Chapter

Relaxation Dynamics of Point Vortices

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Abstract

We study a model describing relaxation dynamics of point vortices, from quasi-stationary state to the stationary state. It takes the form of a mean field equation of Brownian point vortices derived from Chavanis, and is formulated by our previous work as a limit equation of the patch model studied by Robert-Someria. This model is subject to the micro-canonical statistic laws; conservation of energy, that of mass, and increasing of the entropy. We study the existence and nonexistence of the global-in-time solution. It is known that this profile is controlled by a bound of the negative inverse temperature. Here we prove a rigorous result for radially symmetric case. Hence E/M^2 large and small imply the global-in-time and blowup in finite time of the solution, respectively. Where E and M denote the total energy and the total mass, respectively.

Keywords: point vortex, quasi-equilibrium, relaxation dynamics

1. Introduction

Our purpose is to study the system

$$
\begin{aligned}
&\omega_t + \nabla\cdot\omega\nabla^{\perp}\psi = \nabla\cdot(\nabla\omega + \beta\omega\nabla\psi) \quad \text{in} \ \ \Omega\times \ \ (0,T), \\
&\left.\frac{\partial\omega}{\partial\nu} + \beta\omega\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0, \quad \omega|_{t=0} = \omega_0(x)
\end{aligned}
\tag{1}
$$

with

$$
-\Delta\psi = \omega \ \text{ in } \ \Omega, \quad \psi|_{\partial\Omega} = 0, \quad \beta = -\frac{\int_\Omega \nabla\omega\cdot\nabla\psi}{\int_\Omega \omega|\nabla\psi|^2}, \tag{2}
$$

where $\Omega\subset\mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outer unit normal vector on $\partial\Omega$, and

$$
\nabla = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{pmatrix}, \quad \nabla^{\perp} = \begin{pmatrix} \dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_1} \end{pmatrix}, \quad x = (x_1, x_2). \tag{3}
$$

The unknown $\omega = \omega(x,t)\in\mathbf{R}$ stands for a mean field limit of many point vortices,

$$\omega(x,t)dx = \sum_{i=1}^{N} \alpha_i \delta_{x_i(t)}(dx). \tag{4}$$

It was derived, first, for Brownian point vortices by [1, 2], with $\beta = \beta(t)$ standing for the inverse temperature. Then, [3, 4] reached it by the Lynden-Bell theory [5] of relaxation dynamics, that is, as a model describing the movement of the mean field of many point vortices, from quasi-stationary state to the stationary state. This model is consistent to the Onsager theory [6–12] on stationary states and also the patch model proposed by [13, 14], that is,

$$\omega(x,t) = \sum_{i=1}^{N_p} \sigma_i 1_{\Omega_i(t)}(x), \tag{5}$$

where N_p, σ_i, and $\Omega_i(t)$ denote the number of patches, the vorticity of the i-th patch, and the domain of the i-th patch, respectively [15–17].

This chapter is concerned on the one-sided case of

$$\omega_0 = \omega_0(x) > 0. \tag{6}$$

If this initial value is smooth, there is a unique classical solution to (1)–(4) local in time, denoted by $\omega = \omega(x,t)$, with the maximal existence time $T = T_{\max} \in (0, +\infty]$. More precisely, the strong maximum principle to (1) guaranttes

$$\omega = \omega(x,t) > 0 \quad \text{on } \overline{\Omega} \times \quad [0,T). \tag{7}$$

Then, the Hopf lemma to the Poisson equation in (2) ensures

$$\left.\frac{\partial \psi}{\partial \nu}\right|_{\partial\Omega} < 0, \tag{8}$$

and hence the well-definedness of

$$-\beta = \frac{\int_\Omega \nabla\omega \cdot \nabla\psi}{\int_\Omega \omega |\nabla\psi|^2}. \tag{9}$$

We confirm that system (1)–(3) satisfies the requirements of isolated system of thermodynamics. First, the mass conservation is derived from (1) as

$$\frac{d}{dt}\int_\Omega \omega = 0, \tag{10}$$

because

$$\nu \cdot \nabla^\perp \psi\big|_{\partial\Omega} = 0 \tag{11}$$

holds by (2). Second, the energy conservation follows as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_2^2 &= (\nabla\psi, \nabla\psi_t) = (\omega_t, \psi) \\ &= \left(\omega\nabla^\perp\psi, \nabla\psi\right) - (\nabla\omega + \beta\omega\nabla\psi, \nabla\psi) \\ &= -(\nabla\omega, \nabla\psi) - \beta\int_\Omega \omega|\nabla\psi|^2 = 0 \end{aligned} \tag{12}$$

by (1) and (2), because

$$\nabla^{\perp}\psi \cdot \nabla\psi = 0, \tag{13}$$

where (,) denotes the L^2 inner product. Third, the entropy increasing is achieved, writing (1) as

$$\omega_t = \nabla \cdot \omega\left(-\nabla^{\perp}\psi + \nabla(\log\omega + \beta\psi)\right), \quad \frac{\partial}{\partial\nu}(\log\omega + \beta\psi)\Big|_{\partial\Omega} = 0. \tag{14}$$

In fact, it then follows that

$$\int_{\Omega} \omega_t(\log\omega + \beta\psi) = \int_{\Omega} \omega\nabla^{\perp}\psi \cdot \nabla(\log\psi + \beta\psi) - \omega|\nabla(\log\omega + \beta\psi)|^2 \ dx \tag{15}$$

with

$$\begin{aligned} \int_{\Omega} \omega\nabla^{\perp}\psi \cdot \nabla(\log\omega + \beta\psi) &= \int_{\Omega} \nabla\omega \cdot \nabla^{\perp}\psi \\ &= \int_{\partial\Omega} \omega\nu \cdot \nabla^{\perp}\psi - \int_{\Omega} \omega\nabla \cdot \left(\nabla^{\perp}\psi\right) = 0 \end{aligned} \tag{16}$$

from (11) and

$$\nabla^{\perp} \cdot \nabla = \nabla \cdot \nabla^{\perp} = 0. \tag{17}$$

Since

$$\int_{\Omega} \omega_t \log\omega = \frac{d}{dt}\int_{\Omega} \omega(\log\omega - 1), \quad \int_{\Omega} \omega_t\psi = \frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_2^2 = 0, \tag{18}$$

We thus end up with the mass conservation

$$M = \int_{\Omega} \omega, \tag{19}$$

the energy conservation

$$E = \|\nabla\psi\|_2^2 = (\psi, \omega), \tag{20}$$

and the entropy increasing

$$\frac{d}{dt}\int_{\Omega} \omega(\log\omega - 1) = -\int_{\Omega} \omega|\nabla(\log\omega + \beta\psi)|^2 \le 0. \tag{21}$$

Henceforth, $C > 0$ stands for a generic constant. In the previous work [4] we studied radially symmetric solutions and obtained a criterion for the existence of the solution global in time. Here, we refine the result as follows, where $B(0,1)$ denotes the unit ball.

