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Algebraic Algorithms for Image Tomographic Reconstruction from Incomplete Projection Data

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1. Introduction

Technique of computerized tomography has a wide application not only in medicine but also in different fields of technique. In many applications because of some reasons one cannot obtain the full set of projection data of a reconstructed object, e.g. the projection data are not available at each angle of view or they are very limited in number. Sometimes because of the large size of objects and limitation in the size of the scanners it is not possible to obtain the complete set of required projections. In these cases we deal with the problem of image reconstruction from incomplete projection data. In particular, such kind of problems arise in mineral industries and engineering geophysics connected with acid drainage, the stability of mine workers, mineral exploration and others.

When the projection data available are not limited in number and complete, the transform methods of reconstructions are usually used (Herman, 1980), (Natterer, 1986), (Kak & Slaney, 1988). For incomplete projection data these methods cannot be used directly. In this case there often used different kinds of algebraic iterative algorithms the most well-known from which are algorithms of algebraic reconstruction technique (ART) (Gordon et al., 1970), (Herman et al., 1973), (Eggermont et al., 1981). They are generally simple, flexible and permit to use a priori knowledge of the object before its reconstruction that is very important in many practical applications. Recently algebraic iterative algorithms are also used in magnetic resonance imaging (MRI) (Liang & Lauterbur, 2000), (Harshbarger & Twieg, 1999) and low-contrast 3D-cone-beam tomography (Mueller et al., 1997).

However the application of algebraic iterative algorithms to real practical problems has some important obstacles. The main repellant for using these algorithms are their significantly slow reconstruction speed and so a large time of computations for obtaining the good results, and a large memory space required to store the reconstruction image, projection data and the projection matrix. In order to avoid these difficulties there are often used the algebraic algorithms which can be allowed to parallelize and can be realized on the parallel computing systems (PCS). The main general types of parallel iterative algebraic algorithms for computerized tomography were proposed by Y. Censor (Censor, 1988). The efficient performance of some of the parallel algorithms were described in (De Pierro & Iusem, 1985); (Chen & Lee, 1994); (Chen et al., 1990); (Laurent et al., 1996); (Gubareni, 1998a).

One of the most perspective areas of parallel computations is an elaboration of asynchronous realizations of iterative algorithms. The main characteristic of the organization of asynchronous computations is that the solution is obtained during a non-synchronous interaction of processor elements of a parallel structure. Each processor of this PCS updates the values of corresponding components of the solution using available information about other components of the solution, and it obtains this information from local processors or shared memory without waiting their full update. The researches showed that the asynchronous realizations of parallel algorithms are more efficient from the point of view of their speed of convergence in many important cases (Kung, 1976); (Bertsekas & Tsitsiklis, 1989; 1991); (Savari & Bertsekas, 1996). Note that the convergence of asynchronous algorithms and their synchronous prototypes may be different. Some models of asynchronous iterative methods for image reconstruction are considered in (Baudet, 1978); (Chazan & Miranker, 1969); (Bru et al., 1988); (Elsner et al., 1990); (Kaszkurewicz et al., 1990). Some generalization of these models for image reconstruction were considered in (Baran et al., 1996); (Gubareni et al., 1997b); (Gubareni, 1999).

This chapter is devoted to consider the problem of image reconstruction from incomplete projection data for particular reconstruction systems which arise in engineering geophysics and mineral industry. Besides the well-known algorithms such as ART and MART their chaotic, parallel and block-parallel implementations are considered in this chapter. The use of these algorithms to reconstruct high-contrast objects from incomplete data is examined. The influence of various parameters of these algorithms, such as the relaxation coefficients, the number of iterations, the number of projections, and noise in projection data on the reconstruction quality for different schemes of reconstruction are investigated.

Numerical results of image reconstruction from incomplete projection data for some modeling objects, comparing evaluations of errors and the rate of convergence of these algorithms are presented and discussed. It is shown that for some choice of parameters one can obtain a good quality of reconstruction with these algorithms under the noise and incomplete data.

2. Problem of incomplete projection data

The main goal of computerized tomography is to recover an unknown density function from its line integrals. Let $f(x,y)$ be a density function which represents the spatial distribution of a physical parameter. If $L:\ l = x\cos\theta + y\sin\theta$ is a line (ray) in the plane then the line integral

$$p_L = \int_L f(x,y)dL = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(l - x\cos\theta - y\sin\theta)dxdy\,, \qquad (1)$$

which is called a projection, is usually obtained from physical measurements.

From mathematical point of view the problem of reconstruction from projection data is to find an unknown function $f(x,y)$ by means of a given set of projections p_L for all L. Theoretically it is possible to reconstruct the function $f(x,y)$ from the set of projections p_L by means of the Radon inversion formula (Radon, 1917). The classical inversion formula of Radon requires information of all the line integrals in order to recover the function $f(x,y)$ in each point. Unfortunately, this mathematical problem represents only an idealized abstraction of problems which occur in real practical applications. In practice there is given

only a discrete set of projection data that estimate p for a finite number of rays. Since projection data are obtained by physical measurements with limited precision, they are given with some errors. Therefore all these restrictions do not allow to use the Radon inversion formula directly.

The projection data for computerized tomography is said to be complete if they are obtained from every aspect of view angle. In many practical applications projections are often not available at each direction and may be very limited in number. Moreover, because of the large size of objects it is not possible to obtain the complete set of the required projection data. In these cases one says that there is a problem of image reconstruction from incomplete projection data. In particular, such kind of problems arise in mineral industries and engineering geophysics connected with acid drainage, the stability of mine workers, mineral exploration and others (Patella, 1997), (Williams, et al., 2004).

There exist two fundamentally different approaches for solving the image reconstruction problem. In the first approach the problem is formulated for continuous functions f and p and the inversion formula is derived in this continuous model. This method is called the transform method approach (Censor & Zenios, 1997). The second approach is connected with the discretization of functions f and p at the outset. So the object f and measurements p become the vectors in the finite dimensional Euclidean space. In this case the methods of linear algebra and optimization theory are used for solving the problem of image reconstruction. This approach is called the fully discretized model (Censor & Zenios, 1997).

If the projection data can be obtained from every aspect of view angle and their number can be obtained large enough (in medicine, for example), then it is more preferred to use the transform method approach, e.g. the convolution back projection (CBP) algorithm (Ramachandran & Lakhshminarayanan, 1970) or direct Fourier technique.

In dependence on the obtaining system of projections there are many image reconstruction schemes, the main of them are parallel and beam schemes in the two-dimensional space. Both of them are represented in Figure 1.

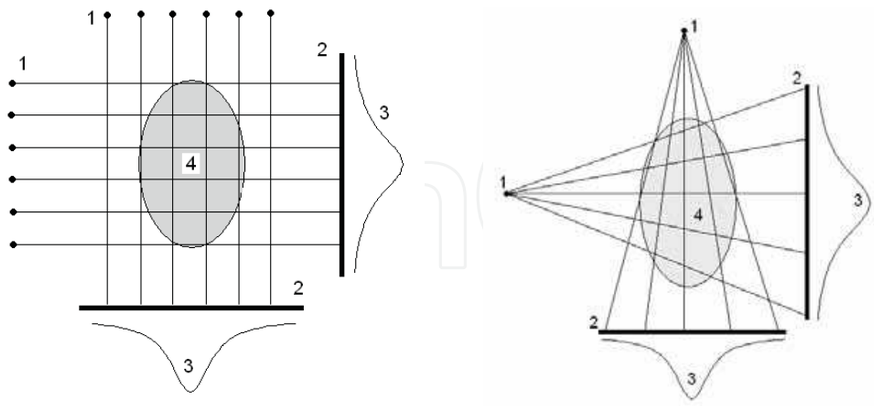

Fig. 1. Parallel and beam schemes of obtaining projection data in image reconstruction. 1- sources; 2- detectors; 3 – projections; 4 – a research object

In some practical problems, in engineering for example, it is impossible to obtain projections from all directions because of the existence of some important reasons (such as situation,

size or impossibility of an access to a research object). This situation arises, for example, in the coal bed working. During the preparing process for working in this coal bed the access to longwalls may be very difficult or impossible at all in dependence on the scheme of obtaining projection data. Sometimes it is impossible to access to one or two sides of longwalls, and sometimes it is impossible only to access to the basis but all the longwalls are accessible. Each this situation has its own scheme of obtaining information.
Some examples of the schemes for obtaining projection data is shown in Figure 2. In the first case there is an access to a research object from only two opposite sides. Therefore the sources of rays can be situated only on one side and the detectors are situated on the opposite side of the research part of a coal bed. This scheme will be called the system (1 × 1). And in the second case there is an access to all four sides of an object. Therefore the sources can be situated, for example, onto two neighboring sides, and the detectors can be situated on the opposite sides. So the projections can be obtained from two pairs of the opposite sides.

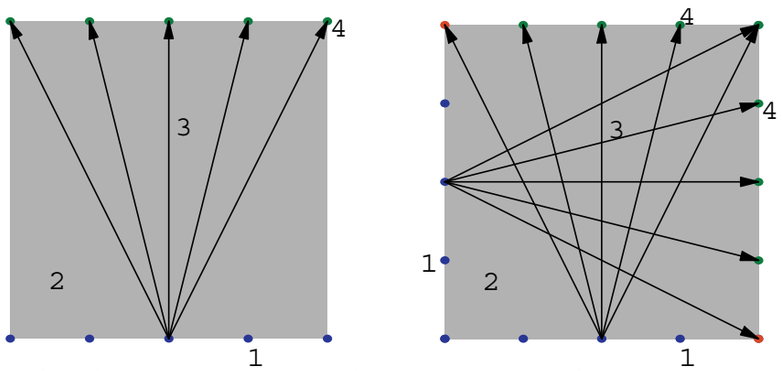

Fig. 2. Schemes for obtaining projections data. 1 – sources of rays; 2- a research object; 3 – rays; 4 – detectors.

