Chapter

Black Holes as Possible Dark Matter

Aloke Kumar Sinha

Abstract

Black holes and Dark matter are two fascinating things that are known very little. They may have non gravitational interactions, but those are definitely extremely feeble in comparison to their gravitational interactions. Nowadays some people think that one may contain the other. In this chapter we will see that some black holes may contain the dark matter. These black holes decay under Hawking radiation, but do not vanish completely. They produce stable end states due to both quantum gravitational effects and thermodynamic reasons. These end states are the replicas of what we call dark matter. We will develop the complete theory for decay of such black holes, starting from some scheme independent assumptions for the quantum mechanical nature of the black holes. We will then consider explicit examples of some black holes to show that they indeed produce replicas of dark matter at their end states. Thus this chapter is going to be a manuscript for theoretical development of black hole decay from a quantum mechanical perspective and its consequences for producing replicas of dark matter.

Keywords: Quasi thermal stability, Thermal black holes, Black hole phase transition, Quantum gravity, Dark matter

1. Introduction

Einstein had first shown, with the help of his classical field equations of general theory of relativity, that black holes accreted everything surrounding them [1, 2]. Hence they are expected to grow in size in an unbounded manner. His theory was entirely classical. But Hawking later invoked quantum mechanics in the context of black hole [3], to study its interaction with matters surrounding it. He proved explicitly that black holes could radiate and as a consequence they decayed away. Thus a black hole radiates along with simultaneous accretion.

Hawking considered only matters as quantum entities, but spacetime was still classical in his theory. Hence in his theory, black holes were still classical. Thus this theory was semi classical as matters were treated differently in comparison to black holes. We had resolved this issue in our earlier works [4, 5]. Semiclassical analysis claimed the thermal instability of asymptotically flat, non extremal black holes under Hawking radiation. They are unstable as their specific heat is negative [6, 7] and have been deduced from semiclassical facts based on their classical metric. Their temperature increases as they lose mass, indicating a complete thermal run away process. It is to be noted that semiclassical analysis explicitly depends on the classical metric of a black hole. Hence it is inherently a 'case-by-case' analysis. This shortcoming implies that such semiclassical analysis cannot give general results about the thermal stability of generic black holes under Hawking radiation. Semiclassical analysis predicted the thermal instability of asymptotically flat black holes from the negativity of their specific heat, defined semi classically from their metric. But this result does not say anything in general about an arbitrary black hole. It is of course true that gravity is yet to be quantized fully. But we realistically expect certain symmetries for that theory [4]. These symmetries are sufficient for us to construct the grand canonical partition function of a generic black hole, if we assume the black hole to be in contact with the rest of the universe, that acts as a heat bath. We derived the criteria for thermal stability of a generic black hole with arbitrary number of parameters in any dimensional spacetime, based on the convergence of the grand canonical partition function [5]. These criteria appeared as a series of inequalities, connecting second order derivatives of black hole mass with respect to its parameters.

These criteria imply that AdS black holes with fixed cosmological constant are stable under Hawking radiation for a certain range of their parameters [4]. We have also noticed that asymptotically flat rotating charged black holes satisfy some of the stability criteria, but not all together, in certain regimes of spacetime [4, 5]. Thus although these black holes decay, they are different from unstable black holes, like asymptotically flat Schwarzschild black holes. These black holes are named as "Quasi Stable" black holes. We will later see that AdS black holes with varying cosmological constant are also quasi stable under Hawking radiation.

We had calculated the fluctuations for the parameters of a stable black hole and they were expectedly turned out to be very small [8]. These tiny fluctuations are actually the indications of the stability for a black hole. We did the same for quasi stable black holes and it resulted in tiny fluctuations for some parameters [9], like stable black holes, in a certain regime of parameter space. This is as quasi stable black holes satisfy some of the stability criteria. This makes them slow down their decay rate in certain regimes of their parameter space [9].

Black holes, like ordinary thermodynamic systems, also have different phases. Stable and unstable black holes respectively possess stable and unstable phases in possible allowed regimes of their parameter spaces. The respective examples are AdS black holes with fixed cosmological constant and asymptotically flat Schwarzschild black holes. Unstable black holes remain in the same phase during their decay. Stable black holes likewise stay in a stable phase, maintaining equilibrium with their surroundings and hence they do not decay under Hawking radiation. But things are changed entirely for quasi stable black holes. We had already shown that quasi stable black holes also have various different phases. The quasi stable black holes undergo phase transitions among these phases during their decay process. The nature of fluctuations change from one phase to another phase. In this way quasi stable black holes decay under Hawking radiation. But at the end states, most of these black holes become tiny balls of the order of Planck size. They settle down to these tiny size balls due to quantum gravity effects. On the other hand, some other parameters of certain quasi stable black holes settle down to their macroscopic values at the end states. Thus we see that these black holes become thermodynamically stable, preventing further decay under Hawking radiation. Hence they stop interacting with the rest of the universe, except gravitational interaction. Thus these black holes seem to behave like dark matter (the way we call it). In fact some of these black holes may have electric charge as well. Hence it may correspond to charged dark matter. But some unknown mechanism must be there to prevent it from interacting with the universe through known electrical interaction.

This chapter is organized as follows: A detailed discussion on thermal stability of black holes is done in Section 2. In the next section, we have discussed quasi stability and phase transitions of quasi stable black holes. In the following section, we have considered some examples of quasi stable black holes and have discussed their quasi stability and hence the possible connection with dark matter. We finished in the next chapter with a special note.

2. Thermodynamic stability criteria for black holes

A rotating, electrically charged black hole is represented classically by four parameters (M, Q, J, A), where M, Q, J, A are respectively the mass, electric charge, angular momentum and horizon area of the black hole. These four quantities are related by a relation on the horizon. Thus these parameters are expected to be promoted as operators if black hole can be treated as a quantum system. Three out of these four parameters are independent and the remaining one depends on the other three. It is certainly not possible to have charged rotating black hole without any mass and horizon area. Thus $\hat{Q}$ and $\hat{J}$ have to play the status of primary operators i.e. role of fundamental observables. We choose $\left(\hat{A}, \hat{Q}, \hat{J}\right)$ to be the primary operators and $\widehat{M}$ to be the secondary operator. Hence $\widehat{M}$ as an operator becomes $\widehat{M} = \widehat{M}\left(\hat{A}, \hat{Q}, \hat{J}\right)$. Now horizon area, like electric charge, is invariant under SO(3) rotations beside its invariance under U(1) gauge transformation. $SO(3)$ generates angular momentum while global gauge group $U(1)$ generates electric charge. These give the following commutation relations,

$$\left[\hat{A}, \hat{J}\right] = \left[\hat{A}, \hat{Q}\right] = \left[\hat{Q}, \hat{J}\right] = 0 \tag{1}$$

Since $\widehat{M}$ is a quantum operator of secondary observable $(M(A, J, Q))$, Eq. (1) can be extended as,

$$\left[\hat{A}, \hat{J}\right] = \left[\hat{A}, \hat{Q}\right] = \left[\hat{A}, \widehat{M}\right] = \left[\hat{Q}, \hat{J}\right] = \left[\widehat{M}, \hat{Q}\right] = \left[\hat{J}, \widehat{M}\right] = 0 \tag{2}$$

Thus $\hat{Q}, \hat{J}, \hat{A}$ can have simultaneous eigenstates. Hence definite values of electric charge, angular momentum and horizon area can be assigned to a black hole up to quantum and thermodynamic fluctuations. The eigenvalues of $\hat{Q}, \hat{J}$ and $\hat{A}$ are precisely the parameters used in the classical metric of a black hole to express its mass (M) as a function of them. We consider the isolated horizon to be the boundary of the black hole.