Theorem 1 *Let*

$$\Omega = B(0,1), \quad \omega_0 = \omega_0(r), \ \ \omega_{0r} < 0, \quad 0 < r = |x| \le 1. \tag{22}$$

Then there is $C_0 > 0$ *such that*

$$C_0 \|\omega_0\|_2^3 \leq E\underline{\omega} \Rightarrow T = +\infty, \ \ \|\omega(\cdot, t)\|_\infty \leq C, \ \ t \geq 0, \tag{23}$$

where

$$\underline{\omega} = \min_{\overline{\Omega}} \omega_0 > 0. \tag{24}$$

Theorem 2 *Under the assumption of* (22) *there is* $\delta_0 > 0$ *such that*

$$\frac{E}{M^2} < \delta_0 \Rightarrow T < +\infty. \tag{25}$$

Remark 1 *Since*

$$\begin{aligned} \|\omega_0\|_2^3 &= \left(\int_\Omega \omega_0^2\right)^{3/2} \geq \left(\underline{\omega}^{2/3} \int_\Omega \omega_0^{4/3}\right)^{3/2} \\ &= \underline{\omega}\left(\int_\Omega \omega_0^{4/3}\right)^{3/2} \geq \underline{\omega}|\Omega|^{-1/2}\left(\int_\Omega \omega_0\right)^2 = \underline{\omega}|\Omega|^{-1/2}M^2 \end{aligned} \tag{26}$$

the assumption (23) implies

$$\frac{E}{M^2} \geq C_0 |\Omega|^{-1/2}. \tag{27}$$

Therefore, roughly, the conditions $E/M^2 \gg 1$ *and* $E/M^2 \ll 1$ *imply* $T = +\infty$ *and* $T < +\infty$*, respectively.*

Remark 2 *The assumption* (22) *implies*

$$\beta = \beta(t) < 0, \quad 0 \leq t < T, \tag{28}$$

and then we obtain Theorem 1. In other words, the conclusion of this theorem arises from (28)*, without* (22).

Remark 3 *Since*

$$\frac{E}{M^2} = \frac{\int_\Omega |\nabla\psi|^2}{\left(\int_\Omega \omega\right)^2} \tag{29}$$

it holds that

$$\frac{E}{M^2} = \|\nabla c\|_2^2, \quad c = \frac{(-\Delta)^{-1}\omega_0}{\int_\Omega \omega_0} = \frac{\psi_0}{\int_{\partial\Omega} -\frac{\partial\psi_0}{\partial\nu}}, \tag{30}$$

where $\psi_0 = (-\Delta)^{-1}\omega_0$.

The system (1)–(4) thus obays a profile of the micro-canonical ensemble. In a system associated with the canonical ensemble, the inverse temperature β is a constant in (1) independent of t, with the third equality in (2) elimiated:

$$\begin{aligned} &\omega_t + \nabla \cdot \omega\nabla^\perp\psi = \nabla \cdot (\nabla\omega + \beta\omega\nabla\psi), \quad \frac{\partial\omega}{\partial\nu} + \beta\omega\frac{\partial\psi}{\partial\nu}\bigg|_{\partial\Omega} = 0, \quad \omega|_{t=0} = \omega_0(x) > 0 \\ &-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0. \end{aligned} \tag{31}$$

Then there arise the mass conservation

$$\frac{d}{dt}\int_\Omega \omega = 0, \tag{32}$$

and the free energy decreasing

$$\frac{d}{dt}\int_\Omega \omega(\log\omega - 1) + \frac{\beta}{2}|\nabla\psi|^2 \ dx = -\int_\Omega \omega|\nabla(\log\omega + \beta\psi)|^2 \leq 0. \tag{33}$$

The system (31) without vortex term,

$$\begin{aligned} &\omega_t = \nabla\cdot(\nabla\omega + \beta\omega\nabla\psi), \quad \left.\frac{\partial\omega}{\partial\nu} + \beta\omega\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0, \quad \omega|_{t=0} = \omega_0(x) > 0 \\ &-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0. \end{aligned} \tag{34}$$

is called the Smoluchowski-Poisson equation. This model is concerned on the thermodynamics of self-gravitating Brownian particles [18] and has been studied in the context of chemotaxis [19–23]. We have a blowup threshold to (34) as a consequence of the quantized blowup mechanism [19, 23]. The results on the existence of the bounded global-in-time solution [24–26] and blowup of the solution in finite time [27] are valid even to the case that β is a function of t as in $\beta = \beta(t)$. provided with the vortex term $\nabla\cdot\omega\nabla^\perp\psi$ on the right-hand side. We thus obtain the following theorems.

Theorem 3 *It holds that*

$$-\beta(t) \leq \delta, \ \ \|\omega_0\|_1 < 8\pi\delta^{-1} \Rightarrow T = +\infty, \ \ \|\omega(\cdot, t)\|_\infty \leq C \tag{35}$$

in (31), *where* $\delta > 0$ *is arbitrary*.

Theorem 4 *It holds that*

$$-\beta(t) \geq \delta, \ \ \|\omega_0\|_1 > 8\pi\delta^{-1} \Rightarrow \exists\omega_0 > 0, \ \ \|\omega_0\|_1 > 8\pi\delta^{-1} \quad \textit{such that} \ \ T < +\infty \tag{36}$$

in (31), *where* $\delta > 0$ *is arbitrary*.

Remark 4 *In the context of chemotaxis in biology, the boundary condition of* ψ *is required to be the form of Neumann zero. The Poisson equation in* (34) *is thus replaced by*

$$-\Delta\psi = \omega - \frac{1}{|\Omega|}\int_\Omega \omega, \quad \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0 \tag{37}$$

or

$$-\Delta\psi + \psi = \omega, \quad \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0 \tag{38}$$

by [28] *and* [29], *respectively. In this case there arises the boundary blowup, which reduces the value* 8π *in Theorems 3–4 to* 4π. *The value* 8π *in Theorems 3–4, therefore, is a consequence of the exclusion of the boundary blowup* [30]. *This property is valid even for* (37) *or* (38) *of the Poisson part, if* (22) *is assumed*.

Remark 5 *The requirement to* ω_0 *in Theorem 4 is the concentration at an interior point, which is not necessary in the case of* (22). *Hence Theorems 3 and 4 are refined as*

$$-\beta(t) \leq \delta, \ \ \|\omega_0\|_1 < 8\pi\delta^{-1} \Rightarrow T = +\infty, \ \ \|\omega(\cdot, t)\|_\infty \leq C \tag{39}$$

and

$$-\beta(t) \geq \delta, \quad \|\omega_0\|_1 > 8\pi\delta^{-1} \Rightarrow T < +\infty, \tag{40}$$

if (22) holds in (35). The main task for the proof of Theorems 1 and 2, therefore, is a control of $\beta = \beta(t)$ in (1).

This paper is composed of four sections and an appendix. Section 2 is devoted to the study on the stationary solutions, and Theorems 1 and 2 are proven in Sections 3 and 4, respectively. Then Theorem 4 is confirmed in Appendix.

2. Stationary states

First, we take the canonical system (31) with β independent of t. By (32) and (33), its stationary state is defined by

$$\log\omega + \beta\psi = \text{constant}, \quad \omega = \omega(x) > 0, \quad \int_\Omega \omega = M. \tag{41}$$

Then it holds that

$$\omega = \frac{Me^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} \tag{42}$$

and hence

$$-\Delta\psi = \frac{Me^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \quad \psi|_{\partial\Omega} = 0. \tag{43}$$

There arises an oredered structure arises in $\beta < 0$, as observed by [11], as a consequence of a quantized blowup mechanism [19, 20, 31]. In the micro-canonical system (1) and (2), the value β in (43) has to be determined by E besides M.

