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On the Estimation of Asymptotic Stability Region of Nonlinear Polynomial Systems: Geometrical Approaches

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1. Introduction

In recent years, the problem of determining the asymptotic stability region of autonomous nonlinear dynamic systems has been developed in several researches. Many methods, usually based on approaches using Lyapunov's candidate functions (Davidson & Kurak, 1971) and (Tesi et al., 1996) which altogether allow for a sufficient stability region around an equilibrium point. Particularly, the method of Zubov (Zubov, 1962) is a vital contribution. In fact, it provides necessary and sufficient conditions characterizing areas which are deemed as a region of asymptotic stability around stable equilibrium points.

Such a technique has been applied for the first time by Margolis (Margolis & Vogt, 1963) on second order systems. Moreover, a numerical approach of the method was also handled by Rodden (Rodden, 1964) who suggested a numerical solution for the determination of optimum Lyapunov function. Some applications on nonlinear models of electrical machines, using the last method, were also presented in the Literature (Willems, 1971), (Abu Hassan & Storey, 1981), (Chiang, 1991) and (Chiang et al., 1995). In the same direction, the work presented in (Vanelli & Vidyasagar, 1985) deals with the problem of maximizing Lyapunov's candidate functions to obtain the widest domain of attraction around equilibrium points of autonomous nonlinear systems. Burnand and Sarlos (Burnand & Sarlos, 1968) have presented a method of construction of the attraction area using the Zubov method.

All these methods of estimating or widening the area of stability of dynamic nonlinear systems, called Lyapunov Methods, are based either on the Characterization of necessary and sufficient conditions for the optimization of Lyapunov's candidate functions, or on some approaches using Zubov's digital theorem. Equally important, however, they also have some constraints that prevented obtaining an exact asymptotic stability domain of the considered systems. Nevertheless, several other approaches nether use Lyapunov's functions nor Zubov's which have been dealt with in recent researches. Among these works cited are those based on topological considerations of the Stability Regions (Benhadj Braiek et al., 1995), (Genesio et al., 1985) and (Loccufier & Noldus, 2000).

Indeed, the first method based on optimization approaches and methods using the consideration of Lasalle have been developed to ensure a practical continuous stability

region of the second order systems. Furthermore, other methods based on interpretations of geometric equations of the model have grabbed an increasing attention to the equivalence between the convergence of the linear part of the autonomous nonlinear system model and whether closed Trajectories in the plan exist. An interesting approach dealing with this subject is called trajectory reversing method (Bacha et al. 1997) and (Noldus et al.1995).

In this respective, an advanced reversing trajectory method for nonlinear polynomial systems has been developed (Bacha et al.,2007).Such approach can be formulated as a combination between the algebraic reversing method of the recurrent equation system and the concept of existence of a guaranteed asymptotic stability region around an equilibrium point. The improvement of the validity of algebraic reversing approach is reached via the way we consider the variation model neighbourhood of points determined in the stability boundary's asymptotic region. An obvious enlargement of final region of stability is obtained when compared by results formulated with other methods. This very method has been tested to some practical autonomous nonlinear systems as Van Der Pool.

2. Backward iteration approaches

We attempt to extend the trajectory reversing method to discrete nonlinear systems. In this way, we suggest two different algebraic approaches so as to invert the recurrent polynomial equation representing the discrete-time studied systems. The enlargement and the exactness of the asymptotic stability region will be considered as the main criterion of the comparison of the two proposed approaches applied to an electrical discrete system.

2.1 Description of the studied systems

We consider the polynomial discrete systems described by the following recurrent state equation:

$$X_{k+1} = F(X_{k+1}) = \sum_{i=1}^{r} F_i . X_k^{[i]} \tag{1}$$

where $F_i, i=1,...,r$ are (n x $n^{[i]}$) matrices and X_k is an n dimensional discrete-time state vector. $X_k^{[i]}$ designates the ith order of the Kronecker power of the state X_k. The initial state is denoted by X_0. Note that this class of polynomial systems (1) may represent various controlled practical processes as electrical machines and robot manipulators.

It is assumed that system (1) satisfies the known conditions for the existence and the uniqueness of each solution $X(k,X_0)$ for all k, with initial condition $X(k=0)=X_0$.

The origin is obviously an equilibrium point which is assumed to be asymptotically stable.

The region of asymptotic stability of the origin is defined as the set Ω of all points X_0 such that:

$$\forall X_0 \in \Omega, \forall k \in \aleph, X(k, X_0) \in \Re \text{ and } \lim_{k \to \infty} X(k, X_0) = 0 \tag{2}$$

So, we can find an open invariant set Ω with boundary Γ such that Ω is a region of asymptotic stability (RAS) of system (1) defined by the property that every trajectory starting from $X_0 \in \Omega$ reaches the equilibrium point of the corresponding system.

Note that determining the global stability region of a given system is a difficult task. In this respect, one often has to be satisfied with an approximation that leads to a guaranteed stable region in which all points belong to the entire stability region.

In the forthcoming sections, we try to estimate a stability region of the system (1) included in the entire RAS. The main way to get a stable region Ω is to use the reversing trajectory method called also backward iteration.