2.1 Quantum geometry

The boundary degrees of freedom and their dynamics of a classical spacetime is determined by the boundary conditions. For a quantum spacetime, fluctuations of the boundary degrees of freedom have a 'life' of their own [10, 11]. Hence the Hilbert space of a quantum spacetime with boundary has the tensor product structure $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_v$, where b (v) denotes the boundary (bulk) component.

So a generic quantum state$(|\Psi\rangle)$ is expandable as,

$$|\Psi\rangle = \sum_{b,v} C_{b,v} |\chi_b\rangle \otimes |\psi_v\rangle \tag{3}$$

where, $|\chi_b\rangle$ and $|\psi_v\rangle$ are respectively the boundary and bulk component of the full quantum state.

The total Hamiltonian operator($\hat{H}$) is given as,

$$\hat{H} \equiv \left(\widehat{H_b} \otimes I_v + I_b \otimes \widehat{H_v}\right) \tag{4}$$

where, respectively, $\widehat{H_b}\left(\widehat{H_v}\right)$ are the Hamiltonian operators on $\mathcal{H}_b(\mathcal{H}_v)$ and $I_b(I_v)$ are the identity operators on $\mathcal{H}_b(\mathcal{H}_v)$.

In presence of rotation and electric charge, $|\psi_v\rangle$ is be the composite bulk state and consequently it is annihilated by the full bulk Hamiltonian i.e.

$$\widehat{H_v}|\psi_v\rangle = 0 \tag{5}$$

This is the quantum analouge of the classical Hamiltonian constraint [12].

The charge operator ($\hat{Q}$) is defined as,

$$\hat{Q} \equiv \left(\hat{Q}_b \otimes \hat{I}_v + \hat{I}_b \otimes \hat{Q}_v\right) \tag{6}$$

where, $\hat{Q}_b$ and $\hat{Q}_v$ are respectively the charge operators for the boundary($|\chi_b\rangle$) and the bulk states ($|\psi_v\rangle$).

Electric charge is defined on the horizon of a classical black hole (e.g. Einstein-Maxwell or Einstein-Yang-Mills theories in [13]) and hence bulk does not carry anything i.e. $Q_v \approx 0$, the Gauss law constraint for electrodynamics. Hence, its quantum version takes the form,

$$\hat{Q}_v|\psi_v\rangle = 0 \tag{7}$$

Similarly angular momentum operator ($\hat{J}$) is defined as,

$$\hat{J} \equiv \left(\hat{J}_b \otimes \hat{I}_v + \hat{I}_b \otimes \hat{J}_v\right) \tag{8}$$

where $\hat{J}_b$ and $\hat{J}_v$ are respectively the angular momentum operators for the boundary ($|\chi_b\rangle$) and the bulk state($|\psi_v\rangle$).

Local spacetime rotation, as a part of local Lorentz invariance, leaves quantum bulk Hilbert space invariant. Hence angular momentum operator, being the generator of spacetime rotation, annihilate the bulk states i.e.

$$\hat{J}_v|\psi_v\rangle = 0 \tag{9}$$

So Eqs. (5), (7) and (9) together imply,

$$\left[\widehat{H_v} - \beta\Phi\widehat{Q_v} - \beta\Omega\widehat{J_v}\right]|\psi_v\rangle = 0 \tag{10}$$

where, Φ, β and Ω are arbitrary functions at this stage.

2.2 Grand Canonical partition function

We will now consider a grand canonical ensemble of quantum spacetimes with horizons as boundaries, in contact with a heat bath, at some (inverse) temperature β. We will assume that this grand canonical ensemble of massive rotating charged black holes can exchange energy, angular momentum and electric charge with the heat bath. Therefore the grand canonical partition function becomes,

$$Z_G = Tr\left(exp\left(-\beta\hat{H} + \beta\Phi\hat{Q} + \beta\Omega\hat{J} \right) \right) \tag{11}$$

where the trace is taken over all states. Φ and Ω are respectively electrostatic potential and angular velocity of the black hole on the horizon.

Hence Eqs. (3), (4), (6), (8), (10) and (11) together yield

$$\begin{aligned} Z_G &= \sum_{b,v} |C_{b,v}|^2 \left\langle \psi_v \right| \otimes \left\langle \chi_b \right| exp\left(-\beta\hat{H} + \beta\Phi\hat{Q} + \beta\Omega\hat{J} \right) \left| \chi_b \right\rangle \otimes \left| \psi_v \right\rangle \\ &= \sum_b |C_b|^2 \left\langle \chi_b \right| exp\left(-\beta\hat{H}_b + \beta\Phi\hat{Q}_b + \beta\Omega\hat{J}_b \right) \left| \chi_b \right\rangle \end{aligned} \tag{12}$$

assuming that the boundary states can be normalized through the squared norm $\sum_v |c_{vb}|^2 \langle \psi_v | \psi_v \rangle = |C_b|^2$. This is analogous to the canonical ensemble scenario described in [14].

The partition function thus turns out to be completely determined by the boundary states (Z_{Gb}), i.e.,

$$Z_G = Z_{Gb} = Tr_b \exp\left(-\beta\hat{H}_b + \beta\Phi\hat{Q}_b + \beta\Omega\hat{J}_b \right) \tag{13}$$

The spectrum of the boundary Hamiltonian operator is assumed to be a function of the discrete electric charge and angular momentum spectrum associated with the horizon[1]. The total electric charge of a black hole is proportional to some fundamental charge from a quantum mechanical point of view and hence the electric charge spectrum is considered to be equispaced [16–20]. In fact the angular momentum spectrum can also be considered as equispaced in the macroscopic spectrum limit of the black hole [21], in which we are ultimately interested.

It has already been seen that electric charge, horizon area and angular momentum operators of a black hole commute among them and hence they can be diagonalized simultaneously. Therefore working in such diagonalized basis, the partition function (13) becomes

$$Z_G = \sum_{k,l,m} g(k,l,m) \quad \exp\left(-\beta(E(A_k, Q_l, J_m) - \Phi Q_l - \Omega J_m) \right) \tag{14}$$

where $g(k,l,m)$ is the degeneracy factor. k,l,m are respectively the quantum numbers corresponding to eigenvalues of horizon area, electric charge and angular momentum. In the macroscopic spectra limit of quantum isolated horizons i.e. regime of the large area, electric charge and angular momentum eigenvalues ($k \gg 1, l \gg 1, m \gg 1$), the Poisson resummation formula [22] implies

$$Z_G = \int dx\, dy\, dz\, g(A(x), Q(y), J(z)) \quad \exp\left(-\beta(E(A(x), Q(y), J(z)) - \Phi Q(y) - \Omega J(z)) \right) \tag{15}$$

where x, y, z are respectively the continuum limit of k, l, m respectively.

