{\theta}} B_j(\boldsymbol{\theta}, \boldsymbol{\eta}, \tau)$ exist and are continuous for all $(\boldsymbol{\theta}, \boldsymbol{\eta}, \tau) \in \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}$, moreover, the partial derivatives $\nabla_{\tau_k} \gamma_i(\tau_0, \boldsymbol{\theta}_0, \tau_1, \boldsymbol{\theta}_1)$, $\nabla_{\boldsymbol{\theta}_k} \gamma_i(\tau_0, \boldsymbol{\theta}_0, \tau_1, \boldsymbol{\theta}_1)$, $\nabla_{\tau_k} \beta_j(\tau_0, \boldsymbol{\theta}_0, \tau_1, \boldsymbol{\theta}_1)$, $\nabla_{\boldsymbol{\theta}_k} \beta_j(\tau_0, \boldsymbol{\theta}_0, \tau_1, \boldsymbol{\theta}_1)$, $k = 0,1$ exist and are continuous for all $(\tau_0, \boldsymbol{\theta}_0, \tau_1, \boldsymbol{\theta}_1) \in \mathrm{R} \times \mathrm{R}^n \times \mathrm{R} \times \mathrm{R}^n$. The shape optimization problem involves the constrained minimization of a cost function. In our case the cost function will be formulated as

$$J_0(\boldsymbol{x}, \boldsymbol{u}, \xi_0, \xi_1) = \int_{\xi_0}^{\xi_1} G_0(\boldsymbol{x}(\xi), \boldsymbol{u}(\xi), \xi) \mathrm{d}\xi + \gamma_0(\xi_0, \boldsymbol{x}(\xi_0), \xi_1, \boldsymbol{x}(\xi_1)). \tag{6}$$

Here, the functions G_0 and γ_0 are from the same class as G_i and γ_i, $i = 1,2,\ldots,p$. The cost function may represent any design requirement of the physical system such as displacement of a chosen point, surface or volume of an element, etc..

The problem is to determine the quadruple $(\boldsymbol{x}, \boldsymbol{u}, \xi_0, \xi_1) \in \mathrm{W}^{1,\infty}([\xi_0, \xi_1], \mathrm{R}^n) \times U_{\mathrm{ad}} \times \mathrm{R} \times \mathrm{R}$ which satisfies the equation (1), constrains (4), (5) and minimizes the cost function (6).

3. General solution of the problem

These types of problems as presented shortly in section 2 can be solved using one of the variants of the Pontryagin maximum principle (Ioffe & Tikhomirov, 1979; Alekseev et al., 1987; Hartl et al., 1995; Bania, 2008). In this approach the key role plays the Hamiltonian $H_0 : \mathrm{R}^n \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R} \times \mathrm{R} \times \mathrm{R}^p \times \mathrm{R}^q \to \mathrm{R}$ defined as

$$H_0(\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{u},\xi,\lambda_0,\boldsymbol{\lambda},\boldsymbol{\mu})=\boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\xi)-\sum_{i=0}^{p}\lambda_i G_i(\boldsymbol{x},\boldsymbol{u},\xi)-\sum_{j=1}^{q}\mu_j B_j(\boldsymbol{x},\boldsymbol{u},\xi), \tag{7}$$

where the function $\boldsymbol{\psi}\in \mathrm{W}^{1,\infty}([\xi_0,\xi_1],\mathrm{R}^n)$, $\lambda_0\in\mathrm{R}$, $\boldsymbol{\lambda}=[\lambda_1,\lambda_2,\ldots,\lambda_p]^{\mathrm{T}}\in\mathrm{R}^p$, $\boldsymbol{\mu}=[\mu_1,\mu_2,\ldots,\mu_q]^{\mathrm{T}}\in\mathrm{R}^q$. The number λ_0 and the vectors $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ are called Lagrange multipliers.

Suppose a quadruple $(\boldsymbol{x}^*,\boldsymbol{u}^*,\xi_0^*,\xi_1^*)$ gives the local minimum in the problem described above. Then according to Pontryagin maximum principle, there exist the Lagrange multipliers $\lambda_0^*\ge 0$, $\boldsymbol{\lambda}^*\ge 0$, $\boldsymbol{\mu}^*$ and the function $\boldsymbol{\psi}^*$ that satisfy the following:

(a) adjoint equations

$$\frac{\mathrm{d}\boldsymbol{\psi}^*(\xi)}{\mathrm{d}\xi}=-\nabla_{\boldsymbol{x}}H_0(\boldsymbol{\psi}^*(\xi),\boldsymbol{x}^*(\xi),\boldsymbol{u}^*(\xi),\xi,\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*), \tag{8}$$

(b) state equations

$$\frac{\mathrm{d}\boldsymbol{x}^*(\xi)}{\mathrm{d}\xi}=\boldsymbol{f}(\boldsymbol{x}^*(\xi),\boldsymbol{u}^*(\xi),\xi), \tag{9}$$

(c) maximum condition

$$H_0(\boldsymbol{\psi}^*(\xi),\boldsymbol{x}^*(\xi),\boldsymbol{u}^*(\xi),\xi,\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)\ge H_0(\boldsymbol{\psi}^*(\xi),\boldsymbol{x}^*(\xi),\boldsymbol{v},\xi,\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*),\quad \boldsymbol{v}\in U, \tag{10}$$

(d) nontriviality conditions

$$\lambda_0^*+|\boldsymbol{\lambda}^*|+|\boldsymbol{\mu}^*|+\|\boldsymbol{\psi}^*\|_\infty>0, \tag{11}$$

(e) complementary conditions

$$\lambda_i^* I_i(\boldsymbol{x}^*,\boldsymbol{u}^*,\xi_0^*,\xi_1^*)=0,\quad i=1,2,\ldots,p, \tag{12}$$

(f) continuity of the function $H_0(\boldsymbol{\psi}^*(\xi),\boldsymbol{x}^*(\xi),\boldsymbol{u}^*(\xi),\xi,\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)$ for $\xi\in[\xi_0^*,\xi_1^*]$,