3. Algebraic iterative algorithms

The numerical solution of equation (1) using ART requires the discretization of the cross-section of an object. To construct a discretized model, a reconstructed domain $D \subset R^2$ is included into a rectangle E and divided into n small elements (pixels). The full discrete model of the problem of image reconstruction is based on the main principal that a research object has the constant distribution inside each pixel. So for any i-th pixel one can correspond an unknown x_i. Secondly, one can assume that sources and detectors are points and the rays between them are lines. Denote by a_{ij} the length of the intersection of the i-th ray with j-th pixel. The length a_{ij} represents the contribution of the j-th pixel to the total attenuation along the i-th ray. Thus, the discretized model of the problem of image reconstruction is reduced to a system of linear algebraic equations:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{p} \ , \tag{2}$$

where:

$\mathbf{A} = (a_{ij}) \in \mathbf{R}^{m,n}$ is the matrix of coefficients,

$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$ is the image vector,

- $\mathbf{p} = (p_1, p_2, \ldots, p_m)^T \in \mathbf{R}^m$ is the measurement vector of projection data.

This system has a few characteristics: it is a rectangular as a rule and it has a very large dimension. For solving this system it is often used different kinds of algebraic iterative algorithms the most well-known of which are the additive algorithm ART (Herman, el. al., 1973), (Herman, 1975), (Herman, 1980), (Eggermont, el. al., 1981). These algorithms are very flexible and allow to apply different *a priory* information about object before its reconstruction that is especially very important when we have incomplete projection data. Denote

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{(\mathbf{a}^i, \mathbf{x}) - p_i}{\left\|\mathbf{a}^i\right\|^2}\mathbf{a}^i, \tag{3}$$

$$\mathbf{P}_i^{\omega} = (1-\omega)\mathbf{I} + \omega\mathbf{P}_i, \tag{4}$$

where $\mathbf{a}^i$ is the i-th row of the matrix **A**, and ω is a relaxation parameter.

Algorithm 1 (ART-1).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k-th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{(k+1)} = \mathbf{C}\mathbf{P}_i^{\omega_k}\mathbf{x}^{(k)} \quad (i = 1,2,...,m), \tag{5}$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (4), ω_k are relaxation parameters, **C** is a constraining operator, and $i(k) = k(\mathrm{mod}\, m) + 1.$

This algorithm was proposed by Kaczmarz (Kaczmarz, 1937) and independently discovered and investigated by G.T.Herman, A.Lent, S.Rowland in (Herman, el. al., 1973). It was used successfully in application of computerized tomography in medicine. This algorithm runs through all equations cyclically with modification of the present estimate $\mathbf{x}^{(k)}$ in such a way that the present equation with index i is fulfilled.
The multiplicative variant of ART, the algorithm MART, is given by the following form.

Algorithm 2 (MART).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector and $\mathbf{x}^{(0)}>0$.
2. The k+1-th iteration is calculated by the following way:

$$x_j^{(k+1)} = \left(\frac{p_i}{(\mathbf{a}^{i(k)}, \mathbf{x}^{(k)})}\right)^{\omega_k^i a_{ij}} x_j^{(k)} \tag{6}$$

where

- $\mathbf{a}^i$ is the i-th row of the matrix **A,**
- ω_k^i is a relaxation parameter,
- p_i is the i-th coordinate of the projection vector **p**,
- $i(k) = k \pmod m + 1$.

This algorithm was invented and reinvented in several fields. It was shown that it is convergent if $0 < \omega_k^i a_{ij} < 1$ for all i, k, j, and its solution gives the solution of the linearly constrained entropy optimization problem (Lent, 1977), (De Pierro, 1990), (Censor & Zenios, 1997).
In practice the vector of projection data is given as a rule with some error. Therefore instead of a system of linear equations (2) there results a system of linear inequalities:

$$\mathbf{p} - \mathbf{e} \le \mathbf{A} \cdot \mathbf{x} \le \mathbf{p} + \mathbf{e} \tag{7}$$

where $\mathbf{e} = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$ is a non-negative vector. And one can consider that the vector $\mathbf{e}$ is given a priory and defines the errors of projection data.
Introduce the following projection operator:

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{((\mathbf{a}^i, \mathbf{x}) - p_i - \varepsilon_i)^+ - (p_i - \varepsilon_i - (\mathbf{a}^i, \mathbf{x}))^+}{\left\|\mathbf{a}^i\right\|^2} \mathbf{a}^i, \tag{8}$$

where

$$s^+ = \begin{cases} s, & \text{if } s \ge 0; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathbf{P}_i^{\omega} = (1 - \omega)\mathbf{I} + \omega \mathbf{P}_i, \tag{9}$$

where $\mathbf{a}^i$ is the i-th row of a matrix $\mathbf{A}$, and ω is a relaxation parameter.
In this case there results the following additive algorithm which is analogous to the algorithm 1.

Algorithm 3 (ART-3).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector.
2. The k+1-th iteration is calculated by the following way:

$$\mathbf{x}^{(k+1)} = \mathbf{C}\mathbf{P}_i^{\omega_k} \mathbf{x}^{(k)} \quad (i = 1,2,\ldots,m), \tag{10}$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (8) and (9), ω_k are relaxation parameters, $i(k) = k(\mathrm{mod}\, m) + 1$, and $\mathbf{C}$ is a constraining operator.

This algorithm was investigated by G.T. Herman (Herman, 1975), and it was used successfully in medicine.

4. Block-parallel iterative algorithms

The convergence rate of algebraic iterative algorithms considered in the previous section is very slow and a lot of iterations should be made to obtain a good reconstruction. It is more efficient to apply the algorithms which use simultaneously all equations (or inequalities) of system (2) (or (3)) at each step of iteration process. The examples of these algorithms are the generalized algorithms of the Cimmino type (Censor, 1988) which can be related to the class

of parallel algorithms. Following Y. Censor (Censor, 1988), the iterative algorithm is considered to be parallel if it can be represented in the following form:

$$\mathbf{y}^{k,i} = \mathbf{R}_i(\mathbf{x}^{(k)}, \mathbf{a}^i, p_i) \tag{11}$$

$$\mathbf{x}^{(k+1)} = \mathbf{S}\left(\left\{\mathbf{y}^{k,i}\right\}_{i\in J}\right) \tag{12}$$

where $\mathbf{R}_i$ is an operator of the row type, $\mathbf{S}$ is an algorithmic operator which uses simultaneously information obtained while solving all equations (inequalities) of system (2) (or (7)) and generalizes them, $J=\{1,2,\ldots,m\}$, k is the number of iteration.
Consider the operator $\mathbf{S}$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ in the following form:

$$\mathbf{S}\left(\left\{\mathbf{y}^{k,i}\right\}_{i\in J}\right) = \sum_{i=1}^{m} \mathbf{B}_i^k \mathbf{y}^{k,i} \tag{13}$$

where $\mathbf{B}_i^k$ are $n\times n$-matrices with real nonnegative elements and $\sum_{i=1}^{m} \mathbf{B}_i^k = \mathbf{E}$ ($\mathbf{E}$ is the identity matrix).
Consider the following class of parallel iterative algorithms.

Algorithm 4 (PART).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\mathbf{y}^{k,i} = \mathbf{P}_i^{\omega_k} \mathbf{x}^{(k)} \qquad (i = 1,2,\ldots,m), \tag{14}$$

$$\mathbf{x}^{(k+1)} = \mathbf{C}\sum_{i=1}^{m} \mathbf{B}_i^k \mathbf{y}^{k,i}, \tag{15}$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (9), ω_k are relaxation parameters, $\mathbf{C}$ is a constraining operator and $\mathbf{B}_i^k$ are matrices of dimension $n\times n$ with real nonnegative elements and

$$\sum_{i=1}^{m} \mathbf{B}_i^k = \mathbf{E}, \qquad \sum_{i=1}^{m} \| \mathbf{B}_i^k \| \le 1, \tag{16}$$

for all $k \in \mathbf{N}$.

Remark 1. Let $\mathbf{B}_i^k = (\gamma_{jj}^i)_{j=1}^n$ be a diagonal matrix with elements $0 < \gamma_{jj}^i < 1$. If $\gamma_{jj}^i = \gamma_i$ for each $j \in J$, $i \in I$, $\omega_k = 1$, $\mathbf{C} = \mathbf{I}$, then there results the Cimmino algorithm (Censor, 1978). If $\gamma_{jj}^i = 1/m$ for each $j \in J$, $i \in I$, $\omega_k = 1$, $\mathbf{C} = \mathbf{I}$, then there results the von Neumann algorithm (Censor, 1978).
The study of different variants of this class of parallel algorithms by analyzing their convergence was conducted by many authors, e.g. (Censor, 1978), (De Pierro & Iusem, 1985a, 1985b), (Censor & Zenious, 1997), (Gubareni, 1997).

For many practical applications $\mathbf{x} \geq 0$, the elements of a matrix $\mathbf{A}=(a_{ij})$ are nonnegative real numbers and $p_i > 0$ for all $i \in I$. In this case one may consider the following parallel multiplicative algorithm for solving system of linear inequalities (7) (Censor, 1974), (De Pierro, 1990).

