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NEW STOCHASTIC DEPENDENCES PARADIGM AND ITS APPLICATION IN PROBABILISTIC MODELING

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Abstract. We present a quite **simple pattern** that, by a properly defined conditioning, allows to describe a wide range of new stochastic dependences **(**virtually, it is the **type of the multivariate normal kind of the dependencies extended to** numerous **other** multivariate cases**).**

We then construct a variety of stochastic models in form of multivariate probability distributions. The key element of the theory is the obtained a quite easy method for constructing various classes of conditional pdfs $\mathbf{g}_k(\mathbf{y} \mid \mathbf{x}_1, \dots, \mathbf{x}_k)$ of random variable's, here denoted by Y, given realizations of some other (explanatory) random variables $X_1, \dots,$ X_k. The latter variables are either independent or posses some known joint probability distribution. The striking easiness in their construction and a significant universality for a variety of anticipated applications, suggest a possibility of employing them in a variety of areas that sometimes seem to be remote from each other.

Some of the main applications of the considered constructions pattern are practical problems associated with the **statistical regression.** Here, the paradigm, relies on **replacing (**or 'extending'**),** whenever possible a regression model, typically used in the form of the conditional expectation $\mathbf{E}[\,\mathbf{Y} \mid \mathbf{x}_1, \dots, \mathbf{x}_k\,]$ by the corresponding (one, but not necessarily unique) conditional pdf $\mathbf{g}_k(\mathbf{y} \mid \mathbf{x}_1, \dots, \mathbf{x}_k)$**.** In this case, the original regression model is simply the expectation of the latter conditional pdf.

Nevertheless, in this work we rather concentrate on modeling of multi component **systems reliability** as well as similar **biomedical** problems. One of the main reasons for this preference is the fact that the reliability examples bring better clarity for demonstration of the common '*stochastic model – real world*' relation.

As for the reliability application, we model (by means of the introduced general stochastic dependence**)** a parallel two component system with respect to its stochastically dependent components life times.

1. On A General Continuous Stochastic Dependences Pattern; Introductory Part

1.1 The pattern for stochastic dependences is a basic ingredient of **stochastic models** that we construct and explore. The motivation for the constructions is both a recognition and anticipation of a strong potential for applications of the obtained models in a variety of everyday' phenomena that are "by nature" random. The **pattern,** determines a general method that is fundamental for all the constructions that are or can be performed in the presented framework**.** Roughly speaking, this method relies on what follows. Given **(any)** two random quantities $\mathbf{X_1}$**,** $\mathbf{X_2}$ representing, say, two objects u_1, u_2 respectively **(**whose **"physical"** meaning is to a large extend "arbitrary"**)** such that the magnitude of a given realization (or outcome) $\mathbf{x_1}$ of the random variable X_1, "stochastically **influences"** magnitude of a possible realization $\mathbf{x_2}$ of the random variable X_2. The way the value $\mathbf{x_1}$ of the quantity $\mathbf{X_1}$ "acts" on a possible outcome $X_2 = x_2$ of the other quantity, is described 'indirectly' in terms of **changes in the $\mathbf{X_2}$'s probability distribution,** rather than in terms of direct changes in X_2 itself.

Our Objective is to present an analytic description of a "measure" that reflects a particular situation in modeled reality. Among several possibilities we basically concentrate on two. The first possibility is to investigate the distribution's changes through changes in its original hazard rate, see [9]. This approach strongly applies to determining dependent life-times distributions in reliability (see [2] and also [4, 9, 10, 12]) or in some **biomedical** problems [3]. Note, however, that the mechanism of changes in the hazard (failure) rate, presented here, is essentially **different than** the one that was described for the first time in 1961 by **Freund** in [9]**.**

The second possibility we consider requires a somewhat more general procedure. In this case we tend to find a description of changes in the probability distribution or the corresponding density through corresponding (hypothetical) **changes of the density's parameters,** see [5 – 8]. Thus, in this case, the pattern relies on (statistical) 'measuring' change of the (original) pdf 's **parameter** value(s) $\boldsymbol{\theta_2}$. The changed value of that parameter is assumed to be functionally **dependent on** a magnitude of the first random realization of quantity ($X_1 = \mathbf{x_1}$)**.**

1.2 In other words, we develop a powerful while still simple, **method** for determining **stochastic dependences** of various kinds of random variables as well as of random vectors. Application of this method to various practical problems of the 'real life' produces a remarkably wide range of flexible **stochastic models.**

Roughly speaking, a huge variety of models and their more specific versions is based on a simple observation that the **stochastic dependence** of a random quantity, say **Y (**whose cdf, in an absence of considered in this work stresses is denoted by $F(y; \theta)$ **)** on a set of other **(explanatory)** random variables $\{ X_1, \ldots ,X_k \}$ can nicely be described in a proper mathematical model.