(g) transversality conditions

$$\boldsymbol{\psi}^*(\xi_0^*)=\nabla_{\boldsymbol{\theta}_0}\varphi(\xi_0^*,\boldsymbol{x}^*(\xi_0^*),\xi_1^*,\boldsymbol{x}^*(\xi_1^*),\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*), \tag{13}$$

$$\boldsymbol{\psi}^*(\xi_1^*)=-\nabla_{\boldsymbol{\theta}_1}\varphi(\xi_0^*,\boldsymbol{x}^*(\xi_0^*),\xi_1^*,\boldsymbol{x}^*(\xi_1^*),\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*), \tag{14}$$

$$H_0(\boldsymbol{\psi}^*(\xi_0^*),\boldsymbol{x}^*(\xi_0^*),\boldsymbol{u}^*(\xi_0^*),\xi_0^*,\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)=-\nabla_{\tau_0}\varphi(\xi_0^*,\boldsymbol{x}^*(\xi_0^*),\xi_1^*,\boldsymbol{x}^*(\xi_1^*),\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*), \tag{15}$$

$$H_0(\boldsymbol{\psi}^*(\xi_1^*),\boldsymbol{x}^*(\xi_1^*),\boldsymbol{u}^*(\xi_1^*),\xi_1^*,\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)=\nabla_{\tau_1}\varphi(\xi_0^*,\boldsymbol{x}^*(\xi_0^*),\xi_1^*,\boldsymbol{x}^*(\xi_1^*),\lambda_0^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*), \tag{16}$$

where $\varphi:\mathrm{R}\times\mathrm{R}^n\times\mathrm{R}\times\mathrm{R}^n\times\mathrm{R}\times\mathrm{R}^p\times\mathrm{R}^q\to\mathrm{R}$ is defined as follows

$$\varphi(\tau_0,\boldsymbol{\theta}_0,\tau_1,\boldsymbol{\theta}_1,\lambda_0,\boldsymbol{\lambda},\boldsymbol{\mu})=\sum_{i=0}^{p}\lambda_i\gamma_i(\tau_0,\boldsymbol{\theta}_0,\tau_1,\boldsymbol{\theta}_1)+\sum_{j=1}^{q}\mu_j\beta_j(\tau_0,\boldsymbol{\theta}_0,\tau_1,\boldsymbol{\theta}_1). \quad (17)$$

The maximum principle has been and remains an important and effective tool in the many areas in which optimal control plays a role. There have been many advances of this principle in the last fifty years that extended its applicability. It should be underlined that the theorem gives only necessary conditions of optimality. Existence of solutions needs separate studies.

4. Optimal design of a single span beam with rectangular cross-section

4.1 Equation of a physical system

Beams are used to support and strengthen structures ranging from silos to bridges to towering skyscrapers. In this section we explore the shape optimization problem associated with the static deformation of beams. The strategy is to mathematically describe the quantities that affect the deformation of a beam, and to relate these quantities through differential equations that describe the bending of a beam. Then using the method based on the Pontryagin maximum principle, we show how to choose the beam's cross-section in order to assure minimum deflection at the end point of a beam.

Consider a single span beam with rectangular cross-section working under self-weight (fig. 1). The statics of the beam can be described using the following equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[EI(\xi)\frac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}\right]=-u(\xi), \quad (18)$$

where $\xi\in[0,l]$, $u(\xi)=\gamma bh(\xi)$, $y(\xi)$ represents vertical displacement of the beam along the interval $[0,l]$, l stands for the length of the beam, b is the width of the beam's cross-section, h is the height of the cross-section, E is the Young's module, $I(\xi)$ is the moment of inertia of the cross-section, γ is the volume mass density of the beam material.

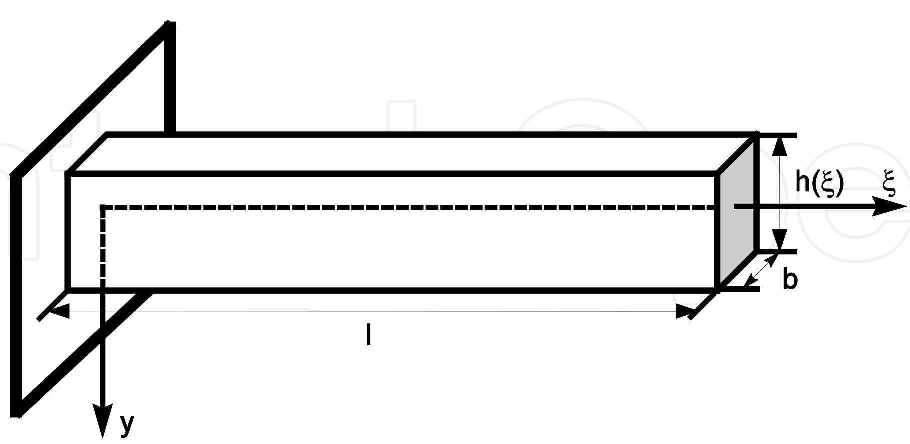

Fig. 1. Single span beam under self-weight

For state notation of the equation (18) we introduce the vector

$$\boldsymbol{x}(\xi) = [x_1(\xi) \quad x_2(\xi) \quad x_3(\xi) \quad x_4(\xi)]^{\mathrm{T}}, \tag{19}$$

where

$$\begin{cases} x_1(\xi) = y(\xi), \\ x_2(\xi) = \dfrac{\mathrm{d}y(\xi)}{\mathrm{d}\xi}, \\ x_3(\xi) = EI(\xi)\dfrac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}, \\ x_4(\xi) = \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left[EI(\xi)\dfrac{\mathrm{d}^2 y(\xi)}{\mathrm{d}\xi^2}\right], \end{cases} \tag{20}$$

$$I(\xi) = \frac{bu(\xi)^3}{12}, \tag{21}$$

and additionally we define the vector

$$\boldsymbol{f}(\boldsymbol{x}(\xi), u(\xi)) = \begin{bmatrix} x_2(\xi) \\ \dfrac{12x_3(\xi)}{Ebu(\xi)^3} \\ x_4(\xi) \\ -u(\xi) \end{bmatrix}. \tag{22}$$

Then our system can be written shortly in the classical form

$$\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi} = \boldsymbol{f}(\boldsymbol{x}(\xi), \boldsymbol{u}(\xi)), \tag{23}$$

with the boundary conditions

$$x_1(0) = x_2(0) = x_3(l) = x_4(l) = 0. \tag{24}$$

The boundary condition $x_1(0) = 0$ says that the base of the beam (at the wall) does not experience any deflection. We also assume that the beam at the wall is horizontal, so that the derivative of the deflection function is zero at that point, i.e. $x_2(0) = 0$. The boundary condition $x_3(l) = 0$ models the assumption that there is no bending moment at the free end of the cantilever. The boundary condition $x_4(l) = 0$ models the assumption that there is no shearing force acting at the free end of the beam (see also Laskowski, 2006). It should be noted that the values $x_3(0)$, $x_4(0)$, $x_1(l)$ and $x_2(l)$ are unknown. Side conditions concerning strength constraints and geometry are imposed on the dimensions of the cross-section, so that

$$U_{\mathrm{ad}} = \{u \in \mathrm{PC}([0, l], \mathrm{R}) : u(\xi) \in U\}, \tag{25}$$

$$U = \{v \in \mathrm{R} : H_1 \le v \le H_2, H_1 < H_2\}, \; H_1, H_2 \in \mathrm{R}. \tag{26}$$