[1] Actually this second assumption follows from [13, 15] for spacetimes admitting weakly isolated horizons where there exists a mass function determined by the area and electric charge associated with the horizon. This is an extension of that assumption to the quantum domain.

Now, A, Q and J are respectively, functions of x, y and z alone. Therefore we have,

$$dx = \frac{dA}{A_x}, dy = \frac{dQ}{Q_y}, dz = \frac{dJ}{J_z}$$

where, $A_x \equiv \frac{dA}{dx}$ and so on.

So, the partition function, in terms of area, electric charge and angular momentum as free variables, can be written as follows

$$Z_G = \int dA\, dQ\, dJ \quad \exp\left[S(A) - \beta(E(A,Q,J) - \Phi Q - \Omega J)\right], \tag{16}$$

where, following [23], the *microcanonical* entropy of the horizon is defined by $\exp S(A) \equiv \frac{g(A(x),Q(y),J(z))}{\frac{dA dQ dJ}{dx\, dy\, dz}}$ and is a function of horizon area(A) alone [10, 11, 24].

2.3 Stability against Gaussian fluctuations

2.3.1 Saddle point approximation

The equilibrium of a black hole is given by the saddle point $(\overline{A}, \overline{Q}, \overline{J})$ in the space of integration over horizon area, electric charge and angular momentum. It is now to study the grand canonical partition function for fluctuations $a = (A - \overline{A})$, $q = (Q - \overline{Q})$, $j = (J - \overline{J})$ around the saddle point to determine the stability of the black hole under Hawking radiation. We as usual restrict ourselves only up to Gaussian fluctuations, in order to extremize the free energy for the most probable configuration. Taylor expanding Eq. (16) about the saddle point, gives

$$\begin{aligned} Z_G = & \exp\left[S(\overline{A}) - \beta M(\overline{A}, \overline{Q}, \overline{J}) + \beta\Phi\overline{Q} + \beta\Omega\overline{J}\right] \\ & \times \int da\, dq\, dj\ \exp\left\{ -\frac{\beta}{2}\left[\left(M_{AA} - \frac{S_{AA}}{\beta}\right)a^2 + (M_{QQ})q^2 + (2M_{AQ})aq \right.\right. \\ & \left.\left. +(M_{JJ})j^2 + (2M_{AJ})aj + (2M_{QJ})qj\right]\right\}, \end{aligned} \tag{17}$$

where $M(\overline{A}, \overline{Q}, \overline{J})$ is the mass of the isolated horizon at equilibrium. Here $M_{AQ} \equiv \frac{\partial^2 M}{\partial A \partial Q}\Big|_{(\overline{A}, \overline{Q}, \overline{J})}$ etc. and they are evaluated on the horizon. We will take the entropy of a black hole as linear in horizon area and hence S_{AA} equals to zero.

Now, in the saddle point approximation the coefficients of terms linear in a, q, j vanish by definition of the saddle point. These imply that

$$\beta = \frac{S_A}{M_A}, \Phi = M_Q, \Omega = M_J \tag{18}$$

Of course these derivatives are evaluated at the saddle point.

2.3.2 Criteria

Convergence of the integral (17) implies that the Hessian matrix (H) has to be positive definite, where

$$H = \begin{pmatrix} \beta M_{AA} & \beta M_{AQ} & \beta M_{AJ} \\ \beta M_{AQ} & \beta M_{QQ} & \beta M_{JQ} \\ \beta M_{AJ} & \beta M_{JQ} & \beta M_{JJ} \end{pmatrix} \tag{19}$$

The necessary and sufficient conditions for a real symmetric square matrix to be positive definite are: determinants of all principal square submatrices, and the determinant of the full matrix, are positive [25]. This condition leads to the following 'stability criteria':

$$M_{AA} > 0 \tag{20}$$

$$M_{QQ} > 0 \tag{21}$$

$$M_{JJ} > 0 \tag{22}$$

$$\left(M_{QQ}M_{JJ} - (M_{JQ})^2\right) > 0 \tag{23}$$

$$\left(M_{JJ}M_{AA} - (M_{AJ})^2\right) > 0 \tag{24}$$

$$\left(M_{QQ}M_{AA} - (M_{AQ})^2\right) > 0 \tag{25}$$

$$\begin{aligned} \Big[M_{AA}\left(M_{QQ}M_{JJ} - (M_{JQ})^2\right) - M_{AQ}\left(M_{AQ}M_{JJ} - M_{JQ}M_{AJ}\right) \\ +M_{AJ}\left(M_{AQ}M_{JQ} - M_{QQ}M_{AJ}\right)\Big] > 0 \end{aligned} \tag{26}$$

Of course, (inverse) temperature β is assumed to be positive for a stable configuration.

Now, the temperature is defined as, $T \equiv \frac{1}{\beta} = \frac{M_A}{S_A}$ (From Eq. (18)).

The relation $T = \frac{M_A}{S_A}$ implies that,

$$\frac{dT}{dA} = \frac{\beta M_A M_{AA}}{(S_A)^2} \tag{27}$$

Hence positivity of M_{AA} implies that a stable black hole becomes hotter as it grows in size. Schwarzschild black hole, violating this, invites its own thermal instability and decays under Hawking radiation [22].

It is obvious from Eq. (18) that, $M_{QQ} = \frac{d\Phi}{dQ}$ and $M_{JJ} = \frac{d\Omega}{dJ}$. Hence positivity of M_{QQ} implies that accumulation of charge increases the electric potential of the black hole, whereas positivity of M_{JJ} implies that accumulation of angular momentum makes the black hole to rotate faster. These are the features of a stable black hole (22).

The conditions for the convergence of grand partition function under Gaussian fluctuation imply the convexity of entropy [22, 23, 26]. Thus the above inequalities are correctly the conditions for thermal stability of a charged rotating black hole. Eqs. (20) and (27) together correctly reproduce that positivity of specific heat is the only criteria for thermal stability of an electrically neutral non rotating black hole [14]. Actually both mass and temperature of such black holes are functions of the horizon area (A) only and hence specific heat(C) is given as,

$$C \equiv \frac{dM}{dT} = \frac{(S_A)^2}{\beta M_{AA}} \tag{28}$$

Eqs. (20), (21) and (25) together describe the thermal stability of a non rotating electrically charged black hole, while (20), (22) and (24) together describe the same for rotating electrically neutral black holes [27]. Thus we find that positivity of specific heat cannot be the only criteria for thermal stability of an electrically charged rotating black hole, unlike Schwarzschild black hole, but the charge and the angular momentum play vital roles as well.

