Chapter

Obscure Qubits and Membership Amplitudes

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Abstract

We propose a concept of quantum computing which incorporates an additional kind of uncertainty, i.e. vagueness (fuzziness), in a natural way by introducing new entities, obscure qudits (e.g. obscure qubits), which are characterized simultaneously by a quantum probability and by a membership function. To achieve this, a membership amplitude for quantum states is introduced alongside the quantum amplitude. The Born rule is used for the quantum probability only, while the membership function can be computed from the membership amplitudes according to a chosen model. Two different versions of this approach are given here: the "product" obscure qubit, where the resulting amplitude is a product of the quantum amplitude and the membership amplitude, and the "Kronecker" obscure qubit, where quantum and vagueness computations are to be performed independently (i.e. quantum computation alongside truth evaluation). The latter is called a double obscure-quantum computation. In this case, the measurement becomes mixed in the quantum and obscure amplitudes, while the density matrix is not idempotent. The obscure-quantum gates act not in the tensor product of spaces, but in the direct product of quantum Hilbert space and so called membership space which are of different natures and properties. The concept of double (obscure-quantum) entanglement is introduced, and vector and scalar concurrences are proposed, with some examples being given.

Keywords: qubit, fuzzy, membership function, amplitude, Hilbert space

1. Introduction

Nowadays, the development of quantum computing technique is governed by theoretical extensions of its ground concepts [1–3]. One of them is to allow two kinds of uncertainty, sometimes called randomness and vagueness/fuzziness (for a review, see, [4]), which leads to the formulation of combined probability and possibility theories [5] (see, also, [6–9]). Various interconnections between vagueness and quantum probability calculus were considered in [10–13], including the treatment of inaccuracy in measurements [14, 15], non-sharp amplitude densities [16] and the related concept of partial Hilbert spaces [17].

Relations between truth values and probabilities were also given in [18]. The hardware realization of computations with vagueness was considered in [19, 20]. On the fundamental physics side, it was shown that the discretization of space–time at small distances can lead to a discrete (or fuzzy) character for the quantum states themselves [21–24].

With a view to applications of the above ideas in quantum computing, we introduce a definition of quantum state which is described by both a quantum probability and a membership function, and thereby incorporate vagueness/fuzziness directly into the formalism. In addition to the probability amplitude we will define a membership amplitude, and such a state will be called an obscure/fuzzy qubit (or qudit).

In general, the Born rule will apply to the quantum probability alone, while the membership function can be taken to be an arbitrary function of all the amplitudes fixed by the chosen model of vagueness. Two different models of "obscure-quantum computations with truth" are proposed below: (1) A "Product" obscure qubit, in which the resulting amplitude is the product (in $\mathbb{C}$) of the quantum amplitude and the membership amplitude; (2) A "Kronecker" obscure qubit for which computations are performed "in parallel", so that quantum amplitudes and the membership amplitudes form "vectors", which we will call obscure-quantum amplitudes. In the latter case, which we call a double obscure-quantum computation, the protocol of measurement depends on both the quantum and obscure amplitudes, and in this case the density matrix need not be idempotent. We define a new kind of "gate", namely, the obscure-quantum gates, which are linear transformations in the direct product (not in the tensor product) of spaces: a quantum Hilbert space and a so-called membership space having special fuzzy properties. We introduce a new concept of double (obscure-quantum) entanglement, in which vector and scalar concurrences are defined and computed for some examples.

2. Preliminaries

To establish notation standard in the literature (see, e.g. [1, 2, 25–27]) we present the following definitions. In an underlying d-dimensional Hilbert space, the standard qudit (using the computational basis and Dirac notation) $\mathcal{H}_q^{(d)}$ is given by

$$\left|\psi^{(d)}\right\rangle = \sum_{i=0}^{d-1} a_i |i\rangle, \qquad a_i \in \mathbb{C}, |i\rangle \in \mathcal{H}_q^{(d)}, \tag{1}$$

where a_i is a probability amplitude of the state $|i\rangle$. (For a review, see, e.g. [28, 29]) The probability p_i to measure the ith state is $p_i = F_{p_i}(a_1, \dots, a_n)$, $0 \le p_i \le 1$, $0 \le i \le d-1$. The shape of the functions F_{p_i} is governed by the Born rule $F_{p_i}(a_1, \dots, a_d) = |a_i|^2$, and $\sum_{i=0}^{d} p_i = 1$. A one-qudit ($L = 1$) quantum gate is a unitary transformation $U^{(d)} : \mathcal{H}_q^{(d)} \to \mathcal{H}_q^{(d)}$ described by unitary $d \times d$ complex matrices acting on the vector (1), and for a register containing L qudits quantum gates are unitary $d^L \times d^L$ matrices. The quantum circuit model [30, 31] forms the basis for the standard concept of quantum computing. Here the quantum algorithms are compiled as a sequence of elementary gates acting on a register containing L qubits (or qudits), followed by a measurement to yield the result [25, 32].

For further details on qudits and their transformations, see for example the reviews [28, 29] and the references therein.

3. Membership amplitudes

We define an obscure qudit with d states via the following superposition (in place of that given in (1))

$$\left|\psi_{ob}^{(d)}\right\rangle = \sum_{i=1}^{d-1} \alpha_i a_i |i\rangle, \tag{2}$$

where a_i is a (complex) probability amplitude $a_i \in \mathbb{C}$, and we have introduced a (real) membership amplitude α_i, with $\alpha_i \in [0,1]$, $0 \le i \le d-1$. The probability p_i to find the ith state upon measurement, and the membership function μ_i ("of truth") for the ith state are both functions of the corresponding amplitudes as follows

$$p_i = F_{p_i}(a_0, \dots, a_{d-1}), \qquad 0 \le p_i \le 1, \tag{3}$$

$$\mu_i = F_{\mu_i}(\alpha_0, \dots, \alpha_{d-1}), \qquad 0 \le \mu_i \le 1. \tag{4}$$

The dependence of the probabilities of the ith states upon the amplitudes, i.e. the form of the function F_{p_i} is fixed by the Born rule

$$F_{p_i}(a_1, \dots, a_n) = |a_i|^2, \tag{5}$$

while the form of F_{μ_i} will vary according to different obscurity assumptions. In this paper we consider only real membership amplitudes and membership functions (complex obscure sets and numbers were considered in [33–35]). In this context the real functions F_{p_i} and F_{μ_i}, $0 \le i \le d-1$ will contain complete information about the obscure qudit (2).

