Chapter

Magnetic Skyrmions: Theory and Applications

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Abstract

Magnetic skyrmions have been subject of growing interest in recent years for their very promising applications in spintronics, quantum computation and future low power information technology devices. In this book chapter, we use the field theory method and coherent spin state ideas to investigate the properties of magnetic solitons in spacetime while focussing on 2D and 3D skyrmions. We also study the case of a rigid skyrmion dissolved in a magnetic background induced by the spin-tronics; and derive the effective rigid skyrmion equation of motion. We examine as well the interaction between electrons and skyrmions; and comment on the modified Landau-Lifshitz-Gilbert equation. Other issues, including emergent electrodynamics and hot applications for next-generation high-density efficient information encoding, are also discussed.

Keywords: Geometric phases, magnetic monopoles and topology, soliton and holonomy, skyrmion dynamics and interactions, med-term future applications

1. Introduction

During the last two decades, the magnetic skyrmions and antiskyrmions have been subject to an increasing interest in connection with the topological phase of matter [1–4], the spin-tronics [5, 6] and quantum computing [7, 8]; as well as in the search for advanced applications such as racetrack memory, microwave oscillators and logic nanodevices making skyrmionic states very promising candidates for future low power information technology devices [9–12]. Initially proposed by T. Skyrme to describe hadrons in the theory of quantum chromodynamics [13], skyrmions have however been observed in other fields of physics, including quantum Hall systems [14, 15], Bose-Einstein condensates [16] and liquid crystals [17]. In quantum Hall (QH) ferromagnets for example [18, 19], due to the exchange interaction; the electron spins spontaneously form a fully polarized ferromagnet close to the integer filling factor $\nu \simeq 1$; slightly away, other electrons organize into an intricate spin configuration because of a competitive interplay between the Coulomb and Zeeman interactions [18]. Being quasiparticles, the skyrmions of the QH system condense into a crystalline form leading to the crystallization of the skyrmions [20–23]; thus opening an important window on promising applications.

In order to overcome the lack of a prototype of a skyrmion-based spintronic devices for a possible fabrication of nanodevices of data storage and logic technologies, intense research has been carried out during the last few years [24, 25]. In this regard, several alternative nano-objects have been identified to host stable skyrmions at room temperature. The first experimental observation of crystalline skyrmionic states was in a three-dimensional metallic ferromagnet MnSi with a B20 structure using small angle neutron scattering [26]. Then, real-space imaging of the skyrmion has been reported using Lorentz transmission electron microscopy in non-centrosymmetric magnetic compounds and in thin films with broken inversion symmetry, including monosilicides, monogermanides, and their alloys, like $Fe_{1-x}Co_xSi$ [27], FeGe [28], and MnGe [29].

One of the key parameters in the formation of these topologically protected non-collinear spin textures is the Dzyaloshinskii-Moriya Interaction (DMI) [30–32]. Originating from the strong spin-orbit coupling (SOC) at the interfaces, the DM exchange between atomic spins controls the size and stability of the induced skyrmions. Depending on the symmetry of the crystal structures and the skyrmion windings number, the internal spins within a single skyrmion envelop a sphere in different arrangements [33]. The in-plane component of the magnetization, in the Néel skyrmion, is always pointed in the radial direction [34], while it is oriented perpendicularly with respect to the position vector in the Bloch skyrmion [26]. Different from these two well-known types of skyrmions are skyrmions with mixed Bloch-Néel topological spin textures observed in Co/Pd multilayers [35]. Magnetic antiskyrmions, having a more complex boundary compared to the chiral magnetic boundaries of skyrmions, exist above room temperature in tetragonal Heusler materials [36]. Higher-order skyrmions should be stabilized in anisotropic frustrated magnet at zero temperature [37] as well as in itinerant magnets with zero magnetic field [38].

In the quest to miniaturize magnetic storage devices, reduction of material's dimensions as well as preservation of the stability of magnetic nano-scale domains are necessary. One possible route to achieve this goal is the formation of topological protected skyrmions in certain 2D magnetic materials. To induce magnetic order and tune DMIs in 2D crystal structures, their centrosymmetric should first be broken using some efficient ways such a (i) generate one-atom thick hybrids where atoms are mixed in an alternating manner [39–41], (ii) apply bias voltage or strain [42–44], (iii) insert adsorbents, impurities and defects [45–47]. In graphene-like materials, fluorine chemisorption is an exothermic adsorption that gives rise to stable 2D structures [48] and to long-range magnetism [49, 50]. In semi-fluorinated graphene, a strong Dzyaloshinskii-Moriya interaction has been predicted with the presence of ferromagnetic skyrmions [51]. The formation of a nanoskyrmion state in a Sn monolayer on a SiC(0001) surface has been reported on the basis of a generalized Hubbard model [52]. Strong DMI between the first nearest magnetic germanium neighbors in 2D semi-fluorinated germanene results in a potential antiferromagnetic skyrmion [53].

In this bookchapter, we use the coherent spin states approach and the field theory method (continuous limit of lattice magnetic models with DMI) to revisit some basic aspects and properties of magnetic solitons in spacetime while focusing on 1d kinks, 2d and 3d spatial skyrmions/antiskyrmions. We also study the case of a rigid skyrmion dissolved in a magnetic background induced by the electronic spins of magnetic atoms like Mn; and derive the effective rigid skyrmion equation of motion. In this regard, we describe the similarity between, on one hand, electrons in the electromagnetic background; and, on the other hand, rigid skyrmions bathing in a texture of magnetic moments. We also investigate the interaction between electrons and skyrmions as well as the effect of the spin transfer effect.

This bookchapter is organized as follows: In Section 2, we introduce some basic tools on quantum SU(2) spins and review useful aspects of their dynamics. In Section 3, we investigate the topological properties of kinks and 2d space solitons while describing in detail the underside of the topological structure of these low-dimensional solitons. In Section 4, we extend the construction to approach topological properties to 3d skyrmions. In Section 5, we study the dynamics of rigid skyrmions without and with dissipation; and in Section 6, we use emergent gauge potential fields to describe the effective dynamics of electrons interacting with the skyrmion in the presence of a spin transfer torque. We end this study by making comments and describing perspectives in the study of skyrmions.

2. Quantum SU(2) spin dynamics

In this section, we review some useful ingredients on the quantum SU(2) spin operator, its underlying algebra and its time evolution while focussing on the interesting spin 1/2 states, concerning electrons in materials; and on coherent spin states which are at the basis of the study of skyrmions/antiskyrmions. First, we introduce rapidly the SU(2) spin operator **S** and the implementation of time dependence. Then, we investigate the non dissipative dynamics of the spin by using semi-classical theory approach (coherent states). These tools can be also viewed as a first step towards the topological study of spin induced 1D, 2D and 3D solitons undertaken in next sections.

2.1 Quantum spin 1/2 operator and beyond

We begin by recalling that in non relativistic 3D quantum mechanics, the spin states $|S_z, S\rangle$ of spinfull particles are characterised by two half integers (S_z, S), a positive $S \geq 0$ and an S_z taking $2S + 1$ values bounded as $-S \leq S_z \leq S$ with integral hoppings. For particles with spin 1/2 like electrons, one distinguishes two basis vector states $\left|\pm\frac{1}{2}, \frac{1}{2}\right\rangle$ that are eigenvalues of the scaled Pauli matrix $\frac{\hbar}{2}\sigma_z$ and the quadratic (Casimir) operator $\frac{\hbar^2}{4}\sum_{a=1}^{3}\sigma_a^2$, here the three $\frac{\hbar}{2}\sigma_a$ with $\sigma_a = \vec{\sigma}.\vec{e}_a$ are the three components of the spin 1/2 operator vector[1] $\vec{\sigma}$. From these ingredients, we learn that the average $<S_z, S|\frac{\hbar}{2}\sigma_z|S_z, S>= \hbar S_z$ (for short $\langle\frac{\hbar}{2}\sigma_z\rangle$) is carried by the z-direction since $S_z = \vec{S}.\vec{e}_z$ with $\vec{e}_z = (0,0,1)^T$. For generic values of the SU(2) spin S, the spin operator reads as $\hbar J_a$ where the three J_a's are $(2S+1)\times(2S+1)$ generators of the SU(2) group satisfying the usual commutation relations $[J_a, J_b] = i\varepsilon_{abc}J^c$ with ε_{abc} standing for the completely antisymmetric Levi-Civita tensor with non zero value $\varepsilon_{123} = 1$; its inverse is ε^{cba} with $\varepsilon^{123} = -1$. The time evolution of the spin $\frac{1}{2}$ operator $\hbar\frac{\sigma_a}{2}$ with dynamics governed by a stationary Hamiltonian operator $(dH/dt = 0)$ is given by the Heisenberg representation of quantum mechanics. In this non dissipative description, the time dependence of the spin $\frac{1}{2}$ operator $\hat{S}_a(t)$ (the hat is to distinguish the operator $\hat{S}_a$ from classical S_a) is given by

$$\hat{S}_a = e^{\frac{i}{\hbar}Ht}\left(\hbar\frac{\sigma_a}{2}\right)e^{-\frac{i}{\hbar}Ht} \qquad (1)$$

where the Pauli matrices σ_a obey the usual commutation relations $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma^c$. For a generic value of the SU(2) spin S, the above relation extends as $\hat{S}_a = e^{\frac{i}{\hbar}Ht}(\hbar J_a)e^{-\frac{i}{\hbar}Ht}$. So, many relations for the spin 1/2 may be straightforwardly generalised for generic values S of the SU(2) spin. For example, for a spin value S_0, the $(2S_0 + 1)$ states are given by $\{|m, S_0\rangle\}$ and are labeled by $-S_0 \leq m \leq S_0$; one of these

[1] For convenience, we often refer to $\vec{\sigma}$, $\vec{e}_i$, $\vec{\sigma}.\vec{e}_i = \sigma_i$ respectively by bold symbols as $\boldsymbol{\sigma}$, $\mathbf{e}_i$, $\boldsymbol{\sigma}.\mathbf{e}_i = \sigma_i$.