Algorithm 5 (MARTP).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ and $\mathbf{x}^{(0)} > 0$.
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$x_j^{(k+1)} = x_j^{(k)} \prod_{i=1}^{m} y_j^{k,i}, \tag{17}$$

where

$$y_j^{k,i} = \left(\frac{p_i}{(\mathbf{a}^i, \mathbf{x}^{(k)})} \right)^{\gamma_{ij}^k a_{ij}}, \tag{18}$$

(i=1,2,..., m; j=1,2,..., n), γ_{ij}^k are positive real numbers for every j, k.

These algorithms may be realized on parallel computing structure consisted of m elementary processors and one central processor. On each (k+1)-th step of iteration every i-th elementary processor computes the coordinates of the vector $\mathbf{y}^{k,i}$ in accordance with formula (14) or (18) and then the central processor computes the (k+1)-th iteration of the image vector $\mathbf{x}$ in accordance with formula (15) or (17).

The main defect of parallel algorithms considered above is their practical realization on parallel computational structures because it needs a lot of local processors in an MPCS. In order to reduce the number of required local processors consider block-iterative additive and multiplicative algorithms considered in (Elfving, 1980), (Eggermont et al. 1981), (Censor, 1988), (Gubareni, 1997).

For this purpose decompose the matrix $\mathbf{A}$ and the projection vector $\mathbf{p}$ into M subsets in accordance with a decomposition

$$\{1,2,...,m\} = H_1 \cup H_2 \cup ... \cup H_M, \tag{19}$$

where

$$H_t = \{m_{t-1}+1,\ m_{t-1}+2,\ \ldots,\ m_t\}, \tag{20}$$

$0=m_0<m_1<\ldots< m_M=m$, $1 \leq t \leq M$.

Algorithm 6 (BPART).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{(k+1)} = \mathbf{C} \sum_{i \in H_{t(k)}} \mathbf{B}_i^k \mathbf{P}_i^{\omega_k} \mathbf{y}^i, \tag{21}$$

where $t(k) = k(\text{mod } M) + 1$, $\mathbf{P}_i^{\omega_k}$ are operators defined by (9), $0 < \omega_k < 2$ are relaxation parameters, $\mathbf{C}$ is a constraining operator and $\mathbf{B}_i^k$ are matrices of dimension $n \times n$ with real nonnegative elements and

$$\sum_{i \in H_{t(k)}}^{m} \mathbf{B}_i^k = \mathbf{E}, \qquad \sum_{i \in H_{t(k)}}^{m} \| \mathbf{B}_i^k \| \leq 1, \tag{22}$$

for all $k \in \mathbf{N}$.

The parallel implementation of this algorithm can be described as follows:

$$\mathbf{y}^{k,i} = \mathbf{P}_i^{\omega_k} \mathbf{x}^{(k)} \qquad (i \in H_{t(k)}),$$

$$\mathbf{x}^{(k+1)} = \mathbf{C} \sum_{i \in H_{t(k)}}^{m} \mathbf{B}_i^k \mathbf{y}^{k,i},$$

The block-iterative algorithms represent examples of sequential-parallel algorithms. They may be considered as intermediate version between sequential algorithms and full parallel ones. In each step of an iterative process the block-iterative algorithm uses simultaneously information about all equations concerning to a given block.
Block-iterative algorithms may be also considered in the case of multiplicative algorithms. In this case there results the following algorithm.

Algorithm 7 (BMART).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ and $\mathbf{x}^{(0)} > 0$.
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$x_j^{(k+1)} := x_j^{(k)} \prod_{i \in H_{t(k)}} \left(\frac{p_i}{(\mathbf{a}^i, \mathbf{x}^k)} \right)^{\gamma_{ij}^k a_{ij}}, \tag{23}$$

where γ_{ij}^k are positive real numbers such that

$$0 < \sum_{i \in H_{t(k)}} a_{ij} \gamma_{ij}^k \leq 1 \tag{24}$$

for every j, k; $H_{t(k)}$ are defined in accordance with (20) and $t(k)$ is almost cycle control sequence.

If $\gamma_{ij}^k = \gamma_i$ for all k,j and $0 \leq a_{ij} \leq 1$, $\sum_{i \in H_{t(k)}} \gamma_i = 1$, then there results the block-iterative multiplicative algorithm proposed in (De Pierro & Iusem, 1985).

5. Chaotic and asynchronous algorithms

First chaotic and asynchronous algorithms for image reconstruction were proposed and studied in (Bru et al., 1988), (Elsner et al., 1990). These algorithms are based on the methods of asynchronous iterations introduced first by Chasan and Miranker (Chazan & Miranker, 1969). The further development of these methods and their generalizations for the case of non-linear operators was obtained by Baudet (Baudet, 1978).
Recall some important notions of the theory of chaotic and asynchronous iterations (Chazan & Miranker, 1969), (Baudet, 1978), Bertsekas D.P. (1983), (Bertsekas & Tsitsiklis, 1989), (Bertsekas & Tsitsiklis, 1991).

Definition 1. A sequence of nonempty subsets $I = \{I_k\}_{k=0}^{\infty}$ of the set {1,2…, m} is **a sequence of chaotic sets** if

$$\limsup_{k\to\infty} I_k = \{1,2,...,m\} \tag{25}$$

(another words, if each integer $j \in \{1,2,...,m\}$ appears in this sequence an infinite number of times).
For the first time such sequences were used by Baudet (Baudet, 1978).

Definition 2. If any subset of a sequence of chaotic sets I has the form $I_k = \{j_k\}$, where $j_k \in \{1,2,...,m\}$ (i.e. each set consists of only one element), then the sequence I is called **acceptable (**or **admissible)**.

Suppose that PCS (Parallel Computing System) consists of m processors working local independently. In this case the notion of the sequence of chaotic sets has a simple interpretation: it sets the time diagram of work of each processor during non-synchronous work of PCS. So the subset I_k is the set of the numbers of those processors which access the central processor at the same time.
Note, that the definition of the sequence of chaotic sets can be given in the following equivalent form:

Lemma 1. Let $I = \{I_k\}_{k=0}^{\infty}$ be a sequence of nonempty subsets of the set {1,2..., m}. Then the following conditions are equivalent:
1) $\limsup_{k\to\infty} I_k = \{1,2,...,m\}$
2) the set $\{ k \mid i \in I_k \}$ is unlimited for each i = 1,2..., m.
3) for each $j \in \mathbf{N}$ there exists $p(j) \in \mathbf{N}$ such that the following condition satisfies:

$$\bigcup_{i=j+1}^{j+p(j)} I_i = \{1,2,...,m\} . \tag{26}$$

For any sequence of chaotic sets the numbers $p(j)$ depends on a number j. In practice and for researching the convergence of asynchronous implementations of iterative processes

there are more important sequences of chaotic sets, for which these numbers do not depend on number *j*.

Definition 3. If for a sequence of chaotic sets $I = \{I_k\}_{k=0}^{\infty}$ the numbers $p(j)$ defined by (26) do not depend on the choice of number *j*, i.e. $p(j) = T = \text{const}$ for each $j \in \mathbf{N}$, then this sequence is called **regular**, and the number *T* is called the **number of regularity o**f the sequence *I*.

Note, that this definition coincides with a concept of a regular sequence, introduced in (El Tarazi, 1984) for the case of an admissible sequence. In this work El Tarazi obtained important results introducing the obvious model for the class of a synchronous algorithms and giving the first correct conditions of convergence in the non-linear case of contraining operators.

Other important concept in the theory of an asynchronous iterations is the concept of a sequence of delays.

Definition 4. A sequence $J = \{\sigma(k)\}_{k=1}^{\infty}$ of *m*-dimensional vectors $\sigma(k) = \{\sigma_1(k), \sigma_2(k), ..., \sigma_m(k)\}$ with integer coordinates, satisfying the following conditions:

$$1) \quad 0 \le \sigma_i(k) \le k - 1; \tag{27}$$

$$2) \quad \lim_{k \to \infty} \sigma_i(k) = \infty, \tag{28}$$

for each i = 1,2...,*m* and $k \in \mathbf{N}$, is called a **sequence of delays**.

In the case, when instead of condition 2) it holds the following condition:

2′) there exists a fixed number $L \in \mathbf{N}$ such that

$$k - \sigma_i(k) \le L \tag{29}$$

for each $k \in \mathbf{N}$ and i = 1,2...,*m*, the sequence is called a **sequence with limited delays** and the number *L* is called a **delay**, or an **asynchronous measure**.

The sequence of delays determines the numbers of using iterations by each fixed processor, and the number *L* shows a depth of used iterations and actually reflects possibilities of the concrete computing system. For synchronous implementation of the iterative process the difference $k - \sigma_i(k)$ is equal to 0 for $\forall i$ = 1,2..., *m* and $k \in \mathbf{N}$.

Consider the definition of some generalized model of asynchronous computational process (Robert, et al., 1975), (Baran et al., 1996), (Gubareni et al., 1997b).