For a more compact definition of the stochastic dependence suppose that an elementary random **event** $(X_1, \dots, X_k) = (x_1, \dots, x_k)$ has happened.
Mathematically, we define the dependence as a result of rather new, but quite "obvious", kind of the **transformations:**

$$(x_1, \dots, x_k) \rightarrow F(y; \theta), \tag{1}$$

that we propose to call **"weak"**, or "stochastic" in a contrast to the ordinary "strong" or **"algebraic" transformation** $(X_1, \dots, X_k) \rightarrow Y$.
Among several ways the weak transformation can be determined, we have chosen possibly the simplest by claiming that the **impact** of the random quantities **(** say, **"stresses")** $X_1, \dots, X_k$ **on the cdf** of the quantity of interest **Y,** exhibits itself as an impact on the scalar or vector parameter θ of the cdf. **F(y; θ).**
Thus the defined above weak transformation can be shorten to the relation:

$$\theta \rightarrow \theta(x_1, \dots, x_k), \tag{2}$$

where the symbol 'θ' alone, represents the cdf's **parameter** original numerical value i.e., the value in an absence of the stresses $X_1, \dots, X_k$.
The new (random) value $\theta(X_1, \dots, X_k) \neq \theta$, of the parameter of the Y's cdf may, in most general case, be considered as just **any continuous function** (!) of its k arguments. However, for the sake of tractability of further **statistical verification** of the so constructed models, it may become necessary to limit the class of the continuous functions to some **parametric classes .** Thus, the statistical parametric procedures will be concerned now with a given functions' class **(**such as the exponential, say $\theta(x) = \{A\exp[\,bx\,]\}$ or power, polynomials etc **…)** parameters, instead of the original numerical values of the previous parameter θ itself. In general, the number of such a function's parameters **(** for example, the **parameters A, b** in the above exponential model for the transformed value of θ, where it also may assumed, that $\theta = A$ **)** should not be much more than double number of the original parameters θ.
Notice, that the so obtained new cdf $F(y; \theta(x_1, \dots, x_k))$, virtually is the defined **conditional distribution of Y** given an elementary event $(X_1, \dots, X_k) = (x_1, \dots, x_k)$ and, actually the so defined stochastic **dependence** of **Y** from the $(X_1, \dots, X_k)$ embraces **the core idea** of an emerging new theory **(**compare this concept with the very similar structure of the classical **multivariate normal pdf).**
Obviously, the ordinary product of the conditional pdf, say :

$$f(y \mid x_1, \dots, x_k) = f(y; \theta(x_1, \dots, x_k)) \tag{3}$$

and the joint pdf $g(x_1, \dots, x_k)$ of the "explanatory" variables $X_1, \dots, X_k$ provides the (k+1) – variate joint pdf $h(y, x_1, \dots, x_k)$. In that way, most of the multivariate distributions can be constructed within the theory presented here. Notice also the important fact that there is no need for any assumptions concerning classes of probability distributions of all underlying random variables such as $Y, X_1, \dots, X_k$. As a matter of fact, any of the variables, may belong to **any class of probability distributions (!).** Moreover, the procedures delineated here are expected to extend the classical **regression** methods. Namely, one can extend the regression

models, typically met in form of the **conditional expectations** **E [Y | x_1, … ,x_k]** to the ones defined as the conditional probability **distributions** **F(y | x_1, … ,x_k),** whose densities (if exist) are given by **(3).** Notice an obvious fact, that their expectations are the same as in classical models for the regression.
Considering the various stochastic models, obtained by the **common paradigm,** given by the scheme expressed in (1), (2) and (3) one finds that the **generality** of the described modeling **method** may lead to a number of possible applications. The applications can likely spread out over many "physically" different areas of the real world. We hope that the considered here method will turn out to be fruitful in solving variety of practical problems.
Nevertheless, regardless of that remoteness of application areas the **mathematical** (probabilistic) methods and the models' **structure is very similar,** sometimes just 'identical' in some of "totally different" fields.
In our opinion, the most "natural" and relatively easy approach to explain the general idea is the following two components system **reliability' modeling** case.

We have decided that in the next section this (first) topic in reliability will be a little more elaborated than some other, mathematically similar, that will follow after.

2. The Reliability of Two Component Parallel System

Now, we start to investigate each of the system components' **failure mechanism** as being subjected to the following **two,** associated each to other, **patterns** of the components interactions. The first scheme we call:
"Micro-shocks → Micro-damages" phenomena.

This relationship is considered in a junction with the second that may be understood as a method of a "translation" some 'physical' phenomena to a proper mathematical model as (the second)**: "Micro-damages → Micro-hazard rate** (probability) **changes"** scheme. We apply a **continuous** approximation approach to this phenomena. In that setting we also describe cumulation **(**in the chosen model) of the stochastic micro-effects **(**or equivalently: the 'probability micro-changes'**)** by means of the calculus Riemann integral formalism.

Generally speaking, we encounter the following two situations.
The (random) **life-times** of two physical units **u_1, u_2** are estimated in **two distinct conditions** by use of the common statistical methods.
At first, both the units u_1, u_2 are tested, in separation of each other in, say "laboratory" **'off-system'** conditions. It is assumed that, as a result one obtains good enough estimations of the (**"baseline"**) probability distributions **$F_1(x_1)$, F_2 (x_2) of** the life-times **T_1, T_2 .**
Because of the physical separation at this stage of the research **(**that eventually may be considered as first stage of "mental experiment"**)** the resulting life-times T_1, T_2, of the units are stochastically **independent.**
At the **second stage** of the procedure the units are considered as components installed (in parallel) into the system. Now, as we assume**, "side-effects"** of various physical phenomena, associated with operating of any of the components, contribute to the failure mechanism of the other. Therefore, unlike in the previous off-system conditions, **additional** physical **stresses** are put on each of the two system components u_1, u_2**.**