states namely $|S_0, S_0\rangle$ is very special; it is commonly known as the highest weight state (HWS) as it corresponds to the biggest value $m = S_0$; from this state one can generate all other spin states $|m, S_0\rangle$; this feature will be used when describing coherent spin states. Because of the property $\sigma_a^2 = I,$ the square $\hat{S}_a^2 = \frac{\hbar^2}{4} I$ is time independent; and then the time dynamics of $\hat{S}_a(t)$ is rotational in the sense that $\frac{d\hat{S}_a}{dt}$ is given by a commutator as follows $\frac{d\hat{S}_a}{dt} = \frac{i}{\hbar}\left(H\hat{S}_a - \hat{S}_a H\right)$. For the example where H is a linearly dependent function of $\hat{S}_a$ like for the Zeeman coupling, the Hamiltonian reads as $H_Z = \sum_a \omega^a \hat{S}_a$ (for short $\omega^a \hat{S}_a$) with the ω^a's are constants referring to the external source[2]; then the time evolution of $\hat{S}_a$ reads, after using the commutation relation $\left[\hat{S}_a, \hat{S}_b\right] = i\hbar\varepsilon_{abc}\hat{S}^c$, as follows

$$\frac{d\hat{S}_a}{dt} = \varepsilon_{abc}\omega^b \hat{S}^c \qquad \Leftrightarrow \qquad \frac{d\hat{\mathbf{S}}}{dt} = \omega \wedge \hat{\mathbf{S}} \tag{2}$$

where appears the Levi-Civita ε_{abc} which, as we will see throughout this study, turns out to play an important role in the study of topological field theory [54, 55] including solitons and skyrmions we are interested in here [56–59]. In this regards, notice that, along with this ε_{abc}, we will encounter another completely antisymmetric Levi-Civita tensor namely $\varepsilon_{\mu_1 \dots \mu_D}$; it is also due to DM interaction which in lattice description is given by $\left(\vec{S}_{\mathbf{r}_{\mu_2}} \wedge \vec{S}_{\mathbf{r}_{\mu_1}}\right).\vec{d}_{\mu_3 \dots \mu_{D-2}} \varepsilon^{\mu_1 \cdots \mu_D}$; and in continuous limit reads as $\varepsilon_{abc} S^b S^c_{\mu_1\mu_2} d^a_{\mu_3 \dots \mu_{D-2}} \varepsilon^{\mu_1 \cdots \mu_D}$ where, for convenience, we have set $S^c_{\mu_1\mu_2} = \mathbf{e}_{\mu_1\mu_2}.\nabla S^c$ with $\mathbf{e}_{\mu_1\mu_2} = \mathbf{e}_{\mu_2} - \mathbf{e}_{\mu_1}$. To distinguish these two Levi-Civita tensors, we refer to ε_{abc} as the target space Levi-Civita with $\mathrm{SO}(3)_{\text{target}}$ symmetry; and to $\varepsilon_{\mu_1 \dots \mu_D}$ as the spacetime Levi-Civita with $\mathrm{SO}(1, D-1)$ Lorentz symmetry containing as subsymmetry the usual space rotation group $\mathrm{SO}(D-1)_{\text{space}}$. Notice also that for the case where the Hamiltonian $H\left(\hat{S}\right)$ is a general function of the spin, the vector ω^a is spin dependent and is given by the gradient $\frac{\partial H}{\partial \hat{S}_a}$.

2.2 Coherent spin states and semi-classical analysis

To deal with the semi-classical dynamics of $\hat{\mathbf{S}}(t)$ evolved by a Hamiltonian $H\left(\hat{\mathbf{S}}\right)$, we use the algebra $\left[\hat{S}_a, \hat{S}_b\right] = i\hbar\varepsilon_{abc}\hat{S}^c$ to think of the quantum spin in terms of a coherent spin state [60] described by a (semi) classical vector $\vec{S} = \hbar S \vec{n}$ (no hat) of the Euclidean $\mathbb{R}^3$; see the **Figure 1**(a). This "classical" 3-vector has an amplitude $\hbar S$ and a direction $\vec{n}$ related to a given unit vector $\vec{n}_0$ as $\vec{n} = R(\alpha, \beta, \gamma)\vec{n}_0;$ and parameterised by α, β, γ. In the above relation, the $\vec{n}_0$ is thought of as the north direction of a 2-sphere $\mathbb{S}^2_{(\mathbf{n})}$ given by the canonical vector $(0, 0, 1)^T$; it is invariant under the proper rotation; i.e. $R_z(\gamma)\vec{n}_0 = \vec{n}_0$; and consequently the generic $\vec{n}$ is independent of γ; i.e.: $\vec{n} = R(\alpha, \beta)\vec{n}_0$. Recall that the 3×3 matrix $R(\alpha, \beta, \gamma)$ is an $\mathrm{SO}(3)$ rotation $[\mathrm{SO}(3) \sim SU(2)]$ generating all other points of $\mathbb{S}^2_{(\mathbf{n})}$ parameterised by (α, β). In this regards, it is interesting to recall some useful properties that we list here after as three points: (1) the rotation matrix $R(\alpha, \beta, \gamma)$ can be factorised like

[2] For an electron with Zeeman field B^a, we have $\omega^a = -g\frac{q_e}{2m_e}B^a$ with $g = 2$ and $q_e = -e$.

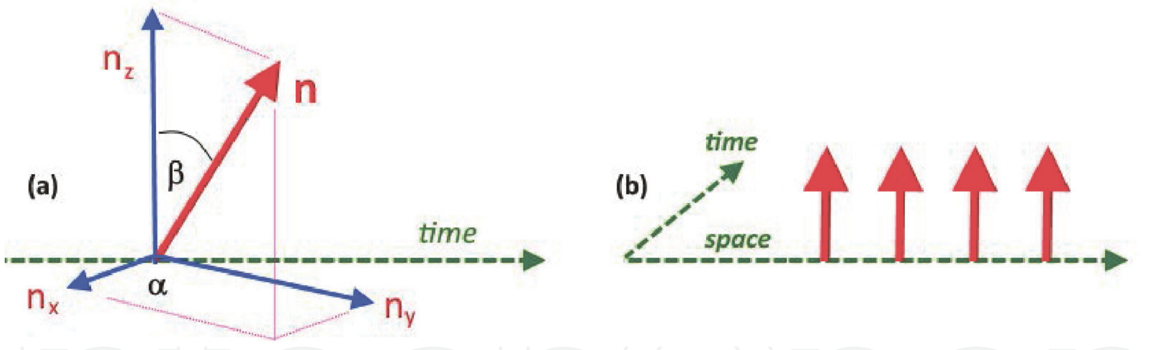

Figure 1.
(a) Components of the spin orientation n; its time dynamics in presence of a magnetic field is given by Larmor precession. (b) A configuration of several spins in spacetime.

$R_z(\alpha)R_y(\beta)R_z(\gamma)$ where each $R_a(\psi_a)$ is a rotation $e^{-i\psi_a J_a}$ around the a- axis with an angle ψ_a and generator J_a. (2) As the unit $\vec{n}_0$ is an eigen vector of $e^{-i\gamma J_z}$; it follows that $\vec{n}$ reduces to $e^{-i\alpha J_z}e^{-i\beta J_y}\vec{n}_0$; this generic vector obeys as well the constraint $\left|\vec{n}\right| \equiv |\mathbf{n}| = 1$ and is solved as follows.

$$\mathbf{n} = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta) \tag{3}$$

with $0 \le \alpha \le 2\pi$ and $0 \le \beta \le \pi$; they parameterise the unit 2-sphere $\mathbb{S}^2_{(\mathbf{n})}$ which is isomorphic to $SU(2)/U(1)$; the missing angle γ parameterises a circle $\mathbb{S}^1_{(\mathbf{n})}$, isomorphic to $U(1)$, that is fibred over $\mathbb{S}^2_{(\mathbf{n})}$. (3) the coherent spin state representation gives a bridge between quantum spin operator and its classical description; it relies on thinking of the average $<\hat{\mathbf{S}}>$ in terms of the classical vector $\vec{S}_0 = \hbar S\vec{n}_0$ considered above ($\vec{S}_0 \leftrightarrow$ HWS $|S_0, S_0\rangle$). In this regards, recall that the $\hat{S}_a$ acts on classical 3-vectors V_b through its 3×3 matrix representation like $\left[\hat{S}_a, V_b\right] = -\hbar(J_c)_{ab}V^c$ with $(J_c)_{ab}$ given by $-i\varepsilon_{abc}$; these J_c's are precisely the generators of the $SU(2)$ matrix representation $R(\alpha,\beta,\gamma)$; by replacing V_b by the operator $\hat{S}_b$, one discovers the SU(2) spin algebra $\left[\hat{S}_a, \hat{S}_b\right] = i\hbar\varepsilon_{abc}\hat{S}^c$. Notice also that the classical spin vector $\vec{S} = \hbar S\vec{n}$ can be also put in correspondence with the usual magnetic moment $\vec{\mu} = -\gamma\vec{S}$ (with $\gamma = \frac{ge}{2m}$ the gyromagnetic ratio); thus leading to $\vec{\mu} = |\mu|\vec{n}$. So, the magnetization vector describes (up to a sign) a coherent spin state with amplitude $\hbar S\gamma$; and a (opposite) time dependent direction $\vec{n}(t)$ parameterizing the 2-sphere $\mathbb{S}^2_{(\mathbf{n})}$.

$$n_x^2(t) + n_y^2(t) + n_z^2(t) = 1 \Leftrightarrow \left|\vec{n}(t)\right| = 1 \tag{4}$$

For explicit calculations, this unit 2-sphere equation will be often expressed like $n^a n_a = 1$; this relation leads in turns to the property $n^a dn_a = 0$ (indicating that $\vec{n}$ and $d\vec{n}$ are normal vectors); by implementing time, the variation $\mathbf{n}.d\mathbf{n}$ gets mapped into $\mathbf{n}.\dot{\mathbf{n}} = 0$ teaching us that the velocity $\dot{\mathbf{n}}$ is carried by $\mathbf{u}$ and $\mathbf{v}$; two normal directions to $\mathbf{n}$ with components;

$$u_a = (\cos\beta\cos\alpha,\ \cos\beta\sin\alpha,\ -\sin\beta), \quad v_a = (-\sin\alpha,\ \cos\alpha,\ 0) \tag{5}$$

and from which we learn that $dn_a = u_a d\beta + v_a \sin\beta d\alpha$, $[(\mathbf{u}, \mathbf{v}, \mathbf{n})$ form an orthogonal vector triad). So, the dynamics of μ_a (and that of $-\overrightarrow{S}$) is brought to the dynamics of the unit n_a governed by a classical Hamiltonian $\mathrm{H}[n_a(\alpha, \beta)]$. The resulting time evolution is given by the so called Landau-Lifshitz (LL) equation [61]; it reads as $\frac{dn_a}{dt} = -\frac{\gamma}{|\mu|}\varepsilon_{abc}\left(\partial^b H\right)n^c$ with $\partial^b H = \frac{\partial H}{\partial n_b}$. By using the relations $d\beta = u^a dn_a$ and $\sin\beta d\alpha = v^a dn_a$ with $u_a = \frac{\partial n_a}{\partial \beta}$; and $v_a \sin\beta = \frac{\partial n_a}{\partial \alpha}$; as well as the expressions $\varepsilon_{abc}u^a n^c = v_b$ and $\varepsilon_{abc}v^a n^c = -u_b$, the above LL equation splits into two time evolution equations $\frac{d\beta}{dt} = -\gamma v_b\left(\partial^b H\right)$ and $\sin\beta\frac{d\alpha}{dt} = \gamma u_b\left(\partial^b H\right)$. These time evolutions can be also put into the form