So far we have considered only the quantum version of a classical charged rotating black hole. But a quantum black hole may have other types of quantum charges as well. Hence we will consider all the charges of a quantum black hole in the same footing including angular momentum and electric charge. We consider a quantum black hole with n charges $C^1, \ldots, C^n$. Now following exactly the same prescription for constructing grand canonical partition function from operator algebra, we get the partition function here as,

$$\begin{aligned} Z_G = &\exp\left[S(\overline{A}) - \beta M\left(\overline{A}, \overline{C}^1, \ldots, \overline{C}^n\right) + \beta P_i \overline{C}^i\right] \\ &\times \int dA \left(\prod_{i=1}^{n} \int dC^i\right) \exp\left\{-\frac{1}{2}\left[\left(M_{AA} a^2 + 2\sum_{i=1}^{n} \beta M_{AC^i} a c^i\right.\right.\right. \\ &\left.\left.\left. + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta M_{C^i C^j} c^i c^j\right]\right\}, \end{aligned} \tag{29}$$

where $M\left(\overline{A}, \overline{C}^1, \ldots, \overline{C}^n\right)$ is the mass of equilibrium isolated horizon and $M_{AC^i} \equiv \partial^2 M / \partial A \partial C^i \big|_{\left(\overline{A}, \overline{C}^1, \ldots, \overline{C}^n\right)}$ etc., are evaluated on the horizon.

Convergence of the above integral (29) implies that the Hessian matrix (H) has to be positive definite, where

$$H = \begin{pmatrix} \beta M_{AA} & \beta M_{AC^1} & \beta M_{AC^2} & \ldots\ldots\ldots & \beta M_{AC^n} \\ \beta M_{AC^1} & \beta M_{C^1C^1} & \beta M_{C^1C^2} & \ldots\ldots\ldots & \beta M_{C^1C^n} \\ \beta M_{AC^2} & \beta M_{C^2C^1} & \beta M_{C^2C^2} & \ldots\ldots\ldots & \beta M_{C^2C^n} \\ .\ldots\ldots & .\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots & .\ldots\ldots \\ \beta M_{AC^n} & \beta M_{C^nC^1} & \beta M_{C^nC^2} & \ldots\ldots\ldots & \beta M_{C^nC^n} \end{pmatrix} \tag{30}$$

Here, all the derivatives are calculated at the saddle point. Hence the stability criteria i.e. the criteria for positive definiteness of Hessian matrix are given as:

$$D_1 > 0, D_2 > 0, \ldots\ldots, D_{n+1} > 0 \tag{31}$$

where,

$$\begin{aligned} D_1 = \beta M_{AA}, \qquad D_2 = \begin{vmatrix} \beta M_{AA} & \beta M_{AC^1} \\ \beta M_{AC^1} & \beta M_{C^1C^1} \end{vmatrix}, \\ D_3 = \begin{vmatrix} \beta M_{AA} & \beta M_{AC^1} & \beta M_{AC^2} \\ \beta M_{AC^1} & \beta M_{C^1C^1} & \beta M_{C^1C^2} \\ \beta M_{AC^2} & \beta M_{C^2C^1} & \beta M_{C^2C^2} \end{vmatrix}, \ldots\ldots, D_{n+1} = |H| \end{aligned} \tag{32}$$

where, $|H|$ = determinant of the Hessian matrix H.

The inverse temperature β is expectantly assumed to be positive for a stable black hole. We again find that temperature must increase with horizon area, inherent in the positivity of M_{AA}. $'n'$ equals two for a charged rotating black hole and hence according to (32) there should be three stability criteria, not seven (20)–(26). It is to note that those seven conditions are not all independent, actually only three of them are independent.

3. Quasi stability, thermal fluctuations and phase transitions of black holes

Some black holes may not satisfy all the stability criteria together everywhere in their parameter spaces. Such regimes are regions of quasi stability for that black hole and the black hole is quasi stable in that regime. Thus quasi stability of a black hole depends entirely on the regime of parameter space where the black hole is. Of course certain stability criteria may not hold anywhere in parameter space for some black holes and they are completely quasi stable. We will see the relationship between quasi stability and thermal fluctuation in this section.

We found for stable black holes that the grand canonical partition function is converging. We can hence define fluctuation of their parameters. The standard deviation of the statistical distribution of a quantity measures the expectation value of its fluctuation. This knowledge along with the grand canonical partition function implies the standard deviation of charge(Q) as,

$$(\Delta Q)^2 = \frac{\int da\, dq\, dj\, q^2 \exp\left\{-\frac{\beta}{2}\left[\left(M_{AA}-\frac{S_{AA}}{\beta}\right)a^2+(M_{QQ})q^2+(2M_{AQ})aq+(M_{JJ})j^2+(2M_{AJ})aj+(2M_{QJ})qj\right]\right\}}{\int da\, dq\, dj\, \exp\left\{-\frac{\beta}{2}\left[\left(M_{AA}-\frac{S_{AA}}{\beta}\right)a^2+(M_{QQ})q^2+(2M_{AQ})aq+(M_{JJ})j^2+(2M_{AJ})aj+(2M_{QJ})qj\right]\right\}} \tag{33}$$

where, ΔQ is the standard deviation for the electric charge of the black hole. Similarly, ΔJ and ΔA are respectively the same for angular momentum and horizon area of the black hole.

Both the numerator and denominator are converging and turns out to be,

$$(\Delta Q)^2 = -\frac{2}{\beta}\cdot\frac{1}{Z_G}\cdot\frac{\partial Z_G}{\partial M_{QQ}} = \frac{1}{|H|}\cdot\frac{\partial|H|}{\partial(\beta M_{QQ})} \tag{34}$$

where, $|H|$ = determinant of Hessian matrix(H).

The above said process is invalid for quasi stable black holes as their grand canonical partition functions diverge. Hence necessary rearrangements are required to express their grand canonical partition function in the diagonal basis of their Hessian matrices and then to look for stable modes. Fortunately fluctuations of these stable modes are calculable and finite, although their grand partition functions diverge.

We can now rewrite the grand canonical partition function (Z_G) in the diagonal basis of the Hessian matrix as,

$$Z_G = \left(\prod_{j=1}^{n+1}\int d\underline{c}^j\right)\exp\left\{-\frac{1}{2}\left[D_1\left(\underline{c}^1\right)^2+\frac{D_2}{D_1}\left(\underline{c}^2\right)^2+\ldots+\frac{D_{n+1}}{D_n}\left(\underline{c}^{n+1}\right)^2\right]\right\} \tag{35}$$

where the expressions of $D_1, D_2, , D_{n+1}$ are the same as given in (20). The new variables $(\underline{c}^1, , \underline{c}^{n+1})$ are related to the old variables $(a, c^1, , c^n)$ by some linear transformation. The linear transformation matrix is a $(n+1)$ dimensional upper triangular square matrix and hence it has unit determinant. The elements of this transformation matrix are functions of the elements of the Hessian matrix H. Thus it is obvious that exactly one of the $\underline{c}^j$ is equal to C^j, but that identification is not unique. This actually helps us to calculate the fluctuation of any parameter of quasi stable black hole that we want. If at least one of D_1, $\frac{D_2}{D_1}$, … , $\frac{D_{n+1}}{D_n}$ is negative, then Z_G blows up.