As a result of the component's "harmful" activities, some **changes** in the other component's physical structure, such as **micro-damages,** occur. These micro-damages accelerate (or, in some cases delay) the processes leading to the component failures.
The objective is to find as "good" as possible stochastic **model** for such a system's reliability.
(For a similar construction's pattern, see [9]**).**
For this sake, one should admit that the physical phenomena associated with processes of the components interaction may turn out to be too complicated to be followed and analyzed adequately in traditional deterministic ways. This is the reason we rely on stochastic description by constructing a **joint probability distribution** of the "in-system" component life-times $\mathbf{X_1, X_2}$ **.**

The **key idea** to start with the construction is**:** Express the **stochastic dependences** in terms of some extra (due to the new 'in-system' conditions) **increments** in the 'original' (off-system) component **failure rates,** say $\lambda_1(\mathbf{x_1})$**,** $\lambda_2(\mathbf{x_2})$**.**
Recall, these failure (hazard) rates are associated with the 'original' life-time cdfs $F_1(x_1)$, $F_2(x_2)$ **of** $\mathbf{T_1, T_2}$ **(**so the changes in one <u>equivalently</u> parallel changes in other**).**
On the physical part of the problem, the mutual impact of any component on the other can be explained in the following manner. During the components' in-system performance either of the two creates such a situation that the other component is "constantly bombarded" by a **string of** harmful (or beneficial) **"micro-shocks".** Each such a "micro-shock" causes a corresponding **"micro-damage"** in the affected unit's physical constitution. Interpreting that **physical processes** 'language' **into** the 'language' of the **corresponding probabilistic** facts, one can say that the micro-damages of the components are equivalent to corresponding small **(micro) changes in** the original **failure (hazard) rates (**<u>probability</u> distribution**)** of their life-times.
On the other hand, each such a micro-damage is very small so that there is no practical possibility as to detect immediately any significant effect. However, after a time period long enough, the **micro-damages cumulate** their effects so that after that time the difference in the corresponding probability distributions may become quite significant.
To express the "smallness" of the micro-effects and then their significant accumulation, in the language related to the constructed **analytical model** we utilize **(**as it is frequently applied in modeling such phenomena like many those considered in physics**)** the familiar calculus notions of an **'infinitesimal quantity'** and that of the <u>Riemann integral</u>**.** Here, the integral will be applied as a "formal tool" to "sum up"(in the approximating analytical model) 'all' of the "infinitely many infinitesimal damages" in terms of the related micro-changes in the given value of the component's baseline failure rate **(**or in a distribution's parameter(s) **).** As a <u>result of that integration</u> over some finite time interval (of the component functioning) of length t one obtains a finite, practically recognizable, probabilistic quantity, say φ **(t;).**
Generally speaking, we apply a **continuous description** as a "smoothing" approximation to that kind of physical reality.
The goal is to **construct** a proper **bi-variate probability distribution of** the system component life-times $\mathbf{X_1, X_2}$ **,** which is a common type of the stochastic models in the system reliability investigations.
For the classical examples, see [9, 12]**;** also see [4 - 8]**.**

3. The Reliability Problem's Analytical Solution

In accordance with the general concept described in section 1 let us consider the previous two component' lifetimes X_1, X_2, as a "tandem" such that each of the two random variables is "explanatory variable" for the other.
In this section we will find the joint probability distribution of the random vector (X_1, X_2) in terms of its **joint survival function** $S(x_1, x_2) = Pr(X_1 > x_1, X_2 > x_2)$.
Recall, that the joint survival function $s^*(x_1, x_2)$ of the independent off-system component life-times T_1, T_2 is given by the following 'product form':

$$s^*(x_1, x_2) = \exp[-\int_0^{x_1} \lambda_1(t_1)\, d t_1 - \int_0^{x_2} \lambda_2(t_2)\, d t_2], \tag{4}$$

where $\lambda_1(t_1)$, $\lambda_2(t_2)$ are the components' off-system (baseline) failure rates.
When the components u_1, u_2 work in the system then, in accordance with the adopted (in the analytical model) assumption, in "every" infinitesimal small time interval $[\tau_k, \tau_k + d\tau_k)$ an occurrence of an infinitesimal **micro-damage** of the component, say u_k, $k = 1, 2$; (that is caused by '**side effects'** accompanying an activity of the component u_m, where $m = 1, 2; m \neq k$) results in an **infinitesimal increment** $\alpha_{km}(\tau_k)\, d\tau_k$ **of** the u_k's failure rate.
For every 'past' time instant τ_k, that increment is given by a predetermined quantity $\alpha_{km}(\tau_k)$ that, in a stochastic way reflects "an amount" of the physical influence of a component u_m on the component u_k. That quantity is chosen (and then must be **statistically estimated and verified** for fit to data available) to be a continuous function of all the past epochs τ_k the micro-damages occurred.
All the **stochastic** effects $\alpha_{km}(\tau_k) d\tau_k$ (of the physical micro-damage's) "sum up" over the time. Each of their "partial accumulation" (created from the time zero, up to the "current" epoch, say t_k) is expressed as the following Riemann integral:

$$\varphi_k(t_k) = (\int_0^{t_k} \alpha_{km}(\tau_k)\, d\tau_k, \ (k \neq m). \tag{5}$$