The deflection at the end point of the beam is the optimality criterion

$$J(u) = x_1(l)\,. \tag{27}$$

The cost function (27) can be also expressed in the following form

$$J(u) = x_1(l) = \int_0^l x_2(\xi)\mathrm{d}\xi. \tag{28}$$

We want to determine such $u \in U_{\mathrm{ad}}$, which minimizes the functional (28) and satisfies the state equations (23) with the boundary conditions (24).
In order to solve the formulated problem, we use the Pontryagin maximum principle. Therefore we introduce the Hamiltonian

$$H_0(\boldsymbol{\psi}, \boldsymbol{x}, u, \lambda_0) = \psi_1 x_2 + \psi_2 \frac{12 x_3}{Ebu^3} + \psi_3 x_4 - \psi_4 u - \lambda_0 x_2\,, \tag{29}$$

where the variable $\lambda_0 \geq 0$ and the function $\boldsymbol{\psi}(\xi) = [\psi_1(\xi) \quad \psi_2(\xi) \quad \psi_3(\xi) \quad \psi_4(\xi)]^{\mathrm{T}}$ satisfies the equation

$$\frac{\mathrm{d}\boldsymbol{\psi}(\xi)}{\mathrm{d}\xi} = -\nabla_x H_0(\boldsymbol{\psi}(\xi), \boldsymbol{x}(\xi), u(\xi), \lambda_0)\,, \tag{30}$$

it is

$$\frac{\mathrm{d}\boldsymbol{\psi}(\xi)}{\mathrm{d}\xi} = \begin{bmatrix} 0 \\ -\lambda_0 - \psi_1(\xi) \\ -\psi_2(\xi)\dfrac{12}{Ebu(\xi)^3} \\ -\psi_3(\xi) \end{bmatrix}. \tag{31}$$

The transversality conditions lead to the following boundary values

$$\psi_3(0) = 0, \quad \psi_4(0) = 0\,, \tag{32}$$

$$\psi_1(l) = 0, \quad \psi_2(l) = 0\,. \tag{33}$$

According to the Pontryagin maximum principle for the optimal control u^* there is

$$H_0(\boldsymbol{\psi}^*, \boldsymbol{x}^*, \boldsymbol{u}^*, \lambda_0^*) = \max_{v \in U} H_0(\boldsymbol{\psi}^*, \boldsymbol{x}^*, v, \lambda_0^*)\,. \tag{34}$$

The optimal control u^* can be obtained from the condition $\partial H_0 / \partial u = 0$ with the help of the system of the adjoint equations (31) and the boundary conditions (32), (33).
We assume that $\lambda_0 = 1$. Then from (31) we can obtain

$$\psi_1(\xi) = 0, \quad \psi_2(\xi) = l - \xi\,. \tag{35}$$

Invoking the condition $\partial H_0 / \partial u = 0$ we have the equation

$$u(\xi)=\sqrt[4]{(\xi-l)\frac{36x_3(\xi)}{Eb\psi_4(\xi)}}\,, \tag{36}$$

from which we shall obtain the optimal control u^*. In the formulated optimization problem the constraints (26) cause that not the whole space is an admissible region. Because

$$\lim_{\xi\to 0_+}\sqrt[4]{(\xi-l)\frac{36x_3(\xi)}{Eb\psi_4(\xi)}}=+\infty\,, \tag{37}$$

$$\lim_{\xi\to l_-}\sqrt[4]{(\xi-l)\frac{36x_3(\xi)}{Eb\psi_4(\xi)}}=0\,, \tag{38}$$

then the optimal solution has the final form

$$u^*(\xi)=\begin{cases} H_2, \text{ for } \xi\in[0,\xi_\mathrm{a}] \\ \sqrt[4]{(\xi-l)\frac{36x_3(\xi)}{Eb\psi_4(\xi)}}, \text{ for } \xi\in(\xi_\mathrm{a},\xi_\mathrm{b}] \\ H_1, \text{ for } \xi\in(\xi_\mathrm{b},l] \end{cases} \tag{39}$$

The optimal control (39) has a purely formal character because we do not know the functions $x_3(\xi)$, $\psi_4(\xi)$ and the ranges $[0,\xi_\mathrm{a}]$, $(\xi_\mathrm{a},\xi_\mathrm{b}]$, $(\xi_\mathrm{b},l]$ in which the individual relations (39) hold. These unknowns can be found in a numerical way.

4.2 Solution method

To find effectively the optimal control $u^*(\xi)$, it is necessary to solve the system which consists of nonlinear ordinary differential equations of the first-order with the boundary conditions defined at initial and end points. The solution of this system is possible only in a numerical way. The following algorithm has been implemented in the MATLAB/Simulink environment. The algorithm for numerical solution of the shape optimization problem for the single span beam with rectangular cross-section uses simple shooting method (see for example Keller, 1971; Roberts & Shipman, 1972; Matauek, 1973; Lastman, 1974). Interesting results regarding solution methods of boundary value problems can be found in (Mufti et al., 1969; Miele et al., 1972; Meyer, 1973; Laporte & Le Tallec, 2003).

- **Assumptions**
  The system (19), (22), (23), (24) has a solution and the optimal control exists.
- **Step 1**
  For arbitrarily chosen values $x_3(0)$ and $x_4(0)$ solve the problem in the interval $[0,\xi_\mathrm{a}]$ using the model created in Simulink. In this model time works as geometrical variable ξ. Then find ξ_a such that $u^*(\xi_\mathrm{a})=H_2$ using the equation (36).
- **Step 2**
  Based on the results from the previous step determine the end conditions $\boldsymbol{x}(\xi_\mathrm{a})$. They will be used as initial conditions in the next step.

- **Step 3**
  Find ξ_{b} such that $u^*(\xi_{\mathrm{b}}) = H_1$ using the equation (36).
- **Step 4**
  Solve the problem in the interval $(\xi_{\mathrm{a}}, \xi_{\mathrm{b}}]$ using the model created in Simulink and determine the end conditions $\boldsymbol{x}(\xi_{\mathrm{b}})$. These values will be used as initial conditions in the next step.
- **Step 5**
  Solve the problem in the interval $(\xi_{\mathrm{b}}, l]$ using the model created in Simulink.
- **Step 6**
  Calculate the norm $|x_3(l)| + |x_4(l)|$ and compare it with 0. First approximation for the optimal control $u^*(\xi)$ can be obtained from (39).
- **Step 7**
  Based on the norm $|x_3(l)| + |x_4(l)|$ recalculate the points $x_3(0)$ and $x_4(0)$ using the Nelder-Mead simplex (direct search) method (fmins function included in MATLAB).
- **Step 8**
  The steps 1-8 are repeated many times until $|x_3(l)| + |x_4(l)| < \varepsilon$.