$$\sin\beta\frac{d\beta}{dt} + \gamma\frac{\partial H}{\partial \alpha} = 0, \quad \sin\beta\frac{d\alpha}{dt} - \gamma\frac{\partial H}{\partial \beta} = 0 \tag{6}$$

and can be identified with the Euler- Lagrange equations following from the variation $\delta\mathcal{S} = 0$ of an action $\mathcal{S} = \int L dt$. Here, the Lagrangian is related to the Hamiltonian like $L = L_B - H[n_a(\alpha, \beta)]$ where L_B is the Berry term [62] known to have the form $<\mathbf{n}|\dot{\mathbf{n}}>$; this relation can be compared with the well known Legendre transform p$\dot{q}$-H(q, p). For later interpretation, we scale this hamiltonian as $\hbar S\gamma H$ such that the spin lagrangian takes the form $L_{spin} = L_B - \hbar S\gamma H$. To determine L_B, we identify the Eq. (6) with the extremal variation $\delta\mathcal{S}/\delta\beta = 0$ and $\delta\mathcal{S}/\delta\alpha = 0$. Straightforward calculations leads to

$$L_B = -\hbar S(1 - \cos\beta)\frac{d\alpha}{dt} \tag{7}$$

showing that α and β form a conjugate pair. By substituting $\sin\beta\frac{d\alpha}{dt} = v^a\frac{dn_a}{dt}$ back into above L_B, we find that the Berry term has the form of Aharonov-Bohm coupling $L_{AB} = q_e A^a\frac{dn_a}{dt}$ with magnetic potential vector A^a given by $A_a = \frac{\hbar S}{q_e}\frac{(1 - \cos\beta)}{\sin\beta}v_a$. However, this potential vector is suggestive as it has the same form as the potential vector $\mathbf{A}^{(monopole)} = \frac{\hbar S}{q_e r}\frac{(1 - \cos\beta)}{\sin\beta}\mathbf{v}$ of a magnetic monopole. The curl of this potential is given by $\overrightarrow{\mathbf{B}} = q_m\frac{\overrightarrow{r}}{r^3}$ with magnetic charge $q_m = -\frac{\hbar S}{q_e}$ located at the centre of the 2-sphere; the flux Φ of this field through the unit sphere is then equal to $-4\pi\frac{\hbar S}{q_e}$; and reads as $-2S\Phi_0$ with a unit flux quanta $\Phi_0 = \frac{h}{q_e}$ as indicated by the value $S = 1/2$. So, because $2S = -n$ is an integer, it results that the flux is quantized as $\Phi = n\Phi_0$.

3. Magnetic solitons in lower dimensions

In previous section, we have considered the time dynamics of coherent spin states with amplitude $\hbar S$ and direction described by $\overrightarrow{n}(t)$ as depicted by the **Figure 1(a)**; this is a 3-vector having with no space coordinate dependence, $grad\overrightarrow{n} = 0$; and as such it can be interpreted as a $(1 + 0)D$ vector field; that is a vector belonging to $\mathbb{R}^{1,d}$ with $d = 0$ (no space direction). In this section, we first turn on 1d space coordinate x and promotes the old unit- direction $\overrightarrow{n}(t)$ to a $(1 + 1)$D field $\overrightarrow{n}(t, x)$. After that, we turn on two space directions (x, y); thus leading to $(1 + 2)$D field $\overrightarrow{n}(t, x, y)$; a picture is depicted by the **Figure 1(b)**. To deal with the dynamics of these local fields and their topological properties, we use the field theory method while focussing on particular solitons; namely the 1d kinks and the 2d skyrmions. In this extension, one encounters two types of spaces: (1) the target space $\mathbb{R}^3_{\mathbf{n}}$ parameterised by $n_a = (n_1, n_2, n_3)$ with Euclidian metric δ_{ab} and topological Levi-Civita ε_{abc}. (2) the spacetime $\mathbb{R}^{1,1}_{\xi}$ parameterised by $\xi^\mu = (t, x)$, concerning the 1d kink evolution; and the spacetime $\mathbb{R}^{1,2}_{\xi}$ parameterised by $\xi^\mu = (t, x, y)$, regarding the 2d skyrmions dynamics. As we have two kinds of evolutions; time and space; we denote the time variable by $\xi^0 = t$; and the space coordinates by $\xi^i = (x, y)$. Moreover, the homologue of the tensors δ_{ab} and ε_{abc} are respectively given by the usual Lorentzian spacetime metric $g_{\mu\nu}$, with signature like $g_{\mu\nu}\xi^\mu\xi^\nu = x^2 + y^2 - t^2$, and the spacetime Levi-Civita $\varepsilon_{\mu\nu\rho}$ with $\varepsilon_{012} = 1$.

3.1 One space dimensional solitons

In $(1+1)$D spacetime, the local coordinates parameterising $\mathbb{R}^{1,1}_{(\xi)}$ are given by $\xi^\mu = (t, x)$; so the metric is restricted to $g_{\mu\nu}\xi^\mu\xi^\nu = x^2 - t^2$. The field variable $n^a(\xi)$ has in general three components (n_1, n_2, n_3) as described previously; but in what follows, we will simplify a little bit the picture by setting $n_3 = 0$; thus leading to a magnetic 1d soliton with two component field variable $\mathbf{n} = (n_1, n_2)$ satisfying the constraint equation $\mathbf{n}.\mathbf{n} = 1$ at each point of spacetime. As this constraint relation plays an important role in the construction, it is interesting to express it as $n_a n^a = 1$. Before describing the topological properties of one space dimensional solitons (kinks), we think it interesting to begin by giving first some useful features; in particular the three following ones. (1) The constraint $(n_1)^2 + (n_2)^2 = 1$ is invariant $SO(2)_{\mathbf{n}}$ rotations acting as $n'^a = \mathcal{R}^a_b n^b$ with orthogonal rotation matrix

$$\mathcal{R}^a_b = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}, \qquad \mathcal{R}^T\mathcal{R} = I \tag{8}$$

The constraint $n_a n^a = 1$ can be also presented like $\overline{N}N = 1$ with N standing for the complex field $n_1 + in_2$ that reads also like $e^{i\alpha}$. In this complex notation, the symmetry of the constraint is given by the phase change acting as $N \to UN$ with $U = e^{i\psi}$ and corresponding to the shift $\alpha \to \alpha + \psi$. Moreover the correspondence $(n_1, n_2) \leftrightarrow n_1 + in_2$ describes precisely the well known isomorphisms $SO(2) \sim U(1) \sim \mathbb{S}^1_{(\mathbf{n})}$ where $\mathbb{S}^1_{(\mathbf{n})}$ is a circle; it is precisely the equatorial circle of the 2-sphere $\mathbb{S}^2_{(\mathbf{n})}$ considered in previous section. (2) As for Eq. (5), the constraint $n_a n^a = 1$ leads to $n_a dn^a = 0$; and so describes a rotational movement encoded in the relation $dn^a = \varepsilon^{ab} n_b$ where ε^{ab} is the standard 2D antisymmetric tensor with $\varepsilon^{21} = \varepsilon_{12} = 1$; this ε_{ab} is related to the previous 3D Levi-Civita like ε_{zab}. Notice also that the constraint $n_a n^a = 1$ implies moreover that $dn_2 = -\frac{n_n}{n_2} dn_1$; and consequently the area $dn_1 \wedge dn_2$, to be encountered later on, vanishes identically. In this regards, recall that we have the following transformation

$$dn_1 \wedge dn_2 = \mathfrak{J} dt \wedge dx, \quad \mathfrak{J} = \varepsilon^{\mu\nu}\partial_\mu n_1 \partial_\nu n_2 \tag{9}$$

where $\varepsilon^{\mu\nu}$ is the antisymmetric tensor in 1+1 spacetime, and $\mathfrak{J}$ is the Jacobian of the transformation $(t, x) \to (n_1, n_2)$. (3) The condition $n_a n^a = 1$ can be dealt in two manners; either by inserting it by help of a Lagrange multiplier; or by solving it in term of a free angular variable like $n_a = (\cos\alpha, \sin\alpha)$ from which we deduce the normal direction $u^a = \frac{dn^a}{d\alpha}$ reading as $u_a = (-\sin\alpha, \cos\alpha)$. In term of the complex field; we have $N = e^{i\alpha}$ and $\overline{N}dN = id\alpha$. Though interesting, the second way of doing hides an important property in which we are interested in here namely the non linear dynamics and the topological symmetry.

3.1.1 Constrained dynamics

The classical spacetime dynamics of $n^a(\xi)$ is described by a field action $\mathcal{S} = \int dtL$ with Lagrangian $L = \int dx\mathcal{L}$ and density $\mathcal{L}$; this field density is given by $-\frac{1}{2}(\partial_\mu n_a)(\partial^\mu n^a) - V(n) - \Lambda(n^a n_a - 1)$ with $\partial_\mu = \frac{\partial}{\partial \xi^\mu}$; it reads in terms of the Hamiltonian density as follows

$$\mathcal{L} = \pi^a \dot{n}_a - \mathcal{H} \tag{10}$$

where $\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{n}_a}$. In the above Lagrangian density, the auxiliary field $\Lambda(\xi)$ (no Kinetic term) is a Lagrange multiplier carrying the constraint relation $n_a n^a = 1$. The $V(n)$ is a potential energy density which play an important role for describing 1d kinks with finite size. Notice also that the variation $\frac{\delta S}{\delta \Lambda} = 0$ gives precisely the constraint $n_a n^a = 1$ while the $\frac{\delta S}{\delta n^a} = 0$ gives the spacetime dynamics of n^a described by the spacetime equation $\partial_\mu \partial^\mu n^a - \frac{\partial V}{\partial n^a} - \Lambda n^a = 0$. By substituting $n_a = (\cos\alpha, \sin\alpha)$, we obtain $\mathcal{L} = -\frac{1}{2}(\partial_\mu \alpha)(\partial^\mu \alpha) - V(\alpha)$. If setting $V(\alpha) = 0$, we end up with the free field equation $\partial_\mu \partial^\mu \alpha = 0$ that expands like $(\partial_x^2 - \partial_t^2)\alpha = 0$; it is invariant under spacetime translations with conserved current symmetry $\partial^\mu T_{\mu\nu} = 0$ with $T_{\mu\nu}$ standing for the energy momentum tensor given by the 2×2 symmetric matrix $\partial_\mu \alpha \partial_\nu \alpha + g_{\mu\nu}\mathcal{L}$. The energy density T_{00} is given by $\frac{1}{2}(\partial_t \alpha)^2 + \frac{1}{2}(\partial_x \alpha)^2$ and the momentum density T_{10} reads as $\partial_x \varphi \partial_x \varphi$. Focussing on T_{00}, the conserved energy E reads then as follows

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} dx \left[(\partial_t \alpha)^2 + (\partial_x \alpha)^2\right] \geq 0 \tag{11}$$

with minimum corresponding to constant field ($\alpha = cte$). Notice that general solutions of $\partial_\mu \partial^\mu \alpha = 0$ are given by arbitrary functions $f(x \pm t)$; they include oscillating and non oscillating functions. A typical non vibrating solution that is interesting for the present study is the solitonic solution given (up to a constant c) by the following expression

$$\varphi(t, x) = \pi \tanh\left(\frac{x + t}{\lambda}\right) \tag{12}$$

where λ is a positive parameter representing the width where the soliton $\alpha(t, x)$ acquires a significant variation. Notice that for a given t, the field varies from $\alpha(t, -\infty) = -\pi$ to $\alpha(t, +\infty) = \pi$ regardless the value of λ. These limits are related to each other by a period 2π.

