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# Coupled Oscillators

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## 1. Introduction

A wave is the propagation of oscillations in space through the coupling between individual oscillators. There is an obvious experiment to hit the right piano key, which sounds the tuning-fork tuned to the same frequency as the key we hit. Another example is breaking a fragile wine glass by the intense sound of the frequency that corresponds to the resonant frequency of the glass.

The crystal of the substance is formed by the regular arrangement of individual atoms, ions, or molecules that are bound together by electrical forces. Since the crystal keeps its form, it means that there are attractive forces between the particles. Each particle has its equilibrium position, which is given by the minimum potential energy in the force field of the surrounding particles. The particles oscillate around the equilibrium position concerning the homogeneity of the substance. Besides, the frequencies of the self-oscillations of all particles are the same. When one particle oscillates, energy transfers to the adjacent particle by resonance, and then the initial excitation travels through space. This excitation propagating is called the *wave*.

First, we show the oscillations of a system consisting of two or three particles. Then, we show the energy transfer between two particles and finally, the wave propagation in an infinite chain of the same oscillators.

### 1.1 Molecular vibration spectra

Radiation passing through the material medium vibrates the particles of the substance that are in resonance with the radiation. The particles then absorb the energy of the radiation, and thus, the radiation is attenuated. This phenomenon is used in spectroscopy to identify the particles with common properties in a specific medium.

#### 1.1.1 The frequency of vibrations of the diatomic molecule

One of the oscillators of a gaseous environment is a molecule. The molecules consist of the individual atoms bound together. Thus, the system of bonded atoms represents an oscillator. The simplest is a diatomic molecule.

Consider a molecule in which two atoms with masses of  $m_1, m_2$  are bound by the force  $F = -k (r - r_0)$ , where the  $k$  is the stiffness coefficient of the molecule, and the  $r_0$  is the equilibrium distance of the centers of the atoms. If we denote  $r_1$  and  $r_2$  the coordinates of atoms, in the coordinate system of a straight line passing through the centers of atoms, the equations of motion of both atoms are.

$$m_1 \ddot{r}_1 = -k (r_2 - r_1) \text{ and } m_2 \ddot{r}_2 = -k (r_1 - r_2).$$

Subtracting the equation, we get.

$$\ddot{r}_1 - \ddot{r}_2 = -\frac{k}{m_1} (r_2 - r_1) - \frac{k}{m_2} (r_2 - r_1),$$

or

$$\ddot{r} = -\frac{k(m_1 + m_2)}{m_1 m_2} r. \quad (1)$$

The solution to the equation is a harmonic function of  $r = r_m \sin \omega t$ . These are vibrations of a molecule with an angular frequency.

$$\omega = \sqrt{\frac{k}{m^*}}, \quad (2)$$

where  $m^* = \frac{m_1 m_2}{m_1 + m_2}$  is the reduced mass of the particles.

The molecule, thus, represents a simple oscillating system.

If we apply an external periodic force, then we can achieve resonance when a molecule or array of molecules absorbs the energy. This phenomenon finds its place in spectroscopy. If we irradiate the gas consisting of the diatomic molecules with electromagnetic radiation, the resonance wavelengths are attenuated in the transmitted radiation.

Thus, we can determine the stiffness coefficient of the molecule by measuring the resonance frequency.

Example 1 The vibration frequency of some molecules.

Oxygen  $O_2$ :  $m = 16 \times 1.67 \times 10^{-27}$  kg,  $\mu = m/2$ ,  $k = 1133$  N/m, hence  $f = 46.3$  THz, wavelength in vacuum  $\lambda = 6.47$   $\mu$ m.

Nitrogen  $N_2$ :  $m = 17 \times 1.67 \times 10^{-27}$  kg,  $\mu = m/2$ ,  $k = 2287$  N/m, hence  $f = 63.9$  THz, wavelength in vacuum  $\lambda = 4.70$   $\mu$ m.

From this, we can see that the air in the atmosphere absorbs infrared radiation.

Example 2 Stiffness coefficient of the molecule.

The HCl molecule has the resonance frequency of  $f = 86.7$  THz (wavelength 3.46  $\mu$ m). For  $m_H = 1.67 \times 10^{-27}$  kg and  $m_{Cl} = 35 \times 1.67 \times 10^{-27}$  kg,  $\mu = 1.62 \times 10^{-27}$  kg, we get the  $k \approx 481$  N/m.

The NO molecule has the resonance frequency of  $f = 56.3$  THz (wavelength 5.33  $\mu$ m). For  $m_N = 17 \times 1.67 \times 10^{-27}$  kg and  $m_O = 16 \times 1.67 \times 10^{-27}$  kg,  $\mu = 13.76 \times 10^{-27}$  kg, we get the  $k \approx 1722$  N/m.

The CO molecule has the resonance frequency of  $f = 66.3$  THz (wavelength 4.52  $\mu$ m). For  $m_C = 12 \times 1.67 \times 10^{-27}$  kg and  $m_O = 16 \times 1.67 \times 10^{-27}$  kg,  $\mu = 11.45 \times 10^{-27}$  kg, we get the  $k \approx 1987$  N/m.