We impose the normalization conditions

$$\sum_{i=0}^{d-1} p_i = 1, \tag{6}$$

$$\sum_{i=0}^{d-1} \mu_i = 1, \tag{7}$$

where the first condition is standard in quantum mechanics, while the second condition is taken to hold by analogy. Although (7) may not be satisfied, we will not consider that case.

For $d = 2$, we obtain for the obscure qubit the general form (instead of that in (2))

$$\left|\psi_{ob}^{(2)}\right\rangle = \alpha_0 a_0 |0\rangle + \alpha_1 a_1 |1\rangle, \tag{8}$$

$$F_{p_0}(a_0, a_1) + F_{p_1}(a_0, a_1) = 1, \tag{9}$$

$$F_{\mu_0}(\alpha_0, \alpha_1) + F_{\mu_1}(\alpha_0, \alpha_1) = 1. \tag{10}$$

The Born probabilities to observe the states $|0\rangle$ and $|1\rangle$ are

$$p_0 = F_{p_0}^{Born}(a_0, a_1) = |a_0|^2, \qquad p_1 = F_{p_1}^{Born}(a_0, a_1) = |a_1|^2, \tag{11}$$

and the membership functions are

$$\mu_0 = F_{\mu_0}(\alpha_0, \alpha_1), \qquad \mu_1 = F_{\mu_1}(\alpha_0, \alpha_1). \tag{12}$$

If we assume the Born rule (11) for the membership functions as well

$$F_{\mu_0}(\alpha_0, \alpha_1) = \alpha_0^2, \qquad F_{\mu_1}(\alpha_0, \alpha_1) = \alpha_1^2, \tag{13}$$

(which is one of various possibilities depending on the chosen model), then

$$|a_0|^2 + |a_1|^2 = 1, \tag{14}$$

$$\alpha_0^2 + \alpha_1^2 = 1. \tag{15}$$

Using (14)–(15) we can parametrize (8) as

$$\left|\psi_{ob}^{(2)}\right\rangle = \cos\frac{\theta}{2}\cos\frac{\theta_\mu}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\sin\frac{\theta_\mu}{2}|1\rangle, \tag{16}$$

$$0 \le \theta \le \pi, \qquad 0 \le \varphi \le 2\pi, \qquad 0 \le \theta_\mu \le \pi. \tag{17}$$

Therefore, obscure qubits (with Born-like rule for the membership functions) are geometrically described by a pair of vectors, each inside a Bloch ball (and not as vectors on the boundary spheres, because "$|\sin|, |\cos| \le 1$"), where one is for the probability amplitude (an ellipsoid inside the Bloch ball with $\theta_\mu = const_1$), and the other for the membership amplitude (which is reduced to an ellipse, being a slice inside the Bloch ball with $\theta = const_2$, $\varphi = const_3$). The norm of the obscure qubits is not constant however, because

$$\left\langle \psi_{ob}^{(2)} | \psi_{ob}^{(2)} \right\rangle = \frac{1}{2} + \frac{1}{4}\cos\left(\theta + \theta_\mu\right) + \frac{1}{4}\cos\left(\theta - \theta_\mu\right). \tag{18}$$

In the case where $\theta = \theta_\mu$, the norm (18) becomes $1 - \frac{1}{2}\sin^2\theta$, reaching its minimum $\frac{1}{2}$ when $\theta = \theta_\mu = \frac{\pi}{2}$.

Note that for complicated functions $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ the condition (15) may be not satisfied, but the condition (7) should nevertheless always be valid. The concrete form of the functions $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ depends upon the chosen model. In the simplest case, we can identify two arcs on the Bloch ellipse for α_0, α_1 with the membership functions and obtain

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arctan\frac{\alpha_1}{\alpha_0}, \tag{19}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arctan\frac{\alpha_0}{\alpha_1}, \tag{20}$$

such that $\mu_0 + \mu_1 = 1$, as in (7).

In [36, 37] a two stage special construction of quantum obscure/fuzzy sets was considered. The so-called classical-quantum obscure/fuzzy registers were introduced in the first step (for $n = 2$, the minimal case) as

$$|s\rangle_f = \sqrt{1-f}|0\rangle + \sqrt{f}|1\rangle, \tag{21}$$

$$|s\rangle_g = \sqrt{1-g}|0\rangle + \sqrt{g}|1\rangle, \tag{22}$$

where $f, g \in [0, 1]$ are the relevant classical-quantum membership functions. In the second step their quantum superposition is defined by

$$|s\rangle = c_f|s\rangle_f + c_g|s\rangle_g, \tag{23}$$

where c_f and c_g are the probability amplitudes of the fuzzy states $|s\rangle_f$ and $|s\rangle_g$, respectively. It can be seen that the state (23) is a particular case of (8) with

$$\alpha_0 a_0 = c_f \sqrt{1-f} + c_g \sqrt{1-g}, \tag{24}$$

$$\alpha_1 a_1 = c_f \sqrt{f} + c_g \sqrt{g}. \tag{25}$$

This gives explicit connection of our double amplitude description of obscure qubits with the approach [36, 37] which uses probability amplitudes and the membership functions. It is important to note that the use of the membership amplitudes introduced here α_i and (2) allows us to exploit the standard quantum gates, but not to define new special ones, as in [36, 37].

Another possible form of $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ (12), with the corresponding membership functions satisfying the standard fuzziness rules, can be found using a standard homeomorphism between the circle and the square. In [38, 39] this transformation was applied to the probability amplitudes $a_{0,1}$, but here we exploit it for the membership amplitudes $\alpha_{0,1}$

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha_0^2 * \operatorname{sign} \alpha_0 - \alpha_1^2 * \operatorname{sign} \alpha_1 + 1}{2}}, \tag{26}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha_0^2 * \operatorname{sign} \alpha_0 + \alpha_1^2 * \operatorname{sign} \alpha_1 + 1}{2}}. \tag{27}$$

So for positive $\alpha_{0,1}$ we obtain (cf. [38])

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha_0^2 - \alpha_1^2 + 1}{2}}, \tag{28}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = 1. \tag{29}$$

The equivalent membership functions for the outcome are

$$\max\left(\min\left(F_{\mu_0}(\alpha_0, \alpha_1), 1 - F_{\mu_1}(\alpha_0, \alpha_1)\right), \min\left(1 - F_{\mu_0}(\alpha_0, \alpha_1)\right), F_{\mu_1}(\alpha_0, \alpha_1)\right), \tag{30}$$

$$\min\left(\max\left(F_{\mu_0}(\alpha_0, \alpha_1), 1 - F_{\mu_1}(\alpha_0, \alpha_1)\right), \max\left(1 - F_{\mu_0}(\alpha_0, \alpha_1)\right), F_{\mu_1}(\alpha_0, \alpha_1)\right). \tag{31}$$

There are many different models for $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ which can be introduced in such a way that they satisfy the obscure set axioms [7, 9].