3.1.2 Topological current and charge

To start, notice that as far as conserved symmetries of (10) are concerned, there exists an exotic invariance generated by a conserved $J_\mu(t, x)$ going beyond the spacetime translations generated by the energy momentum tensor $T_{\mu\nu}$. The conserved spacetime current $J_\mu = (J_0, J_1)$ of this exotic symmetry can be introduced in two different, but equivalent, manners; either by using the free degree of freedom α; or by working with the constrained field n^a. In the first way, we think of the charge density J_0 like $\frac{1}{2\pi}\partial_1\alpha$ and of the current density as $J_1 = -\frac{1}{2\pi}\partial_0\alpha$. This conserved current is a topological $(1+1)D$ spacetime vector J_μ that is manifestly conserved; this feature follows from the relation between J_μ and the antisymmetric $\varepsilon_{\mu\nu}$ as follows [57],

$$J_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\alpha \tag{13}$$

Because of the $\varepsilon_{\mu\nu}$; the continuity relation $\partial^\mu J_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\mu\partial^\nu\alpha$ vanishes identically due to the antisymmetry property of $\varepsilon_{\mu\nu}$. The particularity of the above conserved J_μ is its topological nature; it is due to the constraint $n_a n^a = 1$ without recourse to the solution $n_a = (\cos\alpha, \sin\alpha)$. Indeed, Eq. (13) can be derived by computing the Jacobian $\mathfrak{J} = \det\left(\frac{\partial n^a}{\partial\xi^\mu}\right)$ of the mapping from the 2d spacetime coordinates (t, x) to the target space fields (n_1, n_2). Recall that the spacetime area $dt \wedge dx$ can be written in terms of $\varepsilon_{\mu\nu}$ like $\frac{1}{2}\varepsilon_{\mu\nu}d\xi^\mu \wedge d\xi^\nu$ and, similarly, the target space area $dn_1 \wedge dn_2$ can be expressed in terms of as ε_{ab} follows $\frac{1}{2}\varepsilon_{ab}dn^a \wedge dn^b$. The Jacobian $\mathfrak{J}$ is precisely given by (9); and can be presented into a covariant form like $\mathfrak{J} = \frac{1}{2}\varepsilon^{\mu\nu}\partial_\mu n^a\partial_\nu n^b\varepsilon_{ab}$. This expression of the Jacobian $\mathfrak{J}$ captures important informations; in particular the three following ones. (1) It can be expressed as a total divergence like $\partial_\mu(\pi J^\mu)$ with spacetime vector

$$J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu}n^a\partial_\nu n^b\varepsilon_{ab} \tag{14}$$

and where $\frac{1}{\pi}$ is a normalisation; it is introduced for the interpretation of the topological charged as just the usual winding number of the circle [encoded in the homotopy group relation $\pi_1(\mathbb{S}^1) = \mathbb{Z}$]. (2) Because of the constraint $dn_2 = -\frac{n_n}{n_2}dn_1$ following from $n_a n^a = 1$, the Jacobian $\mathfrak{J}$ vanishes identically; thus leading to the conservation law $\partial_\mu J^\mu = 0$; i.e. $\mathfrak{J} = 0$ and then $\partial_\mu J^\mu = 0$. (3) The conserved charge Q associated with the topological current is given by $\int_{-\infty}^{+\infty}dxJ^0(t,x)$; it is time independent despite the apparent t- variable in the integral $(dQ/dt = 0)$. By using (13), this charge reads also as $\frac{1}{2\pi}\int_{-\infty}^{+\infty}dx\partial_x\alpha(t,x)$ and after integration leads to

$$Q = \frac{1}{2\pi}[\alpha(t,\infty) - \alpha(t,-\infty)] \tag{15}$$

Moreover, seen that $\alpha(t,\infty)$ is an angular variable parameterising $\mathbb{S}^1_{\mathbf{n}}$; it may be subject to a boundary condition like for instance the periodic $\alpha(t,\infty) = \alpha(t,-\infty) + 2\pi N$ with N an integer; this leads to an integral topological charge $Q = N$ interpreted as the winding number of the circle. In this regards, notice that: (i) the winding interpretation can be justified by observing that under compactification of the space variable x, the infinite space line $\mathbb{R}_x =]-\infty, +\infty[$ gets mapped into a circle $\mathbb{S}^1_{(x)}$ with angular coordinate $-\pi \leq \varphi \leq \pi$; so, the integral $\frac{1}{2\pi}\int_{-\infty}^{+\infty}dx\partial_x\alpha(t,x)$ gets replaced by $\frac{1}{2\pi}\int_{-\pi}^{+\pi}d\varphi\frac{\partial\alpha}{\partial\varphi}$; and then the mapping $\alpha_t : \varphi \to \alpha(t,\varphi)$ is a mapping between two circles namely $\mathbb{S}^1_{(x)} \to \mathbb{S}^1_{(\mathbf{n})}$; the field $\alpha(t,\varphi)$ then describes a soliton (one space extended object) wrapping the circle $\mathbb{S}^1_{(x)}$ N times; this propery is captured by $\pi_{\mathbb{S}^1_{(x)}}\left(\mathbb{S}^1_{(\mathbf{n})}\right) = \mathbb{Z}$, a homotopy group property [63]. (ii) The charge Q is independent of the Lagrangian of the system as it follows completely from the field constraint

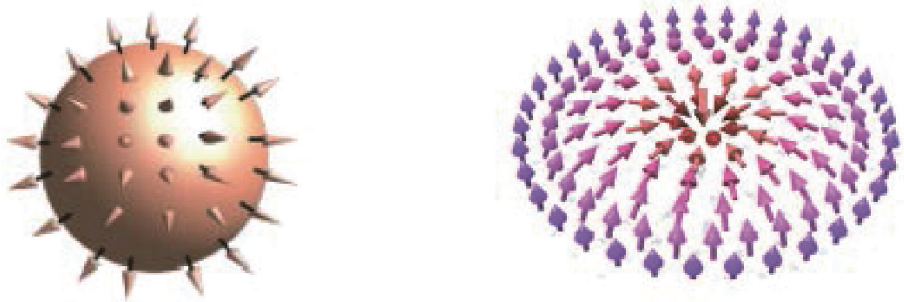

Figure 2.
On left: a spin configuration with $n_1^2 + n_2^2 + n_3^2 = 1$ dispatched on a 2-sphere. On right: a two space dimensional magnetic skyrmion given by the stereographic projection of $\mathbb{S}^2$ to plane.

without any reference to the field action. (iii) Under a scale transformation $\xi' = \xi/\lambda$ with a scaling parameter $\lambda > 0$, the topological charge of the field (12) is invariant; but its total energy (11) get scaled as follows

$$Q' = Q, \qquad E' = \frac{1}{\lambda}E \tag{16}$$

This energy transformation shows that stable solitons with minimal energy correspond to $\lambda \to \infty$; and then to a trivial soliton spreading along the real axis. However, one can have non trivial solitonic configurations that are topologically protected and energetically stable with non diverging λ. This can done by turning on an appropriate potential energy density $V(\mathbf{n})$ in Eq. (10). An example of such potential is the one given by $\frac{g}{8}\left(n_1^4 + n_2^4 - 1\right)$, with positive $g = M^2$, breaking $SO(2)_{\mathbf{n}}$; by using the constraint $n_1^2 + n_2^2 = 1$, it can be put $\frac{g}{4}n_1^2n_2^2$. In terms of the angular field α, it reads as $V(\alpha) = \frac{g}{16}(1 - \cos 4\alpha)$ leading to the well known sine-Gordon Eq. [64, 65] namely $\partial_\mu\partial^\mu\alpha - \frac{g}{4}\sin 4\alpha = 0$ with the symmetry property $\alpha \to \alpha + \frac{\pi}{2}$. So, the solitonic solution is periodic with period $\frac{\pi}{2}$; that is the quarter of the old 2π period of the free field case. For static field $\alpha(x)$, the sine Gordon equation reduces to $\frac{d^2\alpha}{dx^2} - \frac{M^2}{4}\sin 4\alpha = 0$; its solution for $M > 0$ is given by $\arctan[\exp Mx]$ representing a sine- Gordon field evolving from 0 to $\frac{\pi}{2}$ and describing a kink with topological charge $Q = \frac{1}{4}$. For $M < 0$, the soliton is an anti-kink evolving from $\frac{\pi}{2}$ to 0 with charge $Q = -\frac{1}{4}$. Time dependent solutions can be obtained by help of boost transformations $x \to \frac{x \pm vt}{\sqrt{1-v^2}}$.

3.2 Skyrmions in 2d space dimensions

In this subsection, we investigate the topological properties of 2d Skyrmions by extending the field theory study we have done above for 1d kinks to two space dimensions. For that, we proceed as follows: First, we turn on the component n_3 so that the skyrmion field $\mathbf{n}$ is a real 3-vector with three components (n_1, n_2, n_3) constrained as in Eqs. (4) and (5); see **Figure 2**. Second, here we have $\mathbf{n} = \mathbf{n}(t, x, y)$; that is a 3-component field living in the $(2 + 1)$ space time with Lorentzian metric and coordinates $\xi^\mu = (t, x, y)$. This means that $d\mathbf{n} = (\partial_\mu \mathbf{n})d\xi^\mu$; explicitly $d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial t}dt + \frac{\partial \mathbf{n}}{\partial x}dx + \frac{\partial \mathbf{n}}{\partial y}dy$.