Definition 5. Let there exist a set of nonlinear operators $\mathbf{T}_i$: $\mathbf{R}^n \to \mathbf{R}^n$, $i \in \{1,2,...,m\}$ and an initial value $\mathbf{x}^{(0)} \in \mathbf{R}^n$. A **generalized model of asynchronous iterations with limited delays** for the set of operators $\mathbf{T}_i$, i=1,2,...,*m* is called a method of building the sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$, which is given recursively by the following scheme:

$$\begin{aligned} \mathbf{y}^{k,i} &= \begin{cases} \mathbf{T}_i \mathbf{x}^{\left(\sigma^i(k)\right)}, & \text{if } i \in I_k \\ \mathbf{y}^{k-1,i}, & \text{otherwise} \end{cases} \\ \mathbf{x}^{(k)} &= \mathbf{S}\left(\mathbf{x}^{(k-1)}, \left\{\mathbf{y}^{k,i}\right\}_{i \in I_k}\right), \end{aligned} \tag{30}$$

where $I = \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$ and $J_i = \left\{\sigma^i(k)\right\}_{k=1}^{\infty}$ are sequences of limited delays (i=1,2,...,m).

Apply the generalized model of asynchronous iterations for an implementation of the ART algorithm on a non-synchronous computer structure. In this case there results the following asynchronous algorithm, where the numbers of operators are chosen by a chaotic way.

Algorithm 8.

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\begin{aligned} \mathbf{y}^{k+1,i} &= \begin{cases} \mathbf{P}_i^{\omega_k} \mathbf{x}^{\left(\sigma^i(k)\right)}, & \text{if } i \in I_k \\ \mathbf{y}^{k,i}, & \text{otherwise} \end{cases} \\ \mathbf{x}^{(k+1)} &= \sum_{i \in I_k} \gamma_i^k \mathbf{y}^{k+1,i}, \quad (i = 1,2,...,m), \end{aligned} \tag{31}$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (9), ω_k are relaxation parameters, γ_i^k are positive real numbers for each $k \in \mathbf{N}$, $I = \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$, $J_i = \left\{\sigma^i(k)\right\}_{k=1}^{\infty}$ are sequences of delays.

The convergence of this algorithm is given by the following theorem:

Theorem 1. Let system (2) be consistent, $I = \{I_k\}_{k=1}^{\infty}$ a regular sequence of chaotic sets $I_k \subset \{1,2,...,m\}$ with a number of regularity T, $\left\{\sigma^i(k)\right\}_{k=1}^{\infty}$ sequences with limited delays and $\sigma_j^i(k) = \sigma_i(k)$, and let a delay be equal to T. If $0 < \omega_k < 2$, γ_i^k are positive real numbers with property $\sum_{i \in I_k} \gamma_k^i = 1$, then for every point $\mathbf{x}^{(0)} \in \mathbf{R}^n$ the sequence $\left\{\mathbf{x}^k\right\}_{k=0}^{\infty}$ defined by the algorithm 8 converges to some point $\mathbf{x}^* \in H$, which is a fixed point of orthogonal projection operators $\mathbf{P}_i$ (i=1,2,...,m).

Consider the particular case of the algorithm 8 when there are no delays and the sequence of chaotic sets is acceptable.

Algorithm 9 (CHART).

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\mathbf{y}^{k,i} = \begin{cases} \mathbf{P}_i^{\omega_k} \mathbf{x}^{(k-1)}, & \text{if } i \in I_k \\ \mathbf{y}^{k-1,i}, & \text{otherwise} \end{cases}$$
$$\mathbf{x}^{(k)} = \mathbf{C} \sum_{i \in I_k} \gamma_i^k \mathbf{y}^{k,i}, \quad (i = 1,2,...,m), \tag{32}$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (9), ω_k are relaxation parameters, γ_i^k are positive real numbers for each $k \in \mathbf{N}$, $I = \{I_k\}_{k=1}^{\infty}$ is an acceptable sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$ and $\mathbf{C}$ is a constraining operator.

The convergence of the algorithm 9 is given by the following theorem.

Theorem 2. Let system (2) be consistent, $I = \{I_k\}_{k=1}^{\infty}$ be an acceptable sequence of chaotic sets $I_k \subset \{1,2,...,m\}$. If $0 < \omega_k < 2$, γ_i^k are positive real numbers with property $\sum_{i \in I_k} \gamma_k^i = 1$, then for every point $\mathbf{x}^{(0)} \in \mathbf{R}^n$ the sequence $\{\mathbf{x}^k\}_{k=0}^{\infty}$ defined by the algorithm 9 converges to some solution of this system.

6. Block-parallel asynchronous algorithms for computer tomography

Block-parallel asynchronous algorithms with application to tomographic reconstruction from incomplete data were studied by Elsner, Koltracht and Neumann in (Elsner et al. 1990). In this section the generalized model of asynchronous iterations is applied for an implementation of the algorithm BPART on a non-synchronous computer structure. In this case there results the following algorithm, where the numbers of operators are chosen by the chaotic way:

Algorithm 10.

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{k+1,i} = \mathbf{C} \sum_{i \in H_{t(k)}} \mathbf{B}_i^k \mathbf{P}_i^{\omega_k} \mathbf{x}^{(\sigma^i(k))}, \tag{33}$$

where $\mathbf{P}_i^{\omega_k}$ are operators defined by (9), ω_k are relaxation parameters, $\mathbf{C}$ is a constraining operator, $t(k) = I_k$, $I = \{I_k\}_{k=0}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1,2,...,M\}$ and $\mathbf{B}_i^k$ are matrices of dimension $n \times n$ with real nonnegative elements which satisfy conditions (22), $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ are sequences of delays.

Theorem 3. Let system (2) be consistent, $I = \{I_k\}_{k=0}^{\infty}$ a regular sequence of chaotic sets $I_k \subset \{1,2,...,M\}$ with a number of regularity T, $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ sequences with limited delays

and $\sigma_j^i(k) = \sigma_i(k)$, and let the number of delay be equal to T. Then for every point $\mathbf{x}^{(0)} \in \mathbf{R}^n$ the sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ defined by the algorithm 10 converges to some point $\mathbf{x}^* \in$ H, which is a fixed point of orthogonal projection operators $\mathbf{P}_i$ (i = 1,2,..., M).
Let the particular case of algorithm 10 when there are no delays and the sequence of chaotic sets is acceptable. Decompose the matrix **A** and the projection vector **p** into *M* subsets in accordance with decomposition (13) and (20). Consider $s_t = |H_t| = m_t - m_{t-1}$ be the cardinality of H_t .

Algorithm 11 (CHBP).
1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The *k*+1-th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{k+1} = \mathbf{C}\sum_{i=1}^{M} \mathbf{B}_i^k y^{(k+1),i}, \tag{34}$$

where

$$y^{(k+1),i} = \mathbf{Q}_i \mathbf{x}^k,$$
$$\mathbf{Q}_i = \mathbf{P}_{i,s_i} \mathbf{P}_{i,s_i-1} \ldots \mathbf{P}_{i,1},$$
$$\mathbf{P}_{i,j} = \mathbf{P}_j^{\omega} \sum_{i=1}^{M} \mathbf{B}_i^k, \quad j \in I_{i(j)},$$

$\mathbf{P}_i^{\omega}$ are operators defined by (9), ω is a relaxation parameter, **C** is a constraining operator, $I = \{I_{i(k)}\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_{i(k)} \subset \{m_{i-1}+1, m_{i-1}+2, \ldots, m_i\} = H_i$ and $\mathbf{B}_i^k$ are matrices of dimension $n \times n$ with real nonnegative elements which satisfy conditions (22) for each $k \in \mathbf{N}$.

Remark 2. If operator $\mathbf{P}_i$ is defined by (3), algorithm CHART, PART, BPART, CHBP will be called CHART-1, PART-1, BPART-1, CHBP-1 respectively, and if $\mathbf{P}_i$ is defined by (8) these algorithms will be called by CHART-3, PART-3, BPART-3, CHBP-3 respectively.

7. Computer simulation and numerical results

In this section there are presented some numerical results of applying the different algorithms considered in the previous sections for reconstruction of high contrast objects from incomplete projection data in the case when they are not available at each angle of view and they are a few-number limited. There are also studied the influence of various parameters of these algorithms such as a pixel initialization, relaxation parameters, number of iterations and noise in the projection data on reconstruction quality and convergence of these algorithms (Gubareni, 1998a), (Gubareni & Pleszczynski, 2007), (Gubareni & Pleszczynski, 2008).

In order to evaluate the goodness of the computer reconstruction of a high-construct image there were tested different kinds of geometric figures and reconstruction schemes. In this

chapter the results of image reconstruction are presented for two different schemes of obtaining projection data (1 × 1) and (1 × 1, 1 × 1), which are described above.
An important factor in the simulation process of image reconstruction is the choice of modeling objects which describe the density distribution of research objects. In a coal bed, where one searches the reservoirs of compressed gas or interlayers of a barren rock, the density distribution may be considered discrete and the density difference of three environments (coal, compressed gas and barren rock) is significant. Therefore for illustration of the implementation of the algorithms working with incomplete data there were chosen the discrete functions with a high contrast.
The first discrete function $f_1(x,y)$ is given in the following form:

$$f_1(x,y)=\begin{cases}1, & (x,y)\in D\subset E\subset \mathbf{R}^2,\\ 0, & \text{otherwise}\end{cases} \tag{35}$$

where E is a square $E=\{(x,y):-1\le x,y\le 1\}$, and D is a subset of E of the following form:

$$D=[-0.4,-0.2]\times[-0.5,0.5]\cup\ [-0.2,0.2]\times[0.3,0.5]\cup\ [-0.2,0.2]\times[-0.1,0.1]\cup[0,0.2]\times[0.1,0.3]. \tag{36}$$

The second discrete function $f_2(x,y)$ is given in the following form:

$$f_2(x,y)=\begin{cases}1, & (x,y)\in D_1\subset E\subset \mathbf{R}^2,\\ 2, & (x,y)\in D_2\subset E\subset \mathbf{R}^2,\\ 3, & (x,y)\in D_3\subset E\subset \mathbf{R}^2,\\ 4, & (x,y)\in D_4\subset E\subset \mathbf{R}^2,\\ 0, & \text{otherwise}\end{cases} \tag{37}$$

where E is a square $E=\{(x,y):-1\le x,y\le 1\}$, and D_i are subsets of E of the following form:

$$D_1 = [-0.7,-0.4]\times[-0.5,0.2],\ D_2= [-0.2,0.2]\times[-0.1,0.1],$$

$$D_3 = [-0.2,0.2]\times[0.3,0.5],\ D_4 = [0.4,0.7]\times[0.4,0.7].$$

The three-dimensional view of the plots of these functions are given in Figure 3.