For a discrete nonlinear system with a state equation (1) the backward iteration means the running of the reverse of the discrete state equation (1) which requires to explicit the following retrograde recurrent equation:

$$X_k = F^{-1}(X_{k+1}) \tag{3}$$

Note that this reverse system is characterized by the same trajectories in discrete state space as (1). So, it is obvious that the asymptotic behaviour of trajectories starting in the region of asymptotic stability Ω is related to its boundary Γ and always provides information about it. In order to determine the inverted polynomial recurrent equation, we expose in the next section two dissimilar digital backward iteration approaches by using the Kronecker product form and the Taylor expansion development.

2.2 Proposed approaches for the formulation of the discrete model inversion

The exact determination of the reverse polynomial recurrent equation (3) could not be reached. For the achievement of this target, most of the methods that have been suggested are based on approximation ideas.

- First approach

In the first method, we suggest to use the following approximation:

$$X_k = X_{k+1} + \varepsilon(X_{k+1}) \tag{4}$$

where $\varepsilon(X_{k+1})$ is assumed to be a little term, which we will explicit. This assumption requires a suitable choice of the sampling period. Without loss of generality we will develop the approach of expliciting the term $\varepsilon(X_{k+1})$ for the case r=3. The obtained results can be easily generalized for any polynomial degree r by following the same principle.

So, we consider then the following recurrent equation:

$$X_{k+1} = F_1.X_k + F_2.X_k^{[2]} + F_3.X_k^{[3]} \tag{5}$$

Replacing in (5) X_k by its expression (4) and neglecting all terms $\varepsilon^{[n]}(X_{k+1})$ for n>1, one can easily obtain the following expression of $\varepsilon(X_{k+1})$.

$$\varepsilon(X_{k+1}) = \begin{bmatrix} F_1 + F_2.(X_{k+1} \otimes I_n + I_n \otimes X_{k+1}) + \\ F_3.(X_{k+1} \otimes I_n \otimes X_{k+1} + X_{k+1}^{[2]} \otimes I_n + I_n \otimes X_{k+1}^{[2]}) \end{bmatrix}^{-1} \cdot \left[X_{k+1} - \left(F_1.X_{k+1} + F_2.X_{k+1}^{[2]} + F_3.X_{k+1}^{[3]} + F_4.X_{k+1}^{[4]} \right) \right] \tag{6}$$

Then, the RAS may be well estimated by means of a convergent sequence of simply connected domains generated by the backward iterations (4) and (6).

- Second approach

The second proposed technique of the inversion of the model (1) is made up by the characterization of the reverse model:

$$X_k = F^{-1}(X_{k+1}) = G(X_{k+1}) \tag{7}$$

by a r-polynomial vectorial function G(.) i.e.:

$$X_k = \sum_{i=1}^{r} G_i . X_{k+1}^{[i]} \tag{8}$$

where G_i, i=1,...,r are matrices of (nx n[i]) dimensions.

Hence, it is easy to identify the G_i matrices in (8) by writing $F(G(X_{k+1})) = X_{k+1}$ which leads to the following relations:

$$\begin{cases} G_1 = G_1^1 = F_1^{-1} \\ G_2 = G_2^1 = -F_1^{-1} F_2 . G_2^1 \\ \quad\vdots \\ G_r = G_r^1 = -F_1^{-1} \left(\sum_{i=1}^{r-1} F_i . G_r^i \right) \end{cases} \tag{9}$$

where G_p^i, for i=2,...,r and p=2,...,r verify the following recurrent relations:

$$\begin{cases} G_p^2 = \sum_{j=1}^{p-1} \left(G_{p-j}^1 \otimes G_j^1 \right) \\ \quad\vdots \\ G_p^i = \sum_{j=1}^{p-i+1} \left(G_{p-j}^{i-1} \otimes G_j^1 \right) \end{cases} \tag{10}$$

- Evolutionary algorithm of backward iteration method

By using one of the presented approaches of the reversing recurrent equation of system (1) formulated above, the reversing trajectory technique can be run by the following conceptual algorithm.

1. Verify that the origin equilibrium of the system (1) is asymptotically stable i.e. $\left\| eig(F_1) \right\| < 1$.

2. Determine a guaranteed stable region (GSR) noted Ω_0 using the theorem 1 proposed in (Benhadj, 1996a) and presented in page 64.

3. Determine the discrete reverse model of the system (1) using the first or the second approach.

4. Apply the reverse model for different initial states belonging to the boundary Γ_0 of the GSR Ω_0.

The application of the backward iteration k times on the boundary Γ_0 leads to a larger stability region Ω_k such that $\Omega_0 \subset \Omega_1 \subset ... \subseteq \Omega_{k-1} \subset \Omega_k$.

The performance of the backward iteration algorithm depends on the used inversion technique of the polynomial discrete model among the above two proposed approaches.

In order to compare the two formulated approaches, we propose next their implementation on a synchronous generator second order model.

2.3 Simulation study

We consider the simplified model of a synchronous generator described by the following second order differential equation (Willems, 1971):

$$\frac{d^2\delta}{dt^2} + a\frac{d\delta}{dt} + \sin(\delta + \delta_0) - \sin(\delta_0) = 0 \tag{11}$$

where δ_0 is the power angle and δ is the power angle variation.