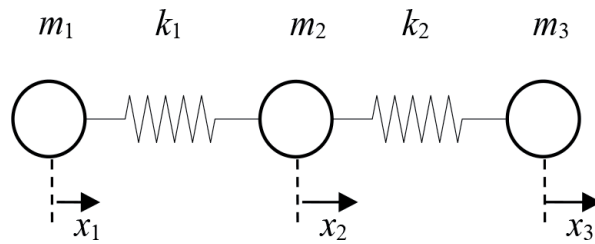
Example 3 Coupling stiffness coefficient.

The bonds of pairs of atoms in more complex molecules also show resonance character. The binding force and hence the frequency of the oscillations vary for different binding strengths, e.g., carbon bonds:  $-C \equiv C-$  ( $\lambda = 4.73$   $\mu$ m),  $=C=C=$  ( $\lambda = 5.90$   $\mu$ m),  $\equiv C-C \equiv$  ( $\lambda = 9.48$   $\mu$ m), heterogeneous bonds  $\equiv C-H$  ( $\lambda = 3.00$   $\mu$ m),  $-O-H$  ( $\lambda = 2.80$   $\mu$ m),  $-C \equiv N$  ( $\lambda = 4.40$   $\mu$ m),  $=C=N-$  ( $\lambda = 6.30$   $\mu$ m).

### 1.1.2 Oscillations of multiatomic molecules

In the case of multiatomic molecules, vibrations of individual pairs of atoms, but also other collective stationary modes will occur. The spectrum of such molecules is more complex, and its theoretical calculation is complicated.

As a simple case, consider a  $CO_2$  molecule that has a simple linear geometry. The  $O=C=O$  atoms are on one line, as opposed, e.g., to the two-dimensional  $H_2O$  molecule, in which the  $H-O-H$  atoms form a "V" structure with an opening angle of  $104.5^\circ$  and an O atom at the midpoint. Another example is the 3-dimensional



**Figure 1.**  
*Linear system of two bound particles.*

molecule of  $\text{NH}_3$ . A similar linear structure has an  $\text{N}_2\text{O}$  molecule having the arrangement of  $\text{N} \equiv \text{N}^+ - \text{O}^-$  or  $\text{N}^- = \text{N}^+ = \text{O}$ .

Consider the general case of a triple atom in a linear arrangement, see **Figure 1**. The equations of motion are.

$$\begin{aligned} m_1 \ddot{x}_1 &= k_1 (x_2 - x_1), \\ m_2 \ddot{x}_2 &= k_2 (x_3 - x_2) - k_1 (x_2 - x_1), \\ m_3 \ddot{x}_3 &= -k_2 (x_3 - x_2). \end{aligned} \quad (3)$$

The stationary mode represents oscillations of the system in which all particles vibrate at the same frequency and constant amplitudes. If we use phasor symbolism, we get.

$$\mathbf{x}_1(t) = \mathbf{X}_1 e^{j\omega t}, \quad \mathbf{x}_2(t) = \mathbf{X}_2 e^{j\omega t}, \quad \mathbf{x}_3(t) = \mathbf{X}_3 e^{j\omega t}.$$

After substituting into differential equations, we get a system of algebraic equations.

$$\begin{pmatrix} -\omega^2 m_1 + k_1 & -k_1 & 0 \\ -k_1 & -\omega^2 m_2 + k_2 + k_1 & -k_2 \\ 0 & -k_2 & -\omega^2 m_3 + k_2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{pmatrix} = 0. \quad (4)$$

The condition of nontrivial solution is zero value of the system determinant (characteristic equation)

$$\left[ \omega^4 - \omega^2 \left( \frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} + \frac{k_2}{m_3} \right) + \left( \frac{m_1 + m_2 + m_3}{m_3 m_2 m_1} \right) k_1 k_2 \right] \omega^2 = 0.$$

This is the equation of the third degree for  $\omega^2$ .

The first solution is  $\omega^2 = 0$  and  $x_1 = x_2 = x_3$ . It represents the translation of the molecule by uniform motion.

The other two solutions are the solution of the quadratic equation for  $\omega^2$

$$\omega^4 - \omega^2 \left( \frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} + \frac{k_2}{m_3} \right) + \left( \frac{m_1 + m_2 + m_3}{m_3 m_2 m_1} \right) k_1 k_2 = 0,$$

which has two solutions.

$$\begin{aligned} \omega_{1,2}^2 &= \frac{1}{2} \left( \frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} + \frac{k_2}{m_3} \right) \\ &\pm \sqrt{\frac{1}{4} \left( \frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} + \frac{k_2}{m_3} \right)^2 - \left( \frac{m_1 + m_2 + m_3}{m_3 m_2 m_1} \right) k_1 k_2}. \end{aligned} \quad (5)$$

Example 4 The CO<sub>2</sub> molecule oscillations.