4. Transformations of obscure qubits

Let us consider the obscure qubits in the vector representation, such that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{32}$$

are basis vectors of $\mathcal{H}_q^{(2)}$. Then a standard quantum computational process in the quantum register with L obscure qubits (qudits (1)) is performed by sequences of unitary matrices $\hat{U}$ of size $2^L \times 2^L$ $(n^L \times n^L)$, $\hat{U}^\dagger \hat{U} = \hat{I}$, which are called quantum gates ($\hat{I}$ is the unit matrix). Thus, for one obscure qubit the quantum gates are 2×2 unitary complex matrices.

In the vector representation, an obscure qubit differs from the standard qubit (8) by a 2×2 invertible diagonal (not necessarily unitary) matrix

$$\left|\psi_{ob}^{(2)}\right\rangle = \hat{M}(\alpha_0, \alpha_1)\left|\psi^{(2)}\right\rangle, \tag{33}$$

$$\hat{M}(\alpha_0, \alpha_1) = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_1 \end{pmatrix}. \tag{34}$$

We call $\hat{M}(\alpha_0, \alpha_1)$ a membership matrix which can optionally have the property

$$\mathrm{tr}\hat{M}^2 = 1, \tag{35}$$

if (15) holds.

Let us introduce the orthogonal commuting projection operators

$$\hat{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{36}$$

$$\hat{P}_0^2 = \hat{P}_0, \quad \hat{P}_1^2 = \hat{P}_1, \quad \hat{P}_0\hat{P}_1 = \hat{P}_1\hat{P}_0 = \hat{0}, \tag{37}$$

where $\hat{0}$ is the 2×2 zero matrix. Well-known properties of the projections are that

$$\hat{P}_0\left|\psi^{(2)}\right\rangle = a_0|0\rangle, \qquad \hat{P}_1\left|\psi^{(2)}\right\rangle = a_1|0\rangle, \tag{38}$$

$$\left\langle\psi^{(2)}\middle|\hat{P}_0\middle|\psi^{(2)}\right\rangle = |a_0|^2, \qquad \left\langle\psi^{(2)}\middle|\hat{P}_1\middle|\psi^{(2)}\right\rangle = |a_1|^2. \tag{39}$$

Therefore, the membership matrix (34) can be defined as a linear combination of the projection operators with the membership amplitudes as coefficients

$$\hat{M}(\alpha_0, \alpha_1) = \alpha_0\hat{P}_0 + \alpha_1\hat{P}_1. \tag{40}$$

We compute

$$\hat{M}(\alpha_0, \alpha_1)\left|\psi_{ob}^{(2)}\right\rangle = \alpha_0^2 a_0|0\rangle + \alpha_1^2 a_1|1\rangle. \tag{41}$$

We can therefore treat the application of the membership matrix (33) as providing the origin of a reversible but non-unitary "obscure measurement" on the standard qubit to obtain an obscure qubit (cf. the "mirror measurement" [40, 41] and also the origin of ordinary qubit states on the fuzzy sphere [42]).

An obscure analog of the density operator (for a pure state) is the following form for the density matrix in the vector representation

$$\rho_{ob}^{(2)} = \left|\psi_{ob}^{(2)}\right\rangle\left\langle\psi_{ob}^{(2)}\right| = \begin{pmatrix} \alpha_0^2|a_0|^2 & \alpha_0 a_0^* \alpha_1 a_1 \\ \alpha_0 a_0 \alpha_1 a_1^* & \alpha_1^2|a_1|^2 \end{pmatrix} \tag{42}$$

with the obvious standard singularity property $\det \rho_{ob}^{(2)} = 0$. But $\mathrm{tr}\rho_{ob}^{(2)} = \alpha_0^2|a_0|^2 + \alpha_1^2|a_1|^2 \neq 1$, and here there is no idempotence $\left(\rho_{ob}^{(2)}\right)^2 \neq \rho_{ob}^{(2)}$, which distincts $\rho_{ob}^{(2)}$ from the standard density operator.

5. Kronecker obscure qubits

We next introduce an analog of quantum superposition for membership amplitudes, called "obscure superposition" (cf. [43], and also [44]).

Quantum amplitudes and membership amplitudes will here be considered separately in order to define an "obscure qubit" taking the form of a "double superposition" (cf. (8), and a generalized analog for qudits (1) is straightforward)

$$|\mathbf{\Psi}_{ob}\rangle = \frac{\hat{A}_0|\hat{0}\rangle + \hat{A}_1|\hat{1}\rangle}{\sqrt{2}}, \tag{43}$$

where the two-dimensional "vectors"

$$\hat{A}_{0,1} = \begin{bmatrix} a_{0,1} \\ \alpha_{0,1} \end{bmatrix} \tag{44}$$

are the (double) "obscure-quantum amplitudes" of the generalized states $|\hat{0}\rangle$, $|\hat{1}\rangle$. For the conjugate of an obscure qubit we put (informally)

$$\langle\mathbf{\Psi}_{ob}| = \frac{\hat{A}_0^{\star}\langle\hat{0}| + \hat{A}_1^{\star}\langle\hat{1}|}{\sqrt{2}}, \tag{45}$$

where we denote $\hat{A}_{0,1}^{\star} = \begin{bmatrix} a_{0,1}^{*} & \alpha_{0,1} \end{bmatrix}$, such that $\hat{A}_{0,1}^{\star}\hat{A}_{0,1} = |a_{0,1}|^2 + \alpha_{0,1}^2$. The (double) obscure qubit is "normalized" in such a way that, if the conditions (14)–(15) hold, then

$$\langle\mathbf{\Psi}_{ob}|\mathbf{\Psi}_{ob}\rangle = \frac{|a_0|^2 + |a_1|^2}{2} + \frac{\alpha_0^2 + \alpha_1^2}{2} = 1. \tag{46}$$