The continuous state equation of the studied process for the state vector: $\begin{cases} x_1 = \delta \\ x_2 = \dfrac{d\delta}{dt} \end{cases}$ is

given by the following couple of equation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_2 - \sin(x_1 + \delta_0) + \sin\delta_0 \end{cases} \tag{12}$$

where a is the damping factor.

This nonlinear system can be approached by a third degree polynomial system:

$$X_{k+1} = A_1.X_k + A_2.X_k^{[2]} + A_3.X_k^{[3]} \tag{13}$$

with $\quad A_1 = \begin{pmatrix} 0 & 1 \\ -\cos\delta_0 & -a \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{\sin\delta_0}{2} & 0 & 0 & 0 \end{pmatrix}$

and $\quad A_3 = \begin{pmatrix} 0 & 0 & & \\ & & 0_{2\times 6} \\ \dfrac{\cos\delta_0}{6} & 0 & \end{pmatrix}$

The discretization of the state equation (13) using Newton-Raphson technique (Jennings & McKeown, 1992), (Bacha et al, 2006a) ,(Bacha et al, 2006b) with a sampling period T leads to the following discrete state equation of the synchronous machine:

$$X_{k+1} = F_1.X_k + F_2.X_k^{[2]} + F_3.X_k^{[3]} \tag{14}$$

with
$$F_1 = \begin{pmatrix} 1 & T \\ -T\cos\delta_0 & 1-aT \end{pmatrix} , \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{T.\sin\delta_0}{2} & 0 & 0 & 0 \end{pmatrix}$$

and
$$F_3 = \begin{pmatrix} 0 & 0 & & \\ & & 0_{2\times 6} \\ \dfrac{T.\cos\delta_0}{6} & 0 & \end{pmatrix}$$

With the following parameters: $\begin{cases} \delta_0 = 0.412 \\ a = 0.5 \\ T = 0.05 \end{cases}$

one obtains the following numerical values of the matrices F_i, i=1, 2, 3.

$$F_1 = \begin{pmatrix} 1 & 0.05 \\ -0.0458 & 0.975 \end{pmatrix} , \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.0076 & 0 & 0 & 0 \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 0 & 0 & & \\ & & 0_{2\times 6} \\ 0.01 & 0 & \end{pmatrix}$$

One can easily verify that the equilibrium Xe=0 is asymptotically stable since we have $|eig(F_1)| < 1$.

Our aim now is the estimation of a local domain of stability of the origin equilibrium Xe=0.

For this goal, we make use of the backward iteration technique with the proposed inversion algorithms of the direct system (14) applied from the boundary Γ_0 of the ball Ω_0 centred in the origin and of radius R_0=0.42 which is a guaranteed stability region (GSR) that characterized the method developed in (Benhadj, 1996a).

- Domain of stability obtained by using the first approach of discrete model inversion :

The implementation of the first approach of the discrete model inversion described by equation (4) leads, after running 2000 iterations, to the region of stability represented in the figure 1.

Figure 2 represents the stability domain of the discrete system (14) obtained after running the backward iteration based on the inversion model (4) 50000 times.

It is clear that the domain obtained after 50000 iterations is larger than that obtained after 2000 iterations and it is included in the exact stability domain of the studied system, which reassures the availability of the first proposed approach of the backward iteration formulation.
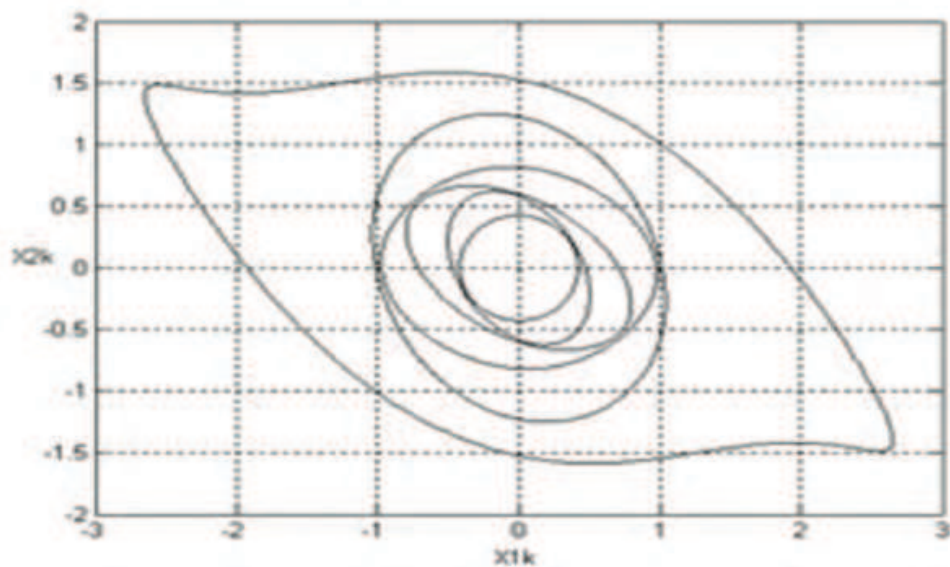
Fig. 1. RAS of discrete synchronous generator model obtained after 2000 backward iterations based on the first proposed approach
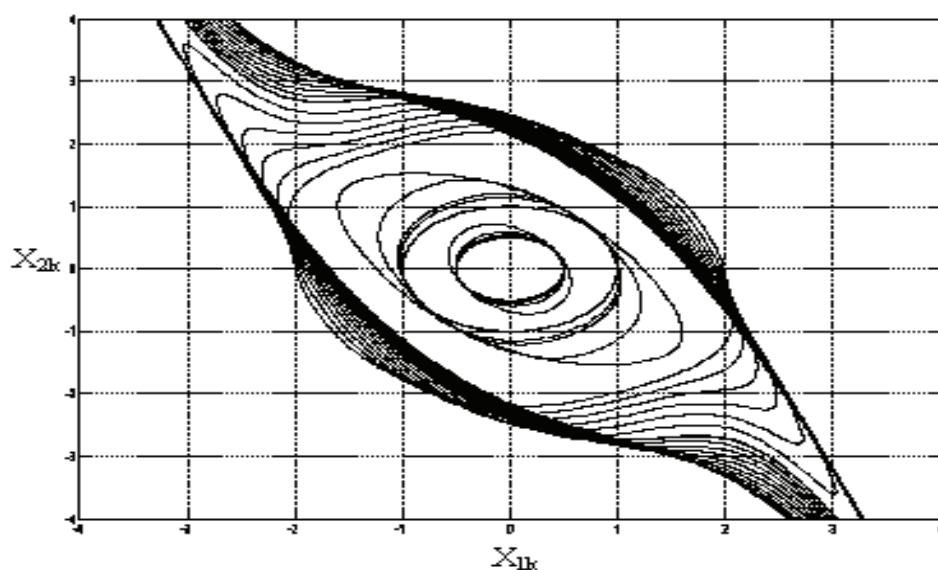