Equality (21), however, still ensures (41) and hence (42) in the stationary state even for (1)–(3). Writing

$$v = -\beta\psi, \quad \mu = \frac{-\beta M}{\int_\Omega e^{-\beta\psi}}, \tag{44}$$

we obtain

$$-\Delta v = \mu e^v \ \text{ in } \ \Omega, \quad v|_{\partial\Omega} = 0, \quad \frac{E}{M^2} = \frac{\|\nabla v\|_2^2}{\left(\int_\Omega -\frac{\partial v}{\partial \nu}\right)^2} \tag{45}$$

by (30) and (43).

This system is the stationary state of (1) and (2) introduced by [4]. The first two equalities

$$-\Delta v = \mu e^v, \quad v|_{\partial\Omega} = 0 \tag{46}$$

comprise a nonlinear elliptic eigenvalue problem and the unknown eigenvalue μ is determined by the third equality,

$$\frac{E}{M^2} = \frac{\|\nabla v\|_2^2}{\left(\int_\Omega -\frac{\partial v}{\partial \nu}\right)^2}. \tag{47}$$

The elliptic theory ensures rather deailed features of the set of solutions to (46). Here we note the following facts [31].

1. There is $\overline{\mu} = \mu(\Omega) > 0$ such that the problem (46) does not admit a solution for $\mu > \overline{\mu}$.

2. Each $\mu \leq 0$ admits a unique solution.

3. Each $0 < \delta < \overline{\mu}$ admits a constant $C = C(\delta) > 0$ such that $\|v\|_\infty \leq C$ for any solution $v = v(x)$.

4. There is a family of solutions $\{(\mu, v)\}$ such that $\mu \downarrow 0$ and $\|v\|_\infty \to +\infty$.

We show the following theorem, consistent to Theorem 2.

Theorem 5 *If* $\Omega = B(0,1) \subset \mathbf{R}^2$*, there is* $\delta > 0$ *such that any solution* (v, μ) *to* (45) *admits*

$$\frac{E}{M^2} \geq \delta. \tag{48}$$

Proof: If $\mu = 0$, it holds that $v = 0$. We have $\pm v > 0$ exclusively in Ω, provided that $\pm\mu > 0$, respectively. By the elliptic theory [32], therefore, any solution v to (46) is radially symmetric as in $v = v(r)$, $r = |x|$. We have, furthermore, $\pm v_r < 0$ in $0 < r \leq 1$, if $\pm\mu > 0$, respectively.

Then it holds that $\psi = \psi(r)$, and hence

$$-\frac{1}{r}(r\psi_r)_r = \omega \text{ in } 0 < r \leq 1, \quad \psi|_{r=1} = 0 \tag{49}$$

by (42) and (43), which implies

$$-r\psi_r(r) = \int_0^r s\omega(s)ds > 0, \quad 0 < r \leq 1. \tag{50}$$

We thus obtain $\mu \neq 0$, in particular.

If $\mu < 0$ we have $\beta > 0$ by (44), and therefore, $\psi_r > 0$ in $0 < r \leq 1$ by $v_r > 0$ there. It is a contradiction, and hence $\mu > 0$. In this case, the solution $v = v(r)$ to (46) is explicit [31]. The numbers of the solution is 0, 1, and 2, according to $\mu > 2$, $\mu = 2$, and $0 < \mu < 2$, respectively, and if $0 < \mu \leq 2$ the solutions $v = v_\pm$ are given as

$$v_\pm(r) = \log \frac{8\gamma_\pm r}{(1 + \gamma_\pm r^2)^2}, \quad \gamma_\pm = \frac{4}{\mu}\left\{1 - \frac{\mu}{4} \pm \left(1 - \frac{\mu}{2}\right)^{1/2}\right\}. \tag{51}$$

In fact, we have $\gamma_+ = \gamma_-$ for $\mu = 2$.

This solution is parametrized by

$$\sigma = \int_\Omega \mu e^v \in (0, 8\pi). \tag{52}$$

Hence each $0 < \sigma < 8\pi$ admits a unique solution (v, μ) to (46), and $v = v_+$ and $v = v_-$ according as $\sigma \geq 4\pi$ and $\sigma \leq 4\pi$, respectively. It holds also that $\mu \downarrow 0$ if either

$\sigma \uparrow 8\pi$ or $\sigma \downarrow 0$. Thus we have only to confirm that E/M^2 is bounded, both as $\sigma \uparrow 8\pi$ and $\sigma \downarrow 0$.

As $\sigma \uparrow 8\pi$, we have

$$v = v_+(x) \quad \to \quad 4 \log \frac{1}{|x|} \quad \text{locally uniformly on } \overline{\Omega} \setminus \{0\} \tag{53}$$

and hence

$$\|\nabla v\|_2^2 \quad \to \quad +\infty, \qquad \int_{\partial\Omega} -\frac{\partial v}{\partial \nu} \quad \to \quad 8\pi, \tag{54}$$

which implies

$$\lim_{\sigma \uparrow 8\pi} \frac{E}{M^2} = +\infty. \tag{55}$$

As $\sigma \downarrow 0$, on the other hand, we have

$$v = v_-(x) \quad \to \quad 0 \quad \text{uniformly in } \overline{\Omega}. \tag{56}$$

Since $\mu \downarrow 0$, furthermore, there arises that

$$\gamma = \gamma_- = \frac{4}{\mu} \left\{ 1 - \frac{\mu}{4} - \left(1 - \frac{\mu}{2} \right)^{1/2} \right\} = \mu(1 + o(1)). \tag{57}$$

It holds also that

$$v(r) = \log \frac{8\gamma}{\mu} - 2 \log \left(1 + \mu r^2 \right) \tag{58}$$

and hence

$$v_r(r) = -\frac{4\mu r}{(1 + \mu r^2)^2} = -4\mu r(1 + o(1)) \quad \text{uniformly on } \overline{\Omega}. \tag{59}$$

Then, (59) implies

$$\begin{aligned} \|\nabla v\|_2^2 &= 2\pi \int_0^1 v_r^2 r \; dr = 2\pi \cdot 16\mu^2 \cdot \int_0^1 r^3 \; dr \cdot (1 + o(1)) \\ &= 8\pi\mu^2(1 + o(1)) \end{aligned} \tag{60}$$

as well as

$$\left(\int_{\partial\Omega} -\frac{\partial v}{\partial \nu} \right)^2 = 16\mu^2 \cdot 2\pi(1 + o(1)). \tag{61}$$

It thus follows that

$$\lim_{\sigma \downarrow 0} \frac{E}{M^2} = \frac{1}{4} \tag{62}$$

and hence the conclusion. □

3. Proof of Theorem 1

The first observation is the following lemma.