A measurement should be made separately and independently in the "probability space" and the "membership space" which can be represented by using an analog of the Kronecker product. Indeed, in the vector representation (32) for the quantum states and for the direct product amplitudes (44) we should have

$$|\mathbf{\Psi}_{ob}\rangle_{(0)} = \frac{1}{\sqrt{2}}\hat{A}_0 \otimes_K \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \hat{A}_1 \otimes_K \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{47}$$

where the (left) Kronecker product is defined by (see (32))

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \otimes_K \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \alpha\begin{pmatrix} c \\ d \end{pmatrix} \end{bmatrix} = \begin{bmatrix} a(c\hat{e}_0 + d\hat{e}_1) \\ \alpha(c\hat{e}_0 + d\hat{e}_1) \end{bmatrix},$$
$$\hat{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \hat{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \hat{e}_{0,1} \in \mathcal{H}_q^{(2)}. \tag{48}$$

Informally, the wave function of the obscure qubit, in the vector representation, now "lives" in the four-dimensional space of (48) which has two two-dimensional spaces as blocks. The upper block, the quantum subspace, is the ordinary Hilbert space $\mathcal{H}_q^{(2)}$, but the lower block should have special (fuzzy) properties, if it is treated as an obscure (membership) subspace $\mathcal{V}_{memb}^{(2)}$. Thus, the four-dimensional space, where "lives" $\left|\Psi_{ob}^{(2)}\right\rangle$, is not an ordinary tensor product of vector spaces, because of (48), and the "vector" $\hat{A}$ on the l.h.s. has entries of different natures, that is the quantum amplitudes $a_{0,1}$ and the membership amplitudes $\alpha_{0,1}$. Despite the unit vectors in $\mathcal{H}_q^{(2)}$ and $\mathcal{V}_{memb}^{(2)}$ having the same form (32), they belong to different spaces (as they are vector spaces over different fields). Therefore, instead of (48) we introduce a "Kronecker-like product" $\tilde{\otimes}_K$ by

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \tilde{\otimes}_K \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a(c\hat{e}_0 + d\hat{e}_1) \\ \alpha(c\varepsilon_0 + d\varepsilon_1) \end{bmatrix}, \tag{49}$$

$$\hat{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{e}_{0,1} \in \mathcal{H}_q^{(2)}, \tag{50}$$

$$\varepsilon_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)}, \quad \varepsilon_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)}, \quad \varepsilon_{0,1} \in \mathcal{V}_{memb}^{(2)}. \tag{51}$$

In this way, the obscure qubit (43) can be presented in the from

$$\begin{aligned} |\boldsymbol{\Psi}_{ob}\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} a_0\hat{e}_0 \\ \alpha_0\varepsilon_0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} a_1\hat{e}_1 \\ \alpha_1\varepsilon_1 \end{bmatrix}. \end{aligned} \tag{52}$$

Therefore, we call the double obscure qubit (52) a "Kronecker obscure qubit" to distinguish it from the obscure qubit (8). It can be also presented using the Hadamard product (the element-wise or Schur product)

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \otimes_H \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} ac \\ \alpha d \end{bmatrix} \tag{53}$$

in the following form

$$|\boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \hat{A}_0 \otimes_H \hat{E}_0 + \frac{1}{\sqrt{2}} \hat{A}_1 \otimes_H \hat{E}_1, \tag{54}$$

where the unit vectors of the total four-dimensional space are

$$\hat{E}_{0,1} = \begin{bmatrix} \hat{e}_{0,1} \\ \varepsilon_{0,1} \end{bmatrix} \in \mathcal{H}_q^{(2)} \times \mathcal{V}_{memb}^{(2)}. \tag{55}$$

The probabilities $p_{0,1}$ and membership functions $\mu_{0,1}$ of the states $|\hat{0}\rangle$ and $|\hat{1}\rangle$ are computed through the corresponding amplitudes by (11) and (12)

$$p_i = |a_i|^2, \quad \mu_i = F_{\mu_i}(\alpha_0, \alpha_1), \quad i = 0, 1, \tag{56}$$

and in the particular case by (13) satisfying (15).

By way of example, consider a Kronecker obscure qubit (with a real quantum part) with probability p and membership function μ (measure of "trust") of the state $|\hat{0}\rangle$, and of the state $|\hat{1}\rangle$ given by $1-p$ and $1-\mu$ respectively. In the model (19)–(20) for μ_i (which is not Born-like) we obtain

$$|\mathbf{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \begin{pmatrix} \sqrt{p} \\ 0 \end{pmatrix} \\ \begin{pmatrix} \cos\frac{\pi}{2}\mu \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \begin{pmatrix} 0 \\ \sqrt{1-p} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \sin\frac{\pi}{2}\mu \end{pmatrix}^{(\mu)} \end{bmatrix} \tag{57}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{e}_0\sqrt{p} \\ \varepsilon_0 \cos\frac{\pi}{2}\mu \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{e}_1\sqrt{1-p} \\ \varepsilon_1 \sin\frac{\pi}{2}\mu \end{bmatrix},$$

where $\hat{e}_i$ and ε_i are unit vectors defined in (50) and (51).

This can be compared e.g. with the "classical-quantum" approach (23) and [36, 37], in which the elements of columns are multiplied, while we consider them independently and separately.

6. Obscure-quantum measurement

Let us consider the case of one Kronecker obscure qubit register $L = 1$ (see (47)), or using (48) in the vector representation (52). The standard (double) orthogonal commuting projection operators, "Kronecker projections" are (cf. (36))

$$\mathbf{P}_0 = \begin{bmatrix} \hat{P}_0 & \hat{0} \\ \hat{0} & \hat{P}_0^{(\mu)} \end{bmatrix}, \qquad \mathbf{P}_1 = \begin{bmatrix} \hat{P}_1 & \hat{0} \\ \hat{0} & \hat{P}_1^{(\mu)} \end{bmatrix}, \tag{58}$$

where $\hat{0}$ is the 2×2 zero matrix, and $\hat{P}_{0,1}^{(\mu)}$ are the projections in the membership subspace $\mathcal{V}_{memb}^{(2)}$ (of the same form as the ordinary quantum projections $\hat{P}_{0,1}$ (36))

$$\hat{P}_0^{(\mu)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{(\mu)}, \qquad \hat{P}_1^{(\mu)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^{(\mu)}, \qquad \hat{P}_0^{(\mu)}, \hat{P}_1^{(\mu)} \in \operatorname{End} \mathcal{V}_{memb}^{(2)}, \tag{59}$$

$$\hat{P}_0^{(\mu)2} = \hat{P}_0^{(\mu)}, \quad \hat{P}_1^{(\mu)2} = \hat{P}_1^{(\mu)}, \quad \hat{P}_0^{(\mu)}\hat{P}_1^{(\mu)} = \hat{P}_1^{(\mu)}\hat{P}_0^{(\mu)} = \hat{0}. \tag{60}$$