Fig. 2. RAS of discrete synchronous generator model obtained after 50000 iterations based on the first proposed approach.

- Domain of stability obtained by using the second approach of discrete model inversion

When applying the discrete backward iteration formulated by using the reverse model (8) we obtain the stability domain shown in figure 3 after 1000 iterations and the domain presented in figure 4 after 50000 iterations.

In figure 4 it seems that the stability domain estimated by the second approach of backward iteration is larger and more precise than that obtained by the first approach. The reached stability domain represents almost the entire domain of stability, which shows the efficiency of the second approach of the backward iteration, particularly when the order of the studied system is not very high as a second order system.
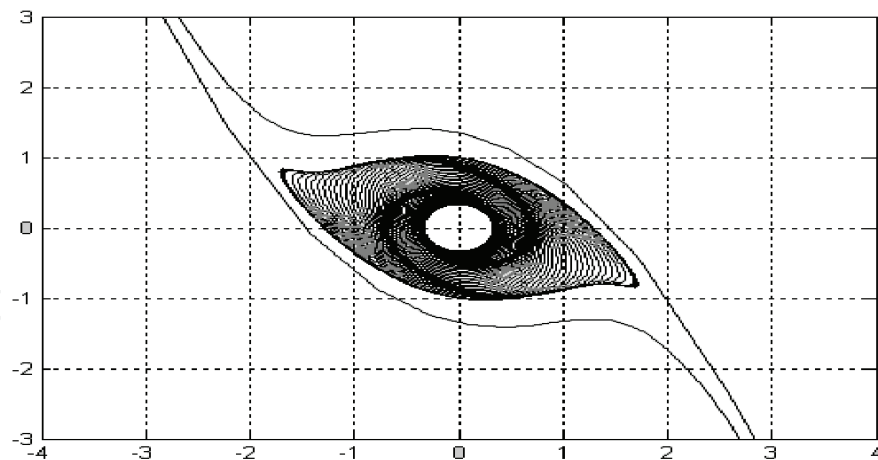
Fig. 3. RAS of discrete synchronous generator model obtained after 1000 backward iterations based on the second proposed approach
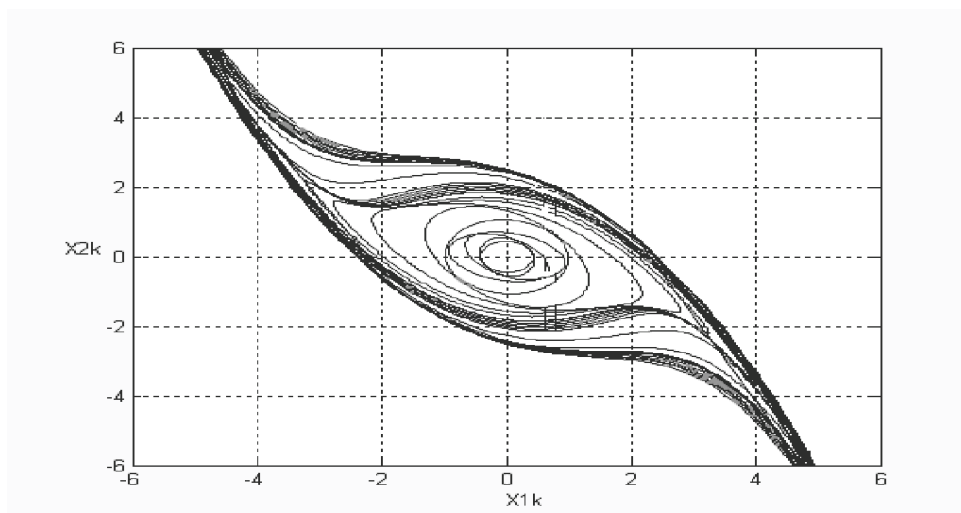

Fig. 4. RAS of discrete synchronous generator model obtained after 50000 backward iterations based on the second proposed approach

2.4 Conclusion

In this work, the extension of the reversing trajectory concept for the estimation of a region of asymptotic stability of nonlinear discrete systems has been investigated.