This integral is a non decreasing continuous function of the current time t_k as is taken over all time intervals $[0, t_k]$, with $0 \leq t_k \leq x_k < \infty$, for $k = 1, 2$.
Both the variables x_k ($k = 1, 2$) as present in the foregoing condition are the arguments of the survival function (4).
At every "current" time instant t_k, the '**in-system' failure rate** $r_k(t_k)$ of the component u_k ($k = 1, 2$) is defined to be a simple arithmetic sum of the 'off-system' baseline failure rate $\lambda_k(t_k)$ and the "additional failure rate" given by the integral

$$\varphi_k(t_k) = \int_0^{t_k} \alpha_{km}(\tau_k)\, d\tau_k, \ (k \neq m).$$

This integral will be thought off as a <u>measure</u> of a "magnitude" of the u_k's micro-damage's accumulation up to the current time epoch t_k.
Thus, at every time epoch $\mathbf{t_k}$ one obtains the following formula for the 'in-system' **failure rate** $\mathbf{r_k(t_k)}$ of the component $\mathbf{u_k}$:

$$\mathbf{r_k(t_k) = \lambda_k(t_k) + \int_0^{t_k} \alpha_{k\,m}(\tau_k)\, d\tau_k}, \tag{6}$$

as k, m = 1, 2, and $k \neq m$.

We consider the failure rate formula (6) to be valid for each time argument $\mathbf{t_k}$ satisfying $0 \leq t_k \leq \mathbf{x}$, where $\mathbf{x}$ = minimum (x_1, x_2) is considered to be the time of <u>the first failure</u> in the system. From the above one obtains the following survival function:

$\mathbf{S_1(x) = Pr(\min(X_1, X_2) > x)}$, **with** $\mathbf{x = \min(x_1, x_2)}$**),**

for the <u>first order statistics</u> **X** of the set of random variables: $\{X_1, X_2\}$. That is:

$$\mathbf{S_1(x) = \exp\left[-\int_0^x r_1(t_1)\, dt_1 - \int_0^x r_2(t_2)\, dt_2\right]}, \tag{7}$$

where $\mathbf{r_1(t_1)}$ and $\mathbf{r_2(t_2)}$ are given by (1) for k = 1, 2.
These two functions represent **the 'in system' failure rates** of the components u_1 and u_2 respectively (at the time instances t_1, t_2), both prior to the time, say x, after which the first failure in the system occurs. In other words, (7) represents the **reliability function $\mathbf{S_1(x)}$ of the system as a whole if its reliability structure is series.**
Continuing with the concept of **parallel** reliability structure, we consider the system's residual life -time's failure rate, say $\mathbf{r_k(t)}$ i.e., the failure rate of either surviving component $\mathbf{c_k}$ at any time **t** satisfying $x \leq \mathbf{t} \leq y$. Recall that **x, y** are the time epochs of the first and the second failure in the system respectively. For that period of time we <u>have</u> chosen the following <u>failure</u> pattern.
Namely, we define the failure rate $\mathbf{r_k(t)}$ in the time interval **[x, y]** as the following arithmetic sum:

$$\mathbf{r_k(t) = \lambda_k(t) + \int_0^x \alpha_{k\,m}(\tau)\, d\tau} \tag{8}$$

In this context, **the integral** $\mathbf{\int_0^x \alpha_{k\,m}(\tau)\, d\tau}$ **is constant** over time past x which 'now' is fixed **(** k, m = 1, 2, and $k \neq m$ **).**
The reason for its constancy is based on a simple observation that in the time interval [x , y] only (one) component $\mathbf{c_k}$ is working in the system, and thus the process of the micro - damages accumulation is <u>terminated</u> as the time x passed. This integral **(** present as a part in (9), (9*) that follow **)** is an additional <u>part of</u> the overall <u>failure rate</u> $r_k(t)$ of component c_k, and may be understood as a **measure of "memory"** of the micro-incentives the c_k received

before the other component c_m, stopped its activity at time x**.** For more on that see Remark 1**.**

The Final Formula for the joint survival function **$S(x_1, x_2) = Pr(X_1 > x_1, X_2 > x_2)$** of the in-system component life-times **X_1, X_2** is given as follows:

$$Pr(X_1 > x_1, X_2 > x_2 \mid X_1 \le X_2) = \exp\Big[-\int_0^{x_1}\{\lambda_1(t_1) + \int_0^{t_1}\alpha_{1,2}(\tau_1)\,d\tau_1\}\,dt_1$$

$$-\int_0^{x_1}\{\lambda_2(t_2) + \int_0^{t_2}\alpha_{2,1}(\tau_2)\,d\tau_2\}\,dt_2\Big]\exp\Big[-\int_{x_1}^{x_2}\{\lambda_2(t_2)\,dt_2\} \tag{9}$$

$$-(x_2 - x_1)\int_0^{x_1}\alpha_{2,1}(\tau_1)\,d\tau_1\Big];$$

$$Pr(X_1 > x_1, X_2 > x_2 \mid X_1 > X_2) = \exp\Big[-\int_0^{x_2}\{\lambda_2(t_2) + \int_0^{t_2}\alpha_{2,1}(\tau_2)\,d\tau_2\}\,dt_2$$