We can now define fluctuations for quasi stable black holes in the same way as we did for stable black holes. If D_1 is positive then the fluctuation of $\underline{c}^1$ $\left(\Delta\left(\underline{c}^1\right)^2\right)$ is finite and equals to $\frac{1}{2D_1}$, otherwise it blows up. Similarly $\Delta\left(\underline{c}^2\right)^2$, $\Delta\left(\underline{c}^3\right)^2$, … ., $\Delta\left(\underline{c}^{n+1}\right)^2$ can be defined and equal to $\frac{D_1}{2D_2}$, $\frac{D_2}{2D_3}$, … ., $\frac{D_n}{2D_{n+1}}$ respectively only if these ratios of the coefficients are positive.

A stable black hole with n charges possesses $(n+1)$ independent thermal stability conditions [5]. But it was already shown that an electrically charged, rotating stable black hole possessed seven conditions for thermal stability [4]. But only three of them are independent, the rest depend on those three conditions. But this conclusion holds only for stable black holes, not for quasi stable black holes. Thus one has to check the positivity of determinants of all $\left(2^{n+1}-1\right)$ submatrices of Hessian matrix H (including itself) to ensure the quasi stability of a black hole.

Thus we see that stability of a black hole is determined by the signs of the functions, appeared in the stability criteria. There will be $(n+1)$ no. of fluctuations for a black hole having $'n'$ no. of charges. These fluctuations are individually related, to be shown later, with some physical quantities of the black hole. Signs of each of these physical quantities designate one distinguished phase. Thus a quasi stable black hole with $'n'$ charges can at most have 2^{n+1} number of phases. Any of these physical quantities can possess the same sign in different regimes of parameter space and hence the black hole can enter in the same phase once again. So a decaying black hole may be lucky enough to enjoy the phases of its younger age once more. These interesting reoccurrence of phase transitions are completely absent in both stable or unstable black holes. The relationship among the boundary degrees of freedom determines these phases in a quasi stable black hole.

Finite, bounded fluctuations of the parameters of both stable and quasi stable black holes are directly connected with their respective stability criteria [8, 9]. These fluctuations will be shown to be related with some physically measurable quantities of the black hole. Flipping of their signs indicate phase transitions, generalization of Hawking's old idea for asymptotically flat Schwarzschild black hole (AFSBH) [6] but in case of quasi stable black holes. Hawking showed that negative specific heat made AFSBH thermally unstable. Divergence in ΔA^2 made it happen for AFSBH [8]. But quasi stable black holes possess too many parameters, other than horizon area. Hence fluctuations of other parameters are similarly expected to be related with other physical quantities of the black hole. We will see soon that this expectation is actually the reality.

We will now use the summation formalism of partition function to build up various physical quantities in connection with quasi stable black holes.

In this formalism, grand canonical partition function is given as [4],

$Z_G = \sum_r \exp\left(-\beta(E_r - \Phi Q_r - \Omega J_r)\right)$; here summation is taken over eigenstates. The various symbolic terms like Φ, Ω etc. are as before.

Define, $\overline{\Phi} \equiv \beta\Phi$ and $\overline{\Omega} \equiv \beta\Omega$. $\overline{\Phi}$ and $\overline{\Omega}$ respectively determines the electrical and rotational equilibrium between two connected systems [28].

Hence the grand canonical partition function becomes,

$$Z_G = \sum_r \exp\left(-\beta E_r + \overline{\Phi} Q_r + \overline{\Omega} J_r\right).$$

Thus, average value of angular momentum can be defined as,

$$\bar{J} \equiv \frac{\sum_r J_r \cdot \exp\left(-\beta E_r + \overline{\Phi} Q_r + \overline{\Omega} J_r\right)}{Z_G} = \partial(\ln(Z_G))/\partial\overline{\Omega}$$

Similarly we can calculate $\bar{J}^2$ and is given as,

$$\bar{J}^2 \equiv \frac{\sum_r J_r^2 \cdot \exp\left(-\beta E_r + \overline{\Phi} Q_r + \overline{\Omega} J_r\right)}{Z_G}$$

We can calculate fluctuation of angular momentum and this turns out to be

$$\Delta(J)^2 \equiv \frac{\sum_r \left(J_r - \bar{J}\right)^2 \cdot \exp\left(-\beta E_r + \overline{\Phi} Q_r + \overline{\Omega} J_r\right)}{Z_G} = \bar{J}^2 - \left(\bar{J}\right)^2 = \partial^2(\ln(Z_G))/\partial\overline{\Omega}^2$$

The convergence of fluctuation for angular momentum is mandatory for the above calculation. Most importantly the above partial derivatives are taken at the constant values of β and $\overline{\Phi}$. Likewise partial derivatives with respect to $\overline{\Phi}$ can be taken at constant values of β, $\overline{\Omega}$ and so on.

The rotational inertia of a black hole (S_J) is defined as,

$S_J \equiv \beta \cdot \partial\bar{J}/\partial\overline{\Omega}$ and is equals to $\beta \cdot \Delta(J)^2$.

It is important to note the following issue:

The quantities β, $\overline{\Phi}$ and $\overline{\Omega}$ are functions of independent variables $\overline{A}$, $\overline{Q}$ and $\bar{J}$ and hence consequently $\overline{A}$, $\overline{Q}$ and $\bar{J}$ are the inverse functions of β, $\overline{\Phi}$ and $\overline{\Omega}$. Hence partial derivatives for example with respect to $\overline{\Omega}$, at constant β, $\overline{\Phi}$, can be evaluated and so on. So S_J and $\Delta(J)^2$ are independently calculable. They are related only when fluctuation in angular momentum is bounded and finite. $\Delta(J)^2$ approaches zero and then suddenly blows up at the point of phase transition. But S_J vanishes there and flips its sign afterwards. It starts to disrespect the above equality afterwards.

Electric capacitance of a black hole(S_Q) is defined as,

$S_Q \equiv \beta \cdot \partial\overline{Q}/\partial\overline{\Phi}$ and is equal to $\beta \cdot \Delta(Q)^2$, only when $\Delta(Q)^2$ is finite and bounded.

S_Q and $\Delta(Q)^2$ respectively are in same footings as that of S_J and $\Delta(J)^2$ regarding their relationship and behavior at the point of phase transition. Hence flipping in signs of electric capacitance and rotational inertia separately mark two different phase transitions.

4. Decay of quasi stable black holes and possible identification with dark matter

4.1 Asymptotically flat Reissner-Nordstrom black hole

The mass(M) of asymptotically flat Reissner-Nordstrom black hole (AFRNBH) depends on its parameters as [29],

$$M = \frac{\sqrt{A}}{4\sqrt{\pi}} + \frac{\sqrt{\pi}Q^2}{\sqrt{A}} \tag{36}$$

We can now calculate the temperature of AFRNBH and it will be function of its electric charge(Q) and horizon area(A). On calculation, it turns out that temperature ($\propto M_A$) is positive only if $\frac{Q^2}{A} < \frac{1}{4\pi}$. This restricts the parameter space.