The polynomial nonlinear systems have been particularly considered.

Since the reversing trajectory method, also called backward iteration, is based on the inversion of the direct discrete model, two dissimilar approaches have been proposed in this work for the formulation of the reverse of a discrete polynomial system.

The application of the backward iteration with both proposed approaches starting from the boundary of an initial guaranteed stability region allows to an important enlargement of the searched stability domain. In the particular case of the second order systems, the studied technique can lead to the entire domain of stability.

The simulation of the developed algorithms on a second order model of a synchronous generator has shown the validity of the two approximation ideas with a little superiority of the second approach of the discrete model inversion, since the RAS obtained by this last one is larger and more precise than the one yielded by the first approximation approach.

3. Technique of a guaranteed stability domain determination

In this part we consider a new advanced approach of estimating a large asymptotic stability domain for discrete time nonlinear polynomial system. Based on the Kronecker product (Benhadj Braiek, 1996a; Benhadj Braiek, 1996b) and the Grownwell-bellman lemma for the estimation of a guaranteed region of stability; the proposed method permits to improve previous results in this field of research.

3.1 Description of the studied systems

We consider the discrete nonlinear systems described by a state equation of the following form

$$X(k + I) = F(x(k)) = \sum_{i=I}^{q} A_i X^{[i]}(k) \tag{15}$$

where k is the discrete time variable, $X(k) \in \Re^n$ is the state vector, $X^{[i]}(k)$ designates the i-th Kronecker power of the vector $X(k)$ and $A_i, i = I, \ldots, q$ are $(n \times n^i)$ matrices. The system (15) can also be written in the following form:

$$X(k + 1) = M(X(k)).X(k) \tag{16}$$

where:

$$M(X(k)) = A_1 + \sum_{j=2}^{q} A_j (I_n \otimes X^{[j-1]}(X(k)) \tag{17}$$

where $\otimes$ is the Kronecker product (Benhadj Braiek, 1996a; Benhadj Braiek, 1996b).
Assumption 1: The linear part of the discrete systems (15) is asymptotically stable i.e. all the eigenvalues of the matrix are of module little than 1.

3.2 Guaranteed stability region

Our purpose is to determine a sufficient domain Ω_0 of the initial conditions variation, in which the asymptotic stability of system (15) is guaranteed, according to the following definition:

$$\begin{aligned} \forall X_0 \in \Omega_0 \quad & X(k, k_0, X_0) \in \Re \quad \forall k \\ and \quad & \lim_{k \to +\infty} X(k, k_0, X_0) = 0 \end{aligned} \tag{18}$$

where $X(k, k_0, X_0)$ designates the solution of the nonlinear recurrent equation (15) with the initial condition $X(k_0) = X_0$.

The stability domain that we propose is considered as a ball of radius R_0 and of centre the origin $X = 0$ i.e.,

$$\Omega_0 = \left\{ X_0 \in \Re^n; \|X_0\| < R_0 \right\} \tag{19}$$

the radius R_0 is called the stability radius of the system (15).

A simple domain ensuring the stability of the system (15) is defined by the following theorem (Benhadj Braiek, 1996b).

Theorem 1. Consider the discrete system (15) satisfying the assumption 1, and let c and α the positive numbers verifying $\alpha \in \left]0 \; 1\right[$,

$$\left\| A_1^{k-k_0} \right\| \le c\alpha^{k-k_0} \forall k \ge k_0 \tag{20}$$

Then this system is asymptotically stable on the domain Ω_0 defined in (19) with R_0 the unique positive solution of the following equation:

$$\sum_{k=2}^{q} \gamma_k R_0^{k-1} - \frac{1-\alpha}{c} = 0 \tag{21}$$

where $\gamma_k, k = 2,...,q$ denote :

$$\gamma_k = c^{k-1} \left\| A_k \right\| \tag{22}$$

Furthermore the stability is exponentially.

Proof. The equation (15) can be written as:

$$X(k+1) = A_1 X(k) + h(X(k))X(k) \tag{23}$$

with

$$h(X(k)) = \sum_{j=2}^{q} A_j (I_n \otimes X^{[j-1]}(k)) \tag{24}$$

Let us consider that:

$$\forall k \ge k_0, \left\| X(k) \right\| \le R \tag{25}$$

then we have, using the matrix norm property of the Kronecker product

$$\left\| h(X(k)) \right\| \le \lambda(R) \tag{26}$$

with:

$$\lambda(R) = \sum_{j=2}^{q} \left\| A_j \right\| R^{j-1} \tag{27}$$

By using the lemma 1(see the appendix), we have:

$$\left\| X(k) \right\| \le c(\alpha + c\lambda(R))^{k-k_0} \left\| X(k_0) \right\| \tag{28}$$

with $g(X) = h(X)X$, we have $\left\| g(X) \right\| \le \lambda(R)\left\| X \right\|$.