The algorithm uses the MATLAB/Simulink environment to represent and solve the system (see fig. 2). State variable formulation allows the use of a wide variety of fixed step and variable step integration algorithms from Simulink. Simulation results can be displayed on Simulink scopes while the simulation is running or sent to workspace or disk file. The user can access a variety of MATLAB functions for processing and plotting of waveforms stored in the MATLAB workspace. It should be noted that time step integration methods are used to solve the mechanical system. In other words, one-dimensional computational domain related to beam's geometry is represented by time domain.

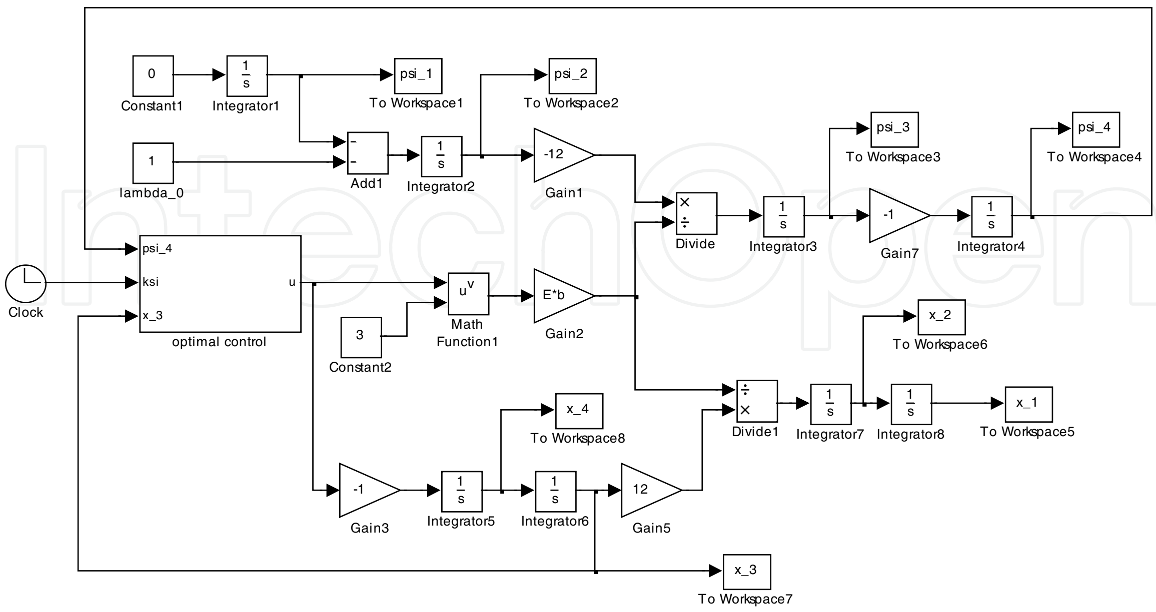

Fig. 2. Model of the system in Simulink environment

4.3 Numerical simulation results

Simulation effects are shown in figs. 3-8. Fig. 3 presents the optimal height $h(\xi)$ of the cross-section. Fig. 4 presents the optimal shape of the beam. Figs. 5 and 6 illustrate the state variables $x_1(\xi)$, $x_2(\xi)$, $x_3(\xi)$ and $x_4(\xi)$ along the interval $[0,l]$. Figs. 7 and 8 show the set of adjoint functions ψ_i, $i = 1,2,3,4$. Calculations were made for the following data: $l = 2.0$ [m], $b = 0.1$ [m], $H_1 = 0.1$ [m], $H_2 = 0.2$ [m], $E = 2.1 \cdot 10^{11}$ [N/m^2], $\gamma = 76500$ [N/m^3].

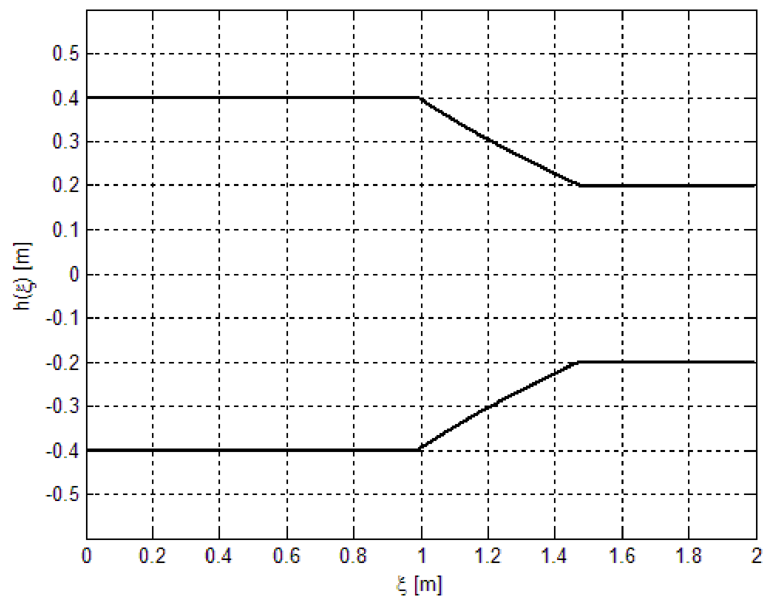

Fig. 3. The height of the cross-section, $\xi_a = 0.99$ [m], $\xi_b = 1.47$ [m]