3.2.1 Dzyaloshinskii-Moriya potential

The field action $\mathcal{S}_{3D} = \int dt L_{3D}$ describing the space time dynamics of $\mathbf{n}(t, x, y)$ has the same structure as Eq. (10); except that here the Lagrangian L_{3D} involves two space variable like $\int dxdy\mathcal{L}_{3D}$ and the density $\mathcal{L}_{3D} = -\frac{1}{2}\left(\partial_\mu \mathbf{n}\right)^2 - V(\mathbf{n}) - \Lambda(\mathbf{n}.\mathbf{n} - 1)$; this is a function of the constrained 3-vector $\mathbf{n}$ and its space time gradient $\partial_\mu \mathbf{n}$; it reads in term of the Hamiltonian density as follows.

$$\mathcal{L}_{3D} = \pi.\dot{\mathbf{n}} - \mathcal{H}_{3D}(\mathbf{n}) \tag{17}$$

In this expression, the $\mathcal{H}_{3D}(\mathbf{n})$ is the continuous limit of a lattice Hamiltonian $\mathrm{H}_{latt}\left(\left[n^a\left(\mathbf{r}_\mu\right)\right]\right.$ involving, amongst others, the Heisenberg term, the Dyaloshinskii-Moriya (DM) interaction and the Zeeman coupling. The $V(\mathbf{n})$ in the first expression of $\mathcal{L}_{3D}$ is the scalar potential energy density; it models the continuous limit of the interactions that include the DM and Zeeman ones [see Eq. (1.22) for its explicit relation]. The field $\Lambda(\xi)$ is an auxiliary 3D spacetime field; it is a Lagrange multiplier that carries the constraint $\mathbf{n}.\mathbf{n} = 1$ which plays the same role as in subSection 3.1. By variating this action with respect to the fields $\mathbf{n}$ and Λ; we get from $\frac{\delta S_{3D}}{\delta \Lambda} = 0$ precisely the field constraint $\mathbf{n}.\mathbf{n} = 1$; and from $\frac{\delta S_{3D}}{\delta \mathbf{n}} = 0$ the following Euler–Lagrange equation $\mathrm{W}\mathbf{n} = \frac{\partial V}{\partial \mathbf{n}} + \Lambda \mathbf{n}$. For later use, we express this field equation like

$$\partial^\mu \partial_\mu n_a = \frac{\partial V}{\partial n^a} + \Lambda n_a \tag{18}$$

The interest into this (18) is twice; first it can be put into the equivalent form $\partial^\mu \partial_\mu n_a = \varepsilon_{abc}\mathcal{D}^b n^c$ where $\mathcal{D}^b$ is an operator acting on n^c to be derived later on [see Eq. (22) given below]; and second, it can be used to give the relation between the scalar potential and the operator $\mathcal{D}^b$. To that purpose, we start by noticing that there are two manners to deal with the field constraint $n^a n_a = 1$; either by using the Lagrange multiplier Λ; or by solving it in terms of two angular field variables as given by Eq. (5). In the second case, we have the triad $n_a = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta)$ and

$$u_a = (\cos\beta\cos\alpha,\ \cos\beta\sin\alpha, -\sin\beta) \quad , \quad v_a = (-\sin\alpha,\ \cos\alpha, 0) \tag{19}$$

but now $\beta = \beta(t,x,y)$ and $\alpha = \alpha(t,x,y)$ with $0 \leq \beta \leq \pi$ and $0 \leq \alpha \leq 2\pi$. Notice also that the variation of the filed constraint leads to $n_a dn^a = 0$ teaching us interesting informations, in particular the two following useful ones. (1) the movement of n_a in the target space is a rotational movement; and so can be expressed like

$$dn_a = \varepsilon_{abc}\omega^b n^c \quad \Leftrightarrow \quad d\mathbf{n} = \omega \wedge \mathbf{n} \quad \Leftrightarrow \quad \omega \sim \mathbf{n} \wedge d\mathbf{n} \tag{20}$$

where the 1-form ω^b is the rotation vector to be derived below. By substituting (20) back into $n^a dn_a$, we obtain $\varepsilon_{bca}\omega^b n^c n^a$ which vanishes identically due to the property $\varepsilon_{bca}n^c n^a = 0$. (2) Having two degrees of freedom α and β, we can expand the differential dn_a like $u_a d\beta + v_a \sin\alpha d\alpha$ with the two vector fields $u_a = \frac{\partial n_a}{\partial \beta}$ and $v_a = \frac{\partial n_a}{\partial \alpha}$ as given above. Notice that the three unit fields $(\mathbf{n}, \mathbf{u}, \mathbf{v})$ plays an important role in this study; they form a vector basis of the field space; they obey the usual cross products namely $\mathbf{n} = \mathbf{u} \wedge \mathbf{v}$ and its homologue which given by cyclic permutations; for example,

$$u_a = \varepsilon_{abc}v^b n^c, \quad v_a = -\varepsilon_{abc}u^b n^c \tag{21}$$

Putting these Eq. (21) back into the expansion of dn_a in terms of $d\alpha,\ d\beta$; and comparing with Eq. (20), we end up with the explicit expression of the 1-form angular "speed" vector ω^b; it reads as follows $\omega^b = v^b d\beta - u^b \sin\alpha d\alpha$. Notice that by using the space time coordinates ξ, we can also express Eq. (20) like $\partial_\mu n_a = \varepsilon_{abc}\omega^b_\mu n^c$ with ω^b_μ given by $v^b(\partial_\mu\beta) - u^b \sin\alpha(\partial_\mu\alpha)$. From this expression, we can compute the Laplacian $\partial^\mu\partial_\mu n_a$; which, by using the above relations; is equal to $\varepsilon_{abc}\partial^\mu\left(\omega^b_\mu n^c\right)$ reading explicitly as $\varepsilon_{abc}\left[\left(\partial^\mu\omega^b_\mu\right)n^c + \omega^b_\mu(\partial^\mu n^c)\right]$ or equivalently like $\partial^\mu\partial_\mu n_a = \varepsilon_{abc}\mathcal{D}^b n^c$ with operator $\mathcal{D}^b = \omega^b_\mu\partial^\mu + \left(\partial^\mu\omega^b_\mu\right)$. Notice that the above operator has an interesting geometric interpretation; by factorising ω^b_μ, we can put it in the form $\omega^d_\mu(\mathcal{D}^\mu)^b_d$ where $(\mathcal{D}^\mu)^b_d$ appears as a gauge covariant derivative $(\mathcal{D}^\mu)^b_d = \delta^b_d\partial^\mu + (A^\mu)^b_d$ with a non trivial gauge potential $(A^\mu)^b_d$ given by $\omega^\mu_d\left(\partial^\nu\omega^b_\nu\right)$. Comparing with (18) with $\partial^\mu\partial_\mu n_a = \varepsilon_{abc}\mathcal{D}^b n^c$, we obtain $\frac{\partial V}{\partial n^a} = \varepsilon_{abc}\mathcal{D}^b n^c - \Lambda n_a$; and then a scalar potential energy V given by $\int\varepsilon_{abc}\left(dn^a\mathcal{D}^b n^c\right) - \Lambda\int n_a dn^a$. The second term in this relation vanishes identically because $n_a dn^a = 0$; thus reducing to

$$V = \int\varepsilon_{abc}\left(dn^a\mathcal{D}^b n^c\right) \tag{22}$$

containing $\varepsilon_{abc}\left(n^a\mathcal{D}^b n^c\right)$ as a sub-term. In the end of this analysis, let us compare this sub-term with the $\varepsilon_{abc}n^b n^c_{\mu_1\mu_2}\Delta^{a\mu_1\mu_2}$ with $\Delta^{a\mu_1\mu_2} = d^a_{\mu_3\dots\mu_{D-2}}\varepsilon^{\mu_1\dots\mu_D}$ giving the general structure of the DM coupling (see end of subSection 2.1). For $(1+2)$D spacetime, the general structure of DM interaction reads $\varepsilon_{abc}d^a_0\left(n^b\nabla n^c\right).\mathbf{e}_{\mu\nu}\varepsilon^{0\mu\nu}$; by setting $\mathbf{e}^0 = \mathbf{e}_{\mu\nu}\varepsilon^{0\mu\nu}$ and $\mathbf{d}^a = d^a_0\mathbf{e}^0$ as well as $D^a = \mathbf{d}^a.\nabla$, one brings it to the form $\varepsilon_{abc}\left(n^a D^b n^c\right)$ which is the same as the one following from (22).

3.2.2 From kinks to 2d Skyrmions

Here, we study the topological properties of the 2d Skyrmion with dynamics governed by the Lagrangian density (17). From the expression of the $(1+1)$D topological current $(J^\mu)_{2D}$ discussed in subSection 2.1, which reads as $\frac{1}{2\pi}\varepsilon^{\mu\nu}n^a\partial_\nu n^b\varepsilon_{ab}$, one can wonder the structure of the $(1+2)$D topological current $(J^\mu)_{3D}$ that is associated with the 2d Skyrmion described by the 3-vector field $n_a(\xi)$. It is given by

$$(J^\mu)_{3D} = \frac{1}{8\pi}\varepsilon^{\mu\nu\rho}n^a\partial_\nu n^b\partial_\nu n^c\varepsilon_{abc} \tag{23}$$

where ε_{abc} is as before and where $\varepsilon^{\mu\nu\rho}$ is the completely antisymmetric Levi-Civita tensor in the $(1+2)$D spacetime. The divergence $\partial_\mu(J^\mu)_{3D}$ of the above spacetime vector vanishes identically; it has two remarkable properties that we want to comment before proceeding. (1) The $\partial_\mu(J^\mu)_{3D}$ is nothing but the determinant of the 3×3 Jacobian matrix $\frac{\partial n^a}{\partial\xi^\mu}$ relating the three field variables n^a to the three spacetime coordinates ξ^μ; this Jacobian $\det\left(\frac{\partial n^a}{\partial\xi^\mu}\right)$ is generally given by $\frac{1}{3!}\varepsilon^{\mu\nu\rho}\partial_\mu n^a\partial_\nu n^b\partial_\nu n^b\varepsilon_{abc}$; it maps the spacetime volume $d^3\xi = dt\wedge dx\wedge dy$ into the target space volume $d^3\mathbf{n} = dn^1\wedge dn^2\wedge dn^3$. In this regards, recall that these two 3D volumes can be expressed in covariant manners by using the completely antisymmetric tensors $\varepsilon_{\mu\nu\rho}$ and ε_{abc} introduced earlier; and as noticed before play a central role in topology. The target space volume $d^3\mathbf{n}$ can be expressed like $\frac{1}{3!}\varepsilon_{abc}dn^a\wedge dn^b\wedge dn^c$; and a similar relation can be also written down for the spacetime volume $d^3\xi$. Notice also that by substituting the differentials dn^a by their expansions $\left(\frac{\partial n^a}{\partial \xi^\mu}\right) d\xi^\mu$; and putting back into $d^3\mathbf{n}$, we obtain the relation $d^3\mathbf{n} = \mathfrak{J}_{3D} d^3\xi$ where $\mathfrak{J}_{3D}$ is precisely the Jacobian $\det\left(\frac{\partial n^a}{\partial \xi^\mu}\right)$. (2) The conservation law $\partial_\mu (J^\mu)_{3D} = 0$ has a geometric origin; it follows from the field constraint relation $n_1^2 + n_2^2 + n_3^2 = 1$ degenerating the volume of the 3D target space down to a surface. This constraint relation describes a unit 2-sphere $\mathbb{S}^2_{(\mathbf{n})}$; and so a vanishing volume $d^3\mathbf{n}\big|_{\mathbb{S}^2_{(\mathbf{n})}} = 0$; thus leading to $\mathfrak{J}_{3D} = 0$ and then to the above continuity equation. Having the explicit expression (23) of the topological current J^μ in terms of the magnetic texture field $n(\xi)$, we turn to determine the associated topological charge $Q = \int dxdyJ^0$ with charge density J^0 given by $\frac{1}{8\pi}\varepsilon_{abc}\varepsilon^{0ij}\left(\partial_i n^b \partial_j n^c\right) n^a$. Substituting $\varepsilon^{0ij} dx \wedge dy$ by $d\xi^i \wedge d\xi^j$, we have $J^0 dx \wedge dy = \frac{1}{8\pi}\varepsilon_{abc} n^a \left(dn^b \wedge dn^c\right)$. Moreover using the differentials $dn^b = u^b d\beta + v^b \sin\alpha d\alpha$, we can calculate the area $dn^b \wedge dn^c$ in terms of the angles α and β; we find $2n_a(\sin\alpha) d\beta \wedge d\alpha$ where we have used $\varepsilon_{abc}\left(u^b v^c - u^c v^b\right) = 2n_a$. So, the topological charge Q reads as $\frac{1}{4\pi}\int_{\mathbb{S}^2_{\mathbf{n}}} (\sin\beta) d\alpha d\beta$ which is equal to 1. In fact this value is just the unit charge; the general value is an integer $Q = N$ with N being the winding number $\pi_2\left(\mathbb{S}^2_{\mathbf{n}}\right)$; see below. Notice that J^0 can be also presented like

$$J^0 = \frac{\varepsilon_{abc}}{8\pi} n^a \left(\frac{\partial n^b}{\partial x}\frac{\partial n^c}{\partial y} - \frac{\partial n^b}{\partial y}\frac{\partial n^c}{\partial x}\right) \tag{24}$$