Lemma 1 *Under the assumption of* (22)*, it holds that*

$$\beta = \beta(t) < 0, \quad \omega_r(r,t) < 0, \quad 0 < r \leq 1, \ \ 0 \leq t < T. \tag{63}$$

Proof: We have (7) and hence

$$\psi_r(r,t) < 0, \quad 0 < r \leq 1, \ \ 0 \leq t < T \tag{64}$$

by (49), which implies, in particular,

$$\beta = -\frac{(\nabla\omega, \nabla\psi)}{\int_\Omega \omega|\nabla\psi|^2} < 0 \tag{65}$$

at $t = 0$ by (22).

Since $\omega = \omega(r,t)$ and $\psi = \psi(r,t)$, we obtain $\nabla^\perp\psi = 0$, and hence

$$\omega_t = \omega_{rr} + \frac{1}{r}\omega_r + \beta\psi_r\omega_r - \beta\omega^2 \tag{66}$$

by (1). Then $z = \omega_r$ satisfies

$$\begin{aligned} & z_t = z_{rr} - \frac{1}{r^2}z + \frac{1}{r}z_r + \beta\psi_{rr}z + \beta\psi_r z_r - 2\beta\omega z, \quad 0 < r \leq 1, \ \ 0 \leq t < T \\ & z|_{r=0} = 0, \qquad z|_{t=0} = \omega_{0r}(r) < 0, \quad 0 < r \leq 1 \end{aligned} \tag{67}$$

and

$$z = -\beta\omega\psi_r, \quad r = 1, \ \ 0 \leq t < T. \tag{68}$$

Putting

$$m(t) = \min_{\partial\Omega} z(\cdot,t) = \omega_r(\cdot,t)|_{r=1}, \tag{69}$$

we obtain $m(0) < 0$ from the assumption. If there is $0 < t_0 <$ such that

$$m(t) < 0, \ \ 0 \leq t < t_0 < T, \quad m(t_0) = 0, \tag{70}$$

we obtain $z(r,t) > 0$ for $0 \leq t < t_0$, $0 < r \leq 1$, and $t = t_0$, $0 < r < 1$ by the strong maximum principle. By (64), we have (65) for $0 \leq t \leq t_0$, that is,

$$\beta = -\frac{\int_0^1 \psi_r z r \ dr}{\int_0^1 \omega\psi_r^2 r \ dr} < 0, \quad 0 \leq t \leq t_0, \tag{71}$$

and hence

$$z = -\beta\omega\psi_r < 0 \quad r = 1, \ \ t = t_0, \tag{72}$$

a contradiction. It holds that $z = \omega_r < 0$ for $0 \leq t < T$, $r = 1$, and hence

$$\beta = -\frac{\int_0^1 \psi_r \omega_r r dr}{\int_0^1 \omega\psi_r^2 r \ dr} < 0, \quad 0 \leq t < T. \qquad \square \tag{73}$$

The proof of Theorem 3 relies on the fact

$$\beta \geq -C, \ \int_\Omega \omega(\log \omega - 1) \leq C \Rightarrow T = +\infty, \ \ \|\omega(\cdot, t)\|_\infty \leq C. \tag{74}$$

This property is known for the Smoluchoski-Poisson equation (34), but the proof is valid even to (31) with vortex term. Having (21), therefore, we have to provide the inequality $\beta \geq -C$.

The inequality $\beta < 0$, on the other hand, is sufficient for the following arguments.

Lemma 2 *If $\beta \leq 0, \ 0 \leq t < T$, it holds that*

$$\omega \geq \underline{\omega} \equiv \min_{\overline{\Omega}} \omega_0 > 0 \quad \text{on } \overline{\Omega} \times [0, T). \tag{75}$$

Proof: Since (17) we obtain

$$\begin{aligned} \omega_t + \nabla^\perp \psi \cdot \nabla \omega &= \Delta \omega + \beta \nabla \psi \cdot \nabla \omega + \beta \Delta \psi \\ &= \Delta \omega + \beta \nabla \psi \cdot \nabla \omega - \beta \omega^2 \\ &\geq \Delta \omega + \beta \nabla \psi \cdot \nabla \omega \quad \text{in } \ \Omega \times (0, T) \end{aligned} \tag{76}$$

with

$$-\frac{\partial \omega}{\partial \nu} = \beta \omega \frac{\partial \psi}{\partial \nu} > 0 \quad \text{on } \partial\Omega \times [0, T) \tag{77}$$

by (8). Then the result follows from the comparison theorem. □

Lemma 3 *Under the assumption of the previous lemma, there is $C_0 = C_0(\Omega) > 0$ such that*

$$C_0 \|\omega_0\|_2^3 \leq E\underline{\omega} \Rightarrow \|\omega(\cdot, t)\|_2 \leq \|\omega_0\|_2, \quad -\beta(t) \leq \alpha \equiv \frac{\|\omega_0\|_2^2}{E\underline{\omega}}, \quad 0 \leq t < T. \tag{78}$$

Proof: Using (11) and (17), we obtain

$$\begin{aligned} \int_\Omega \left[\nabla \cdot \left(\omega \nabla^\perp \psi\right)\right] \omega &= \int_\Omega \omega \nabla \omega \cdot \nabla^\perp \psi = \frac{1}{2} \int_\Omega \nabla \omega^2 \cdot \nabla^\perp \psi \\ &= -\frac{1}{2} \int_\Omega \omega^2 \nabla \cdot \nabla^\perp \psi = 0. \end{aligned} \tag{79}$$

Hence (1) with (2) implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 &= -\beta \int_\Omega \omega \nabla \psi \cdot \nabla \omega = -\frac{\beta}{2} \left(\nabla \psi, \nabla \omega^2\right) \\ &= -\frac{\beta}{2} \int_{\partial\Omega} \omega^2 \frac{\partial \psi}{\partial \nu} + \frac{\beta}{2} \left(\Delta \psi, \omega^2\right) \leq -\frac{\beta}{2} \|\omega\|_3^3 \end{aligned} \tag{80}$$

by $\beta \leq 0$ and (88). Since

$$\int_\Omega \nabla \omega \cdot \nabla \psi = \int_{\partial\Omega} \omega \frac{\partial \psi}{\partial \nu} + \int_\Omega \omega(-\Delta \psi) \leq \int_\Omega \omega^2 \tag{81}$$

follows from (8), furthermore, it holds that

$$-\beta = \frac{\int_\Omega \nabla\omega \cdot \nabla\psi}{\int_\Omega \omega|\nabla\psi|^2} \le \underline{\omega}^{-1} \cdot \frac{\|\omega\|_2^2}{\|\nabla\psi\|_2^2} = \frac{1}{E\underline{\omega}}\|\omega\|_2^2. \tag{82}$$

Then ineqality (80) induces

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \le \frac{1}{2E\underline{\omega}}\|\omega\|_2^2 \cdot \|\omega\|_3^3. \tag{83}$$

Here we use the Gagliardo-Nirenberg inequality (see (4.16) of [19]) in the form of

$$\|\omega\|_3^3 \le C\|\omega\|_{H^1} \cdot \|\omega\|_2^2 = C\|\omega\|_2^2(\|\nabla\omega\|_2 + \|\omega\|_2), \tag{84}$$

to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 &\le \frac{C}{E\underline{\omega}}\|\omega\|_2^4(\|\nabla\omega\|_2 + \|\omega\|_2) \\ &\le \frac{1}{2}\|\nabla\omega\|_2^2 + \frac{C^2}{8(E\underline{\omega})^2}\|\omega\|_2^8 + \frac{C}{2E\underline{\omega}}\|\omega\|_2^5 \end{aligned} \tag{85}$$

and hence

$$\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \le \frac{C}{E\underline{\omega}}\|\omega\|_2^5\left(\frac{C}{E\underline{\omega}}\|\omega\|_2^3 + 1\right). \tag{86}$$