$$\int_0^{x_2}\{\lambda_1(t_1) + \int_0^{t_1}\alpha_{1,2}(\tau_1)\,d\tau_1\}\,dt_1\Big]\exp\Big[-\int_{x_2}^{x_1}\{\lambda_1(t_1)\,dt_1\} \tag{9*}$$

$$-(x_1 - x_2)\int_0^{x_2}\alpha_{1,2}(\tau_1)\,d\tau_1\Big].$$

If in both the formulas (9), (9*) one sets $x_2 = 0$, then one obtains the **marginal** probability distribution **of X_1** to be the same as the original probability distribution **$F_1(x_1)$** of the off-system life-time T_1, **related to** the given in advance original failure rate **$\lambda_1(t_1)$.** The similar result one obtains when imposing in (9), (9*) the condition $x_1 = 0$.
The latter condition yields the marginal distribution of the **X_2** to be equal **$F_2(x_2)$.**
As the conclusion one derives the following surprising property , shared by all the models that obey the pattern expressed by (9), (9*).

Property 1. For any joint probability distribution **$S(x_1, x_2)$** that satisfies the pattern, defined by (9), (9*), the given in advance original probability distributions $F_1(x_1)$, $F_2(x_2)$ of the off-system life-times T_1, T_2 **are preserved ! ** as **the marginal distributions** of the joint probability distribution of the in-system life-times X_1, X_2 (of the considered units u_1, u_2).

From the Property 1, the following conclusion can be derived.

Corollary. Suppose we are given a pair of probability distributions $G_1(x_1)$, $G_2(x_2)$ that belong to any class of probability distribution functions, whose all members posses continuous failure (hazard) rates, say $\lambda_1(t_1)$, $\lambda_2(t_2)$.
If one puts any arbitrary single pair of such distributions into the scheme defined by (9), (9*) then, as a result one can generate a wide class of the bivariate survival functions $S(x_1, x_2)$, whose marginals remain to be the $G_1(x_1)$, $G_2(x_2)$. The class of the so obtained bivariate

probability distributions "given the (fixed) marginals $G_1(x_1)$, $G_2(x_2)$" is determined by the family of all the continuous functions $\alpha_{i,j}(\tau_i)$, (i, j = 1, 2, with $i \neq j$) that produce all the integrals in (9), (9*) finite.

So, in this particular sense one can consider **the "bivariate Weibull, gamma (in particular, exponential), the extreme value"** and other joint probability distributions.

Realize, however that the marginal distributions **$G_1(x_1)$, $G_2(x_2)$,** in Corollary 1 also may represent two **distinct distribution classes.** The last possibility may be utilized in modeling reliability of two stochastically dependent units (such as system components) each one subjected to a different failure mechanism. Apparently such cases often are realistic.

Example 1 As a particular class of bivariate survival functions $S(x_1, x_2)$, satisfying the pattern given by (9), (9*), we now choose a class of bivariate exponential distributions given by two arbitrary constant failure rates λ_1, λ_2 for the marginals.
We also restrict the dependence structure. In this case it is assumed to be determined only by two arbitrary **constant** functions $\alpha_{1,2}(\ )$, $\alpha_{2,1}(\ )$. Recall, they represent the rates of increment in the failure (hazard) rate of the unit u_1 caused by u_2 and that failure rate increment of u_2, caused by u_1, respectively.
The resulting class of the **joint exponential survival functions** can be expressed by the more specific following formulas**:**

$$\Pr(X_1 > x_1, X_2 > x_2 \mid X_1 \leq X_2) = \exp\Big[-\int_0^{x_1}\{\lambda_1 + \int_0^{t_1}\alpha_{1,2}\, d\tau_1\}\, dt_1 - \int_0^{x_1}\{\lambda_2 + \int_0^{t_2}\alpha_{2,1}\, d\tau_2\}\, dt_2\Big] \exp\Big[-\int_{x_1}^{x_2}\{\lambda_2\, dt_2\} - (\alpha_{2,1} x_1)(x_2 - x_1)\Big]; \tag{10}$$

$$\Pr(X_1 > x_1, X_2 > x_2 \mid X_1 > X_2) = \exp\Big[-\int_0^{x_2}\{\lambda_2 + \int_0^{t_2}\alpha_{2,1}\, d\tau_2\}\, dt_2 - \int_0^{x_2}\{\lambda_1 + \int_0^{t_1}\alpha_{1,2}\, d\tau_1\}\, dt_1\Big] \exp\Big[-\int_{x_2}^{x_1}\{\lambda_1\, dt_1\} - (\alpha_{1,2}\, x_2)(x_1 - x_2)\Big]. \tag{10*}$$

(Recall, that any power transformation applied to the random variables X_1, X_2 yields the corresponding bivariate **Weibull model.)** Upon simplifying assumption that

$\alpha_{1,2}(\ ) = \alpha_{2,1}(\ ) = \alpha =$ **constant,** both the formulas (10), (10*) reduce to the following single one**:**

$$\Pr(X_1 > x_1, X_2 > x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \alpha x_1 x_2]. \tag{11}$$

Thus, as a special case one obtains the first bivariate exponential **Gumbel** probability distribution **as (perhaps a first time)** the system **reliability model** (see [11])**.**

<u>Remark 1.</u> Return, for a while to the integral $\int_0^x \alpha_{k\,m}(\tau)\,d\tau$, discussed in this section in association with the formula (8). In the models presented in this paper this additional **(**constant over the time interval [x , y], where **x, y** denote the times of the first and the second failure in the system, respectively **)** value of the e_k's failure rate can be interpreted as a measure of **"an amount of memory"** of the two components past interaction. The memory is assumed to be kept by any survived component $\mathbf{u}_k$ after the failure of the other component $\mathbf{u}_m$ (k, m = 1, 2).
Actually, the assumption of <u>preserving the whole memory</u> for all the residual time is rather strong and for many 'physical' (real) entities not realistic.
As usually, the reason it was adopted in this work was the common need to preserve a reasonable level of the simplicity.