Then, if:

$$\lambda(R) < \frac{1-\alpha}{c} \tag{29}$$

Now, to ensure the hypothesis (25) it is sufficient to have (from (21)):

$$c\left\| X(k_0) \right\| \le R_1 \quad \text{or} \quad \left\| X(k_0) \right\| \le R_0 = \frac{R_1}{c} \tag{30}$$

R_1 satisfies the equation (29) implies that R_0 satisfies the equation (21) of the theorem 1.

3.3 Enlargement of the guaranteed stability region (GSR)

Our object in this section is to enlarge the Guaranteed Stability Region Ω_0 characterized in the section 3.2. For this goal, we consider the boundary Γ_0 of the obtained GSR of radius R_0. Let X^i_0 be a point belonging in Γ_0, and X^i_k the image of X^i_0 by the $F(\,.\,)$ function characterizing the considered system, k times.

$$X^i_k = F^k \left(X^i_0 \right) \tag{31}$$

X^i_k is then a point belonging in the stability domain Ω_0;

$$\left\| X^i_k \right\| < R_0 \quad , \qquad \left\| F^k \left(X^i_0 \right) \right\| < R_0 \tag{32}$$

To enlarge the GSR, we will look for a radius $r_{0,i}$ such that for any initial state X_0 verifying

$$\left\| X_0 - X^i_0 \right\| \le r_{0,i}$$

one has

$$X_k = F^k \left(X_0 \right) \in \Omega_0 \tag{33}$$

and the fact that after k iterations the state of the system attends the domain Ω_0 ensures that X_0 is a state belonging in the stability domain.

Let us note:

$$\delta X_0 = X_0 - X^i_0 \tag{34}$$

And for $k \ge 1$

$$\delta X_k = X_k - X^i_k = F^k \left(X_0 \right) - F^k \left(X^i_0 \right) \tag{35}$$

δX_k can be expressed in terms of δX_0 as a polynomial function of degree $s=q^k$ where q is the degree of the $F(\,.\,)$ polynomial characterizing the system:

$$\delta X_k = E_1.\delta X_0 + E_2.\delta X_0^{[2]} + ... + E_s.\delta X_0^{[s]}; \quad s = q^k \tag{36}$$

E_1, E_2…, E_s are matrices depending on k and X_0^i and they can easily expressed in terms of A_i and X_0^i.

In the particular case where $q = 3$ and $k = 1$ one has:

$$\delta X_1 = X_1 - X_1^i = F(X_0) - F(X_0^i)$$
$$= D_1.\delta X_0 + D_2.\delta X_0^{[2]} + ... + D_r.\delta X_0^{[r]} \tag{37}$$

where

$$\begin{cases} D_1 = \begin{bmatrix} A_1 + A_2\left(X_{j,k} \otimes I_n\right) + A_2\left(I_n \otimes X_{j,k}\right) + \\ A_3\left(\left(X_{j,k} \otimes I_n\right) \otimes X_{j,k}\right) + \\ A_3\left(X_{j,k}^{[2]} \otimes I_n\right) + \\ A_3\left(\left(I_n \otimes X_{j,k}\right) \otimes X_{j,k}\right) \end{bmatrix} \\ \\ D_2 = \begin{bmatrix} A_2 + A_3\left(\left(X_{j,k} \otimes I_n\right) \otimes I_n\right) + \\ A_3\left(\left(I_n \otimes X_{j,k}\right) \otimes I_n\right) + \\ A_3\left(I_n^{[2]} \otimes X_{j,k}\right) \end{bmatrix} \\ \\ D_3 = A_3 \end{cases}$$

From the relation:

$$X_k = \delta X_k + F^k\left(X_0^i\right) \tag{38}$$

one has:

$$\|X_k\| \le \|\delta X_k\| + \left\|F^k\left(X_0^i\right)\right\| \tag{39}$$

From (36) we have:

$$\|\delta X_k\| \le e_1.\|\delta X_0\| + e_2.\left\|\delta X_0^{[2]}\right\| + ... + e_s.\left\|\delta X_0^{[s]}\right\| \tag{40}$$

with:

$$e_j = \|E_j\|, \quad j = 1, 2, ..., s$$

Hence we have:

$$\|X_k\| \leq \sum_{j=1}^{s} e_j \cdot \left\| \delta X_0^{[j]} \right\| + \left\| F^k \left(X_0^i \right) \right\|$$

$$\leq \sum_{j=1}^{s} e_j \cdot r_{0,i}^{j} + \left\| F^k \left(X_0^i \right) \right\| \tag{41}$$