Replacing n_a by their expression in terms of the angles $(\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta)$, we can bring the above charge density J^0 into two equivalent relations; first into the form like $\frac{\sin\beta}{4\pi}\left(\frac{\partial\beta}{\partial x}\frac{\partial\alpha}{\partial y} - \frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x}\right)$; and second as $\frac{1}{4\pi}\frac{\partial[\alpha,\ \cos\beta]}{\partial[x, y]}$ which is nothing but the Jacobian of the transformation from the (x, y) space to the unit 2-sphere with angular variables (α, β). The explicit expression of (n_1, n_2, n_3) in terms of the (x, y) space variables is given by

$$n_1 = \frac{2x}{x^2 + y^2 + 1} \quad , \quad n_2 = \frac{2y}{x^2 + y^2 + 1} \quad , \quad n_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \tag{25}$$

but this is nothing but the stereographic projection of the 2-sphere $\mathbb{S}^2_\xi$ on the real plane. So, the field n^a defines a mapping between $\mathbb{S}^2_\xi$ towards $\mathbb{S}^2_{\mathbf{n}}$ with topological charge given by the winding number $\mathbb{S}^2_{\mathbf{n}}$ around $\mathbb{S}^2_{\mathbf{n}}$; this corresponds just to the homotopy property $\pi_2\left(\mathbb{S}^2_{\mathbf{n}}\right) = N$.

4. Three dimensional magnetic skyrmions

In this section, we study the dynamics of the 3d skyrmion and its topological properties both in target space $\mathbb{R}^4_n$ (with euclidian metric δ_{AB}) and in 4D spacetime $\mathbb{R}^{1,3}_\xi$ parameterised by $\xi^\mu = (t, x, y, z)$ (with Lorentzian metric $g_{\mu\nu}$). The spacetime dynamics of the 3d skyrmion is described by a four component field $\boldsymbol{n}_A(\xi)$ obeying a constraint relation $f(\boldsymbol{n}) = 1$; here the $f(\boldsymbol{n})$ is given by the quadratic form $\boldsymbol{n}_A \boldsymbol{n}^A$ invariant under SO(4) transformations isomorphic to SU(2)×SU(2). The structure of the topological current of the 3d skyrmion is encoded in two types of Levi-Civita tensors namely the target space ε_{ABCD} and the spacetime $\varepsilon_{\mu\nu\rho\tau}$ extending their homologue concerning the kinks and 2d skyrmions.

4.1 From 2d skyrmion to 3d homologue

As for the 1d and 2d solitons considered previous section, the spacetime dynamics of the 3d skyrmion in $\mathbb{R}^{1,3}$ is described by a field action $\mathcal{S}_{4D} = \int dtL_{4D}$ with Lagrangian realized as the space integral $\int dxdydz\mathcal{L}_{4D}$. Generally, the Lagrangian density $\mathcal{L}_{4D}$ is a function of the soliton $\boldsymbol{n}(t,x,y,z)$ which is a real 4-component field $[\boldsymbol{n} = (n_1, n_2, n_3, n_4)]$ constrained like $f[\boldsymbol{n}(\xi)] = 1$. For self ineracting field, the typical field expression of $\mathcal{L}_{4D}$ is given by $-\frac{1}{2}\left(\partial_\mu \boldsymbol{n}\right)^2 - V(\boldsymbol{n}) - \Lambda[f(\boldsymbol{n}) - 1]$ where $V(\boldsymbol{n})$ is a scalar potential; and where the auxiliary field $\Lambda(\xi)$ is a Lagrange multiplier carrying the field constraint. This density $\mathcal{L}_{4D}$ reads in terms of the Hamiltonian as $\Pi.\frac{\partial \mathbf{n}}{\partial t} - \mathcal{H}(\mathbf{n})$. Below, we consider a 4-component skyrmionic field constrained as $\boldsymbol{n}.\boldsymbol{n} = 1$; and focuss on a simple Lagrangian density $\mathcal{L}_\circ = -\frac{1}{2}\left(\partial_\mu \boldsymbol{n}\right)\left(\partial^\mu \boldsymbol{n}\right) - \Lambda[\boldsymbol{n}.\boldsymbol{n} - 1]$ to describe the degrees of freedom of $\boldsymbol{n}$. Being a unit 4-component vector, we can solve the constraint $\boldsymbol{n}.\boldsymbol{n} = 1$ in terms of three angular angles (α, β, γ); by setting

$$\boldsymbol{n} = (\mathbf{m}\sin\gamma,\ \cos\gamma), \quad \mathbf{m} = (\ \sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta) \tag{26}$$

where $\mathbf{m}$ is a unit 3-vector parameterising the unit sphere $\mathbb{S}^2_{[\alpha]}$. Putting this field realisation back into $\mathcal{L}_\circ$, we obtain $\frac{-\cos 2\gamma}{2}\left(\partial_\mu \gamma\right)^2 - \frac{1-\cos 2\gamma}{4}\left(\partial_\mu \mathbf{m}\right)^2 - \Lambda[\mathbf{m}.\mathbf{m} - 1]$. Notice that by restricting the 4D spacetime $\mathbb{R}^{1,3}$ to the 3D hyperplane z=const; and by fixing the component field γ to $\frac{\pi}{2}$, the above Lagrangian density reduces to the one describing the spacetime dynamics of the 2d skyrmion. Notice also that we can expand the differential $d\boldsymbol{n}_A$ in terms of $d\gamma, d\beta, d\alpha$; we find the following

$$d\boldsymbol{n}_a = m_a \cos\gamma d\gamma + \ \sin\gamma(u_a d\beta + v_a \sin\beta d\alpha), \quad d\boldsymbol{n}_4 = -\sin\gamma d\gamma \tag{27}$$

For convenience, we sometimes refer to the three (α, β, γ) collectively like $\alpha_a = (\alpha_1, \alpha_2, \alpha_3)$; so we have $d\boldsymbol{n}^A = E^A_a d\alpha^b$ with $E^A_a = \frac{\partial \mathbf{n}^A}{\partial \alpha^a}$.

4.2 Conserved topological current

First, we investigate the topological properties of the 3d skyrmion from the target space view; that is without using the spacetine variables $(t,x,y,z) = \xi^\mu$. Then, we turn to study the induced topological properties of the 3d skyrmion viewed from the side of the 4D space time $\mathbb{R}^{1,3}$.

4.2.1 Topological current in target space

The 3d skyrmion field is described by a real four component vector $\boldsymbol{n}_A$ subject to the constraint relation $\boldsymbol{n}_A\boldsymbol{n}^A = 1$; so the soliton has $\mathrm{SO}(4) \sim SO(3)_1 \times SO(3)_2$ symmetry leaving invariant the condition $\boldsymbol{n}_A\boldsymbol{n}^A = 1$ that reads explicitly as $(n_1)^2 + (n_2)^2 + (n_3)^2 + (n_4)^2 = 1$. The algebraic condition $f[\boldsymbol{n}] = 1$ induces in turns the constraint equation $df = 0$ leading to $\mathbf{n}^A d\mathbf{n}_A = 0$ and showing that $\boldsymbol{n}_A$ and $d\boldsymbol{n}_A$ orthogonal 4-vectors in $\mathbb{R}^4_{(\boldsymbol{n})}$. From this constraint, we can construct $\left(\boldsymbol{n}^A d\boldsymbol{n}^B - \boldsymbol{n}^B d\boldsymbol{n}^A\right)/2$ which is a 4×4 antisymmetric matrix $\Omega^{[AB]}$ generating the SO(4) rotations; this $\Omega^{[AB]}$ contains 3+3 degrees of freedom generating the two $SO(3)_1$ and $SO(3)_2$ making SO(4); the first three degrres are given by $\Omega^{[ab]}$ with $a,b = 1,2,3$; and the other three concern $\Omega^{[a4]}$. Notice also that, from the view of the target space, the algebraic relation $\boldsymbol{n}_A\boldsymbol{n}^A = 1$ describes a unit 3-sphere $\mathbb{S}^3_{\boldsymbol{n}}$ sitting in $\mathbb{R}^4_{\boldsymbol{n}}$; as such its volume 4-form $d^4\boldsymbol{n}$, which reads as $\frac{1}{4!}\varepsilon_{ABCD}d\boldsymbol{n}^A \wedge d\boldsymbol{n}^B \wedge d\boldsymbol{n}^C \wedge d\boldsymbol{n}^D$, vanishes identically when restricted to the 3-sphere; i.e.: $d^4\boldsymbol{n}\big|_{\mathbb{S}^3_{\boldsymbol{n}}} = 0$. This vanishing property of $d^4\boldsymbol{n}$ on $\mathbb{S}^3_{\boldsymbol{n}}$ is a key ingredient in the derivation of the topological current $\boldsymbol{J}$ of the 3D skyrmion and its conservation $d\boldsymbol{J} = 0$. Indeed, because of the property $d^2 = 0$ (where we have hidden the wedge product $\wedge$), it follows that $d^4\boldsymbol{n}$ can be expressed as $d\boldsymbol{J}$ with the 3-form $\boldsymbol{J}$ given by

$$\boldsymbol{J} = \frac{1}{4!}\varepsilon_{ABCD}\boldsymbol{n}^A d\boldsymbol{n}^B d\boldsymbol{n}^C d\boldsymbol{n}^D \tag{28}$$

This 3-form describes precisely the topological current in the target space; this is because on $\mathbb{S}^3_{\boldsymbol{n}}$, the 4-form $d^4\boldsymbol{n}$ vanishes; and then $d\boldsymbol{J}$ vanishes. By solving, the skyrmion field constraint $\boldsymbol{n}_A\boldsymbol{n}^A = 1$ in terms of three angles α_a as given by Eq. (26); with these angular coordinates, we have mapping $\boldsymbol{f} : \mathbb{R}^4_{\boldsymbol{n}} \to \mathbb{S}^3_{\boldsymbol{n}}$ with $\mathbb{S}^3_{\boldsymbol{n}} \simeq \mathbb{S}^3_{\alpha}$. By expanding the differentials like $d\boldsymbol{n}^A = E^A_a d\alpha^a$ with $E^A_a = \frac{\partial \boldsymbol{n}^A}{\partial \alpha^a}$; then the conserved current on the 3-sphere $\mathbb{S}^3_{\alpha}$ reads as follows

$$\boldsymbol{J} = \frac{1}{4!3!}\varepsilon_{ABCD}\left(\boldsymbol{n}^A E^B_b E^C_c E^D_d\right)\varepsilon^{abc} d^3\alpha \tag{29}$$

where we have substituted the 3-form $d\alpha^b d\alpha^c d\alpha^d$ on the 3-sphere $\mathbb{S}^3_{\alpha}$ by the volume 3-form $\varepsilon^{abc}d^3\alpha$. In this regards, recall that the volume of the 3-sphere is $\int_{\mathbb{S}^3_{(\boldsymbol{n})}} d^3\alpha = \frac{\pi^2}{2}$.