For the double projections we have (cf. (37))

$$\boldsymbol{P}_0^2 = \boldsymbol{P}_0, \;\; \boldsymbol{P}_1^2 = \boldsymbol{P}_1, \;\; \boldsymbol{P}_0\boldsymbol{P}_1 = \boldsymbol{P}_1\boldsymbol{P}_0 = \mathbf{0}, \tag{61}$$

where $\mathbf{0}$ is the 4×4 zero matrix, and $\boldsymbol{P}_{0,1}$ act on the Kronecker qubit (58) in the standard way (cf. (38))

$$\mathbf{P}_0|\mathbf{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \alpha_0\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0\hat{e}_0 \\ \alpha_0\varepsilon_0 \end{bmatrix} = \frac{1}{\sqrt{2}} \hat{A}_0 \otimes_H \hat{E}_0, \tag{62}$$

$$\mathbf{P}_1|\boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\begin{pmatrix}0\\1\end{pmatrix} \\ \alpha_1\begin{pmatrix}0\\1\end{pmatrix}^{(\mu)}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\hat{e}_1 \\ \alpha_1\varepsilon_1\end{bmatrix} = \frac{1}{\sqrt{2}}\hat{A}_1 \otimes_H \hat{E}_1. \tag{63}$$

Observe that for Kronecker qubits there exist in addition to (58) the following orthogonal commuting projection operators

$$\mathbf{P}_{01} = \begin{bmatrix} \hat{P}_0 & \hat{0} \\ \hat{0} & \hat{P}_1^{(\mu)}\end{bmatrix}, \qquad \mathbf{P}_{10} = \begin{bmatrix} \hat{P}_1 & \hat{0} \\ \hat{0} & \hat{P}_0^{(\mu)}\end{bmatrix}, \tag{64}$$

and we call these the "crossed" double projections. They satisfy the same relations as (61)

$$\boldsymbol{P}_{01}^2 = \boldsymbol{P}_{01}, \ \ \boldsymbol{P}_{10}^2 = \boldsymbol{P}_{10}, \ \ \boldsymbol{P}_{01}\boldsymbol{P}_{10} = \boldsymbol{P}_{10}\boldsymbol{P}_{01} = \mathbf{0}, \tag{65}$$

but act on the obscure qubit in a different ("mixing") way than (62) i.e.

$$\mathbf{P}_{01}|\boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_0\begin{pmatrix}1\\0\end{pmatrix} \\ \alpha_1\begin{pmatrix}0\\1\end{pmatrix}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_0\hat{e}_0 \\ \alpha_1\varepsilon_1\end{bmatrix}, \tag{66}$$

$$\mathbf{P}_{10}|\boldsymbol{\Psi}_{ob}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\begin{pmatrix}0\\1\end{pmatrix} \\ \alpha_0\begin{pmatrix}1\\0\end{pmatrix}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\hat{e}_1 \\ \alpha_0\varepsilon_0\end{bmatrix}. \tag{67}$$

The multiplication of the crossed double projections (64) and the double projections (58) is given by

$$\boldsymbol{P}_{01}\boldsymbol{P}_0 = \boldsymbol{P}_0\boldsymbol{P}_{01} = \begin{bmatrix} \hat{P}_0 & \hat{0} \\ \hat{0} & \hat{0}\end{bmatrix} \equiv \boldsymbol{Q}_0, \qquad \boldsymbol{P}_{01}\boldsymbol{P}_1 = \boldsymbol{P}_1\boldsymbol{P}_{01} = \begin{bmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{P}_1^{(\mu)}\end{bmatrix} \equiv \boldsymbol{Q}_1^{(\mu)}, \tag{68}$$

$$\boldsymbol{P}_{10}\boldsymbol{P}_0 = \boldsymbol{P}_0\boldsymbol{P}_{10} = \begin{bmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{P}_0^{(\mu)}\end{bmatrix} \equiv \boldsymbol{Q}_0^{(\mu)}, \qquad \boldsymbol{P}_{10}\boldsymbol{P}_1 = \boldsymbol{P}_1\boldsymbol{P}_{10} = \begin{bmatrix} \hat{P}_1 & \hat{0} \\ \hat{0} & \hat{0}\end{bmatrix} \equiv \boldsymbol{Q}_1, \tag{69}$$

where the operators $\boldsymbol{Q}_0, \boldsymbol{Q}_1$ and $\boldsymbol{Q}_0^{(\mu)}, \boldsymbol{Q}_1^{(\mu)}$ satisfy

$$\mathbf{Q}_0^2 = \mathbf{Q}_0, \ \ \mathbf{Q}_1^2 = \mathbf{Q}_1, \ \ \mathbf{Q}_1\mathbf{Q}_0 = \mathbf{Q}_0\mathbf{Q}_1 = \mathbf{0}, \tag{70}$$

$$\mathbf{Q}_0^{(\mu)2} = \mathbf{Q}_0^{(\mu)}, \quad \mathbf{Q}_1^{(\mu)2} = \mathbf{Q}_1^{(\mu)}, \quad \mathbf{Q}_1^{(\mu)}\mathbf{Q}_0^{(\mu)} = \mathbf{Q}_0^{(\mu)}\mathbf{Q}_1^{(\mu)} = \mathbf{0}, \tag{71}$$

$$\mathbf{Q}_1^{(\mu)}\mathbf{Q}_0 = \mathbf{Q}_0^{(\mu)}\mathbf{Q}_1 = \mathbf{Q}_1\mathbf{Q}_0^{(\mu)} = \mathbf{Q}_0\mathbf{Q}_1^{(\mu)} = \mathbf{0}, \tag{72}$$

and we call these "half Kronecker (double) projections".

The relations above imply that the process of measurement when using Kronecker obscure qubits (i.e. for quantum computation with truth or membership) is more complicated than in the standard case.