As an example, we will find the stationary mode frequencies of the linear molecule O=C=O (carbon dioxide), for which the  $k_1 = k_2$  and  $m_1 = m_3$ . If we substitute it to the corresponding relation, we get.

$$\omega_{1,2}^2 = k_1 \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \pm \frac{1}{m_2}, \quad (6)$$

from where  $\omega_1 = \sqrt{k_1 \left( \frac{1}{m_1} + \frac{2}{m_2} \right)}$ , and  $\omega_2 = \sqrt{\frac{k_1}{m_1}}$ .

$f_1 = 40.92$  THz ( $\lambda_1 = 7.33$   $\mu\text{m}$ ),  $k_1 = 1766$  N/m,  $f_2 = 78.35$  THz ( $\lambda_2 = 3.83$   $\mu\text{m}$ ).

The first oscillating mode with the angular frequency  $\omega_1$  represents the simultaneous movement of the outer atoms  $x_1 = x_3$  and the movement of the central atom in the opposite direction. At the same time, the mass center remains at rest. In the second mode  $\omega_2$ , the middle C atom is at rest, and the other O atoms oscillate against each other ( $x_1 = -x_3$ ). In the CO<sub>2</sub> absorption spectrum, these absorption lines correspond to two wavelengths of  $\lambda_1$  and  $\lambda_2$ .

In the case of two or three-dimensional molecules, there are several stationary modes. The complex structure of the spectrum of resonance frequencies is unique to each molecule and thus allows identifying the substance by measuring the resonance spectrum. This method represents spectral analysis. As can be seen from the above examples, the vibration modes have frequencies that belong to the region of infrared radiation (in the wavelength range from one to hundreds of  $\mu\text{m}$ ).

Simple molecules of CO<sub>2</sub>, SO<sub>2</sub>, N<sub>x</sub>O<sub>y</sub>, and others, which enter the atmosphere as a result of human activity, absorb thermal radiation and create a *greenhouse effect*. These gases are so-called *greenhouse gases*.

The gases such as H<sub>2</sub>, O<sub>2</sub>, and N<sub>2</sub> have non-polar molecules. These molecules have zero dipole moment as opposed to the polar molecules of H<sub>2</sub>O, CO<sub>2</sub>, and others. Therefore, they interact very weakly with electromagnetic radiation, and they do not absorb the heat. Thus, the elementary atmospheric gases O<sub>2</sub>, N<sub>2</sub> do not cause a greenhouse effect.

## 1.2 Energy transfer in the system of the coupled oscillators

As stated earlier, the nature of the wave is the propagation of oscillations in space. Let us consider a pair of coupled oscillators and then an extension to an infinite line chain of oscillators.

### 1.2.1 Two coupled oscillation systems

Consider a mechanical model of a pair of particles of an equal mass of the  $m$  bound to each other and stationary walls by springs with a stiffness of the  $k$ , **Figure 2**.

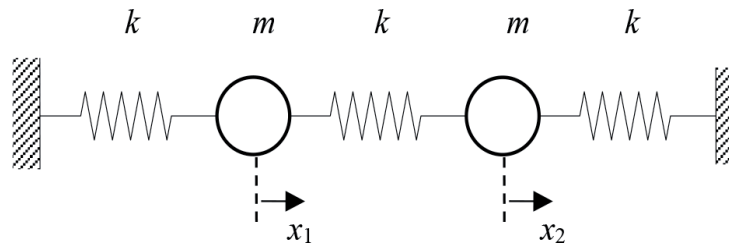
Suppose that particles can deviate from equilibrium positions in the longitudinal  $x$ -direction. Then, denote the displacements as  $x_1$  and  $x_2$ . The following equations describe the motion of the particles:

$$m \ddot{x}_1 = k (x_2 - x_1) - k x_1 \quad m \ddot{x}_2 = -k (x_2 - x_1) - k x_2, \quad (7)$$

or

$$\ddot{x}_1 = \omega_0^2 (x_2 - x_1) - \omega_0^2 x_1 \quad \ddot{x}_2 = -\omega_0^2 (x_2 - x_1) - \omega_0^2 x_2, \quad (8)$$

where  $\omega_0^2 = k/m$ .



**Figure 2.**  
*System of coupled oscillators.*

Consequently, we rewrite the equations to the form.

$$x_{1,2}^{(4)} + 4\omega_0^2 \ddot{x}_{1,2} + 3\omega_0^4 x_{1,2} = 0,$$

which has a harmonic solution

$$x_1 = A_{11} \sin(\omega_1 t + \alpha_{11}) + A_{12} \sin(\omega_2 t + \alpha_{12}) \quad (9)$$

$$x_2 = A_{21} \sin(\omega_1 t + \alpha_{21}) + A_{22} \sin(\omega_2 t + \alpha_{22}), \quad (10)$$

where  $\omega_1 = \omega_0$ , and  $\omega_2 = \sqrt{3} \omega_0$ .

The  $A_{ij}$  and  $\alpha_{ij}$  are constants, which values are determined by the initial conditions such as displacements of the  $x_{10}$ ,  $x_{20}$  and velocities of the  $v_{10}$ ,  $v_{20}$  at the time of  $t = 0$  and Eq. (8).

#### 1.2.1.1 Stationary modes

Stationary oscillations are oscillations with a single frequency and a constant amplitude. There are two stationary modes in the system - one with a frequency of  $\omega_1$  and the other with a frequency of  $\omega_2$ .