4.2.2 Topological symmetry in spacetime

In the spacetime $\mathbb{R}^{1,3}$ with coordinates $\xi^\mu = (t, x, y, z)$, the 3d skyrmion is described by a four component field $\boldsymbol{n}_A(\xi)$ and is subject to the local constraint relation $\boldsymbol{n}_A\boldsymbol{n}^A = 1$. A typical static configuration of the 3d skyrmion is obtained by solving the field sonctraint in terms of the space coordinates; it is given by Eq. (26) with the local space time fields $\mathbf{m}(\xi)$ and $\gamma(\xi)$ thought of as follows

$$\mathbf{m}(\xi) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right), \quad \gamma(\xi) = \arcsin\frac{2rR}{r^2+R^2} = \arccos\frac{r^2-R^2}{r^2+R^2} \tag{30}$$

with $r = \sqrt{x^2+y^2+z^2}$, giving the radius of $\mathbb{S}^2_{\xi}$, and R associated with the circle $\mathbb{S}^1_{\xi}$ fibered over $\mathbb{S}^2_{\xi}$; the value $R = r$ corresponds to $\gamma = \frac{\pi}{2}$ and $R >> r$ to $\gamma = \pi$. Notice that $\gamma(\xi)$ in Eq. (30) has a spherical symmetry as it is a function only of r (no angles α, β, γ). Moreover, as this configuration obeys $\sin\gamma(0) = 0$ and $\sin\gamma(\infty) = 0$; we assume $\gamma(0) = n_0\pi$ and $\gamma(\infty) = n_\infty\pi$. Putting these relations back into (26), we obtain the following configuration

$$\tilde{\boldsymbol{n}} = \left(\frac{2xR}{r^2+R^2}, \frac{2yR}{r^2+R^2}, \frac{2zR}{r^2+R^2}, \frac{r^2-R^2}{r^2+R^2}\right) \tag{31}$$

describing a compactification of the space $\mathbb{R}^3_{\xi}$ into $\mathbb{S}^3_{\xi}$ which is homotopic to $\mathbb{S}^3_{\boldsymbol{n}}$. From this view, the $\tilde{\boldsymbol{n}} : \xi \to \tilde{\boldsymbol{n}}(\xi)$ is then a mapping from $\mathbb{S}^3_{\xi}$ into $\mathbb{S}^3_{\mathbf{n}}$ with topological charge given by the winding number characterising the wrapping $\mathbb{S}^3_{\boldsymbol{n}}$ on $\mathbb{S}^3_{\xi}$; and for which we have the property $\pi_3(\mathbb{S}^3_n) = \mathbb{Z}$. In this regards, recall that 3-spheres $\mathbb{S}^3$

have a Hopf fibration given by a circle $\mathbb{S}^1$ sitting over $\mathbb{S}^2$ (for short $\mathbb{S}^3 \sim \mathbb{S}^1 \ltimes \mathbb{S}^2$); this non trivial fibration can be viewed from the relation $\mathbb{S}^3 \sim SU(2)$ and the factorisation $U(1) \times SU(2)/U(1)$ with the coset $SU(2)/U(1)$ identified with $\mathbb{S}^2$; and $U(1)$ with $\mathbb{S}^1$. Applying this fibration to $\mathbb{S}^3_{\boldsymbol{\xi}}$ and $\mathbb{S}^3_{\boldsymbol{n}}$, it follows that $\tilde{\boldsymbol{n}} : \mathbb{S}^3_\xi \to \mathbb{S}^3_{\boldsymbol{n}}$; and the same thing for the bases $\mathbb{S}^2_\xi \to \mathbb{S}^2_{\boldsymbol{n}}$ and for the fibers $\mathbb{S}^1_\xi \to \mathbb{S}^1_{\boldsymbol{n}}$. Returning to the topological current and the conserved topological charge $Q = \int_{\mathbb{R}^3} d^3\mathbf{r} J^0(t, \mathbf{r})$, notice that in space time the differential $d\boldsymbol{n}^A$ expands like $(\partial_\mu \boldsymbol{n}^A) d\xi^\mu$; then using the duality relation $J_{[\nu\rho\tau]} = \varepsilon_{\mu\nu\rho\tau} J^\mu$, we find, up to a normalisation by the volume of the 3-sphere $\pi^2/2$, the expression of the topological current $J^\mu(\xi)$ in terms of the 3D skyrmion field

$$J^\mu = \frac{1}{12\pi^2} \varepsilon^{\mu\nu\rho\tau} n^a \partial_\nu n^b \partial_\rho n^c \partial_\tau n^c \varepsilon_{abcd} \tag{32}$$

In terms of the angular variables, this current reads like $\mathcal{N} \partial_\nu \alpha \partial_\rho \beta \partial_\tau \gamma \varepsilon^{\mu\nu\rho\tau}$ with $\mathcal{N} = \frac{1}{2\pi^2} (\sin\beta)(\sin\gamma)^2$. From this current expression, we can determine the associated topological charge Q by space integration over the charge density

$$J^0(t, \mathbf{r}) = -\frac{\sin^2\gamma}{2\pi^2 r^2} \frac{d\gamma}{dr} \tag{33}$$

Because of its spherical symmetry, the space volume $d^3\mathbf{r}$ can be substituted by $4\pi r^2 dr$; then the charge Q reads as the integral $-\frac{4\pi}{2\pi^2} \int_{\gamma(0)}^{\gamma(\infty)} \sin^2\gamma d\gamma$ whose integration leads to the sum of two terms coming from the integration of $\sin^2\gamma = \frac{1}{2} - \frac{1}{2}\cos 2\gamma$. The integral first reads as $\frac{1}{\pi}[\gamma(0) - \gamma(\infty)]$; by substituting $\gamma(0) = n_0\pi$, it contributes like $N\pi$. The integral of the second tem gives $\frac{1}{2\pi}[\sin 2\gamma(0) - \sin 2\gamma(\infty)]$; it vanishes identically. So the topological charge is given by

$$Q = \frac{\gamma(0) - \gamma(\infty)}{\pi} = N \tag{34}$$

5. Effective dynamics of skyrmions

In this section, we investigate the effective dynamics of a point-like skyrmion in a ferromagnetic background field while focussing on the 2d configuration. First, we derive the effective equation of a rigid skyrmion and comment on the underlying effective Lagrangian. We also describe the similarity with the dynamics of an electron in a background electromagnetic field. Then, we study the effect of dissipation on the skyrmion dynamics.

5.1 Equation of a rigid skyrmion

To get the effective equation of motion of a rigid skyrmion, we start by the spin $(0+1)D$ action $\mathcal{S}_{spin} = \int dt L_{spin}$ describing the time evolution of a coherent spin vector modeled by a rotating magnetic moment $\mathbf{n}(t)$ with velocity $\dot{\mathbf{n}} = \frac{d\mathbf{n}}{dt}$; and make some accommodations. For that, recall that the Lagrangian L_{spin} has the structure $L_B - \hbar S\gamma H$ where L_B is the Berry term having the form $L_B = q_e \mathbf{A}.\dot{\mathbf{n}}$ with geometric (Berry) potential $\mathbf{A} \sim \langle \mathbf{n} | \dot{\mathbf{n}} \rangle$; and where H is the Hamiltonian of the magnetic moment $\mathbf{n}(t)$ obeying the constraint $\mathbf{n}^2 = 1$. This magnetisation constraint is solved by two free angles $\alpha(t), \beta(t)$; they appear in the Berry term $L_B = -\hbar S(1 - \cos\beta)\frac{d\alpha}{dt}$. Below, we think of the above magnetisation as a ferromagnetic background $\mathbf{n}(\mathbf{r})$ filling the spatial region of $\mathbb{R}_\xi^{1,d}$ with coordinates $\xi = (t, \mathbf{r})$; and of the skymion as a massive point- like particle $\mathbf{R}(t)$ moving in this background.