To show this, let us calculate the "obscure" analogs of expected values for the projections above. Using the notation

$$\overline{\mathbf{A}} \equiv \langle \mathbf{\Psi}_{ob} | \mathbf{A} | \mathbf{\Psi}_{ob} \rangle. \tag{73}$$

Then, using (43)–(45) for the projection operators $\boldsymbol{P}_i, \boldsymbol{P}_{ij}, \mathbf{Q}_i, \mathbf{Q}_i^{(\mu)}, i, j = 0, 1, i \neq j$, we obtain (cf. (39))

$$\overline{\boldsymbol{P}}_i = \frac{|a_i|^2 + \alpha_i^2}{2}, \qquad \overline{\boldsymbol{P}}_{ij} = \frac{|a_i|^2 + \alpha_j^2}{2}, \tag{74}$$

$$\overline{\mathbf{Q}}_i = \frac{|a_i|^2}{2}, \qquad \overline{\mathbf{Q}}_i^{(\mu)} = \frac{\alpha_i^2}{2}. \tag{75}$$

So follows the relation between the "obscure" analogs of expected values of the projections

$$\overline{\boldsymbol{P}}_i = \overline{\mathbf{Q}}_i + \overline{\mathbf{Q}}_i^{(\mu)}, \qquad \overline{\boldsymbol{P}}_{ij} = \overline{\mathbf{Q}}_i + \overline{\mathbf{Q}}_j^{(\mu)}. \tag{76}$$

Taking the "ket" corresponding to the "bra" Kronecker qubit (52) in the form

$$\langle \mathbf{\Psi}_{ob} | = \frac{1}{\sqrt{2}} [a_0^* (1 \quad 0), \quad \alpha_0 (1 \quad 0)] + \frac{1}{\sqrt{2}} [a_1^* (0 \quad 1), \quad \alpha_1 (0 \quad 1)], \tag{77}$$

a Kronecker (4×4) obscure analog of the density matrix for a pure state is given by (cf. (42))

$$\rho_{ob}^{(2)} = |\mathbf{\Psi}_{ob}\rangle \langle \mathbf{\Psi}_{ob}| = \frac{1}{2} \begin{pmatrix} |a_0|^2 & a_0 a_1^* & a_0 \alpha_0 & a_0 \alpha_1 \\ a_1 a_0^* & |a_1|^2 & a_1 \alpha_0 & a_1 \alpha_1 \\ \alpha_0 a_0^* & \alpha_0 a_1^* & \alpha_0^2 & \alpha_0 \alpha_1 \\ \alpha_1 a_0^* & \alpha_1 a_1^* & \alpha_0 \alpha_1 & \alpha_1^2 \end{pmatrix}. \tag{78}$$

If the Born rule for the membership functions (13) and the conditions (14)–(15) are satisfied, the density matrix (78) is non-invertible, because $\det \rho_{ob}^{(2)} = 0$ and has unit trace $\mathrm{tr} \rho_{ob}^{(2)} = 1$, but is not idempotent $\left(\rho_{ob}^{(2)}\right)^2 \neq \rho_{ob}^{(2)}$ (as it holds for the ordinary quantum density matrix [1]).

7. Kronecker obscure-quantum gates

In general, (double) "obscure-quantum computation" with L Kronecker obscure qubits (or qudits) can be performed by a product of unitary (block) matrices $\boldsymbol{U}$ of the (double size to the standard one) size $2 \times \left(2^L \times 2^L\right)$ (or $2 \times \left(n^L \times n^L\right)$), $\boldsymbol{U}^\dagger \boldsymbol{U} = \boldsymbol{I}$ (here $\boldsymbol{I}$ is the unit matrix of the same size as $\boldsymbol{U}$). We can also call such computation a "quantum computation with truth" (or with membership).

Let us consider obscure-quantum computation with one Kronecker obscure qubit. Informally, we can present the Kronecker obscure qubit (52) in the form

$$|\mathbf{\Psi}_{ob}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^{(\mu)} \end{bmatrix}. \tag{79}$$

Thus, the state $|\mathbf{\Psi}_{ob}\rangle$ can be interpreted as a "vector" in the direct product (not tensor product) space $\mathcal{H}_q^{(2)} \times \mathcal{V}_{memb}^{(2)}$, where $\mathcal{H}_q^{(2)}$ is the standard two-dimenional Hilbert space of the qubit, and $\mathcal{V}_{memb}^{(2)}$ can be treated as the "membership space" which has a different nature from the qubit space and can have a more complex structure. For discussion of such spaces, see, e.g. [5, 6, 8, 9]. In general, one can consider obscure-quantum computation as a set of abstract computational rules, independently of the introduction of the corresponding spaces.

An obscure-quantum gate will be defined as an elementary transformation on an obscure qubit (79) and is performed by unitary (block) matrices of size 4×4 (over $\mathbb{C}$) acting in the total space $\mathcal{H}_q^{(2)} \times \mathcal{V}_{memb}^{(2)}$

$$\boldsymbol{U} = \begin{pmatrix} \hat{U} & \hat{0} \\ \hat{0} & \hat{U}^{(\mu)} \end{pmatrix}, \qquad \boldsymbol{U}\boldsymbol{U}^{\dagger} = \boldsymbol{U}^{\dagger}\boldsymbol{U} = \boldsymbol{I}, \tag{80}$$

$$\hat{U}U^{\dagger} = \hat{U}^{\dagger}\hat{U} = \hat{I}, \quad \hat{U}^{(\mu)}\hat{U}^{(\mu)\dagger} = \hat{U}^{(\mu)\dagger}\hat{U}^{(\mu)} = \hat{I}, \qquad \hat{U} \in \operatorname{End}\mathcal{H}_q^{(2)}, \hat{U}^{(\mu)} \in \operatorname{End}\mathcal{V}_{memb}^{(2)}, \tag{81}$$

where $\boldsymbol{I}$ is the unit 4×4 matrix, $\hat{I}$ is the unit 2×2 matrix, $\hat{U}$ and $\hat{U}^{(\mu)}$ are unitary 2×2 matrices acting on the probability and membership "subspaces" respectively. The matrix $\hat{U}$ (over $\mathbb{C}$) will be called a quantum gate, and we call the matrix $\hat{U}^{(\mu)}$ (over $\mathbb{R}$) an "obscure gate". We assume that the obscure gates $\hat{U}^{(\mu)}$ are of the same shape as the standard quantum gates, but they act in the other (membership) space and have only real elements (see, e.g. [1]). In this case, an obscure-quantum gate is characterized by the pair $\left\{\hat{U}, \hat{U}^{(\mu)}\right\}$, where the components are known gates (in various combinations), e.g., for one qubit gates: `Hadamard`, `Pauli-X` (NOT),`Y`,`Z` (or two qubit gates e.g. CNOT, SWAP, etc.). The transformed qubit then becomes (informally)

$$\mathbf{U}|\mathbf{\Psi}_{ob}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\hat{U}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}}\hat{U}^{(\mu)}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^{(\mu)} \end{bmatrix}. \tag{82}$$