For the first mode, we have.

$$x_1 = A_{11} \sin(\omega_1 t + \alpha_{11}) \text{ and } x_2 = A_{21} \sin(\omega_1 t + \alpha_{21})$$

and after substituting to the (8), we get.

$$(2\omega_0^2 - \omega_1^2) x_1 = \omega_0^2 x_2 \text{ and thus, } x_1 = x_2. \quad (11)$$

The first stationary mode is characterized in that both particles vibrate with the same amplitude and the same phase, i.e., together.

The second stationary mode is then.

$$x_1 = A_{12} \sin(\omega_2 t + \alpha_{12}), \text{ and } x_2 = A_{22} \sin(\omega_2 t + \alpha_{22})$$

and after substituting into the (8), we get.

$$(2\omega_0^2 - \omega_2^2) x_1 = \omega_0^2 x_2 \text{ and thus, } x_1 = -x_2. \quad (12)$$

The second stationary mode characterizes two particles oscillating with the same amplitude but the opposite phase, which means opposite direction to each other.

*Note: If the particles have an electric charge, then the dipole moment of the pair does not change at the mechanical oscillations of the first mode, while the second mode changes the electrical dipole moment of the pair. For this reason, the first oscillating mode only weakly interacts with the external EM field, while the second mode interacts with the EM*



relatively strongly. The resonant excitation of mechanical oscillations by the EM field leads to the absorption of electromagnetic power, especially at the resonant frequency of  $\omega_2$ , which is represented by a significant minimum in the spectral transmission characteristic of the substance.

### 1.2.1.2 Stationary mode interference

Any movement of the described system is a superposition of stationary modes.

$$\begin{aligned} x_1 &= A_1 \sin(\omega_1 t + \alpha_1) + A_2 \sin(\omega_2 t + \alpha_2) \\ x_2 &= A_1 \sin(\omega_1 t + \alpha_1) - A_2 \sin(\omega_2 t + \alpha_2). \end{aligned}$$

The oscillation character depends on the initial conditions such as the displacement of  $x_{10}$ ,  $x_{20}$ ,  $v_{10}$  and  $v_{20}$  and the particle velocities at the time  $t = 0$ .

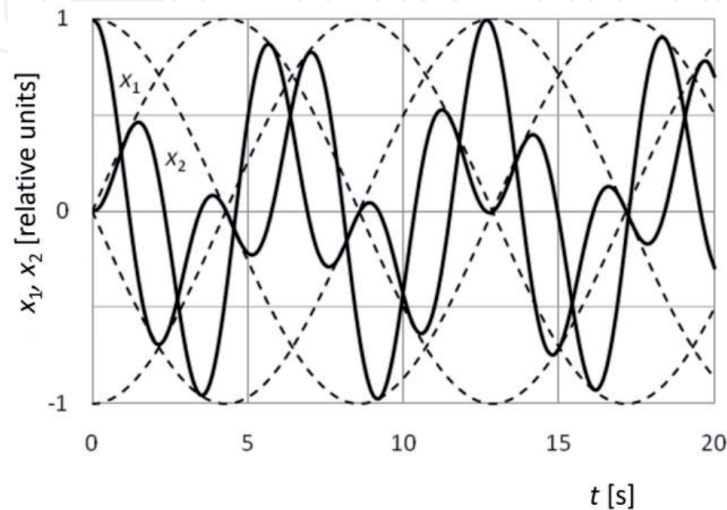
As an example, we assume the case, when the first particle is shifted from the equilibrium and the second one is quiescent. The initial conditions are  $x_{10} = x_m$ ,  $x_{20} = 0$ ,  $v_{10} = v_{20} = 0$ . After substituting the initial conditions, we get.

$$\begin{aligned} x_{10} &= A_1 \sin \alpha_1 + A_2 \sin \alpha_2 = x_m \\ v_{10} &= \omega_1 A_1 \cos \alpha_1 + \omega_2 A_2 \cos \alpha_2 = 0 \\ x_{20} &= A_1 \sin \alpha_1 - A_2 \sin \alpha_2 = 0 \\ v_{20} &= \omega_1 A_1 \cos \alpha_1 - \omega_2 A_2 \cos \alpha_2 = 0, \end{aligned}$$

from where, we obtain amplitudes of  $A_1 = A_2 = A = x_m/2$  and phases  $\alpha_1 = \alpha_2 = \pi/2$  rad.

$$\begin{aligned} x_1 &= A (\cos \omega_1 t + \cos \omega_2 t) = x_m \cos \left( \frac{\sqrt{3}-1}{2} \omega_0 t \right) \cos \left( \frac{\sqrt{3}+1}{2} \omega_0 t \right) \\ x_2 &= A (\cos \omega_1 t - \cos \omega_2 t) = x_m \sin \left( \frac{\sqrt{3}-1}{2} \omega_0 t \right) \sin \left( \frac{\sqrt{3}+1}{2} \omega_0 t \right). \end{aligned} \quad (13)$$

Time courses of displacements are in **Figure 3**.