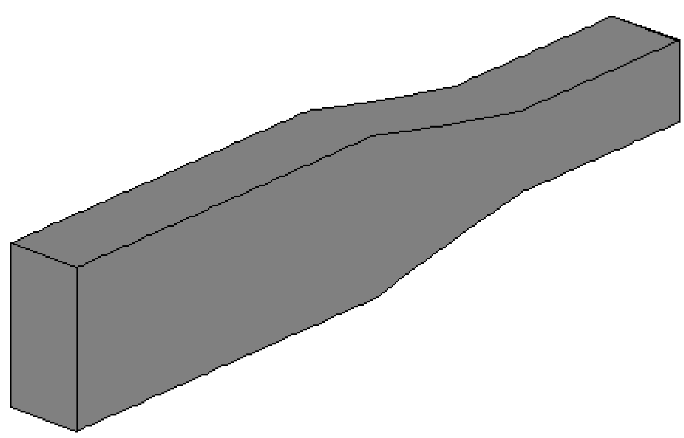

Fig. 4. Optimal shape of the beam

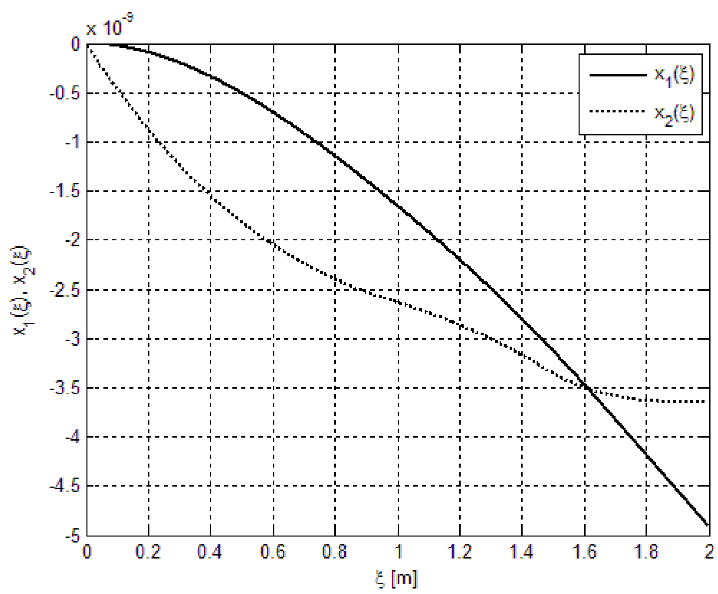

Fig. 5. The state variables $x_1(\xi)$, $x_2(\xi)$

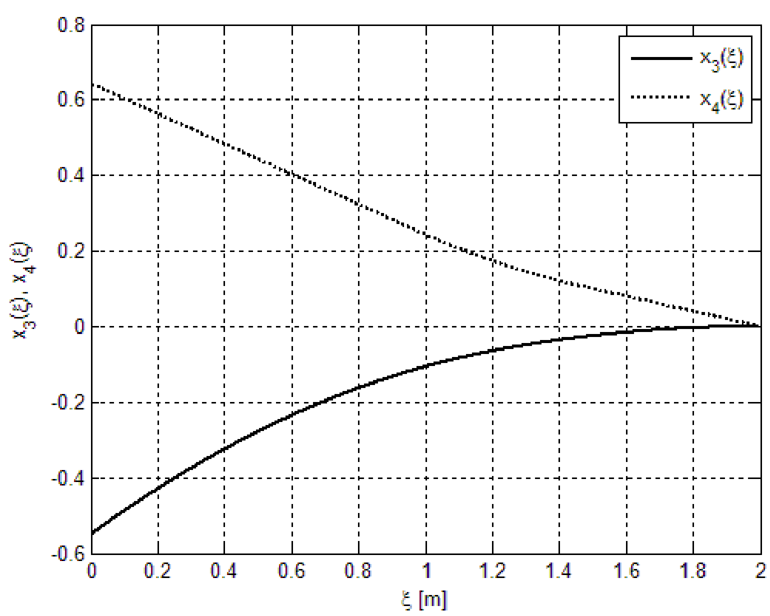

Fig. 6. The state variables $x_3(\xi)$, $x_4(\xi)$

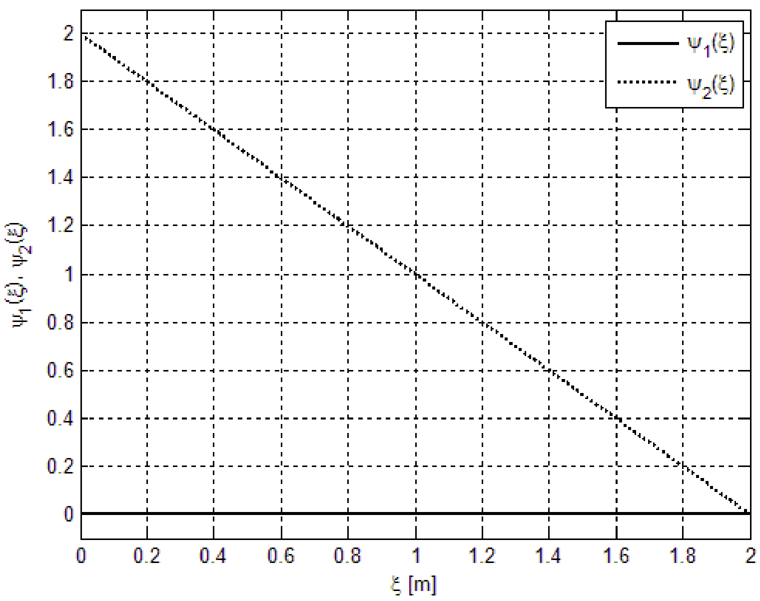

Fig. 7. The adjoint functions $\psi_1(\xi)$, $\psi_2(\xi)$

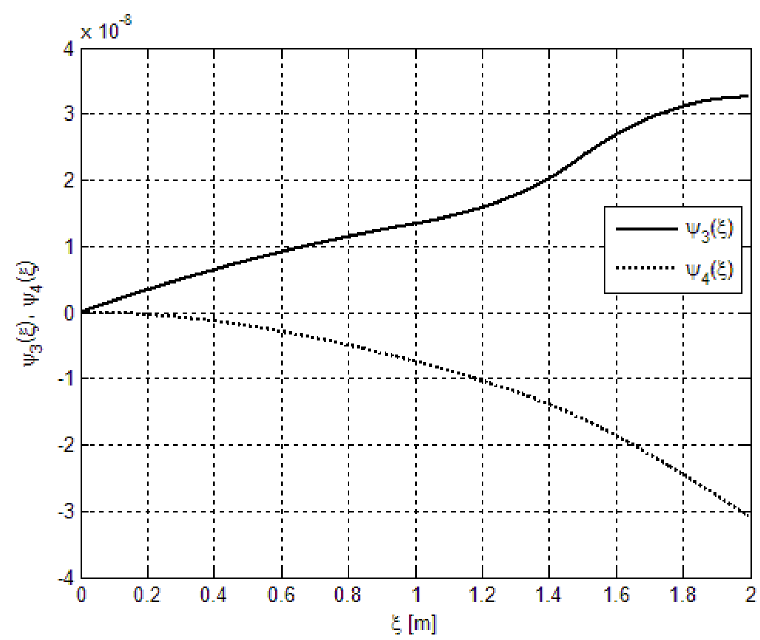

Fig. 8. The adjoint functions $\psi_3(\xi)$, $\psi_4(\xi)$

4.4 Study of the existence of local minimum

The Pontryagin maximum principle gives a necessary condition for an optimum. It does not assure that the solution of the problem really exists and is unique. A general proof of existence, uniqueness and stability is usually impossible. This needs an extensive study and will not be provided in this paper. However, these issues are important from theoretical point of view. Some attempts have been made by (Skruch, 2001; Skruch & Mitkowski, 2008) how to handle this numerically. In some neighbourhood of the candidate for optimal shape K0 (see fig. 9) we choose other shapes K1, K2, K3 and K4. These shapes are described by the following equations:

$$\mathrm{K1}:\quad y(\xi) = -0.7576\xi + 0.9606\,, \tag{40}$$

$$\mathrm{K2}:\quad y(\xi) = 0.5628\xi^{-1} - 0.3606\,, \tag{41}$$

$$\mathrm{K3}:\quad y(\xi) = 0.2398\xi^{-2} - 0.0379\,, \tag{42}$$

$$\mathrm{K4}:\quad y(\xi) = 0.1252\xi^{-3} + 0.0664\,. \tag{43}$$

Then for every shape we need to calculate the cost function J that is the deflection at the end point of the beam. Fig. 10 presents results of this calculation; for the shapes in the neighbourhood of the candidate for optimal shape K0 we obtain worse values of the cost function J.