We can calculate various second derivatives of the black hole mass (M) with respect to its parameters from the above relation. On calculation, this turns out that.

$$M_{QQ} = \frac{2\sqrt{\pi}}{\sqrt{A}}, M_{AQ} = -\frac{\sqrt{\pi}Q}{A^{3/2}}, M_{AA} = -\frac{1}{16\sqrt{\pi}A^{3/2}} + \frac{3\sqrt{\pi}Q^2}{4A^{5/2}}, \left(M_{QQ}M_{AA} - (M_{AQ})^2\right) = \left(-\frac{1}{8A^2} + \frac{\pi Q^2}{2A^3}\right)$$

Thus $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)$ is positive only if $\frac{Q^2}{A} > \frac{1}{4\pi}$. But this region of parameter space is not accessible to any real AFRNBH as it is excluded due to negativity of temperature. Hence $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)$ is negative throughout its physically accessible regime of parameter space. Now, M_{QQ} is always positive while M_{AA} is negative if $\frac{Q^2}{A} < \frac{1}{12\pi}$. Thus AFRNBH can never be thermally stable as it never satisfies any of the above two stability criteria completely. So AFRNBH is actually a quasi stable black hole [9].

Now, $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)$ is always negative for AFRNBH. Keeping this in mind, We can conclude that,

1. $\Delta(A)^2$ always blows up as M_{QQ} is always positive.

2. $\Delta(Q)^2$ converges and equals to the $\frac{M_{AA}}{2\beta\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)}$ only if $M_{AA} < 0$ i.e. $\frac{Q^2}{A} < \frac{1}{12\pi}$.

AFRNBH gradually becomes smaller in size due to unbounded area fluctuation and hence ultimately decays. Thus $\frac{Q^2}{A}$, even if it is less than $\frac{1}{12\pi}$ at the beginning, increases as area(A) decreases. But it cannot go beyond $\frac{1}{4\pi}$. In the regime $\frac{1}{4\pi} > \frac{Q^2}{A} > \frac{1}{12\pi}$, electric charge$(Q)$ of this black hole fluctuates appreciably enough to reduce the value of Q. Thus this ratio becomes lower than the bench mark value $\frac{1}{12\pi}$. Hence we see that this toggling keeps on going around the value $\frac{1}{12\pi}$. In this process the black hole will continue to lose its electric charge and horizon area and consequently moves forward to its end state with a certain minimum area [30], having almost no electric charge. At this point, the black hole will not decay any further and becomes thermodynamically isolated. Only gravitational interaction remains active. This is quite similar to the nature of dark matter. This correspondence is possible only if we are ready to accept that what we think of as dark matter is actually some region of the spacetime of our universe. Thus this region pretends to be neutral Planck dark matter as the size of black hole is now of the order of Planck length.

4.2 Asymptotically flat Kerr-Newman black hole

The mass(M) of this black hole depends on its parameters as [31],

$$M^2 = \frac{A}{16\pi} + \frac{\pi}{A}\left(4J^2 + Q^4\right) + \frac{Q^2}{2}$$

So the parameter space is restricted by the inequality $\left(4J^2 + Q^4\right) < \frac{A^2}{16\pi^2}$ as temperature($\propto M_A$) of a non extremal black hole is always positive. Hence both electric charge and angular momentum are bounded for a given horizon area of the black hole. $|H|$ can be shown to be always negative and hence this black hole would decay under Hawking radiation. It will consequently lose its area. Hence charge and angular momentum have to adjust them respectively through their fluctuations to maintain the above bound. This bounded region is shown in the **Figure 1**.

Now, it can be easily shown that $\left(M_{QQ}(\beta M_{AA} - S_{AA}) - \beta\left(M_{AQ}\right)^2\right)$ is negative in the upper portion of the shaded region of the above figure. Thus this is the region for bounded fluctuation of angular momentum. So, the higher values of $\frac{J}{A}$ make the fluctuation of angular momentum large. As the area of this black hole always decreases, the ratio $\frac{J}{A}$ increases. Thus the fluctuation of angular momentum becomes appreciably large and hence angular momentum is reduced to maintain the non extremality bound. So, $\frac{J}{A}$ ratio again comes to the regime where J does not fluctuate much. But area(A) as usual decreases continuously and consequently $\frac{J}{A}$ ratio again becomes large enough such that J starts to fluctuate appreciably again. Thus this flipping of $\frac{J}{A}$ ratio from larger to smaller value and vice versa keeps on going. Hence angular momentum gradually decreases and consequently KN black hole proceeds to transform into a non rotating black hole.

On the other hand, it can be easily shown that $\left(M_{JJ}(\beta M_{AA} - S_{AA}) - \beta\left(M_{AJ}\right)^2\right)$ is negative in the lower portion of the shaded region of the above figure. Thus this is the region for bounded fluctuation of charge. So, higher values of $\frac{Q^2}{A}$ make the fluctuation of charge bounded only if the ratio $\frac{J}{A}$ is sufficiently high. But we have just seen that $\frac{J}{A}$ ratio cannot always be high, along with the fact that the ratio $\frac{Q^2}{A}$ is itself bounded. Thus Q reduces gradually as the area of the black hole decreases. So, the ratio $\frac{Q^2}{A}$ oscillates between higher and lower values, exactly in the same manner as $\frac{J}{A}$ ratio does the same and gradually discharges all its charges. Consequently it proceeds to transform into a chargeless, non rotating black hole. Thus it resembles neutral Planck dark matter due to the fact explained in the last section. The difference between this sort of dark matter and the earlier one is only that their origins are different.

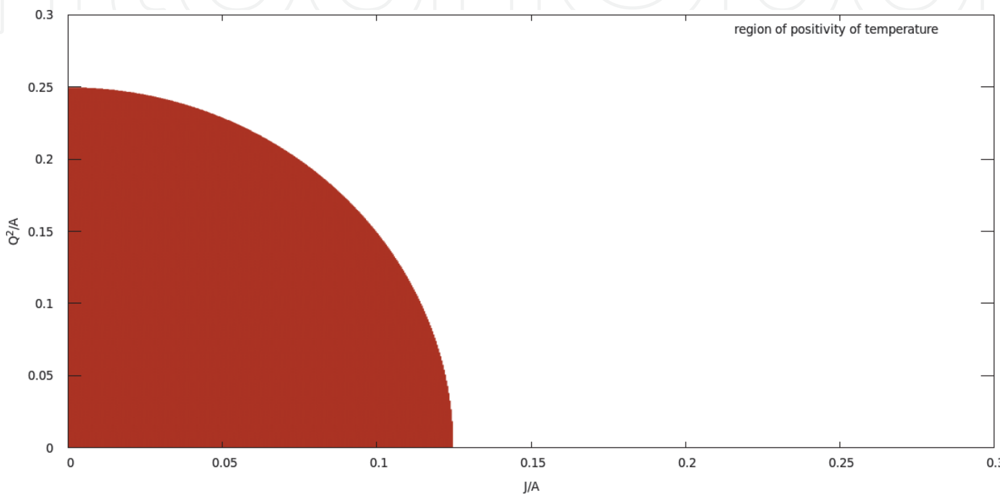

Figure 1.
Pictorial representation of region of positivity of temperature.