**Figure 3.**  
Displacement of coupled oscillators for  $\omega_0 = 1 \text{ rad}\cdot\text{s}^{-1}$ .

It is seen from relations (13) and **Figure 3** that oscillations are periodically transmitted from one body to another and backward. In the figure, the envelope curves of both oscillators are indicated by a dashed line. As a result of the coupling, there is a periodic exchange of energy between oscillators. At the time of  $t = 0$  s, the first particle has the entire energy of the oscillations. On the other hand, in time of.

$$t_1 = \frac{2\pi}{(\sqrt{3} - 1)\omega_0} \approx 8.6 \text{ s (for } \omega_0 = 1.0 \text{ s}^{-1}\text{),}$$

the whole energy is transferred to the second particle.

Example 2.5 Oscillation energy transfer rate.

Let us assume the aluminum ions from Example 1.5 have the angular frequency of  $\omega_0 \approx 6.1 \times 10^{13}$  rad/s and a mutual distance  $a \approx 2.8 \times 10^{-10}$  m. If we suppose the time of transfer  $t_1 \approx 1.4 \times 10^{-13}$  s, the transfer rate is  $v = a / t_1 \approx 2.0 \times 10^3$  m/s, which corresponds to the order of the ultrasound velocity in aluminum ( $5.1 \times 10^3 \text{ m}\cdot\text{s}^{-1}$ ).

As we see from a simple example, the energy of oscillations in a substance is transmitted due to mutual bonds, which are manifested externally as a coherent propagation of mechanical waves, e.g., sound and ultrasound, or incoherent heat propagation.

1.2.2 Propagation of oscillations in a long particle chain

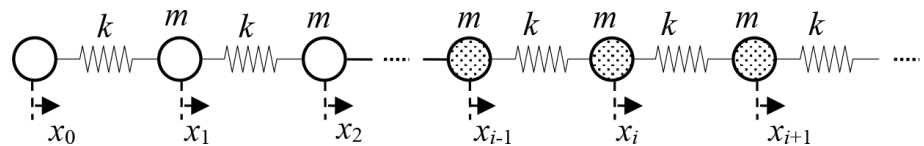
Substances consist of a great number of less or more ordered fundamental particles, which can be molecules, atoms, or ions. Solids and liquids are formed by particles whose attraction to each other is sufficiently large to form a compact body. Each particle of solid or liquid substance has its equilibrium position, which is caused by the force of the surrounding particles. These forces are mainly electrical. The equilibrium position corresponds to the minimum potential energy, and around the minimum of the potential energy, each particle performs an oscillating movement. As a simplified crystal model, we will use an infinite chain of bound particles arranged in a single line.

1.2.2.1 Propagation of longitudinal oscillations along the chain

We see the line chain model in **Figure 4** suppose that at the beginning, only the leftmost particle is put into oscillating motion, and the remaining particles of the chain are quiescent. Let us further consider that an external time-dependent force keeps the extreme left particle in a steady oscillating motion with time dependence.

$$x_0(t) = x_{0m} \sin \omega t. \tag{14}$$

The vibrations of the extreme left particle are gradually transferred to the other particles of the chain via bonds. After some time, all the particles of the system oscillate in harmonic oscillating motion with the same angular frequency of  $\omega$ , but different amplitudes and different phases.



**Figure 4.**  
Linear chain of coupled oscillators.



Consider three adjacent particles with indexes  $i - 1, i, i + 1$ , indicated by a dotted fill in the figure.

The equation of motion of the  $i$ -th particle has the form.

$$m\ddot{x}_i = -k(x_i - x_{i-1}) - k(x_i - x_{i+1}), \text{ for } i > 0,$$

which can be expressed as.

$$\ddot{x}_i = -2\omega_0^2 x_i + \omega_0^2 x_{i-1} + \omega_0^2 x_{i+1}, \quad (15)$$

where  $\omega_0 = \sqrt{\frac{k}{m}}$ .

If the particle chain is very long (theoretically infinite), then, the relationship between neighboring particles does not depend on  $i$  index.

We can use phasor annotation for the harmonic solution of the equation:

$$\mathbf{x}_i(t) = \mathbf{X}_i e^{j\omega t}.$$

Subsequently, we express the relationship between complex particle displacements by the constant using complex exponential form.

$$\frac{\mathbf{x}_{i+1}(t)}{\mathbf{x}_i(t)} = e^{-\gamma}, \text{ resp. } \mathbf{x}_{i+1}(t) = \mathbf{X}_i e^{-\gamma} e^{j\omega t}, \quad (16)$$

where  $\gamma = \beta + j\alpha$ .

Substituting to (15) we get.

$$2\omega_0^2 - \omega^2 = \omega_0^2 (e^{\gamma} + e^{-\gamma}) = 2\omega_0^2 \cosh \gamma,$$

and from there.

$$\cosh \gamma = 1 - \frac{\omega^2}{2\omega_0^2}. \quad (17)$$

Let us consider  $\cosh \gamma = \cosh \beta \cos \alpha + j \sin \alpha \sinh \beta$ . As right side of the (17) is real, the imaginary part of  $\cosh \gamma$  is zero. Thus, we get  $\sin \alpha = 0$  or  $\sinh \beta = 0$ .