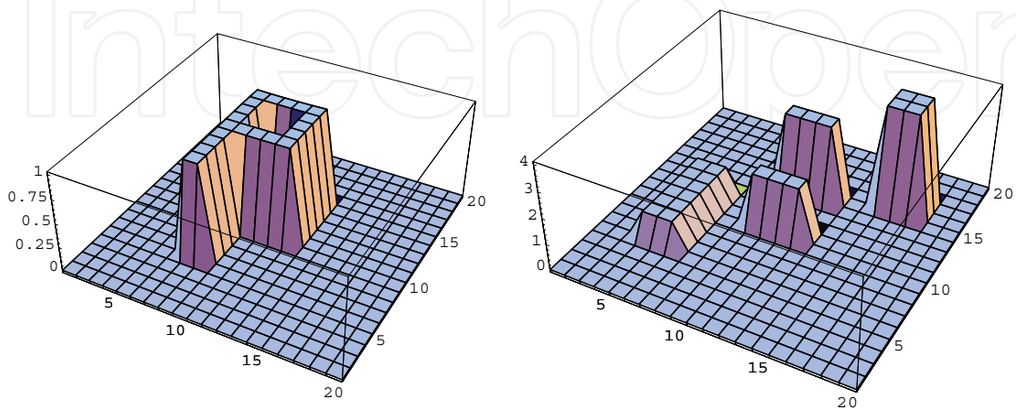

Fig. 3. The original functions $f_1(x,y)$ (on the left side) and $f_2(x,y)$ (on the right side).

As was shown earlier (see, e.g. (Eggermont, el. al., 1981), (Herman, 1980)), the image reconstruction of such objects from complete data gives a good enough results after 6-7 full iterations.
The main criteria which are used to evaluate the quality of reconstruction are the following reconstruction errors: the maximal absolute error, the mean root square error, the maximum relative error, the mean absolute error, the absolute error.
In this paper the convergence characteristics of image reconstruction are presented in a view of plots for the following measures of errors:

- the absolute error:

$$\delta(x,y) = | f(x,y) - \tilde{f}(x,y) |$$

- the maximal absolute error:

$$\Delta = \max_i | f_i - \tilde{f} |$$

- the maximal relative error

$$\delta_1 = \frac{\max_i \left| f_i - \tilde{f}_i \right|}{\max_i | f_i |} \cdot 100\% ,$$

- the mean absolute error

$$\delta_2 = \frac{1}{n} \sum_i \left| f_i - \tilde{f}_i \right|$$

where f_i is the value of a given modeling function in the center of the i-th pixel and $\tilde{f}_i$ is the value of the reconstructed function in the i-th pixel.
In all considered computer simulations it was assumed that **C**= C_1 C_2, where

$$\left(C_1[\mathbf{x}]\right)_i = \begin{cases} a, & \text{if } x_i < a; \\ x_i, & \text{if } a \le x_i \le b; \\ b, & \text{if } x_i > b; \end{cases} \tag{38}$$

$$\left(C_2[\mathbf{x}]\right)_j = \begin{cases} 0, & \text{if } p_i = 0 \text{ and } a_{ij} \neq 0; \\ x_j, & \text{otherwise} \end{cases} \tag{39}$$

From the conducted research it follows that the optimal value of a relaxation parameter ω is equal to 1.1 for the system (1 ×1, 1 ×1) and it is equal to 1.3 for the system (1 × 1) in the case where there is no noise in projections. In the case when there is a noise in the projection data the optimal value of ω is changed in dependence on the value of the noise.
In all presented numerical results it was assumed, that

- n - is the number of pixels, i.e. the number of variables,
- m - is the number of rays, i.e. the number of equations,
- M - is the number of blocks,
- *iter* - is the number of full iterations.

In all experiments it was also assumes that

- M is equal to the number of detectors;
- the sequence of chaotic sets I_k has the form { ξ_k }, where ξ_k is an integer random variable in the interval [1,m] with uniform distribution.

- the reconstruction domain $E = \{(x,y) : -1 \le x, y \le 1\}$ was divided into n = 20 × 20 pixels.
- the number of projections m in the system (1 × 1) is equal to 788, and in the system (1 × 1, 1 × 1) the number m= 644.

7.1 Reconstruction with algorithms ART-3 and MART-3

The reconstruction of $f_1(x,y)$ with ART-3 after 15 iterations in the scheme (1 × 1, 1 × 1) is presented in Figure 4. The plot of the reconstruction function is shown on the left side, and the plot of the absolute errors for this image reconstruction is shown on the right side.

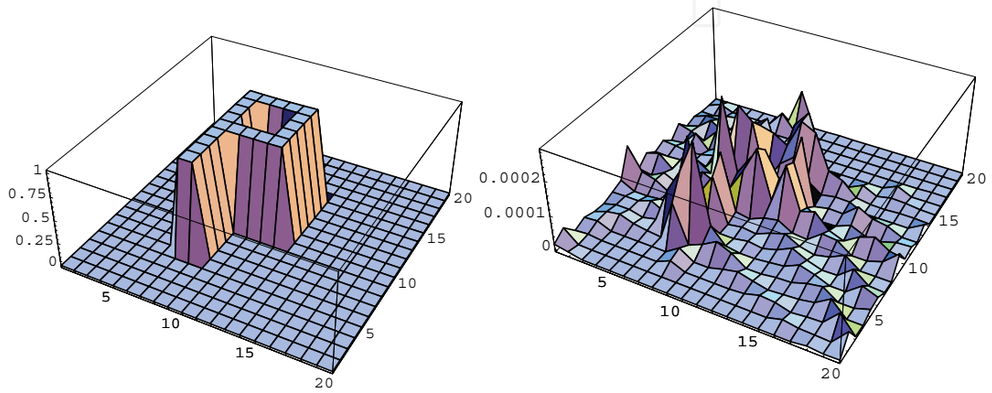

Fig. 4. Reconstruction and the the absolute error $\delta(x,y)$ of $f_1(x,y)$ with ART-3 for n=20 ×20, m= 644, *iter*=15 in the scheme (1 × 1, 1 × 1).

For comparison this function $f_1(x,y)$ was reconstructed with the multiplicative algorithm MART-3 for the same parameters and ω=6.9 and the plots, which are presented in Figure 5, illustrate the dependence of the maximum relative error δ_1 and the mean absolute error δ_2 on number of iterations with ART-3 and MART-3 in the system (1 × 1, 1 × 1).

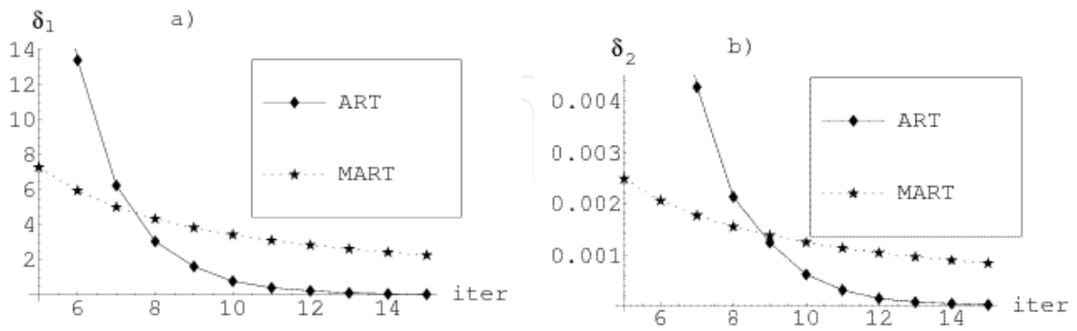

Fig. 5. Dependence of the maximum relative error δ_1 (on the left side) and the mean absolute error δ_2 (on the right side) on the number of iterations for image reconstruction of $f_1(x,y)$ with ART-3 and MART-3 for n=400, m= 644 in the system (1 × 1, 1 × 1).

The same function $f_1(x,y)$ was also reconstructed in the system (1×1). The result of this reconstruction with ART-3 for n = 400, m = 788 and 100 iterations is shown in Figure 6.