Since it is desired that:

$$\|X_k\| \leq R_0; \quad (X_k \in \Omega_0) \tag{42}$$

it will be sufficient to have:

$$\sum_{j=1}^{s} e_j \cdot r_{0,i}^{j} = R_0 - \left\| F^k \left(X_0^i \right) \right\|$$

$$e_1 \cdot r_{0,i} + e_2 \cdot r_{0,i}^2 + \ldots + e_s \cdot r_{0,i}^s = R_0 - \left\| F^k \left(X_0^i \right) \right\| > 0 \tag{43}$$

which yields:

$$\left\| \delta X_0 \right\| = \left\| X_0 - X_0^i \right\| \leq r_{0,i} \tag{44}$$

where r$_{0,i}$ is the unique positive solution of the polynomial equation:

$$e_1 \cdot r_{0,i} + e_2 \cdot r_{0,i}^2 + \ldots + e_2 \cdot r_{0,i}^s = R_0 - \left\| F^k \left(X_0^i \right) \right\| \tag{45}$$

and this result can be stated in the following theorem.

Theorem 2
Let the following polynomial discrete system described by:

$$X_{k+1} = F\left(X_k \right) = A_1 \cdot X_k + A_2 \cdot X_k^{[2]} + \ldots + A_r \cdot X_k^{[r]} \tag{46}$$

and let Ω_0 the GSR of radius R_0 given in theorem 1, and Γ_0 the boundary of the GSR, then:
For any point $X_0^i \in \Gamma_0$, the ball Ω_i centred on X_0^i and of radius $r_{0,i}$ the unique positive solution of the equation (45) is also a domain of stability of the considered system.
In the particular case where consider *k=1*, one has the following corollary.
Corollary 1
The ball B_i of radius $r_{0,i}$ solution of the equation:

$$\left\| D_1 \right\| \cdot r_{0,i} + \left\| D_2 \right\| \cdot r_{0,i}^2 + \ldots + \left\| D_q \right\| \cdot r_{0,i}^q = R_0 - \left\| F \left(X_0^i \right) \right\| > 0 \tag{47}$$

is a domain of asymptotic stability of the considered system.

After considering all the points $X_0^i \in \Gamma_0$ (varying i), a new domain of stability is obtained by collecting all the little balls Ω_i to Ω_0 :

$$D = \bigcup_i \Omega_i \qquad (48)$$

For all the considered points X_i^k and the associated balls Ω_i, we can construct a new domain of stability Ω_{i+1} with a boundary Γ_{i+1}, and we have $\Omega_i \subset \Omega_{i+1}$. This procedure can be repeated with these new data Ω_{i+1} and Γ_{i+1} until obtaining a sufficiently large stability domain of the considered system equilibrium points.
This idea is illustrated in Figure 5.
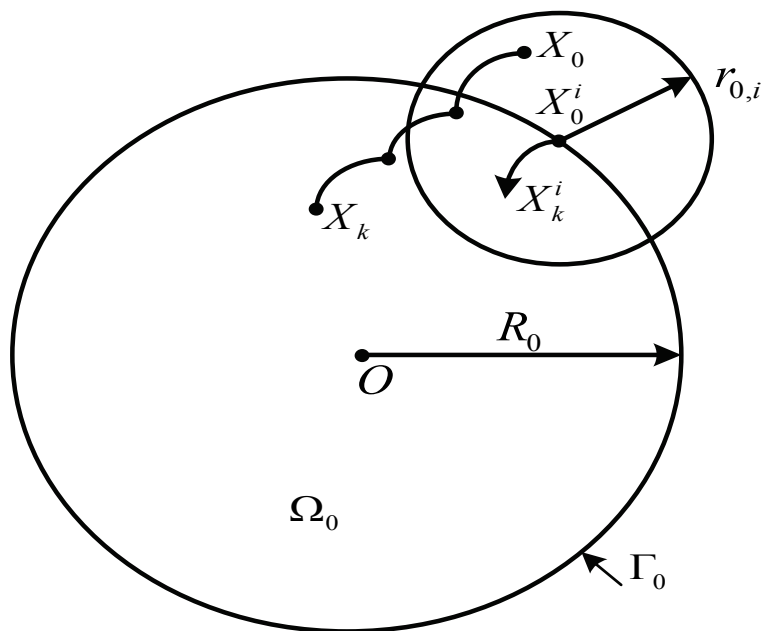
Fig. 5. Illustration of the principle of the proposed method

3.4 Simulation results: application to Van Der pool model
Let us consider the following discrete polynomial Van Der Pool model obtained from the Raphson-newton approximation: (Jening & Mc Keown, 1992)

$$X_{k+1} = A_1 X_k + A_3 X_k^{[3]} \qquad (49)$$

Where $X_k = \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix}$ $A_1 = \begin{pmatrix} 0.9988 & -0.0488 \\ 0.0488 & 0.950 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & -0.0012 & & \\ & & & 0_{2\times6} \\ 0 & 0.0488 & & \end{pmatrix}$

Equation (49) has a linear asymptotically stable matrix A_1, which verifies the inequalities (20) with c=1.7 and α=0.65. Then, we may conclude that the origin is exponentially stable for each initial state X_0 included in the disc Ω_0 centered in the origin and of radius R_0=0.33.