In the case that  $\sinh \beta = 0$ , then  $\beta = 0$ ,  $\cosh \beta = 1$  and  $\cosh \gamma = \cos \alpha$ :

$$\cos \alpha = 1 - \frac{\omega^2}{2\omega_0^2}, \text{ or } \tan \alpha = \frac{\omega}{\sqrt{2\omega_0^2 - \omega^2}}. \quad (18)$$

From  $-1 \leq \cos \alpha \leq 1$ , we get the corresponding interval of angular frequencies:

$$0 \leq \omega \leq \sqrt{2}\omega_0 = \omega_m, \quad (19)$$

where  $\omega_m = \sqrt{\frac{2k}{m}}$  is the cut-off angular frequency.

From (16), we get  $\mathbf{x}_{i+1}(t) = \mathbf{x}_i(t) e^{-j\alpha}$  and thus, for amplitudes  $x_{i+1} = x_i$ .

All particles oscillate with the same amplitude, but the neighboring particles have different phase shift  $\alpha$  of their oscillations. Thus, there is propagating *the undamped traveling harmonic wave* along the chain.

The phase shift  $\alpha$  corresponds to a time shift of  $\Delta t = \alpha / \omega$ . If the distance between the adjacent particles is  $a$ , the phase velocity is

$$v = \frac{a}{\Delta t} = \frac{a}{\alpha} \quad \omega = \frac{\omega_m a}{\arctan \frac{\omega}{\sqrt{\omega_m^2 - \omega^2}}}. \quad (20)$$

For small angular frequencies  $\omega \ll \omega_m$ , we can use an approximation for  $x \ll 1$ , and thus,  $v_0 \approx a \omega_m = a \sqrt{\frac{2k}{m}} = a \omega_0 \sqrt{2}$ .

The low-frequency waves propagate along the chain having a constant phase velocity of  $v_0$ .

In the case of  $\omega > \omega_m$ , then  $\cosh \gamma < -1$ . As comes from  $\cosh x \geq 1$ , we can fulfill (17), if  $\sin a = 0$  and  $\cos a = -1$ .

The (17) gets then.

$$\cosh \beta = \frac{\omega^2}{2 \omega_0^2} - 1, \text{ and } \alpha = \pi \text{ rad.} \quad (21)$$

The relationship between the displacements of two adjacent particles has the form.

$$x_{i+1} = -x_i e^{-\beta}. \quad (22)$$

The adjacent particles oscillate with opposite phases and the amplitude of the oscillation exponentially decreases along the chain. Thus, the *exponentially damped standing wave* is arising in the chain.

#### 1.2.2.2 Wave energy transfer in chain of particles

To investigate oscillation propagation along the particle chain, it is important to observe the transfer of energy in this bound system. Let us assume that an external source generates oscillations according to (14). We denote these particle oscillations as  $x_0$ .

For subcritical frequencies  $\omega < \omega_k$ , the power of the source is.

$$\begin{aligned} P &= F v_0 \\ &= k (x_0 - x_1) \omega x_0 \cos \omega t = k \omega x_0^2 \cos \omega t [\sin \omega t - \sin (\omega t - a)] \\ &= \omega \frac{k x_{0m}^2}{2} [(1 - \cos a) \sin 2\omega t + \sin a (1 + \cos 2\omega t)] \end{aligned}$$

The power has time-dependent parts and constant parts. The constant part represents a steady energy flow.

$$\langle P \rangle = \omega \frac{k x_{0m}^2}{2} \sin a = \frac{\sqrt{k m}}{2} \sqrt{1 - \left( \frac{\omega}{2 \omega_0} \right)^2} \omega^2 x_{0m}^2. \quad (23)$$

At angular frequencies of  $\omega < \omega_m$ , the chain can transfer energy without any losses.

In the case of supercritical angular frequencies of  $\omega > \omega_k$ , the power is.

$$\begin{aligned} P &= F v_0 \\ &= k (x_0 - x_1) \omega x_0 \cos \omega t = k \omega x_{0m}^2 \cos \omega t \sin \omega t (1 + e^{-b}) \\ &= \omega \frac{k x_{0m}^2}{2} (1 + e^{-b}) \sin 2\omega t. \end{aligned} \quad (24)$$

We can see that the power mean value is  $\langle P \rangle = 0$ . At supercritical frequencies of  $\omega > \omega_m$ , the chain cannot transfer energy, and thus, the power of the source is reactive.