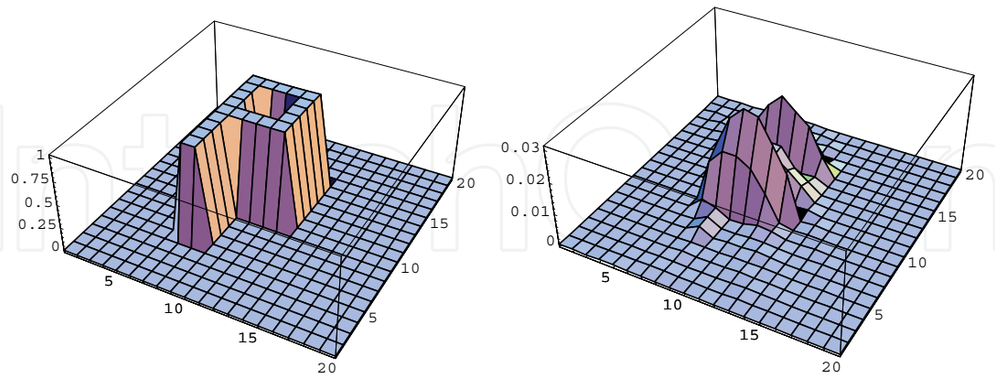

Fig. 6. Reconstruction of $f_1(x,y)$ and the absolute error $\delta(x,y)$ obtained with algorithm ART for n=400, m=788, iter=100 in the system (1×1).

The plots, which are presented in Figure 7, illustrate the dependence of the maximum relative error δ_1 and the mean absolute error δ_2 on the number of iterations of image reconstruction of $f_1(x,y)$ with ART-3 in the systems $(1 \times 1, 1 \times 1)$ and (1×1).

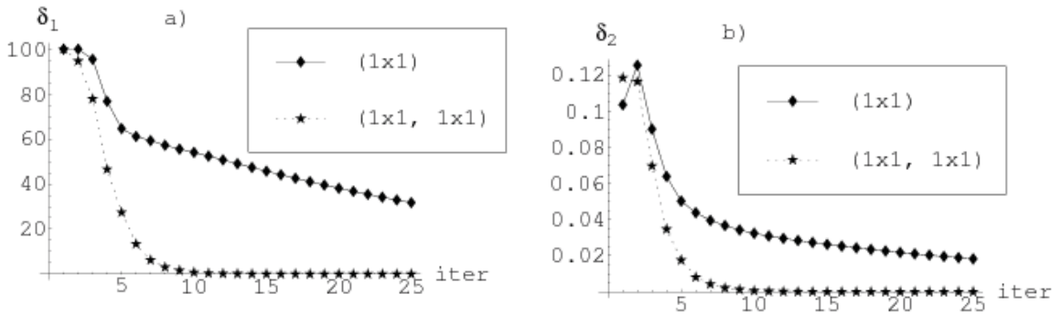

Fig. 7. Dependence of the maximum relative error δ_1 (on the left side) and the mean absolute error δ_2 (on the right side) on the number of iterations of image reconstruction of $f_1(x,y)$ in the systems $(1 \times 1, 1 \times 1)$ and (1×1).

The analogous reconstructions were obtained for the function $f_2(x,y)$. The result of reconstruction of the function $f_2(x,y)$ in the system (1×1) is shown in Figure 8.

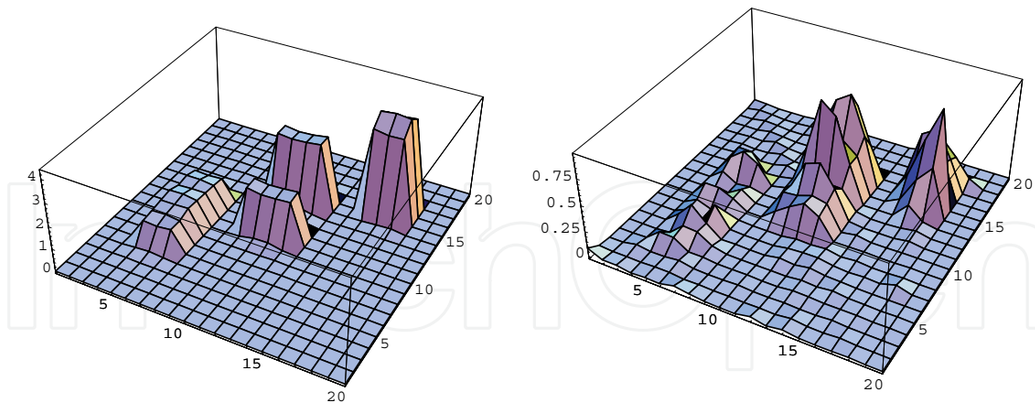

Fig. 8. Reconstruction and the absolute error $\delta(x,y)$ of $f_2(x,y)$ obtained with algorithm ART-3 for n=400, m=788, $iter$=25 in the system (1×1).

The plots presented in Figure 9 illustrate the dependence of the maximum relative error δ_1 and the mean absolute error δ_2 on the number of iterations of image reconstruction of $f_2(x,y)$ with ART-3 in the systems $(1 \times 1, 1 \times 1)$ and (1×1).

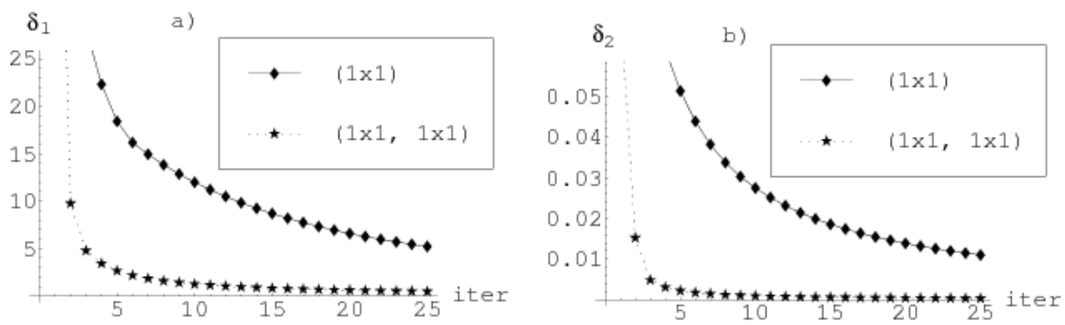

Fig. 9. The dependence of the maximum relative error δ_1 (on the left side) and the mean absolute error δ_2 (on the right side) of image reconstruction of $f_2(x,y)$ in the systems $(1 \times 1, 1 \times 1)$ and (1×1).

Note that the analogous results were obtained for ART-1, MART-1 and MART-3. For the multiplicative algorithms MART-1 and MART-3 the velocity of convergence is considerably less by comparison with the additive algorithms ART-1 and ART-3.

7.2 Reconstruction with chaotic algorithm CHART-1

In this section there are represented some numerical results of reconstructions of the function $f_1(x,y)$ with the chaotic algorithm CHART-1 using the same parameters as in the case of ART-3. The reconstruction result of the function $f_1(x,y)$ with CHART-1 after 100 iterations in the system $(1 \times 1, 1 \times 1)$ is shown in Figure 10.

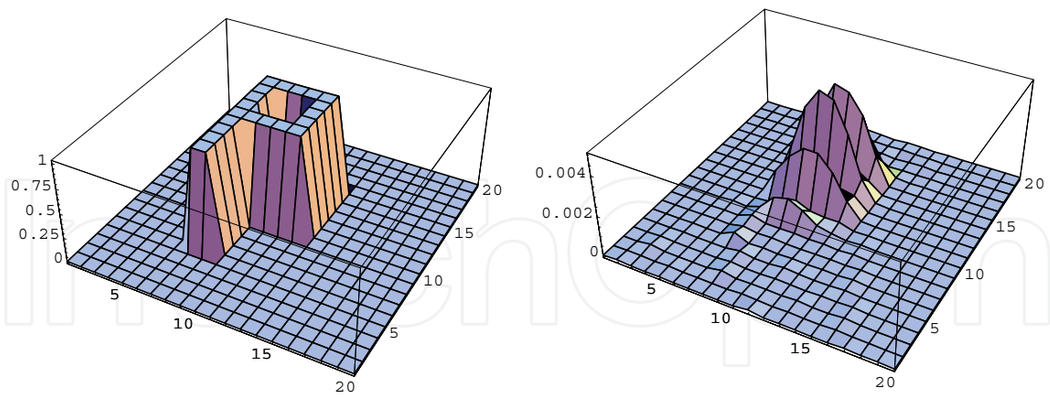

Fig. 10. Reconstruction and the absolute error $\delta(x,y)$ of $f_1(x,y)$ obtained with CHART-1 for n=20 ×20, m= 788, *iter*=100 in the system (1 × 1).

The plots, which are presented in Figure 11, illustrate the dependence of the mean absolute error δ_2 on the number of iterations for image reconstruction of $f_1(x,y)$ with ART-1 and CHART-1 in the systems (1 × 1, 1 × 1) and (1 × 1).

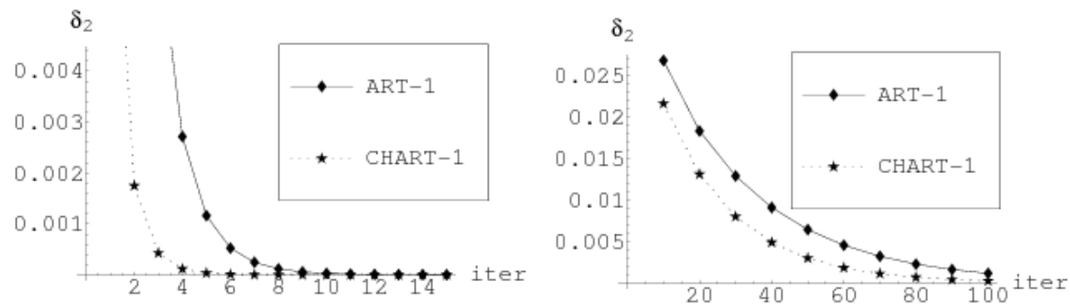

Fig. 11. Dependence of the mean absolute error δ_2 on the number of iterations for image reconstruction of $f_1(x,y)$ with ART-1 and CHART-1 in the system (1 × 1, 1 × 1) (on the left side) and in the system (1 × 1) (on the right side).