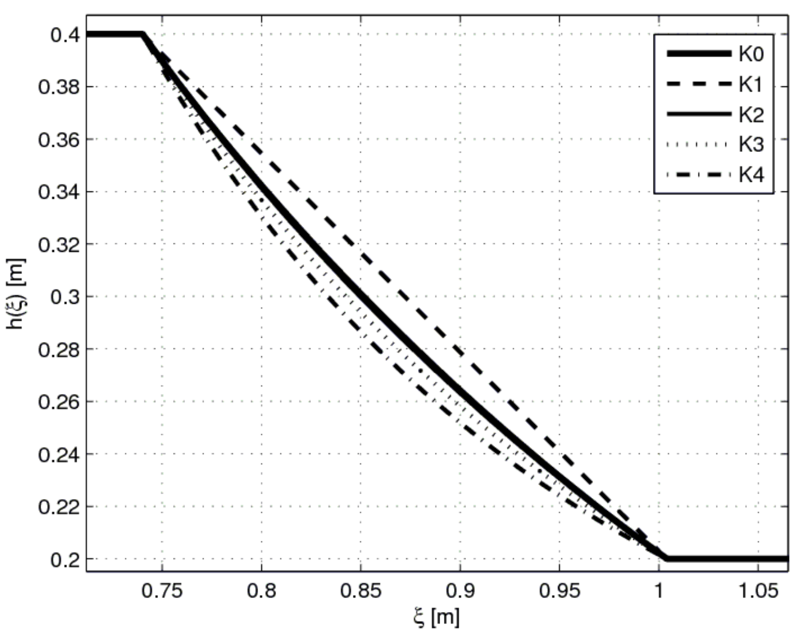

Fig. 9. The candidate for optimal shape K0 and the shapes in neighbourhood K1, K2, K3 and K4

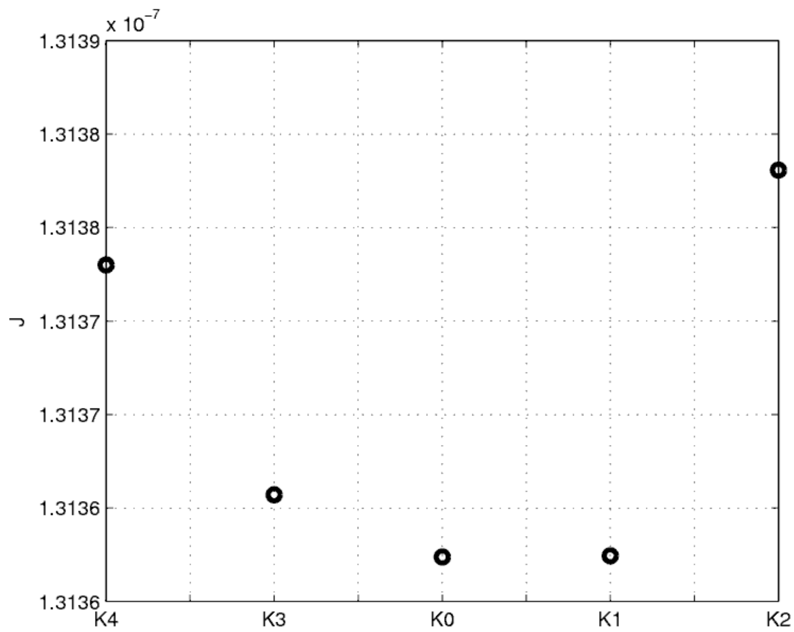

Fig. 10. The cost function J in the neighbourhood of the candidate for optimal control K0

Other neighbourhood of the candidate for optimal shape is presented in fig. 11. The candidate for optimal shape is depicted using bold line. The neighbourhood contains shapes in the form of straight lines with different points ξ_a and ξ_b (thin lines). Then for every shape we calculated the deflection of the beam at the end point. The results of these calculations are shown in fig. 12. For the shapes in the neighbourhood we obtain worse values of the cost function J than for the shape K0 .

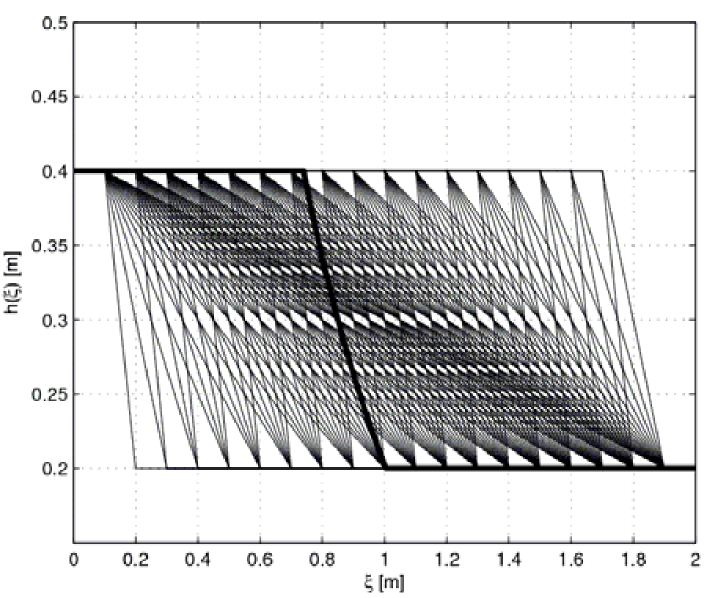

Fig. 11. The neighbourhood of the candidate for optimal shape

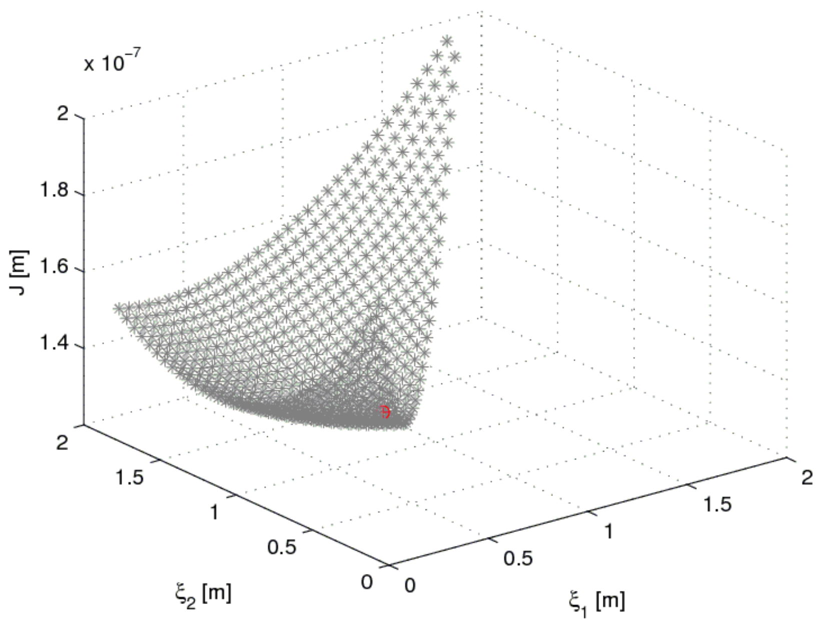

Fig. 12. The cost function J in the neighbourhood of the candidate for optimal shape

5. Optimal design of a single span beam with I cross-section

5.1 Equation of a physical system

The methodology presented in section 3 can be used for solving other types of shape optimization problems. Also the algorithm presented in section 4 can be easily adapted to other types of problems.