### 1.2.2.3 Heat transfer in substances

A substance consists of many particles (atoms, ions, molecules) interacting with each other. If any of them oscillate, the oscillations spread to the surrounding medium, and energy is transmitted. If the surface particles of the body vibrate coherently (together with the same phase), the wave propagates through the medium as described in paragraph 1.2.2.1. In this way, a coherent wave arises in the medium. However, if the particles vibrate randomly (uncoordinated), there is also energy transfer, but the disturbance propagation process is not a coherent wave. Such transfer of energy from higher energy particles to lower energy particles is called thermal transfer. Since the energy of the particles is related to the temperature at a given location of the substance, we observe the transfer of energy from the higher temperature locations to the lower temperature locations. This phenomenon is called heat conduction in the substance. The interaction of particles in different substances varies, and therefore some substances conduct heat better (thermal conductors), others worse (thermal insulators). The power density is proportional to the temperature gradient.

$$\frac{P}{S} = -\lambda \frac{\partial T}{\partial x}, \quad (25)$$

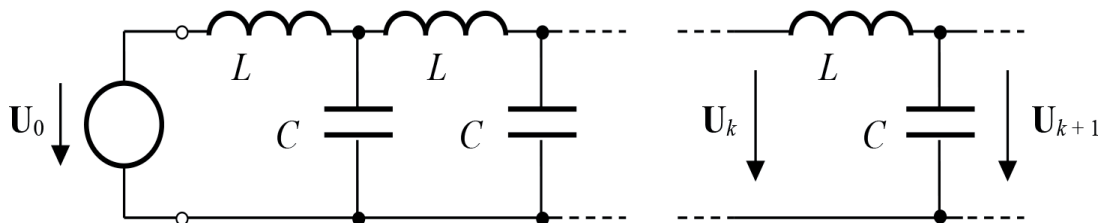
where the  $\lambda$  is the specific thermal conductivity of the substance.

High thermal conductivity is typical for substances consisting of an ordered structure of small particles, especially metals. In contrast, disordered structures and substances consisting of large molecules (glass, plastics, etc.) are mostly thermal insulators. E.g., pure metal is a good conductor of heat, but steel, which contains many different impurities and structure failures, has a significantly lower thermal conductivity. As an example, compare an aluminum spoon and a stainless-steel spoon immersed in a hot tea - the aluminum end is soon hot, but the stainless-steel end remains cool.

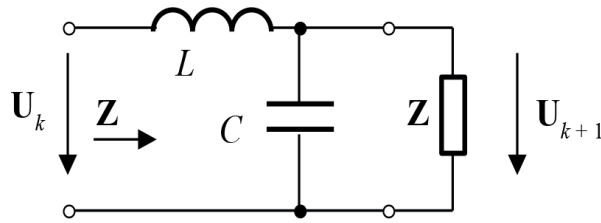
### 1.2.3 Oscillation's propagation in a long LC chain

The electrical system analogous to the long chain of bound particles is a chain consisting of a homogeneous series of LC segments. The harmonic voltage  $u_0 = U_0 \sin \omega t$  is connected to the input, **Figure 5**.

If a chain is very long (theoretically infinite), its properties do not change when we add another segment to it. If the input impedance is  $Z$ , it does not change by adding an LC segment to the beginning, **Figure 6**.



**Figure 5.**  
Long-chain of identical LC segments.



**Figure 6.**  
Equivalent circuit using the total impedance of Z.

$$Z = j\omega L + \frac{Z \frac{1}{j\omega C}}{Z + \frac{1}{j\omega C}}. \quad (26)$$

Then we get  $Z = \frac{j\omega L}{2} + \sqrt{\frac{L}{C} - \left(\frac{\omega L}{2}\right)^2}$ , while  $|Z| = \sqrt{\frac{L}{C}} = R$ .

Complex voltage transfer of the Z loaded segment is then.

$$A_k = \frac{U_{k+1}}{U_k} = \frac{Z}{Z(1 - \omega^2 CL) + j\omega L} = \frac{\sqrt{\frac{L}{C} - \left(\frac{\omega L}{2}\right)^2} - \frac{j\omega L}{2}}{\sqrt{\frac{L}{C} - \left(\frac{\omega L}{2}\right)^2} + \frac{j\omega L}{2}}. \quad (27)$$

If we define a cut-off angular frequency of  $\omega_m$  and characteristic resistance of  $R_0$  as,

$$\omega_m = \frac{2}{\sqrt{LC}}, \text{ and } R_0 = \sqrt{\frac{L}{C}}, \quad (28)$$

then, we can express the impedance of the chain and complex voltage transfer as.

$$Z = R_0 \left( \sqrt{1 - \frac{\omega^2}{\omega_m^2}} + j \frac{\omega}{\omega_m} \right), \text{ and } A_k = \frac{\sqrt{1 - \left(\frac{\omega}{\omega_m}\right)^2} - j \frac{\omega}{\omega_m}}{\sqrt{1 - \left(\frac{\omega}{\omega_m}\right)^2} + j \frac{\omega}{\omega_m}}. \quad (29)$$

If  $\omega < \omega_m$  (low-frequency oscillations), the absolute value of voltage transfer and voltage phase shift on one segment are.

$$|A_k| = 1, \text{ and } \varphi_k = -2 \arctan \frac{\omega}{\sqrt{\omega_m^2 - \omega^2}}. \quad (30)$$