Then, Poincaré-Wirtinger's inequality ensures

$$\frac{d}{dt}\|\omega\|_2^2 + \mu\|\omega\|_2^2 \le \frac{C}{E\underline{\omega}}\left(\frac{C}{E\underline{\omega}}\|\omega\|_2^6 + \|\omega\|_2^3\right)\|\omega\|_2^2, \tag{87}$$

where $\mu = \mu(\Omega) > 0$ is a constant.
Writing

$$y(t) = \frac{C}{E\underline{\omega}}\|\omega\|_2^3, \tag{88}$$

we obtain

$$\frac{d}{dt}\|\omega\|_2^2 + \mu\|\omega\|_2^2 \le (y^2 + y)\|\omega\|_2^2, \tag{89}$$

and therefore, if

$$y^2 + y < \mu/2 \tag{90}$$

holds at $t = 0$, it keeps to hold that

$$\frac{d}{dt}\|\omega\|_2^2 \le 0 \tag{91}$$

and (90) for $0 \le t < T$. Then, we obtain

$$\|\omega(\cdot,t)\|_2 \leq \|\omega_0\|_2, \quad 0 \leq t < T, \tag{92}$$

and hence

$$-\beta(t) \leq \frac{\|\omega_0\|_2^2}{E\underline{\omega}} = \alpha, \quad 0 \leq t < T \tag{93}$$

by (82).

The condition $y(0) < \frac{\mu}{2}$ means

$$C_0\|\omega_0\|_2 \leq E\underline{\omega} \tag{94}$$

for $C_0 > 0$ sufficiently large, and hence we obtain the conclusion. □

Proof of Theorem 1: By the parabolic regularity, it suffices to show that

$$\|\omega(\cdot,t)\|_\infty \leq C, \quad 0 \leq t < T \tag{95}$$

under the assumption. We have readily shown

$$\|\omega(\cdot,t)\|_2 \leq C, \ \ 0 \leq -\beta(t) \leq C, \quad 0 \leq t < T \tag{96}$$

by Lemma 3. Then, the conclusion (95) is obtained similarly to (34). See [26] for more details.

In fact, we have

$$\begin{aligned}\int_\Omega \left[\nabla\cdot\left(\omega\nabla^\perp\psi\right)\right]\omega^p &= -\int_\Omega \omega\nabla^\perp\psi\cdot\nabla\omega^p = -p\int_\Omega \omega^p\nabla^\perp\psi\cdot\nabla\omega \\ &= -\frac{p}{p+1}\int_\Omega \nabla^\perp\psi\cdot\nabla\omega^{p+1} = \frac{p}{p+1}\int_\Omega \omega^{p+1}\nabla\cdot\left(\nabla^\perp\psi\right) = 0\end{aligned} \tag{97}$$

for $p > 0$ by (11) and (34). Then it follows that

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega \omega^{p+1} + \frac{4p}{(p+1)^2}\|\nabla\omega^{\frac{p+1}{2}}\|_2^2 &= -\beta\int_\Omega \omega\nabla\psi\cdot\nabla\omega^p \\ &= -\beta\cdot\frac{p}{p+1}\int_\Omega \nabla\psi\cdot\nabla\omega^{p+1} \leq -\beta\frac{p}{p+1}\int_\Omega \omega^{p+1}(-\Delta\psi) \\ &= -\beta\frac{p}{p+1}\int_\Omega \omega^{p+1} \leq C\int_\Omega \omega^{p+2}\end{aligned} \tag{98}$$

by $\beta < 0$ and (8). Then, Moser's iteration scheme ensures (95) as in [33].

4. Proof of Theorem 2

We begin with the following lemma.

Lemma 4 *Under the assumption of* (22), *it holds that*

$$-\beta(t) \geq \delta, \ \ 0 \leq t < T, \ \ M = \|\omega_0\|_1 > \frac{8\pi}{\delta} \Rightarrow T < +\infty \tag{99}$$

in (31), *where* $\delta > 0$ *is a constant.*

Proof: We have $\omega = \omega(r,t)$ and $\psi = \psi(r,t)$ for $r = |x|$ under the assumption, which implies $\nabla^\perp \psi = 0$. Then we obtain

$$\nabla \cdot \omega \nabla^\perp \psi = \nabla \omega \cdot \nabla^\perp \psi = 0 \tag{100}$$

by (17). It holds also that

$$\begin{aligned} \nabla \cdot (\omega \nabla \psi) &= \nabla \cdot \left(\omega \psi_r \frac{x}{r} \right) = \left(\nabla \cdot \frac{x}{r} \right) \omega \psi_r + \frac{x}{r} \cdot \nabla(\omega \psi_r) \\ &= \frac{1}{r} \omega \psi_r + (\omega \psi_r)_r = \frac{1}{r} (r \omega \psi_r)_r, \end{aligned} \tag{101}$$

and therefore, there arises that

$$\omega_t = \frac{1}{r} (r\omega_r + \beta r \omega \psi_r)_r, \quad \omega_r + \beta \omega \psi_r|_{r=1} = 0. \tag{102}$$

from (31).
Then (102) implies

$$\begin{aligned} \frac{d}{dt} \int_0^1 \omega r^3 \ dr &= \int_0^1 \omega_t r^3 \ dr = \int_0^1 (r\omega_r + \beta r \omega \psi_r)_r r^2 \ dr \\ &= -\int_0^1 2r^2 (\omega_r + \beta \omega \psi_r) \ dr \\ &= -2r^2 \omega \big|_{r=0}^{r=1} + \int_0^1 4r\omega - 2\beta \omega \psi_r r^2 \ dr. \end{aligned} \tag{103}$$

Here we use (50) derived from the Poisson part of (31), that is,

$$-r\psi_r(r,t) = A(r,t) \equiv \int_0^r s\omega(s,t) ds. \tag{104}$$

Putting

$$\lambda = \int_0^1 \omega r \ dr = \frac{M}{2\pi}, \tag{105}$$

we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^1 \omega r^3 \ dr &= -2\omega|_{r=1} + 4\lambda + 2\beta \int_0^1 A A_r \ dr \\ &= -2\omega|_{r=1} + 4\lambda + \beta A^2 \big|_{r=0}^{r=1} \\ &= -2\omega|_{r=1} + 4\lambda + \beta \lambda^2 \\ &< 4\lambda \left(\beta + \frac{M}{8\pi} \right) \le 4\lambda \left(-\delta + \frac{M}{8\pi} \right). \end{aligned} \tag{106}$$

Since $-\delta + \frac{M}{8\pi} < 0$, therefore, $T = +\infty$ is impossible, and we obtain $T < +\infty$. □
Lemma 5 *Under the assumption* (22), *there is* $\delta > 0$ *such that*

$$\frac{E}{M^2} < \delta, \quad \beta(t) \le 0, \ \ 0 \le t < T \Rightarrow \beta(t) \le -\frac{1}{CE^{1/2}} \ \ 0 \le t < T. \tag{107}$$