4.3 Asymptotically flat Kerr-Sen black hole

The mass(M) of this black hole depends on its parameters as [32],

$$M^2 = \frac{A}{16\pi} + \frac{Q^2}{2} + \frac{4\pi J^2}{A}$$

The parameter space here is restricted by the inequality $\frac{J}{A} < \frac{1}{8\pi}$ as temperature $(\propto M_A)$ of a non extremal black hole is always positive. It is important to notice that the electric charge of this black hole, unlike AFKNBH, is not bounded by the non extremality of this black hole. We will see its interesting consequences soon. The quantity $\left(M_{QQ}(\beta M_{AA} - S_{AA}) - \beta(M_{AQ})^2\right)$ is negative in the regime $\frac{J}{A} < \frac{0.4}{8\pi}$, but $\left((\beta M_{AA} - S_{AA})M_{JJ} - \beta(M_{JA})^2\right)$ is always negative. Hence both $\Delta(J)^2$ and $\Delta(Q)^2$ are bounded in the regime $\frac{J}{A} < \frac{0.4}{8\pi}$ for KS black Hole, maintaining a perfect balance between the incoming and outgoing quanta of angular momentum and electric charge respectively. But this balance is lost only for angular momentum in the regime $\frac{0.4}{8\pi} < \frac{J}{A} < \frac{1}{8\pi}$, whereas the same for electric charge is maintained everywhere in the parameter space. But the KS black hole ultimately decays due to unbounded nature of $\Delta(A)^2$.

Suppose the angular momentum(J) is such that $\frac{J}{A} < \frac{0.4}{8\pi}$ and hence J does not fluctuate much as its fluctuation is bounded in this region. But area(A) as usual decreases and hence the ratio $\frac{J}{A}$ increases and becomes greater than $\frac{0.4}{8\pi}$. Once this ratio crosses that value, J starts to fluctuate rapidly. But this ratio, due to non extremality, cannot be greater than $\frac{1}{8\pi}$ with decreasing area(A). Thus J ultimately reduces and hence the ratio $\frac{J}{A}$ becomes lesser than $\frac{0.4}{8\pi}$. This process will go on. This means that KS black hole tries to reduce the angular momentum, in order to satisfy its extremality bound, during the Hawking decay. Hence the black hole gradually loses its area and angular momentum, keeping the charge unchanged. Thus it proceeds to transform into a black hole with charge only. This transformation is purely thermodynamical in nature. Thus we find the difference between KS and KN black hole in terms of their end states.

It is important to note that KS black hole, unlike KN black hole, hardly discharges throughout its life. One has to go back to the construction of grand canonical partition to understand this. We in this analysis have assumed the mass of a rotating charged black hole as a function of its area, charge and angular momentum. It is a fact in any theory of quantum gravity that area, charge and angular momentum are good self-adjoint operators. But mass is not a good primary operator. We still can represent it as a secondary operator in terms of other primary operators. Hence we here consider fluctuations of area, charge and angular momentum only. In semiclassical analyses, one gets various restrictions on the parameter space from the condition of avoiding the naked singularity. We, in thermodynamical analysis, equivalently obtain various restrictions on the parameter space from the condition of avoiding the absolute zero temperature. Semi classically, it had been shown [33] that a charged rotating KN black hole should lose its charge and angular momentum, just from the condition of various restrictions on the parameter space. We also obtain similar results for KN black holes, just from the condition of various restrictions on the parameter space imposed by positivity of the temperature. But this analysis is a bit interesting for KS black hole. Positivity of temperature does not put any bound on its electric charge. Close to the end state, this black hole loses almost all its angular momentum. The area also becomes comparable with the Planck area

[30]. Hence mass of the black hole is approximated given there as, $M^2 \approx \frac{Q^2}{2}$. This is very much similar to stable extremal black holes with magnetic monopoles. Of course the last example is the outcome of semiclassical analysis, where the mass of this black hole in the limiting case is given as, $M^2 \approx P^2$, P is magnetic charge. We compare this thermodynamical analysis with well known semiclassical analyses not to establish our analysis, but to show the simplicity as well as superiority of this analysis. B.carter, through his semi classical analysis [34], had shown that charged black hole with initial mass of order of 10^{15} kg does negligibly discharge throughout its life. This, if translated for KS black hole, implies KS black hole almost does not discharge if its initial charge is roughly one mole of electrons. In fact charged black holes with sufficient initial mass, under certain idealized conditions, had been shown semi classically [35] not to discharge. This again supports our conclusion regarding stability of electric charge for decaying KS black hole.

The end state of this black hole can now be identified as charged Planck dark matter. Thus we get a possible scenario for obtaining a charged black hole through our line of thoughts.

4.4 (2 + 1) dimensional charged BTZ black hole

The mass(M) of $(2+1)$ dimensional charged BTZ black hole $(\Lambda 3BTZBH)$ depends on its parameters as [36],

$$M = \frac{r^2}{8l^2} - \frac{Q^2}{16}\, ln\,(r/l).$$

Here l is known as cosmic length and is related with Λ as $\Lambda = 1/l^2$. r is the radius of the circular horizon. Hence area of it, which is actually its perimeter, is given as $A = 2\pi r$. So the mass(M) of $\Lambda 3BTZBH$ can be expressed in terms of Λ and A as,

$$M = \frac{A^2\Lambda}{32\pi^2} - \frac{Q^2}{32}\, ln\left(\frac{A^2\Lambda}{4\pi^2}\right) \tag{37}$$

We can now calculate the temperature of $\Lambda 3BTZBH$ from above relationship and it becomes a function of its charge(Q), area(A) and Λ. On calculation, it turns out that temperature$(= M_A)$ is positive if $A^2 > \pi^2 Q^2/\Lambda$. This restricts the parameter space.

We can calculate various second order derivatives of the black hole mass(M) with respect to its parameters from the above relationship. On calculation, this turns out that.

$$M_{QQ} = -\frac{1}{16}\, ln\left(\frac{A^2\Lambda}{4\pi^2}\right), M_{AQ} = -\frac{Q}{8A}, M_{Q\Lambda} = -\frac{Q}{16\Lambda}, M_{\Lambda\Lambda} = \frac{Q^2}{32\Lambda^2}, M_{\Lambda A} = \frac{A}{16\pi^2}, M_{AA} = \left(\frac{\Lambda}{16\pi^2} + \frac{Q^2}{16A^2}\right)$$

We, with the help of the above six second order derivatives of M, can show that.