Once it was done, **next step** in a process of the models construction would be adopting the models to more realistic situations when the **memory varies.**
Consider the following two phenomena. In the first, one assumes that due to some kind of "elasticity" of the units, **no memory** at all is kept by any component after the first failure.
To adopt our previous models to this case we set the extra failure rates given by the integrals
$\int_0^{x_1} \alpha_{2,1}(\tau_1)\,d\tau_1$, $\int_0^{x_2} \alpha_{1,2}(\tau_2)\,d\tau_2$, present in (9), (9*), respectively **to zero.**

In the second case, typical, for example in **biomedical phenomena,** the **memory** does not vanish but instead **decays** with the time **(**as result of 'rest' or **'recovery').** To describe that case we can multiply the considered here integrals by continuous functions **(**called the **"forgetting factors")** decreasing with the time, from one to zero, and then <u>integrate</u> the so obtained products over the remaining time. Thus, in expressions (9), (9*) we <u>replace</u> the integrals:

$$(x_2 - x_1) \int_0^{x_1} \alpha_{2,1}(\tau_1)\,d\tau_1, \text{ and } (x_1 - x_2)\,\}\int_0^{x_2} \alpha_{1,2}(\tau_2)\,d\tau_2,$$

by the following expressions**:**

$$\gamma^{-1} \exp\{-\gamma\,(x_2 - x_1)\} \int_0^{x_1} \alpha_{2,1}(\tau_1)\,d\tau_1, \text{ and } \gamma^{-1} \exp\{-\gamma\,(x_1 - x_2)\} \int_0^{x_2} \alpha_{1,2}(\tau_2)\,d\tau_2,$$

where γ is **(** to be estimated **)** a positive constant that may be called the **"coefficient of decay".**
In such a way one obtains <u>other variants</u> of the stochastic models defined earlier in this work.

<u>Remark 2</u> Observe that the dependence of the units u_1, u_2 failure mechanisms and its stochastic description in section 2 somehow <u>resembles</u> the multiple shock models pretty frequently met in literature. See for example "<u>a successive damage model</u>" in [3] . As for the <u>difference</u>, in all such models the shocks were considered to form a discrete kind of sets (mostly finite). Also unlike with the failure mechanisms we consider, every single shock from that (discrete) set is a significant, in the sense it always could cause the unit's failure with a positive probability. Nevertheless, these differences may only be regarded as a usual conceptual difference between discrete and continuous approach to basically similar phenomena. The continuous model is only thought off as an approximating limiting transition if a number of weak "shocks" becomes very large and is "densely" redistributed over the time. Therefore, the ("smoothed") model of the failure mechanism as described in section 2, we propose to call "Continuous 'micro shock – micro damage' model", whenever in reliability framework. In more general settings we propose to call it Continuous "micro action" → "micro effect" model that might, for example, be applied for a joint description of such random quantities like 'level of employment versus rate of inflation ' in a macro economy investigations or in other similar circles of practical problems.

4. Other Applications of the Dependence Paradigm

The pattern of the stochastic dependence, applied to reliability in previous sections, can naturally be extended to the (mathematically similar) modeling of a variety **bio-medical** phenomena and related, in many cases random, quantities, (for that see [3]). A simple but vivid illustration of that kind of the modeling problems provides the following example of actuary investigations.

Example 2. One of the pretty recently considered **actuary problems** (see [11]) is to improve **stochastic predictions** on residual life-times for potential clients of a given age, who possibly widowed in a recent time.
Two situations are considered and compared. Either a candidate for a given insurance plan was living all the past life as a single or spent a significant of part of life in a marriage. It is assumed that "at present" she or he is widowed.
The problem is based on **statistical** data **evidence,** which indicates that the two life-styles significantly result in statistically **different (**in the sense of expected value or of an underlying probability distribution**)** residual life-times for the two groups of the persons. The persons besides are supposed to be "identical" with respect to any other essential criteria that may influence the life-times. The statistical findings suggest that some physical, psychological or mental **interactions** between the spouses in marriage produce some additional stresses, which positively or negatively affect the client's residual life-time or, more precisely, its probability distribution. Also they suggest that, even if the person widows, a "memory" of past experiences remains in a person's psychophysical structure, affecting her/his residual life-time.

Phenomena, like the one above, seem to be typical in many other similar situations that can be present in a variety aspects of the real world, especially those of human' life conditions and accompanying events.
The problem of estimating the (random) length of a human's residual life time is vital for life **insurance** companies, and requires a use of advanced statistical methods. It is well known that the average residual life time or well being of a person at a specific age (besides of his / her "genetics") depends on a variety of factors, the essential ones being "**stresses**" (and corresponding times of their duration) such as smoking tobacco, excessive drinking alcohol, the length of time being exposed to especially harsh conditions such as prison, war or other.