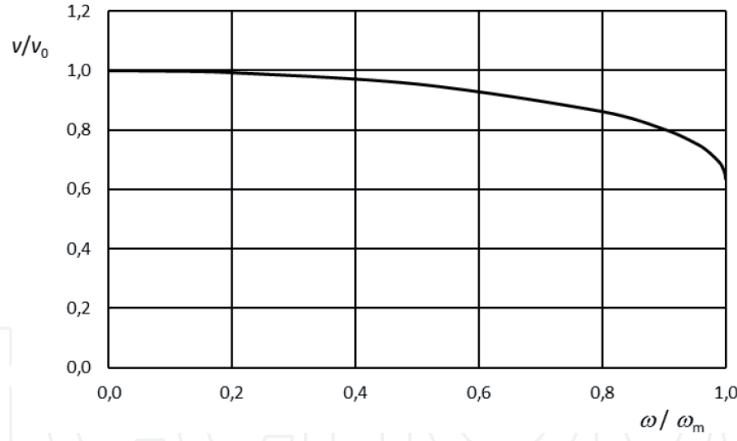
The voltage of the  $n$ -th segment in chain is then.

$$u_n(t) = U_0 \sin(\omega t + n \varphi_k). \quad (31)$$

It follows from this result that the voltage amplitude along the chain remains constant and the phase shift gradually increases:  $\varphi_n = n \varphi_k$ . Assuming harmonic excitation at the input, the harmonic undamped wave propagates along the chain.

If we know the segment length  $d$  and we express the phase difference as  $\varphi_k = \omega \Delta t$ , then, we can determine the velocity of the phase propagation (phase velocity of the wave):

$$v = \frac{d}{\Delta t} = \frac{d}{-\varphi_k / \omega} = \frac{\omega d}{2 \arctan \frac{\omega}{\sqrt{\omega_m^2 - \omega^2}}}. \quad (32)$$



**Figure 7.**

The  $v/v_0$  normalized velocity as a function of normalized angular frequency of  $\omega/\omega_m$ .

For very low frequencies of excitation  $\omega \ll \omega_m$  (taken an approximation  $\arctan x \approx x$  for  $x \ll 1$ ), the velocity is constant and equals

$$v_0 \approx \frac{d}{2} \omega_m = \frac{d}{\sqrt{LC}}. \quad (33)$$

For frequencies  $\omega \rightarrow \omega_m$  the velocity  $v \rightarrow \frac{\omega d}{\pi}$ .

Normalized dependency of velocity  $v$  versus angular frequency  $\omega$  is in **Figure 7**.

Low frequency oscillations having the angular frequency of  $\omega \ll \omega_m$  propagate along the chain as an *undamped harmonic wave* with constant velocity of  $v_0$ , for  $\omega \rightarrow \omega_m$  the velocity decreases to the  $v_m$ .

At  $\omega > \omega_m$  we get.

$$Z = j R_0 \left( \sqrt{\frac{\omega^2}{\omega_m^2} - 1} + \frac{\omega}{\omega_m} \right), \text{ and } A_k = - \frac{\frac{\omega}{\omega_m} - \sqrt{\left(\frac{\omega}{\omega_m}\right)^2 - 1}}{\frac{\omega}{\omega_m} + \sqrt{\left(\frac{\omega}{\omega_m}\right)^2 - 1}} = -b. \quad (34)$$

Absolute value of voltage transfer is  $A_k = b = e^{-\beta} < 1$ , while  $\beta = -\ln b > 0$ .

The impedance has inductive character (reactive character). Voltage transfer is less than one, which means that the voltage along the chain exponentially decreases.

$$u_n(t) = -U_0 e^{-n\beta} \sin \omega t. \quad (35)$$

Since the complex voltage transfer is a negative real number, adjacent segments oscillate with the opposite phase, but the phase constant of the segment does not change. It is, therefore, a *damped standing wave*.

When a harmonic voltage source is connected to the chain, its complex power supplied to the chain is.

$$S = U_0 \cdot I^* = \frac{U_0^2}{Z^*} = \frac{U_0^2}{R} \left( \sqrt{1 - \frac{\omega^2}{\omega_m^2}} - j \frac{\omega}{\omega_m} \right) = P + j Q.$$

For frequencies  $\omega < \omega_m$  the complex power of source consists of the active power part of  $P$  and the reactive part of  $Q$ .

$$P = \frac{U_0^2}{R} \sqrt{1 - \frac{\omega^2}{\omega_m^2}}, \text{ and } Q = \frac{U_0^2}{R} \frac{\omega}{\omega_m}. \quad (36)$$

Thus, the chain transfers the power (permanently draws energy from the source).

For frequencies  $\omega > \omega_m$ .

$$P = 0, \text{ and } Q = \frac{U_0^2}{R} \left( \sqrt{\frac{\omega^2}{\omega_m^2} - 1} - \frac{\omega}{\omega_m} \right). \quad (37)$$

The chain is unable to transmit energy. The power source is only reactive, i.e., the mean value of the energy supplied by the source into the chain is zero.


We can see from these two cases that the oscillating events of mechanical and electrical are analogous, and thus there is electro-mechanical duality. One of the practical possibilities of using this feature is to model mechanical systems using electrical circuits. In the field of biomedicine, we can consider an example of blood flow modeling in blood vessels using electrical circuits [1].

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## **References**

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