The result of image reconstruction of the function $f_2(x,y)$ with CHART-1 after 25 iterations in the system (1 × 1) is shown in Figure 12.

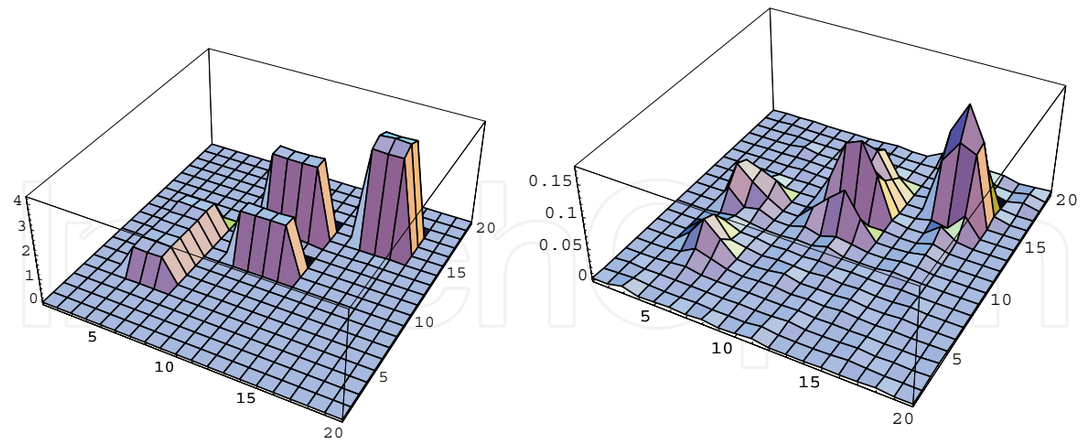

Fig. 12. Reconstruction and the absolute error $\delta(x,y)$ of $f_2(x,y)$ obtained with chaotic algorithm CHART-1 for n=20 ×20, m= 788, *iter*=25 in the system (1 × 1).

Table 1 (resp. Table 2) shows the dependence of the maximum absolute error Δ on the number of iterations for ART-1 and CHART-1 for the system (1 × 1) (resp., the system (1 × 1, 1 × 1)) and the same set of parameters which were chosen above.

iter	ART-1	CHART-1
100	0.0306	0.0073
200	0.00201	0.0001
500	1.209×10^{-6}	4.098×10^{-9}
1000	6.435×10^{-12}	6.328×10^{-15}
2000	5.44×10^{-15}	6.106×10^{-15}

Table 1. The dependence of the maximum absolute error Δ on the number of iterations for ART-1 and CHART-1 in the system (1 × 1).

iter	ART-1	CHART-1
10	0.0077	0.00002
20	9.83×10^{-6}	3.568×10^{-9}
40	3.12×10^{-11}	1.221×10^{-15}
50	3.98×10^{-14}	1.11×10^{-15}
100	8.88×10^{-16}	8.88×10^{-16}

Table 2. The dependence of the maximum absolute error Δ on the number of iterations for ART-1 and CHART-1 in the system (1 × 1, 1 × 1).

The plots, which are presented in Figure 13, illustrates the dependence of the mean absolute error on the number of iterations of image reconstruction of $f_2(x,y)$ with ART-1 and CHART-1 in the system (1 × 1).

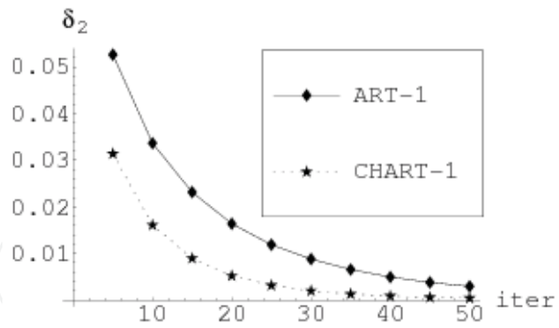

Fig. 13. Comparison between the mean absolute error δ_2 for image reconstruction of $f_2(x,y)$ with ART-1 and CHART-1 in the system (1 × 1).

7.3 Reconstruction with block-parallel algorithms

In this section there are represented some numerical results of reconstructions of the function $f_2(x,y)$ with the block-parallel algorithm BPART-3 and the chaotic block-parallel algorithm CHBP-3. The reconstruction result of the function $f_2(x,y)$ with BPART-3 and CHBP-3 after 75 iterations in the system (1 × 1, 1 × 1) is shown in Figure 14 and Figure 15, respectively.

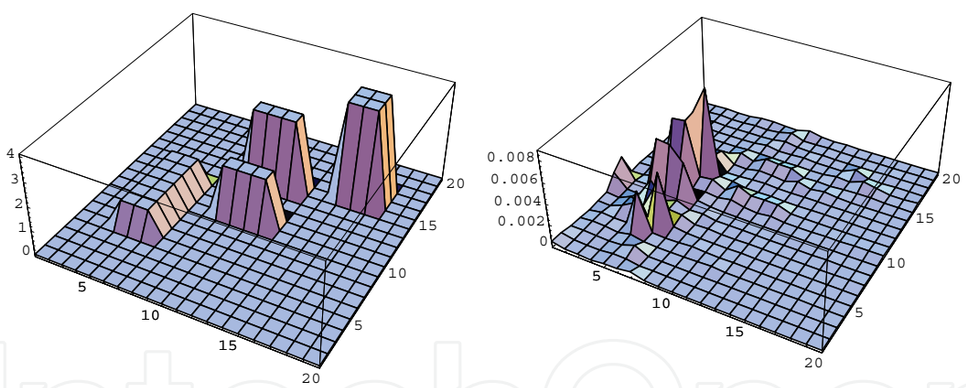

Fig. 14. Reconstruction and the absolute error $\delta(x,y)$ of $f_2(x,y)$ obtained with algorithm BPART-3 for n =20 × 20, m=644, M=36, $iter$=75 in the system (1 × 1, 1 × 1).

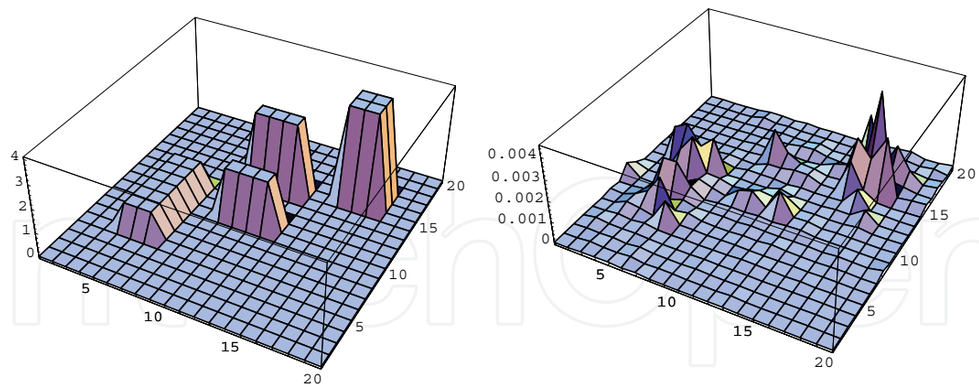

Fig. 15. Reconstruction and the absolute error $\delta(x,y)$ of $f_2(x,y)$ obtained with CHBP-3 for n =20 × 20, m=644, M=36, $iter$=75 in the system (1 × 1, 1 × 1).

The plots, which are presented in Figure 16, illustrate the dependence of the maximum relative error and the mean absolute error on the number of iterations of image reconstruction of $f_2(x,y)$ with BPART-3 and CHBP-3 in the system (1 × 1, 1 × 1):

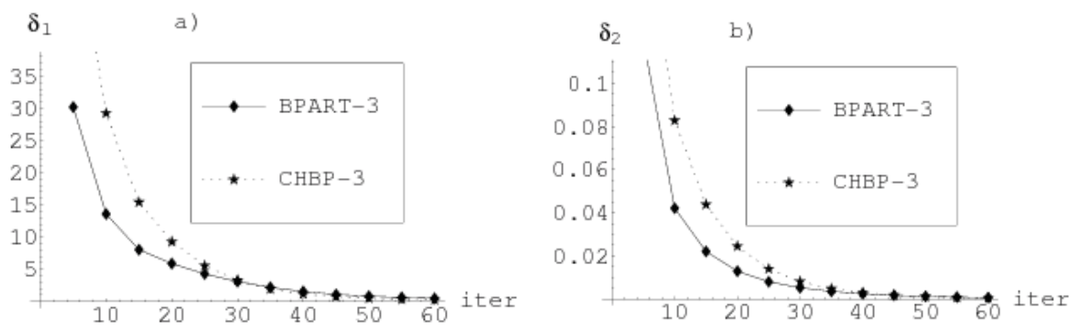

Fig. 16. Dependence of the maximum relative error δ_1 and the mean absolute error δ_2 on the number of iterations for image reconstruction of $f_2(x,y)$ with BPART-3 and CHBP-3 in the system (1 × 1, 1 × 1).

Table 3 shows the dependence of the maximum absolute error Δ on the number of iterations for BPART-3 and CHBP-3 in the system (1 × 1, 1 × 1).

iter	BPART-3	CHBP-3
10	0.4640	0.2112
20	0.1973	0.0478
40	0.0293	0.0054
50	0.0113	0.0018
100	0.0001	0.000001

Table 3. Dependence of the maximum absolute error Δ on the number of iterations for BPART-3 and CHBP-3 in the system (1 × 1, 1 × 1).