In the vast majority of cases where statistical methods are employed, it is customary that the conclusions derived from, say, randomized experiments or other tests, are reduced (in the case of smoking versus nonsmoking, for example) to dichotomous statements on the existence of the influence of a given stress on the residual life time, or lack of such influence (given a significance level). In these cases, there is often **lack of information on any quantified** relationship between the **life time**'s stochastic characteristics **(**such as, for example, expectations, quantiles, hazard rates or other**)** and an 'amount' of stress a person endured. In the case of tobacco smoking this amount can be measured as, for example, a product (or some other function) of the time period of duration and the intensity (amount of nicotine per day) of the given stress.

5. The 'Micro-Incentives → Micro-Effects' Scheme and the Related Stochastic Dependences' Pattern. General Formulation.

A comparison of the, given above, reliability and bio-medical examples exhibits a common idea, in stochastic modeling procedures. The range of possible applications turns out to be remarkably wide, entering far beyond the context of the reliability investigations.
The essence of the considered, in this work general pattern , first of all relies on reviling a **relation** (if exists) **of,** say, "**parallelism"** between some ('continuous') "**physical** interactions" of two or more observed entities, on one side, **and** the corresponding infinitesimal changes in the **mathematical** (probabilistic) model's parameters on the other side.
This, rather a new pattern of the (physical versus stochastic) dependence can be considered as an extension of the **reliability scheme's (**described in Example 1**)** that, as a whole, could be illustrated, by the following 'diagram'**:**

"('micro-shocks' – 'micro-damages') versus ('micro-probability -changes')", (12)

where the 'physical' parallels the 'stochastic'. In (12), the meaning the word "versus" is the same as the meaning of the "parallels". The choice between the two words was dictated by the (partly 'esthetic') structure of the above diagram.
Realize that the above pattern for reliability problems can be extended to a significantly more general domain of the phenomena. Consequently, the diagram (12) can be replaced by the following, more general, diagram**:**

"('micro-incentives' – 'micro-physical changes') versus

('micro-probability -changes')". (13)

The pattern described by (12) is special case of that given by (13).
To explain the **similarities** between the reliability settings and other (if appropriate) types of 'real world' phenomena we refer to Example 2, as well as to a huge number of particular situation ranging from **econometric** problems **(**like, possibly, stock market predictions**)** to the bio-medicals.

Look, for example, at the relationship between a human's smoking tobacco (or other stress), during a given time period and a probability distribution of his/ her **life time: Y.**
Physically, this problem may be "imagined" as "constant attacks" on the human's body by a sequence of very small "pieces" of the harmful substance that result in a corresponding sequence of small micro-changes (damages) in some parts of the human's body. An effect of any single piece of the substance's (in this case, say the nicotine) activity is usually too small to be observed or even admitted. However, these effects cumulate and after some, sufficiently long period of time, the total sum of them makes a significant **contribution** (a "change") to the biochemical processes that are "responsible" for an illness (such as hard attack) or death by accelerating them.
As a corresponding stochastic effect the probability distribution of the life time **Y** may be **changed** making the **probabilities (**or equivalently the **hazard rates)** of shorter life times **higher** than the corresponding probabilities **(**the baseline hazard rate**)** for, the statistically the same but differing by that "**smoking** factor", persons.
Probably, in most of the cases, the 'physics' of the described above phenomena may be too fugitive or complicated to be reasonably describable in terms of deterministic scientific **(**including a 'biomedical description**')** language.
The introduced in this paper **probabilistic approach** is thought off as a **shortcut,** that eventually may lead to a useful stochastic model.
For the stochastic model we have chosen a **continuous** one, despite **discrete nature of** the physical **reality.** As this is a common procedure, applied, first of all, in science and engineering, a continuous approximating description, of being considered 'real-life' phenomena, stand for analytically nice and sufficiently precise models (both, deterministic and stochastic) if some (smoothing) conditions are satisfied.
In our case, the two such conditions are assumed to be satisfied**: 1.** Every string of micro-incentives as well as that of the resulting 'micro-physical changes', only contains very small pieces of, say "micro-actions". **2.** The time-distance between any such two consecutive actions is also very small, so that a number of such actions in some reasonably long time interval is "large".

That smallness and a large number of the micro-changes that occur during any sufficiently long time interval allows for averaging and smoothing the phenomena.
According to this possibility we treat, within the mathematical stochastic model (only) all the micro-changes as '**infinitesimals',** which are "continuously redistributed" along the time.
That reasoning and the analytical efficiency of the mathematical calculus facilities, made us to chose the **continuous** model(s) as the approximation(s) of the 'real life' phenomena, that unfortunately are, by the nature, discrete.

In the following section we give an example of the one more and very important application of the introduced in this work, kind of the stochastic dependences.

6. On the 'Extended Regression' Paradigm

Consider, once more, the conditional density $\mathbf{g_j\ (x_j \mid x_1, \dots, x_{j-1})}$ that represents the main object **(**as the general model**)** of the presented theory. Now, we admit the meaning of the random variables **Y** and the $X_1, \dots, X_k$, to be 'arbitrary', with $\mathbf{X_1, \dots, X_k}$ considered as 'any' **explanatory** (random) variables for the **Y,** that is considered to be the actual variable (or random vector) of interest. The random variables $\mathbf{X_1, \dots ,X_k}$ may either be independent or, if not, their joint probability distribution is, in general, assumed to be known.