Consider for example a single span beam with I cross-section working under self-weight (fig. 13).

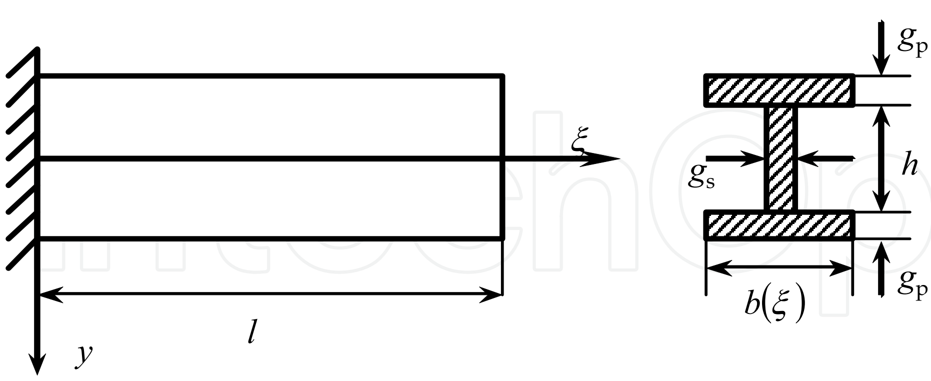

Fig. 13. Single span beam with I cross-section

The statics of the beam can be described using the following equation

$$\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi} = \boldsymbol{f}(\boldsymbol{x}(\xi), u(\xi)), \tag{44}$$

where $\boldsymbol{x}(\xi)=[x_1(\xi) \quad x_2(\xi) \quad x_3(\xi) \quad x_4(\xi)]^{\mathrm{T}}$, $u(\xi)=\gamma b(\xi)h$, $\xi \in [0,l]$, and

$$\boldsymbol{f}(\boldsymbol{x}(\xi),u(\xi))=\begin{bmatrix} x_2(\xi) \\ \dfrac{x_3(\xi)}{a+cu(\xi)} \\ x_4(\xi) \\ -u(\xi) \end{bmatrix}, \tag{45}$$

$$a=E\frac{g_{\mathrm{s}}h^3}{12}, \quad c=2Eg_{\mathrm{p}}\left(\frac{h}{2}+\frac{g_{\mathrm{p}}}{2}\right)^2. \tag{46}$$

Here $x_1(\xi)$ represents vertical displacement of the beam along the interval $[0,l]$, $x_2(\xi)$ indicates the slope of the beam at ξ, $x_3(\xi)$ can measure in physical terms the bending moment of the beam at ξ, $x_4(\xi)$ can measure the shearing force on the beam at ξ, l stands for the length of the beam, $b(\xi)$ is the width of the beam's cross-section, h is the height of the cross-section, E is the Young's module, γ is the volume mass density of the beam material. The static beam equation is fourth-order (it has a fourth derivative) and the mechanism for supporting the beam gives rise to four boundary conditions

$$x_1(0)=x_2(0)=x_3(l)=x_4(l)=0\,. \tag{47}$$

For the control variable u we introduce geometrical and strength constraints that define a set of admissible controls

$$U_{\mathrm{ad}}=\{u\in \mathrm{PC}([0,l],\mathrm{R}):u(\xi)\in U\}, \tag{48}$$

$$U=\{v\in \mathrm{R}:H_1\le v\le H_2, H_1<H_2\},\ H_1,H_2\in \mathrm{R}\,. \tag{49}$$

The cost function denotes the deflection at the end point of the beam and it is defined by the functional

$$J(u)=x_1(l). \tag{50}$$

We want to determine $u\in U$ which minimizes the functional (50) and satisfies the state equation (44) with the boundary conditions (47).

5.2 Solution method

In order to solve the formulated problem we can use the Pontryagin maximum principle and follow the method presented in sections 3 and 4. The Hamiltonian constructed for this case together with the system of adjoint equations lead to the conditions which determine the candidate for the optimal control. The constraints appearing in this problem are identical as in the problem from section 4, therefore the complete analysis for the optimal control is very similar.

5.3 Numerical simulation results

The numerical solution of the formulated problem has been carried out using the program designed in MATLAB/Simulink environment. Fig. 14 presents optimal design of the single span beam with I cross-section. Calculations were made for the following data: $l = 2.0$ [m], $h = 0.2$ [m], $H_1 = 0.2$ [m], $H_2 = 0.4$ [m], $E = 2.1 \cdot 10^{11}$ [N/m^2], $\gamma = 76500$ [N/m^3], $g_{\mathrm{p}} = 0.002$ [m], $g_{\mathrm{s}} = 0.0013$ [m].

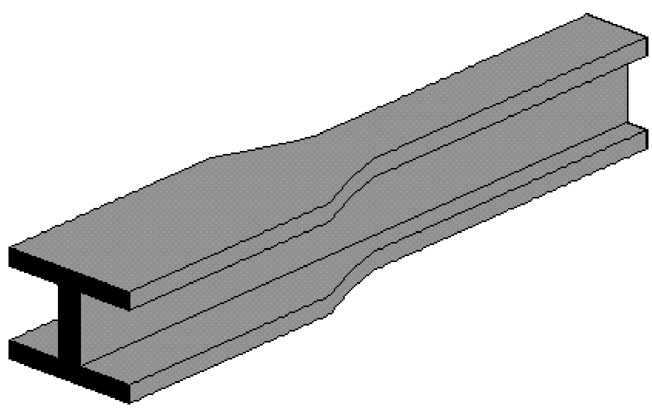

Fig. 14. Optimal shape of the beam with I cross-section

6. Optimal design of a clamped beam with rectangular cross-section

6.1 Equation of a physical system

Consider a clamped beam with rectangular cross-section (fig. 15).