Thus the quantum and the membership parts are transformed independently for the block diagonal form (80). Some examples of this can be found, e.g., in [36, 37, 45]. Differences between the parts were mentioned in [46]. In this case, an obscure-quantum network is "physically" realised by a device performing elementary operations in sequence on obscure qubits (by a product of matrices), such that the quantum and membership parts are synchronized in time (for a discussion of the obscure part of such physical devices, see [19, 20, 47, 48]). Then, the result of the obscure-quantum computation consists of the quantum probabilities of the states together with the calculated "level of truth" for each of them (see, e.g. [18]).

For example, the obscure-quantum gate $\boldsymbol{U}_{\hat{H},\mathrm{NOT}} = \{\mathtt{Hadamard}, \mathrm{NOT}\}$ acts on the state $\hat{E}_0$ (55) as follows

$$\boldsymbol{U}_{\hat{H},\mathrm{NOT}}\hat{E}_0 = \boldsymbol{U}_{\hat{H},\mathrm{NOT}}\begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(\hat{e}_0 + \hat{e}_1) \\ \varepsilon_1 \end{bmatrix}. \tag{83}$$

It would be interesting to consider the case when $\boldsymbol{U}$ (80) is not block diagonal and try to find possible "physical" interpretations of the non-diagonal blocks.

8. Double entanglement

Let us introduce a register consisting of two obscure qubits ($L = 2$) in the computational basis $\left|\hat{i}j'\right\rangle = \left|\hat{i}\right\rangle \otimes \left|\hat{j}'\right\rangle$ as follows

$$\left|\boldsymbol{\Psi}_{ob}^{(n=2)}(L=2)\right\rangle = |\boldsymbol{\Psi}_{ob}(2)\rangle = \frac{\hat{B}_{00'}\left|\hat{0}0'\right\rangle + \hat{B}_{10'}\left|\hat{1}0'\right\rangle + \hat{B}_{01'}\left|\hat{0}1'\right\rangle + \hat{B}_{11'}\left|\hat{1}1'\right\rangle}{\sqrt{2}}, \tag{84}$$

determined by two-dimensional "vectors" (encoding obscure-quantum amplitudes)

$$\hat{B}_{ij'} = \begin{bmatrix} b_{ij'} \\ \beta_{ij'} \end{bmatrix}, \qquad i,j = 0,1, \qquad j' = 0', 1', \tag{85}$$

where $b_{ij'} \in \mathbb{C}$ are probability amplitudes for a set of pure states and $\beta_{ij'} \in \mathbb{R}$ are the corresponding membership amplitudes. By analogy with (43) and (46) the normalization factor in (84) is chosen so that

$$\langle \boldsymbol{\Psi}_{ob}(2) | \boldsymbol{\Psi}_{ob}(2) \rangle = 1, \tag{86}$$

if (cf. (14)–(15))

$$|b_{00'}|^2 + |b_{10'}|^2 + |b_{01'}|^2 + |b_{11'}|^2 = 1, \tag{87}$$

$$\beta_{00'}^2 + \beta_{10'}^2 + \beta_{01'}^2 + \beta_{11'}^2 = 1. \tag{88}$$

A state of two qubits is "entangled", if it cannot be decomposed as a product of two one-qubit states, and otherwise it is "separable" (see, e.g. [1]). We define a product of two obscure qubits (43) as

$$|\boldsymbol{\Psi}_{ob}\rangle \otimes |\boldsymbol{\Psi}'_{ob}\rangle = \frac{\hat{A}_0 \otimes_H \hat{A}'_0\left|\hat{0}0'\right\rangle + \hat{A}_1 \otimes_H \hat{A}'_0\left|\hat{1}0'\right\rangle + \hat{A}_0 \otimes_H \hat{A}'_1\left|\hat{0}1'\right\rangle + \hat{A}_1 \otimes_H \hat{A}'_1\left|\hat{1}1'\right\rangle}{2}, \tag{89}$$

where $\otimes_H$ is the Hadamard product (53). Comparing (84) and (89) we obtain two sets of relations, for probability amplitudes and for membership amplitudes

$$b_{ij'} = \frac{1}{\sqrt{2}} a_i a_{j'}, \tag{90}$$

$$\beta_{ij'} = \frac{1}{\sqrt{2}} \alpha_i \alpha_{j'}, \qquad i,j = 0,1, \qquad j' = 0', 1'. \tag{91}$$

In this case, the relations (14)–(15) give (87)–(88).

Two obscure-quantum qubits are entangled, if their joint state (84) cannot be presented as a product of one qubit states (89), and in the opposite case the states are called totally separable. It follows from (90)–(91), that there are two general conditions for obscure qubits to be entangled

$$b_{00'} b_{11'} \neq b_{10'} b_{01'}, \qquad \text{or } \det \boldsymbol{b} \neq 0, \;\; \boldsymbol{b} = \begin{pmatrix} b_{00'} & b_{01'} \\ b_{10'} & b_{11'} \end{pmatrix}, \tag{92}$$

$$\beta_{00'} \beta_{11'} \neq \beta_{10'} \beta_{01'}, \qquad \text{or } \det \boldsymbol{\beta} \neq 0, \;\; \boldsymbol{\beta} = \begin{pmatrix} \beta_{00'} & \beta_{01'} \\ \beta_{10'} & \beta_{11'} \end{pmatrix}. \tag{93}$$

The first Eq. (92) is the entanglement relation for the standard qubit, while the second condition (93) is for the membership amplitudes of the two obscure qubit joint state (84). The presence of two different conditions (92)–(93) leads to new additional possibilities (which do not exist for ordinary qubits) for "partial" entanglement (or "partial" separability), when only one of them is fulfilled. In this case, the states can be entangled in one subspace (quantum or membership) but not in the other.