We may restrict our investigation to each **single unit (**as characterized by the quantity **'Y'),** separately, just for 'practical' needs, while resigning from developing a more general theory, that would involve classes or "populations" of those units. In such a 'practical' framework instead of the model that contains the probability distribution of the explanatory random variables $\mathbf{X_1, \dots ,X_k}$ one must rely on their deterministic realizations $x_1, \dots ,x_k$, as they can be known, in each single case, 'post factum', by direct observations followed by proper measurements. The way out from this limitation could be, whenever possible, an (approximating) assumption on stochastic independence of the r. variables $\mathbf{X_1, \dots ,X_k}$.

The crucial fact, behind the above description, is that the so defined and relatively easily achievable (!) **stochastic dependences** of a considered random variable **Y** on the random variables $\mathbf{X_1, \dots ,X_k}$, may directly lead to a modification and **enrichment of the classical** **regression methodology.**

Recall, that **the dependence** of the **Y** on the random variables $X_1, \dots ,X_k$ is not a direct 'functional' one. **Instead of** usually considered direct (explicit) influence of the realizations $\mathbf{x_1, \dots ,x_k}$ **on value of Y,** here the given realizations **"only" influence** (explicitly) the **probability distribution** of the **Y.**
The new (conditional) probability distribution of **Y (**after it "enters to interaction with some physical impacts" characterized by the random quantities $X_1, \dots ,X_k$ **)** is obtained by a **direct transformation of** the original (baseline) distribution' **parameters** into their new values, that are continuously dependent on the realizations $\mathbf{x_1, \dots ,x_k}$ of the "impacts".

This is obvious that the pattern for the class of all the obtained conditional pdfs $\mathbf{g\,(y \mid x_1, \dots ,x_k)}$ may provide significantly more information on the random quantity **Y** than just when one considers its conditional **expectation only.** Nevertheless the latter **is** included in the **"extended regression model": $\mathbf{g\,(y \mid x_1, \dots ,x_k)}$,** simply as its expectation. Also realize that in the considered paradigm the normality assumption for the $g\,(y \mid x_1, \dots ,x_k)$ is quite not necessary as it becomes natural that $g(y \mid \dots)$ can be **arbitrary probability density** in the variable **y.**
The progress, we believe has been made in comparison to the present state of the art, first of all relies on allowing any cdf's parameter **(**in particular, an expectation and variance in the case of normal distribution **)** to be **arbitrary continuous function** of the **explanatory** (random) **variables.** We find that there is no necessity to restrict these functions to linear or to simple polynomials only. The use (as the models) of most of the common (parameter)-functions (such as the power, exponential, logarithmic, some trigonometric or their combinations), in general **does not involve** estimation **of much more unknown parameters** than it is used with the simple polynomial or even linear function' application.

The kind of the "regression techniques", here suggested, may too be compared with a more familiar (to reliability oriented readers), procedure associated with the well known **accelerated test models** for the life-time of some technical devices**.** In that case the

explanatory variables $X_1, \ldots, X_k$, that here we consider in a more general setting, can be interpreted as various kinds of, say, random **loads** or stresses that may eventually be multiplied by (also random) times these loads are to be endured by the tested units.
This subject is, for example, very well elaborated in [13].

Other illustration is associated with the common **actuary** problem as to estimate residual life-time of a given age client who smoked tobacco during a time period **T** with an intensity of **X** milligrams of nicotine per day. Supposing the client' **residual life-time Y** is approximate by a normal pdf **N(μ, σ)** in an absence of smoking. If, however, the smoking took place, then we may hypothetically assume that the average life-time μ "somehow" depends on the **(random**, in case of the whole population of the clients **or** a known **deterministic** in each individual case**) load Z** that is strictly proportional to the product of **TX** associated with smoking **tobacco.**
This stochastic relationship between the life-time and the 'smoking amount', may be explicitly given by the following conditional **(**normal in **y) pdf of the residual life-time Y:**

$$\mathbf{g}(y \mid \mathbf{x}, \mathbf{t}) = [\sigma(\mathbf{xt})\sqrt{(2\pi)}]^{-1} \mathbf{exp}[-(y - \mu(\mathbf{xt}))^2 / 2[\sigma(\mathbf{x\,t})]^2], \tag{14}$$

where **μ(xt) and σ(xt)** are arbitrary (reasonable) continuous functions of the **product xt.** Having known joint pdf f(x, t) of the T and X (across the smokers population) we may find the unconditioned pfd of the time Y just by integrating out the variables t, x from the product **g(y | x, t)** f(x, t) that represents the trivariate joint pdf h(x, t, y) of the random vector (**Y,** T, X).
As hypothetical parameter functions, implicitly present in the model (14), we may, for example, consider the functions $\mu\,(\mathbf{xt}) = \mu + \mathbf{a}\,(\mathbf{xt}) + \mathbf{A}\,(\mathbf{xt})^{\mathbf{r}}$ **, (with possibly a = 0, while A, r being positive real numbers), and** $\sigma\,(\mathbf{x\,t}) =$ **constant** (in this particular case)**.** For many more examples on that see [5,6,8]**.**

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