Proof: First, Lemma 1 implies

$$\omega \geq \omega_* \equiv \omega|_{r=1}. \tag{108}$$

Second, we have

$$\begin{aligned}\int_\Omega \nabla\psi \cdot \nabla\omega &= \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\omega + \int_\Omega (-\Delta\psi)\omega = \omega_* \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} + \|\omega\|_2^2 \\ &= \omega_* \int_\Omega \Delta\psi + \|\omega\|_2^2 = \|\omega\|_2^2 - \omega_* M,\end{aligned} \tag{109}$$

and hence

$$-\beta = \frac{\int_\Omega \nabla\psi \cdot \nabla\omega}{\int_\Omega \omega|\nabla\psi|^2} = \frac{\|\omega\|_2^2 - \omega_* M}{\int_\Omega \omega|\nabla\psi|^2}. \tag{110}$$

Here, we use the Gagliardo-Nirenberg inequality in the form of

$$\|w\|_4^2 \leq C\|w\|_2\|w\|_{H^1}, \tag{111}$$

which implies

$$\begin{aligned}\int_\Omega \omega|\nabla\psi|^2 &\leq \|\omega\|_2\|\nabla\psi\|_4^2 \leq C\|\omega\|_2\|\nabla\psi\|_2\|\nabla\psi\|_{H^1} \\ &\leq CE^{1/2}\|\omega\|_2^2\end{aligned} \tag{112}$$

by the elliptic estimate of the Poisson equation in (2),

$$\|\psi\|_{H^2} \leq C\|\omega\|_2. \tag{113}$$

We have, on the other hand,

$$\omega_* M \leq \frac{M}{E}\int_\Omega \omega|\nabla\psi|^2 \tag{114}$$

by (110), and therefore,

$$-\beta \geq \frac{1}{CE^{1/2}} - \frac{E}{M} \geq \frac{1}{2CE^{1/2}}, \tag{115}$$

provided that

$$\frac{E}{M^2} < \left(\frac{1}{2C}\right)^2. \tag{116}$$

Then the conclusion follows. □

Proof of Theorem 2: By Lemma 5, there is $\delta_0 >$ such that

$$\frac{E}{M^2} < \delta \Rightarrow -\beta \geq \frac{1}{CE^{1/2}} \equiv \delta_1, \tag{117}$$

and then, Lemma 4 ensures

$$M > \frac{8\pi}{\delta_1} \Rightarrow T < +\infty. \tag{118}$$

The assumption in (118) means

$$\frac{E}{M^2} < \left(\frac{1}{8\pi c}\right)^2, \tag{119}$$

and hence we obtain the conclusion. □

Appendix Proof of Theorem 4

This theorem is valid to the general case of Ω and ω_0 without (22). We assume $\delta = 1$ without loss of generation, so that

$$\beta \leq -1. \tag{120}$$

We follow the argument [27] concerning (34) with the Poisson part replaced by (42) or (43). Thus we have to take case of the vortex term $\nabla \cdot \omega \nabla^{\perp} \psi$, time varying $\beta = \beta(t)$, and the Dirichlet boundary condition in (31).

We recall the cut-off function used in [34] (see also Chapter 5 of [19]). Hence each $x_0 \in \overline{\Omega}$ and $0 < R \leq 1$ admit $\varphi = \varphi_{x_0,R} \in C^2(\overline{\Omega})$ with

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0, \quad 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ in } \Omega \cap B(x_0, R/2), \quad \varphi = 0 \text{ in } \Omega \setminus B(x_0, R), \tag{121}$$

and

$$|\nabla \varphi| \leq C R^{-1} \varphi^{1/2}, \quad |\nabla^2 \varphi| \leq C R^{-2} \varphi^{1/2}. \tag{122}$$

In more details, we take a cut-off function, denoted by ψ, satisfying (121), using a local conformal mapping, and then put $\varphi = \psi^4$.

Let

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0. \tag{123}$$

be given. First, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega \omega\varphi &= \int_\Omega \omega \nabla^{\perp}\psi \cdot \nabla\varphi - (\nabla\omega + \beta\omega\nabla\psi) \cdot \nabla\varphi \ dx \\ &= \int_\Omega \omega \nabla^{\perp}\psi \cdot \nabla\varphi + \omega\Delta\varphi - \beta\omega\nabla\psi \cdot \nabla\varphi \ dx \end{aligned} \tag{124}$$

by (11). It holds that

$$
\begin{aligned}
\int_\Omega \omega \nabla \psi \cdot \nabla \varphi &= \iint_{\Omega\times\Omega} \omega(x,t)[\nabla_x G(x,x') \cdot \nabla\varphi(x)]\omega(x',t)\ dxdx' \\
&= \iint_{\Omega\Omega} \omega(x,t)\varphi_{x_0,2R}(x')[\nabla_x G(x,x') \cdot \nabla\varphi(x)]\omega(x',t)\ dxdx' \\
&+ \iint_{\Omega\times\Omega} \omega(x,t)\left(1-\varphi_{x_0,2R}(x')\right)[\nabla_x G(x,x') \cdot \nabla\varphi(x)]\omega(x',t)\ dxdx' \\
&= I + II.
\end{aligned}
\tag{125}
$$

Let, furthermore, $x_0 \in \Omega$ and $0 < R \ll 1$ in the above equality. Then,

$$
\varphi = |x - x_0|^2 \varphi_{x_0,.R} \tag{126}
$$

satisfies the requirement (123).
It holds that

$$
\nabla\varphi = 2(x - x_0)\varphi_{x_0,R} + |x - x_0|^2 \nabla\varphi_{x_0,R} \tag{127}
$$

and hence

$$
|\nabla\varphi| \le C|x - x_0|\left(\varphi_{x_0,R} + |x - x_0|R^{-1}\varphi_{x_0,R}^{1/2}\right) \le C|x - x_0|\varphi_{x_0,R}^{1/2}. \tag{128}
$$

We obtain, furthermore,

$$
|x' - x_0| \ge 2R, \ \ |x - x_0| \le R \Rightarrow |x - x'| \ge R, \tag{129}
$$

and hence

$$
|\nabla_x G(x,x')| \le CR^{-1} \tag{130}
$$

in this case. Then it follows that

$$
|II| \le CR^{-1}M\int_\Omega |x - x_0|\varphi_{x_0,R}^{1/2}\omega(x,t)\ dx \le CR^{-1}M^{3/2}A^{1/2}, \tag{131}
$$

where

$$
A = \int_\Omega |x - x_0|^2 \varphi_{x_0,R}\omega. \tag{132}
$$

We have, on the other hand,

$$
\begin{aligned}
I &= \iint_{\Omega\times\Omega} \omega(x,t)\varphi_{x_0,2R}(x')[\nabla_x G(x,x') \cdot \nabla\varphi(x)]\omega(x',t)\ dxdx' \\
&= \frac{1}{2}\iint_{\Omega\Omega} \left[\varphi_{x_0,2R}(x')\nabla\varphi(x) \cdot \nabla_x G(x,x') + \varphi_{x_0,2R}(x)\nabla\varphi(x') \cdot \nabla_{x'} G(x,x')\right]\omega\otimes\omega,
\end{aligned}
\tag{133}
$$

where $G = G(x,x')$ is the Green's function to

$$
-\Delta\psi = \omega, \quad \omega|_{\partial\Omega} = 0 \tag{134}
$$

and

$$\omega \otimes \omega = \omega(x,t)\omega(x',t) \ \ dxdx'. \tag{135}$$

Here we use the local property of the Green's function

$$G(x,x') = \Gamma(x-x') + K(x,x'), \quad K \in C^2\left(\overline{\Omega} \times \Omega\right) \cap C^2\left(\Omega \times \overline{\Omega}\right), \tag{136}$$

where

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|} \tag{137}$$

stands for the fundamental solution to $-\Delta$.
Let

$$\rho^2_{x_0,R}\left(x, x()\right) = \varphi_{x_0,2R}(x')\nabla\varphi(x) \cdot \nabla_x K(x,x') + \varphi_{x_0,2R}\nabla\varphi(x') \cdot \nabla_{x'} K(x.x'). \tag{138}$$