1. $\left(M_{\Lambda\Lambda}M_{AA} - (M_{A\Lambda})^2\right) = \frac{A^2}{256\pi^4}\left(\frac{\pi^2 Q^2}{2\Lambda A^2} + \frac{\pi^4 Q^4}{2\Lambda^2 A^4} - \pi^2\right)$. Now positivity of temperature implies $A^2 > \pi^2 Q^2/\Lambda$. Hence $\left(M_{\Lambda\Lambda}M_{AA} - (M_{A\Lambda})^2\right)$ is always negative.

2. $|H| = ln\left(\frac{A^2\Lambda}{4\pi^2}\right) \cdot \frac{A^2}{256 \cdot 32\pi^4} \cdot \left(2 - \frac{\pi^2 Q^2}{2\Lambda A^2} - \frac{\pi^4 Q^4}{2\Lambda^2 A^4}\right) + \frac{6Q^2}{265 \cdot 32\pi^2\Lambda}\left(1 - \frac{\pi^2 Q^2}{A^2\Lambda}\right)$. Now again the positivity of temperature fixes the sign of $|H|$, but here it is positive.

Thus we explicitly see that $\left(M_{\Lambda\Lambda}M_{AA} - (M_{A\Lambda})^2\right)$ is always negative for Λ3BTZBH, whereas $|H|$ is always positive for it. Hence Λ3BTZBH is actually quasi stable under Hawking radiation. The most interesting point to note that $|H|$ is always positive here, unlike other quasi stable black holes [9, 28, 37].

We have earlier shown that [9, 37] quasi stable black holes possess tiny fluctuations for some of their parameters in certain regions of parameter space. So, the same is expected in case of Λ3BTZBH. We already knew [9] how fluctuations were related to stability criteria. In fact we also knew [9] how to calculate fluctuations in case of quasi stable black holes. Now, $|H|$ is always positive. Keeping this in mind, we can conclude[2] that,

1. $\Delta(A)^2$ is bounded only if $\left(M_{QQ}M_{\Lambda\Lambda} - (M_{\Lambda Q})^2\right)$ is positive. On calculation it turns out that, $\left(M_{QQ}M_{\Lambda\Lambda} - (M_{\Lambda Q})^2\right) = -\frac{Q^2}{256\Lambda^2}\left(1 + \frac{1}{2} \cdot ln\left(\frac{A^2\Lambda}{4\pi^2}\right)\right)$ and is positive if $A^2\Lambda < \frac{4\pi^2}{e^2}$.

2. $\Delta(Q)^2$ is always unbounded as $\left(M_{\Lambda\Lambda}M_{AA} - (M_{A\Lambda})^2\right)$, has already been shown, is always negative.

3. $\Delta(\Lambda)^2$ is bounded only if $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)$ is positive. On calculation it turns out that, $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right) = -\frac{1}{16}\left(ln\left(\frac{A^2\Lambda}{4\pi^2}\right)\left(\frac{\Lambda}{16\pi^2} + \frac{Q^2}{16A^2}\right) + \frac{Q^2}{4A^2}\right)$ and is positive if $ln\left(\frac{A^2\Lambda}{4\pi^2}\right) < -\frac{4}{1+\frac{A\Lambda}{\pi^2 Q^2}}$. Now positivity of temperature gives $A^2 > \pi^2 Q^2/\Lambda$ and consequently this implies $\left(-\frac{4}{1+\frac{A\Lambda}{\pi^2 Q^2}}\right)$ is greater than -2. Thus $A^2\Lambda < \frac{4\pi^2}{e^2}$ is the region for positivity of $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)$. In fact this upper limit is greater than the estimated value as $\frac{A^2\Lambda}{\pi^2 Q^2}$ is greater than unity. So, $\pi^2 Q^2 < A^2\Lambda < \frac{4\pi^2}{e^2}$ is a legitimate regime in parameter space, where $\Delta\Lambda^2$ is bounded.

We have just seen that charge always fluctuates with large magnitude. Now, suppose area(A) is initially so large that it satisfies both the inequalities $\pi^2 Q^2 < A^2\Lambda$ and $A^2\Lambda > \frac{4\pi^2}{e^2}$ by far. In this regime of parameter space all the parameters charge (Q), area (A) and cosmological constant(Λ) together fluctuate appreciably. Area gradually decreases due to Hawking radiation. Cosmological constant also gradually decreases due to bubble emission [38]. Hence charge has to decrease sufficiently fast to maintain the positivity of temperature, as otherwise zero temperature would

[2] $\Delta(\Lambda)^2$ measures the fluctuation of cosmological constant from its equilibrium value and is mathematically expressed as [8, 9], $\Delta(\Lambda)^2 = \frac{\int da\, dq\, d\lambda\, \lambda^2 f(a,\lambda)}{\int da\, dq\, d\lambda}$, where $f(a,\lambda) =$
$exp\left(-\frac{\beta}{2}\left[\left(M_{AA} - \frac{S_{AA}}{\beta}\right)a^2 + (M_{QQ})q^2 + (M_{\Lambda\Lambda})\lambda^2 + (2M_{AQ})aq + (2M_{A\Lambda})a\lambda + (2M_{Q\Lambda})q\lambda\right]\right)$. Similarly, $\Delta(A)^2$ and $\Delta(Q)^2$ are defined.

cause the thermodynamic death of the black hole. Thus the black hole would once cross the curve $A^2\Lambda = \frac{4\pi^2}{e^2}$, making the term $\left(M_{QQ}M_{\Lambda\Lambda} - (M_{\Lambda Q})^2\right)$ positive. Hence $\Delta(A)^2$ becomes bounded, suppressing its large unbounded magnitude exponentially. In fact $\left(M_{QQ}M_{AA} - (M_{AQ})^2\right)$ becomes positive even before $A^2\Lambda$ becomes equal to $\frac{4\pi^2}{e^2}$. This consequently makes $\Delta(\Lambda)^2$ bounded, like $\Delta(A)^2$, suppressing its large unbounded magnitude exponentially. Thus we find that in the regime $\pi^2Q^2 < A^2\Lambda < \frac{4\pi^2}{e^2}$, both area and cosmological constant do not fluctuate appreciably. But charge gradually decreases as before and hence the last inequality holds good. Thus once the black hole loses almost all its charge, it transforms into a stable chargeless BTZ black hole, having negative cosmological constant. This end state of Λ3BTZBH, as we have seen, is different from other AdS black holes as their horizon areas become close to the Planck area in their end states [39]. Thus we now get a non Planck sized dark matter from our line of thoughts.

5. Note

The readers may wonder how this chapter can be something about dark matter? We have hardly used the word "dark matter" so far, at most have used it on a few occasions. But theoretically the connection, which is discussed here, between black holes and dark matter is extremely appealing. There are some experimental evidences that mostly rule out the possibility of connection between dark matter and black hole [40], that we have described. On the other hand the recent observations of LIGO and VIRGO now suggest that black holes are much more common than once imagined and hence they could very well be the missing dark matter [41]. Anyway this chapter is written with the belief that dark matter is the possible end state of the quasi stable black holes. Many more future experiments are required to conclude definitively about the validity of our predictions.

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Author details

Aloke Kumar Sinha
Haldia Government College, West Bengal, India

*Address all correspondence to: akshooghly@gmail.com

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