Figure 6 shows the guaranteed stability domain Ω_0 obtained by the application of the theorem 1, and the enlarged region resulting from the application of the theorem 2 for one iteration (k=1), and for 22 points X_0^i on the boundary Γ_0. It comes out that the new result stated in the theorem 2 leads to an important enlargement of the guaranteed stability domain.
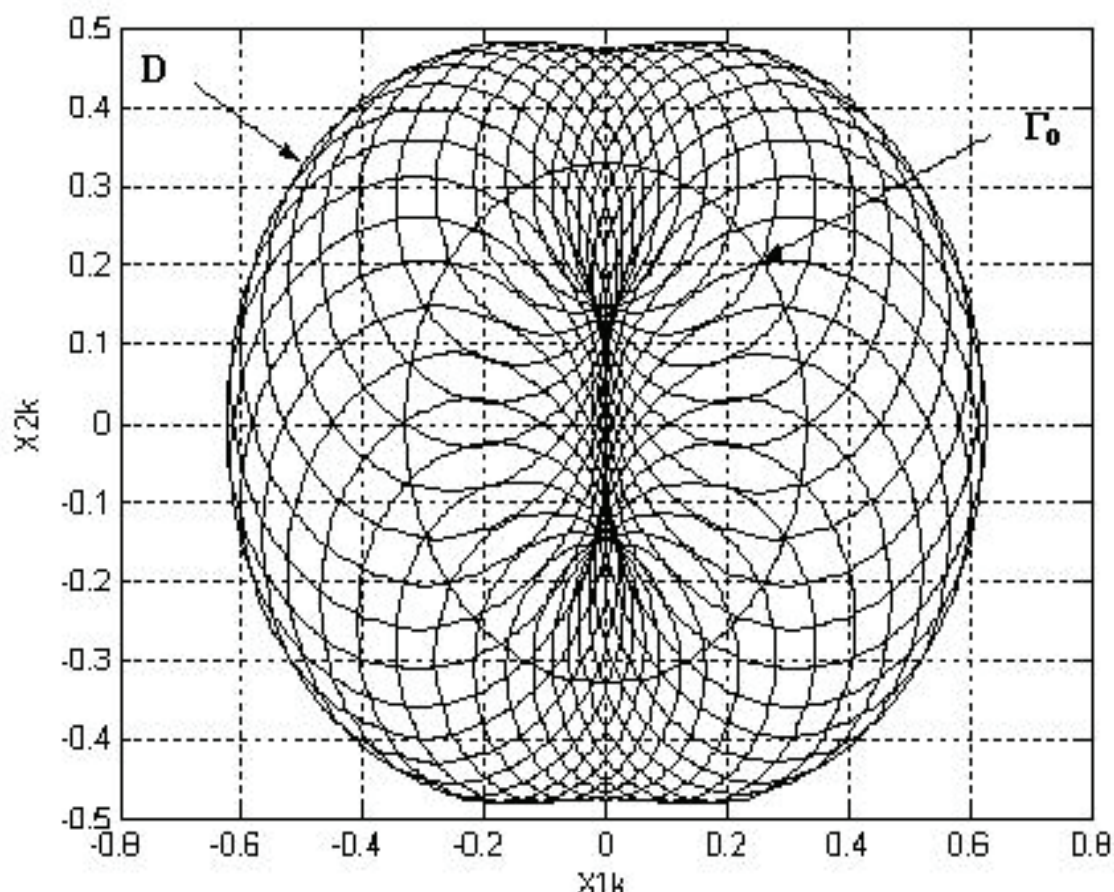

Fig. 6. Enlargement of a guaranteed RAS estimate of Van Der Pool discrete model

4. Conclusion

An advanced discrete algebraic method has been developed to determine and enlarge the region of asymptotic stability for autonomous nonlinear polynomial discrete time systems.
The exactness of the obtained RAS in this case constitutes the main advantage of the proposed approach.
The proposed technique is proved theoretically and tested via numerical simulation on the discrete polynomial Van Der Pool model.
The original discrete developed method is equivalent to the reversing trajectory method which used to determine the RAS for continuous systems.
Further research will be focused on the development and the implementation of an optimal numerical tool which allows to reach the larger region of asymptotic stability for discrete nonlinear systems.

5. Appendix

Lemma 1 (Benhadj Braiek, 1996b):
Let a discrete nonlinear system defined by the state equation:

$$X(k+1) = A_1 X(k) + g(k, X(k)) \tag{50}$$

where the linear part satisfies the assumption 1, and the nonlinear part $g(k, X(k))$ verifies the following inequality :

$$g(k, X(k)) \le \beta \|X'k)\| \tag{51}$$

where β is a positive constant.
Let $\Phi(k, k_0)$ denotes the transition matrix of the linear part of the discrete system (50):

$$\Phi(k, k_0) = A_1^{k-k_0} \tag{52}$$

and let c and α the positive numbers verifying $\alpha \in \,]0\ 1[\,$,

$$\|\Phi(k, k_0)\| \le c\alpha^{k-k_0} \qquad \forall k \ge k_0 \tag{53}$$

Then the solution $X(k)$ of the system (50) verifies the following inequality:

$$\|X(k)\| \le c(\alpha + c\beta)^{k-k_0} \|X(k_0)\| \tag{54}$$

So if $\beta < 1 - \dfrac{\alpha}{c}$, the system (50) is exponentially stable.

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