Since (128) implies

$$\begin{aligned} |\varphi_{x_0,2R}(x')\nabla\varphi(x)| &\leq C\varphi_{x_0,2R}(x')|x-x_0|\varphi^{1/2}_{x_0,R}(x) \\ &\leq C|x-x_0|\varphi^{1/2}_{x_0,R}(x), \end{aligned} \tag{139}$$

it holds that

$$|\rho^1_{x_0,R}(x,x')| \leq C\left(|x-x_0|\varphi^{1/2}_{x_0,R}(x) + |x'-x_0|\varphi^{1/2}_{x_0,R}(x')\right). \tag{140}$$

Then, we obtain

$$I = \frac{1}{2}\iint_{\Omega\times\Omega} \rho^0_{x_0,R}(x,x')\omega \otimes \omega + III \tag{141}$$

with

$$|III| \leq CM^{3/2}A^{1/2} \leq CR^{-1}M^{3/2}A^{1/2}, \tag{142}$$

where

$$\rho^0_{x_0,R}(x,x') = \nabla\Gamma(x-x') \cdot \left(\varphi_{x_0,2R}(x')\nabla\varphi(x) - \varphi_{x_0,2R}(x)\nabla\varphi(x')\right). \tag{143}$$

Here, we have

$$\nabla\Gamma(x) = -\frac{x}{2\pi|x|^2}, \tag{144}$$

and therefore,

$$\rho^0_{x_0,R}(x,x') = \rho^2_{x_0,R}(x,x') + \rho^3_{x_0,R}(x,x') \tag{145}$$

fo

$$\rho^2_{x_0,R}(x,x') = -\frac{1}{2\pi}\frac{x-x'}{|x-x'|^2}\varphi_{x_0,2R}(x') \cdot (\nabla\varphi(x) - \nabla\varphi(x')) \tag{146}$$

$$\rho^3_{x_0,R}(x,x') = -\frac{1}{2\pi}\frac{x-x'}{|x-x'|^2}\left(\varphi_{x_0,2R}(x') - \varphi_{x_0,2R}(x)\right) \cdot \nabla\varphi(x). \tag{147}$$

Since (128) implies

$$|\rho^3_{x_0,R}(x,x')| \leq CR^{-1}|\nabla\varphi(x)| \leq CR^{-1}|x-x_0|\varphi^{1/2}_{x_0,R}(x), \tag{148}$$

there arises that

$$I = \frac{1}{2}\iint_{\Omega\times\Omega} \rho^2_{x_0,R}(x,x') \ \ \omega\otimes\omega + IV, \tag{149}$$

with

$$|IV| \leq CR^{-1}M^{3/2}A^{1/2}, \tag{150}$$

similarly.
We have, furthermore,

$$\begin{aligned}\nabla\varphi(x) - \nabla\varphi(x') &= 2(x-x')\varphi_{x_0,R}(x) + 2(x'-x_0)\left(\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')\right) \\ &+|x'-x_0|^2\left(\nabla\varphi_{x_0,R}(x) - \nabla\varphi_{x_0,R}(x')\right) + \left(|x-x_0|^2 - |x'-x_0|^2\right)\nabla\varphi_{x_0,R}(x),\end{aligned} \tag{151}$$

and hence

$$\rho^2_{x_0,R}(x,x') = -\frac{1}{\pi}\varphi_{x_0,2R}(x')\varphi_{x_0,R}(x) + \rho^4_{x_0,R}(x,x') + \rho^5_{x_0,R}(x,x') + \rho^6_{x_0,R}(x,x') \tag{152}$$

with

$$\begin{aligned}|\rho^4_{x_0,R}(x,x')| &\leq C|x-x'|^{-1}\varphi_{x_0,2R}(x')|x'-x_0||\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')| \\ &\leq CR^{-1}|x'-x_0|\varphi_{x_0,2R}(x'),\end{aligned} \tag{153}$$

$$\begin{aligned}|\rho^5_{x_0,R}(x,x')| &\leq C|x-x'|^{-1}\varphi_{x_0,2R}(x')|x'-x_0|^2|\nabla\varphi_{x_0,R}(x) - \varphi_{x_0,R}(x')| \\ &\leq CR^{-2}|x'-x_0|^2\varphi_{x_0,2R}(x') \\ &\leq CR^{-1}|x'-x_0|\varphi_{x_0,2R}(x'),\end{aligned} \tag{154}$$

and

$$\begin{aligned}|\rho^6_{x_0,R}(x,x')| &\leq C|x-x'|\varphi_{x_0,2R}(x')||x-x_0|^2 - |x'-x_0|^2|\cdot|\nabla\varphi_{x_0,R}(x)| \\ &\leq CR^{-1}(|x-x_0| + |x'-x_0|)\varphi_{x_0,R}(x)\varphi_{x_0,2R}(x') \\ &\leq C\left(R^{-1}|x-x_0|\varphi_{x_0,R}(x) + R^{-1}|x'-x_0|\varphi_{x_0,2R}(x')\right)\end{aligned} \tag{155}$$

by

$$||x-x_0|^2 - |x'-x_0|^2| = |(x-x', x+x'-2x_0)| \leq |x-x'|(|x-x_0| + |x'-x_0|). \tag{156}$$

The residual terms are thus treated similarly, and it follows that

$$\left|I + \frac{1}{2\pi}\int_\Omega \omega\varphi_{x_0,R}\cdot\int_\Omega \omega\varphi_{x_0,2R}\right| \leq CR^{-1}M^{3/2}A^{1/2}, \tag{157}$$

which results in

$$\left|\int_\Omega \omega\nabla\psi\cdot\nabla\varphi + \frac{1}{2\pi}\int_\Omega \omega\varphi_{x_0,R}\cdot\int_\Omega \omega\varphi_{x_0,2R}\right| \le CR^{-1}M^{3/2}A^{1/2}. \tag{158}$$

We can argue similarly to the vortex term in (124). This time, from

$$\nabla^\perp\Gamma(x)\cdot x = 0 \tag{159}$$

it follows that

$$\left|\int_\Omega \omega\nabla^\perp\psi\cdot\nabla\varphi\right| \le CR^{-1}M^{3/2}A^{1/2}. \tag{160}$$