The analogous results were obtained for reconstruction of the function $f_2(x,y)$ with BPART-3 and CHBP-3 in the system (1 × 1). The plots presented in Figure 17 illustrate the dependence of the maximum relative error and the mean absolute error on the number of iterations of image reconstruction of $f_2(x,y)$ with BPART-3 and CHBP-3 in the system (1 × 1):

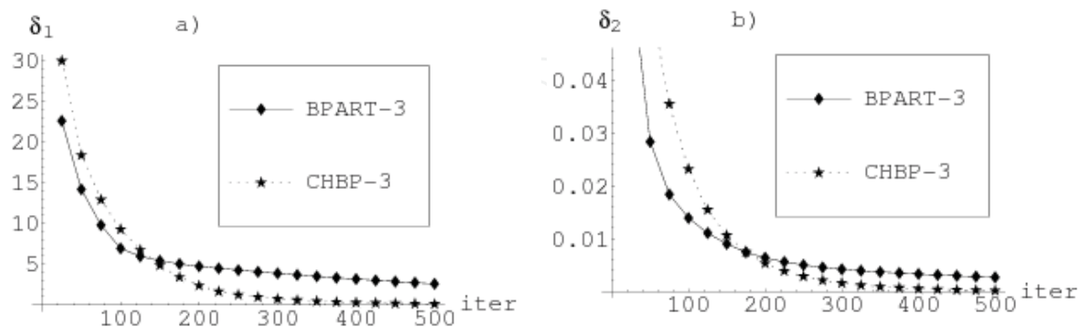

Fig. 17. Dependence of the maximum relative error δ_1 and the mean absolute error δ_2 on the number of iterations for image reconstruction of $f_2(x,y)$ with BPART-3 and CHBP-3 in the system (1 × 1).

Remark 3. The parallel reconstruction algorithms BPART and CHBP were implemented in a simulated parallel environment on a sequential machine. Calculations of all elementary processors that could be run in parallel were times together and the results were stored in temporary arrays. The results of all such calculations were updated into global arrays while all computations were completed.

7.4 Reconstruction with noisy projection data

It is very important for the research of reconstruction quality to consider the effect of noise in projection data on a reconstructed image. For this purpose the various levels of Gaussian (normal) noise are added to projections. The noise model was proposed by Herman (Herman, 1975). In this model the noisy data are simulated by the addition of Gaussian noise. Here each projection is multiplied by a Gaussian distributed random number with mean 1.0 and standard deviation s. Usually the noise level is taken less that 10% in computed tomography for non-medical imaging application. In considered experiments the noise level was in the interval from 1% to 5%.

The plots of image reconstruction of the function $f_1(x, y)$ with ART-1 after 10 iterations in the system (1 × 1, 1 × 1) and the normal noise of projection data s=2% and s=5% are given in Figure 18.

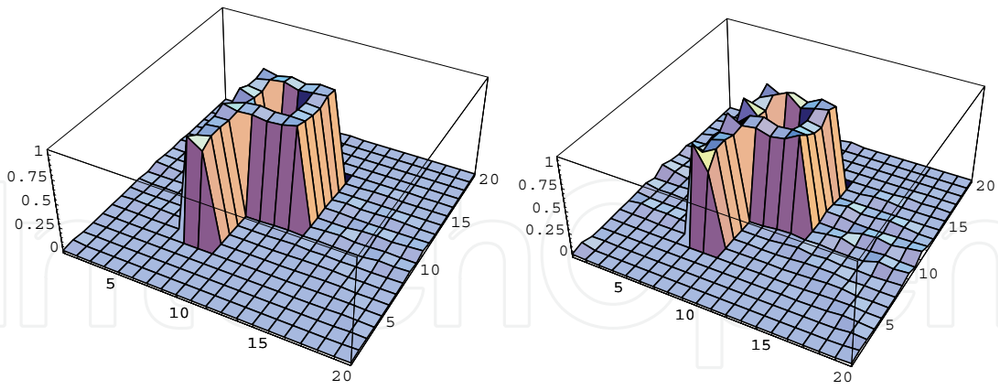

Fig. 18. Reconstruction of $f_1(x,y)$ obtained with ART-1 after 10 iterations in the system (1 × 1, 1 × 1) and the normal noise of projection data s= 2% (on the left side) and s=5% (on the right side).

Analogous results were obtained for the algorithm CHART-1. The plots of image reconstruction of the function $f_1(x,y)$ with CHART-1 after 10 iterations in the system (1 × 1, 1 × 1) and the normal noise of projection data s=2% and s=5% are given in Figure 19.

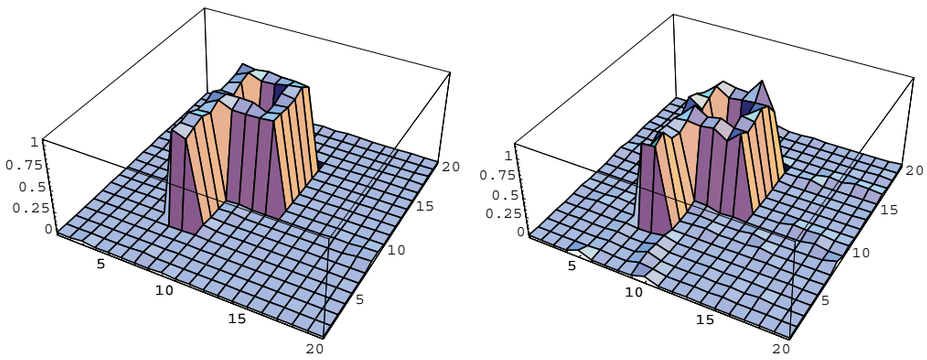

Fig. 19. Reconstruction of $f_1(x,y)$ obtained with CHART-1 after 10 iterations in the system (1 × 1, 1 × 1) and the normal noise of projection data s= 2% (on the left side) and s=5% (on the right side).

The plots, which are represented in Figure 20, show dependence of the mean absolute error δ_2 on the number of iterations for image reconstruction of $f_1(x,y)$ with ART-1 (on the left side) and algorithm CHART-1 (on the right side) in the system (1 × 1, 1 × 1) and the normal noise in projections s= 0% and s= 2%. These results show that both algorithms are stable with regard to noise in projections and they have the same robustness.

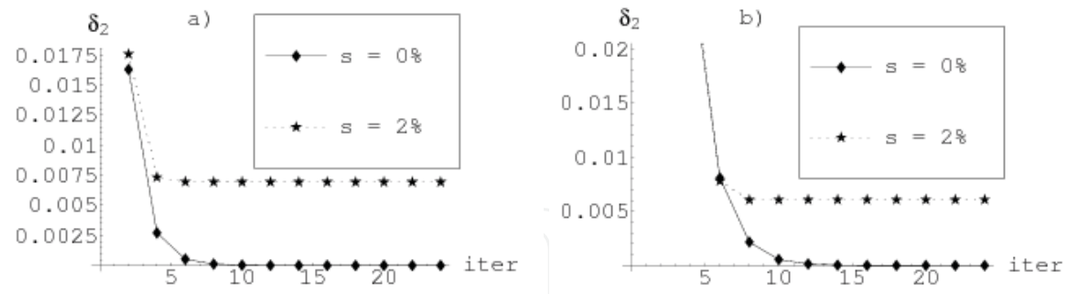

Fig. 20. Dependence of the mean absolute error δ_2 on the number of iterations for image reconstruction of $f_1(x, y)$ with ART-1 (on the left side) and CHART-1 (on the right side) in the system (1 × 1, 1 × 1) and the normal noise of projection data s = 0% and s= 2%.

Remark 4. All algorithms were implemented on IBM/PC (procesor AMD Duron XP, 1600 MHz) by means of C++ and MATHEMATICA 5.1. For reconstruction of examples with small number of pixels (n=20 ×20 and m < 800) one iteration by means of Mathematica 5.1 was implemented approximately 0.5s for all algorithms, and in C++ one iteration for all algorithms is implemented in a real time (< 0.1s).

8. Conclusion

The aim of this paper was to represent and study the different iterative algebraic algorithms for reconstruction of high-contrast objects from incomplete projection data. There were studied the quality and convergence of these algorithms by simulation on a sequential computer.

For each considered system of reconstruction there exist the parameters which allow to obtain a good quality of reconstruction after some number of iterations but this number is considerably larger than for reconstruction with the complete projection data. This number of iterations is approximately 10 times more for the scheme (1 × 1, 1 × 1) by comparison with the case of the complete data.

The obtained results show also that for obtaining the good reconstruction quality (with δ_1 < 1% and δ_2 < 0.001) by means of the chaotic algorithm CHART-1 it suffices 4-6 iterations in the system (1 × 1, 1 × 1) and 20-30 iterations in the system (1 × 1). These results are much better by comparison with algorithm ART-1. The results of reconstruction also show that the chaotic algorithm CHBP-3 gives better results with comparing the block-parallel algorithm BPART-3.

All experimental results show that the errors of reconstruction of considered objects from limited projection data with all considered algorithms are constantly reduced with increasing the number of iterations. It was also shown that all considered algorithms are stable with regard to noise (< 5%) in projections.

It should be note that the parallel and block-parallel algorithms were implemented only on simulated parallel environment on sequential machine. Their implementation on real parallel structure may give some other results. So it will be very interesting to conduct the simulation of these algorithms on such parallel structures.

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