The measure of entanglement is numerically characterized by the concurrence. Taking into account the two conditions (92)–(93), we propose to generalize the notion of concurrence for two obscure qubits in two ways. First, we introduce the "vector obscure concurrence"

$$\hat{C}_{vect} = \begin{bmatrix} C_q \\ C^{(\mu)} \end{bmatrix} = 2 \begin{bmatrix} |\det \boldsymbol{b}| \\ |\det \beta| \end{bmatrix}, \tag{94}$$

where $\boldsymbol{b}$ and β are defined in (92)–(93), and $0 \leq C_q \leq 1$, $0 \leq C^{(\mu)} \leq 1$. The corresponding "scalar obscure concurrence" can be defined as

$$C_{scal} = \sqrt{\frac{|\det \boldsymbol{b}|^2 + |\det \beta|^2}{2}}, \tag{95}$$

such that $0 \leq C_{scal} \leq 1$. Thus, two obscure qubits are totally separable, if $C_{scal} = 0$. For instance, for an obscure analog of the (maximally entangled) Bell state

$$|\boldsymbol{\Psi}_{ob}(2)\rangle = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} |\hat{0}0'\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} |\hat{1}1'\rangle \right) \tag{96}$$

we obtain

$$\hat{C}_{vect} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_{scal} = 1. \tag{97}$$

A more interesting example is the "intermediately entangled" two obscure qubit state, e.g.

$$|\boldsymbol{\Psi}_{ob}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\left|\hat{0}0'\right\rangle + \begin{bmatrix}\frac{1}{4}\\ \frac{\sqrt{5}}{4}\end{bmatrix}\left|\hat{1}0'\right\rangle + \begin{bmatrix}\frac{\sqrt{3}}{4}\\ \frac{1}{2\sqrt{2}}\end{bmatrix}\left|\hat{0}1'\right\rangle + \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{4}\end{bmatrix}\left|\hat{1}1'\right\rangle\right), \quad (98)$$

where the amplitudes satisfy (87)–(88). If the Born-like rule (as in (13)) holds for the membership amplitudes, then the probabilities and membership functions of the states in (98) are

$$p_{00'} = \frac{1}{4}, \quad p_{10'} = \frac{1}{16}, \quad p_{01} = \frac{3}{16}, \quad p_{11'} = \frac{1}{2}, \quad (99)$$

$$\mu_{00'} = \frac{1}{2}, \quad \mu_{10'} = \frac{5}{16}, \quad \mu_{01'} = \frac{1}{8}, \quad \mu_{11'} = \frac{1}{16}. \quad (100)$$

This means that, e.g., the state $\left|\hat{1}0'\right\rangle$ will be measured with the quantum probability 1/16 and the membership function ("truth" value) 5/16. For the entangled obscure qubit (98) we obtain the concurrences

$$\hat{C}_{vect} = \begin{bmatrix}\frac{1}{2}\sqrt{2} - \frac{1}{8}\sqrt{3}\\ \frac{1}{8}\sqrt{2}\sqrt{5} - \frac{1}{4}\sqrt{2}\end{bmatrix} = \begin{bmatrix}0.491\\ 0.042\end{bmatrix}, \qquad C_{scal} = \sqrt{\frac{53}{128} - \frac{1}{16}\sqrt{5} - \frac{1}{16}\sqrt{2}\sqrt{3}} = 0.348. \quad (101)$$

In the vector representation (49)–(52) we have

$$\left|\hat{i}j'\right\rangle = \left|\hat{i}\right\rangle \otimes \left|\hat{j}'\right\rangle = \begin{bmatrix}\hat{e}_i \otimes_K \hat{e}_{j'}\\ \varepsilon_i \otimes_K \varepsilon_{j'}\end{bmatrix}, \qquad i,j = 0,1, \qquad j' = 0', 1', \quad (102)$$

where $\otimes_K$ is the Kronecker product (48), and $\hat{e}_i$, ε_i are defined in (50)–(51). Using (85) and the Kronecker-like product (49), we put (informally, with no summation)

$$\hat{B}_{ij'}\left|\hat{i}j'\right\rangle = \begin{bmatrix}b_{ij'}\hat{e}_i \otimes_K \hat{e}_{j'}\\ \beta_{ij'}\varepsilon_i \otimes_K \varepsilon_{j'}\end{bmatrix}, \qquad i,j = 0,1, \qquad j' = 0', 1'. \quad (103)$$

To clarify our model, we show here a manifest form of the two obscure qubit state (98) in the vector representation

$$|\boldsymbol{\Psi}_{ob}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}\frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}^{(\mu)}\end{bmatrix} + \begin{bmatrix}\frac{1}{4}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\\ \frac{\sqrt{5}}{4}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}^{(\mu)}\end{bmatrix} + \begin{bmatrix}\frac{\sqrt{3}}{4}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\\ \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}^{(\mu)}\end{bmatrix} + \begin{bmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\\ \frac{1}{4}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}^{(\mu)}\end{bmatrix}\right). \quad (104)$$

The states above may be called "symmetric two obscure qubit states". However, there are more general possibilities, as may be seen from the r.h.s. of (103) and (104), when the indices of the first and second rows do not coincide. This would allow more possible states, which we call "non-symmetric two obscure qubit states". It would be worthwhile to establish their possible physical interpretation.

The above constructions show that quantum computing using Kronecker obscure qubits can involve a rich structure of states, giving a more detailed description with additional variables reflecting vagueness.

9. Conclusions

We have proposed a new scheme for describing quantum computation bringing vagueness into consideration, in which each state is characterized by a "measure of truth" Ȧ membership amplitude is introduced in addition to the probability amplitude in order to achieve this, and we are led thereby to the concept of an obscure qubit. Two kinds of these are considered: the "product" obscure qubit, in which the total amplitude is the product of the quantum and membership amplitudes, and the "Kronecker" obscure qubit, where the amplitudes are manipulated separately. In latter case, the quantum part of the computation is based, as usual, in Hilbert space, while the "truth" part requires a vague/fuzzy set formalism, and this can be performed in the framework of a corresponding fuzzy space. Obscure-quantum computation may be considered as a set of rules (defining obscure-quantum gates) for managing quantum and membership amplitudes independently in different spaces. In this framework we obtain not only the probabilities of final states, but also their membership functions, i.e. how much "trust" we should assign to these probabilities. Our approach considerably extends the theory of quantum computing by adding the logic part directly to the computation process. Future challenges could lie in the direction of development of the corresponding logic hardware in parallel with the quantum devices.

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