Concerning the principal term of (124), we use

$$\Delta\varphi = 4\varphi_{x_0,R} + 4(x-x_0)\cdot\nabla\varphi_{x_0,R} + |x-x_0|^2\Delta\varphi_{x_0,R}. \tag{161}$$

From

$$|(x-x_0)\cdot\nabla\varphi_{x_0,R}| \le CR^{-1}|x-x_0|\varphi_{x_0,R}^{1/2} \tag{162}$$

and

$$\begin{aligned} \left||-x_0|^2\Delta\varphi_{x_0,R}\right| &\le CR^{-2}|x-x_0|^2\varphi_{x_0,R}^{1/2} \\ &\le CR^{-1}|x-x_0|\varphi_{x_0,R}^{1/2}, \end{aligned} \tag{163}$$

it follows that

$$\begin{aligned} \left|\int_\Omega \omega\Delta\varphi - 4\int_\Omega \omega\varphi_{x_0,R}\right| &\le C\int_\Omega R^{-1}|x-x_0|\varphi_{x_0,R}\omega \\ &\le CR^{-1}M^{1/2}A^{1/2}. \end{aligned} \tag{164}$$

Let $M_1 = M_{x_0,R}$ and $M_2 = M_{x_0,2R}$ for

$$M_{x_0,R} = \int_\Omega \omega\varphi_{x_0,R}. \tag{165}$$

Then, using (120), we end up with

$$\frac{dA}{dt} \le 4M_1 - \frac{M_1^2}{2\pi} + CR^{-1}\left(M^{3/2}+M^{1/2}\right)A^{1/2} + C(M_2-M_1). \tag{166}$$

Inequalilty (166) implies $T < +\infty$ if $A(0) \ll 1$, as is observed by [27] (see also Chapter 5 of [19]). Here we describe the proof for completeness.

The first observation is the monotoniity formula

$$\left|\frac{d}{dt}\int_\Omega \omega\varphi\right| \le C\left(M+M^2\right)\|\nabla\varphi\|_{C^1}, \tag{167}$$

derived from (124) and the symmetry of the Green's function: $G(x,x') = G(x',x)$. The proof is the same as in (34) and is omitted.

Second, we put $I_1 = I_{x_0,R}$ and $I_2 = I_{x_0,2R}$ for

$$I_{x_0,R} = \int_\Omega |x - x_0|^2 \omega \varphi_{x_0,R}. \tag{168}$$

Then it holds that

$$\begin{aligned} M_2 - M_1 &\le \int_{R<|x-x_0|<2R} \varphi_{x_0,2R}\omega \\ &\le 2R^{-1}\int_\Omega |x - x_0|\varphi_{x_0,2R}\omega \le 2M^{1/2}R^{-1}I_2^{1/2} \end{aligned} \tag{169}$$

and

$$\begin{aligned} A_2 &= A_1 + \int_\Omega |x - x_0|^2 \left(\varphi_{x_0,2R} - \varphi_{x_0,R}\right)\omega \\ &\le A_1 + 4R^2\int_\Omega \left(\varphi_{x_0,2R} - \varphi_{x_0,R}\right)\omega, \end{aligned} \tag{170}$$

which implies

$$\begin{aligned} \frac{dA_1}{dt} &\le 4M_1 - \frac{M_1^2}{2\pi} + CR^{-1}\left(M^{3/2} + M^{1/2}\right)A_1^{1/2} \\ &+ C\left(M^{3/2} + M^{1/2}\right)\left\{\int_\Omega \left(\varphi_{x_0,2R} - \varphi_{x_0,R}\right)\omega\right\}^{1/2}. \end{aligned} \tag{171}$$

Here, we use (167) to ensure

$$\left|\frac{d}{dt}\left(4M_1 - \frac{M^2}{2\pi}\right| \le C\left(M + M^2\right)R^{-2} \tag{172}$$

and

$$\left|\frac{d}{dt}\int_\Omega \left(\varphi_{x_0,2R} - \varphi_{x_0,R}\right)\omega\right| \le C\left(M + M^2\right)R^{-2}. \tag{173}$$

Then, it follows that

$$4M_1 - \frac{M_1^2}{2\pi} \le 4M_1(0) - \frac{M_1(0)^2}{2\pi} + CBa\left(R^{-1}t^{1/2}\right) \tag{174}$$

and

$$\begin{aligned} \int_\Omega \left(\varphi_{x_0,2R} - \varphi_{x_0,R}\right)\omega &\le \int_\Omega \left(\varphi_{x_0,2R} - \varphi_{x_0,R}\right)\omega_0 + CBa\left(R^{-1}t^{1/2}\right) \\ &\le 2R^{-2}A_2(0) + CBa\left(R^{-1}t^{1/2}\right) \end{aligned} \tag{175}$$

for

$$B = M^{3/2} + M^{1/2}, \quad a(s) = s^2 + s. \tag{176}$$

Thus we obtain

$$\begin{aligned}\frac{dA_1}{dt} &\leq 4M_1(0) - \frac{M_1(0)^2}{2\pi} + CR^{-1}BA_1^{1/2} + CBA_2(0)^{1/2} + CBa\left(R^{-1}t^{1/2}\right) \\ &= J(0) + CBa\left(R^{-1}t^{1/2}\right) + CBR^{-1}A_1^{1/2}\end{aligned} \tag{177}$$

for

$$J = 4M_1 - \frac{M_1^2}{4\pi} + CBR^{-1}A_2^{1/2}. \tag{178}$$

Assume $M_1(0) > 8\pi$, and put

$$-4\delta = 4M_1(0) - \frac{M_1(0)^2}{2\pi} < 0. \tag{179}$$

Let, furthemore,

$$\frac{1}{R^2}\int_\Omega |x - x_0|^2 \varphi_{x_0,2R}\omega_0 \leq \eta. \tag{180}$$

Now we define s_0 by

$$CBa(s_0) = \delta \tag{181}$$

in (177), and take $0 < \eta \ll 1$ such that

$$\eta \leq \delta s_0^2. \tag{182}$$

Then, if R and T_0 satisfy $R^{-2}T_0 = \eta\delta^{-1}$, it holds that

$$A_1(0) \leq R^2\eta < 2\delta T_0. \tag{183}$$

Making $0 < \eta \ll 1$, furthermore, we may assume

$$\begin{aligned}J(0) + CBR^{-1}A_1(0)^{1/2} &\leq -4\delta + CBR^{-1}A_2(0)^{1/2} \\ &\leq -4\delta + CB\eta^{1/2} \leq -3\delta,\end{aligned} \tag{184}$$

which results in

$$\begin{aligned}\frac{dA_1}{dt} &\leq J(0) + CBa\left(R^{-1}T_0^{1/2}\right) + BR^{-1}A_1(t)^{1/2} \\ &= J(0) + \delta + CBR^{-1}A_1^{1/2}, \quad 0 \leq t < T_0,\end{aligned} \tag{185}$$

provided that $T \geq T_0$.
A continuation argument to (184)–(185) guarantees

$$\frac{dA_1}{dt} \leq -2\delta, \quad 0 \leq t < T_0, \tag{186}$$

and then we obtain

$$A_1(T_0) \le A_1(0) - 2\delta T_0 < 0 \tag{187}$$

by (183), a contradiction. □

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