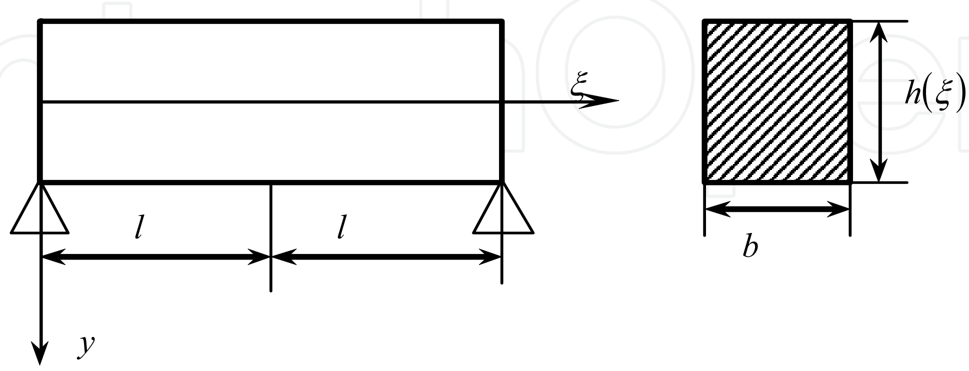

Fig. 15. Clamped beam with rectangular cross-section

The state equation describing statics of the beam has the form

$$\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}=\boldsymbol{f}(\boldsymbol{x}(\xi),u(\xi)), \tag{51}$$

where $\boldsymbol{x}(\xi)=[x_1(\xi) \quad x_2(\xi) \quad x_3(\xi) \quad x_4(\xi)]^{\mathrm{T}}$, $u(\xi)=\gamma bh(\xi)$, $\xi\in[0,l]$, and

$$\boldsymbol{f}(\boldsymbol{x}(\xi),u(\xi))=\begin{bmatrix} x_2(\xi) \\ \dfrac{12x_3(\xi)}{Ebu(\xi)^3} \\ x_4(\xi) \\ -u(\xi) \end{bmatrix}. \tag{52}$$

All parameters have the same meaning as for the single span beam with rectangular cross-section working under self weight. The beam cannot experience deflection neither at the left-hand nor at right-hand support, therefore $x_1(0)=0$ and $x_1(2l)=0$. The beam does not experience also any torque what means that $x_3(0)=0$ and $x_3(2l)=0$. The boundary conditions can be rewritten equivalently to the form

$$x_1(0)=x_2(l)=x_3(0)=x_4(l)=0\,. \tag{53}$$

Strength constraints and geometry are imposed on the dimensions of the cross-section defining the set of admissible controls

$$U_{\mathrm{ad}}=\{u\in \mathrm{PC}([0,l],\mathrm{R}):u(\xi)\in U\}, \tag{54}$$

$$U=\{v\in\mathrm{R}:H_1\le v\le H_2, H_1<H_2\},\ H_1,H_2\in\mathrm{R}\,. \tag{55}$$

The deflection at the middle point of the beam can be the optimality criterion

$$J(u)=x_1(l). \tag{56}$$

We want to determine $u\in U$ which minimizes the functional (56) and satisfies the state equation (51) with the boundary conditions (53).

6.2 Solution method

In order to solve the formulated problem we can use the Pontryagin maximum principle and follow the method presented in sections 3 and 4.

6.3 Numerical simulation results

Fig. 16 presents optimal shape of the clamped beam with rectangular cross-section. Calculations were made for the following data: $l=2.0$ [m], $b=0.05$ [m], $H_1=0.25$ [m], $H_2=0.5$ [m], $E=2.1\cdot10^{11}$ [N/m^2], $\gamma=22000$ [N/m^3].

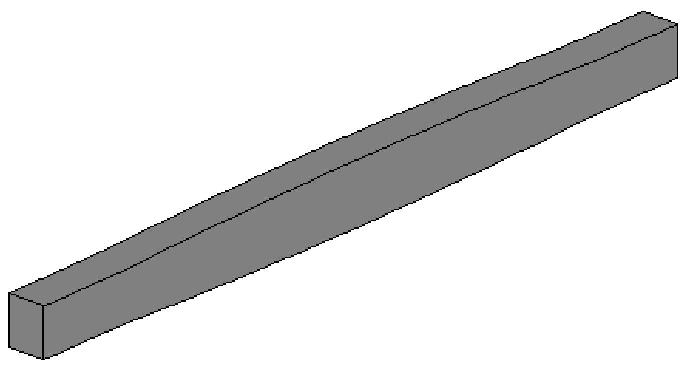

Fig. 16. Optimal shape of the clamped beam

7. Conclusions

We have investigated a shape optimization problem. As it has been shown the problem is not always trivial and the general proof can be very difficult. By using effective Pontryagin's method of optimization, the numerical algorithm has been designed and implemented. The simulation results show the effectiveness of the proposed method.

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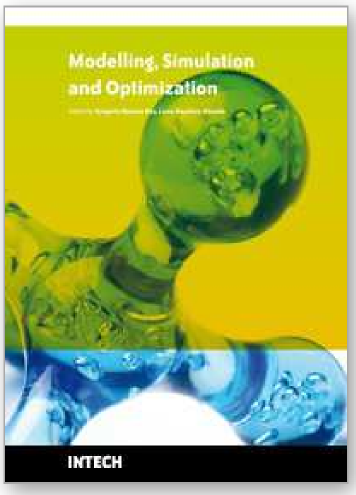

Modelling Simulation and Optimization

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Computer-Aided Design and system analysis aim to find mathematical models that allow emulating the behaviour of components and facilities. The high competitiveness in industry, the little time available for product development and the high cost in terms of time and money of producing the initial prototypes means that the computer-aided design and analysis of products are taking on major importance. On the other hand, in most areas of engineering the components of a system are interconnected and belong to different domains of physics (mechanics, electrics, hydraulics, thermal...). When developing a complete multidisciplinary system, it needs to integrate a design procedure to ensure that it will be successfully achieved. Engineering systems require an analysis of their dynamic behaviour (evolution over time or path of their different variables). The purpose of modelling and simulating dynamic systems is to generate a set of algebraic and differential equations or a mathematical model. In order to perform rapid product optimisation iterations, the models must be formulated and evaluated in the most efficient way. Automated environments contribute to this. One of the pioneers of simulation technology in medicine defines simulation as a technique, not a technology, that replaces real experiences with guided experiences reproducing important aspects of the real world in a fully interactive fashion [iii]. In the following chapters the reader will be introduced to the world of simulation in topics of current interest such as medicine, military purposes and their use in industry for diverse applications that range from the use of networks to combining thermal, chemical or electrical aspects, among others. We hope that after reading the different sections of this book we will have succeeded in bringing across what the scientific community is doing in the field of simulation and that it will be to your interest and liking. Lastly, we would like to thank all the authors for their excellent contributions in the different areas of simulation.

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