5.1.1 Rigid skyrmion

We begin by introducing the variables describing the skyrmion in the magnetic background field $\mathbf{n}(\mathbf{r})$. We denote by M_s the mass of the skyrmion, and by $\mathbf{R}$ and $\dot{\mathbf{R}}$ its space position and its velocity. For concreteness, we restrict the investigation to the spacetime $\mathbb{R}_\xi^{1,2}$ and refer to $\mathbf{R}$ by the components $X_i = (X, Y)$ and to $\mathbf{r}$ by the components $x_i = (x, y)$. Because of the Euclidean metric δ_{ij}; we often we use both notations X^i and $X_i = \delta_{ij}X^j$ without referring to δ_{ij}. Furthermore; we limit the discussion to the interesting case where the only source of displacements in $\mathbb{R}_\xi^{1,2}$ is due to the skyrmion $\mathbf{R}(t)$ (rigid skyrmion). In this picture, the description of the skyrmion $\mathbf{R}(t)$ dissolved in the background magnet $\mathbf{n}(\mathbf{r})$ is given by

$$\mathbf{n}(\mathbf{r}, t) = \mathbf{n}[\mathbf{r} - \mathbf{R}(t)] \tag{35}$$

In this representation, the velocity $\dot{\mathbf{n}}$ of the skyrmion dissolved in the background magnet can be expressed into manners; either like $-\dot{X}^i\frac{\partial \mathbf{n}}{\partial X^i}$; or as $\dot{X}^i\frac{\partial \mathbf{n}}{\partial x^i}$; this is because $\frac{\partial}{\partial X^i} = -\frac{\partial}{\partial x^i}$. With this parametrisation, the dynamics of the skyrmion is described by an action $\mathcal{S}_s = \int dtL_s$ with Lagrangian given by a space integral $L_s = \frac{\hbar s}{\mathrm{a}^2}\int d^2\mathbf{r}\mathcal{L}_s$ and spacetime density as follows

$$\mathcal{L}_s = \gamma\hbar S\mathcal{H} - \hbar S\mathcal{L}_B \tag{36}$$

In this relation, the density $\mathcal{L}_B = -(1 - \cos\beta)\frac{\partial\alpha}{\partial t}$ where now the angular variables are spacetime fields $\beta(t, \mathbf{r})$ and $\alpha(t, \mathbf{r})$. Similarly, the density $\mathcal{H}$ is the Hamiltonian density with arguments as $\mathcal{H}[\mathbf{n}, \partial_\mu\mathbf{n}, \mathbf{r}]$ and magnetic $\mathbf{n}$ as in Eq. (35). In this field action $\mathcal{S}_s$, the prefactor a^{-2} is required by the continuum limit of lattice Hamiltonians H_{latt} living on a square lattice with spacing parameter a. Recall that for these H_{latt}'s, one generally has discrete sums like $\sum_\mu(\dots)$, $\sum_{\mu,\nu}(\dots)$ and so on; in the limit where a is too small, these sums turn into 2D space integrals like $\mathrm{a}^{-2}\int d^2\mathbf{r}(\dots)$. To fix ideas, we illustrate this limit on the typical hamiltonian H_{HDMZ}, it describes the Heisenberg model on the lattice $\mathbb{Z}^2$ augmented by the Dzyaloshinskii-Moriya and the Zeeman interactions [66, 67]

$$H_{HDMZ} = -J\sum_{\langle\mu,\nu\rangle}\mathbf{n}(\mathbf{r}_\mu)\mathbf{n}(\mathbf{r}_\nu) - D\sum_{\mu,\nu,\rho}\mathbf{d}_\mu.\left[\mathbf{n}(\mathbf{r}_\nu)\wedge\mathbf{n}(\mathbf{r}_\rho)\right]\varepsilon^{\mu\nu\rho} - \sum_\mu\mathbf{B}.\mathbf{n}(\mathbf{r}_\mu) \tag{37}$$

with $\mathbf{r}_\nu = \mathbf{r}_\mu + \mathrm{a}\mathbf{e}_{\nu\mu}$; that is $\mathbf{e}_{\nu\mu} = (\mathbf{r}_\nu - \mathbf{r}_\mu)/\mathrm{a}$ where a is the square lattice parameter. So, the continuum limit $\mathcal{H}$ of this lattice Hamiltonian involves the target space metric δ_{ab} and the topological Levi-Civita tensor ε_{abc} of the target space $\mathbb{R}_\mathbf{n}^3$; it involves as well the metric $g_{\mu\nu}$ and the completely antisymmetry $\varepsilon_{\mu\nu\rho}$ of the space time $\mathbb{R}_\xi^{1,2}$. In terms of δ_{ab} and ε_{abc} tensors, the continuous hamiltonian density reads as follows

$$\mathcal{H} = \frac{J\mathrm{a}^2}{2}\delta_{ab}\partial^i n^a\partial_i n^b + \mathrm{a}\varepsilon_{abc}d_\mu^a\left(n^b D_{\nu\rho}n^c\right)\varepsilon^{\mu\nu\rho} - \mathbf{B}.\mathbf{n} \tag{38}$$

with $\nabla_{\nu\rho} = \mathbf{e}_{\nu\mu}.\nabla$. Below, we set J=1 and, to factorise out the normalisation factor a^2, scale the parameters of the model like $d^a_\mu = a\tilde{d}^a_\mu$ and $\mathbf{B} = a^2\tilde{\mathbf{B}}$. For simplicity, we sometimes set as well a= 1.

5.1.2 Skyrmion equation without dissipation

To get the effective field equation of motion of the point-like skyrmion without dissipation, we calculate the vanishing condition of the functional variation of the action; that is $\delta\left(\int dtd^2\mathbf{r}\mathcal{L}\right) = 0$. General arguments indicate that the effective equation of the skyrmion with topological charge q_s in the background magnet has the form

$$M_s\ddot{X}_i = q_s\mathcal{E}_i + q_s\varepsilon_{ijz}\dot{X}^j\mathcal{B}^z \tag{39}$$

from which one can wonder the effective Lagrangian describing the effective dynamics of the skyrmion. It is given by

$$L_s = \frac{M_s}{2}\delta_{ij}\dot{X}^i\dot{X}^j - q_s\varepsilon_{zij}\mathcal{B}^z X^i\dot{X}^j - q_s\mathcal{V}(X) \tag{40}$$

Notice that the right hand of Eq. (39) looks like the usual Lorentz force $(q_e\mathbf{E} + q_e\dot{\mathbf{r}}\wedge\mathbf{B})$ of a moving electron with q_e in an external electromagnetic field (E_i, B_i); the corresponding Lagrangian is $\frac{m}{2}\dot{\mathbf{r}}^2 + q_e\mathbf{B}.(\mathbf{r}\wedge\dot{\mathbf{r}}) - q_e\mathbf{E}.\mathbf{r}$. This similarity between the skyrmion and the electron in background fields is because the skyrmion has a topological charge q_s that can be put in correspondence with q_e; and, in the same way, the background field magnet $\mathcal{E}_i, \mathcal{B}_i$ can be also put in correspondence with the electromagnetic field (E_i, B_i). To rigourously derive the spacetime Eqs. (39) and (40), we need to perform some manipulations relying on computing the effective expression of $\mathcal{S}_s = \hbar S\int dt\left(\int d^2\mathbf{r}\mathcal{L}_s\right)$ and its time variation $\delta\mathcal{S}_s = 0$. However, as $\mathcal{L}_s$ has two terms like $\gamma\hbar S\mathcal{H} - \hbar S\mathcal{L}_B$, the calculations can be split in two stages; the first stage concerns the block $\gamma\hbar S\int d^2\mathbf{r}\mathcal{H}$ with $\mathcal{H}\left[\mathbf{n}, \partial_\mu\mathbf{n}, \mathbf{r}\right]$ which is a function of the magnetic texture (35); that is $\mathbf{n}(\mathbf{r} - \mathbf{R})$. The second stage regards the determination of the integral $\hbar S\int d^2\mathbf{r}\mathcal{L}_B$. The computation of the first term is straightforwardly identified; by performing a space shift $\mathbf{r} \to \mathbf{r} + \mathbf{R}$, the Hamiltonian density becomes $\mathcal{H}\left[\mathbf{n}, \partial_\mu\mathbf{n}, \mathbf{r} + \mathbf{R}\right]$ with $\mathbf{n}(\mathbf{r})$ and where the dependence in $\mathbf{R}$ becomes explicit; thus allowing to think of the integral $\gamma\hbar S\int d^2\mathbf{r}\mathcal{H}$ as nothing but the scalar energy potential $\mathcal{V}(\mathbf{R}) = \hbar S\gamma\int d^2\mathbf{r}\mathcal{H}(t, \mathbf{r}, \mathbf{R})$. So, we have

$$\frac{\delta}{\delta X^a}\left(\hbar S\gamma\int d^2\mathbf{r}\mathcal{H}\right) = \frac{\partial\mathcal{V}}{\partial X^a} \tag{41}$$

Concerning the calculation of the $\hbar S\int d^2\mathbf{r}\mathcal{L}_B$, the situation is somehow subtle; we do it in two steps; first we calculate the $\left(\delta\int d^2\mathbf{r}\mathcal{L}_B\right)$ because we know the variation $\frac{\delta\mathcal{L}_B}{\delta n^a}$ which is equal to $\frac{1}{2}\varepsilon_{abc}n^b\dot{n}^c$. Then, we turn backward to determine $\hbar S\int d^2\mathbf{r}\mathcal{L}_B$ by integration. To that purpose, recall also that the Berry term $\mathcal{L}_B$ is given by $-(1 - \cos\beta)\frac{\partial\alpha}{\partial t}$; and its variation $\frac{\delta\mathcal{L}_B}{\delta n^a}\frac{\partial n^a}{\partial X^j}\partial X^j$ is equal to $\frac{1}{2}\varepsilon_{abc}n^b\dot{n}^c$. To determine the time variation $\delta L_B = \delta\int d^2\mathbf{r}\delta\mathcal{L}_B$, we first expand it like $\int d^2\mathbf{r}\frac{\delta\mathcal{L}}{\delta n^a}\delta n^a$; and use $\delta n^a = -\frac{\partial n^a}{\partial X^j}\partial X^j$ to put it into the form -$\int d^2\mathbf{r}\frac{\delta\mathcal{L}_B}{\delta n^a}\frac{\partial n^a}{\partial X^j}\partial X^j$. Then, substituting $\frac{\delta\mathcal{L}_B}{\delta n^a}$ by its expression $\frac{1}{2}\varepsilon_{abc}n^b\dot{n}^c$ with $\dot{n}^c$ expanded like -$\frac{\partial n^c}{\partial X^i}\dot{X}^i$, we end up with

$$\delta L_B = 2\hbar S\left(\int d^2\mathbf{r}\frac{1}{2}\varepsilon_{abc}n^b\frac{\partial n^c}{\partial x^i}\frac{\partial n^a}{\partial x^j}\right)\varepsilon^{ij}\left(\dot{X}\delta Y - \dot{Y}\delta X\right) \tag{42}$$

Next, using the relation $\varepsilon^{ij}d^2\mathbf{r} = dx^i \wedge dx^j$, the first factor becomes $\int \frac{1}{2}\varepsilon_{abc}n^a dn^b \wedge dn^c$ gives precisely the skyrmion topological charge q_s. So, the resulting δL_B reduces to $2q_s\hbar S(\dot{X}\delta Y - \dot{Y}\delta X)$ that reads also like

$$\delta L_B = -2q_s\hbar S\varepsilon_{ij}\dot{X}^i\delta X^j \tag{43}$$

This variation is very remarkable because it is contained in the variation of the effective coupling $L_B^{int} = -2q_s\hbar S\varepsilon_{ij}\dot{X}^i X^j$ which can be presented like $L_B^{int} = -q_s\mathcal{A}_i\dot{X}^i$ where we have set $\mathcal{A}_i = 2\hbar S\varepsilon_{zij}X^j$; this vector can be interpreted as the vector potential of an effective magnetic field $\mathcal{B}^z = \frac{1}{2}\varepsilon^{zij}\partial_i\mathcal{A}_i$. By adding the kinetic term $\frac{M_s}{2}\dot{X}^i\dot{X}_i$, we end up with an effective Lagrangian L_B associated with the Berry term; it reads as follows $L_B = \frac{M_s}{2}\delta_{ij}\dot{X}^i\dot{X}^j - q_s\mathcal{A}_i\dot{X}^i$. So, the effective Lagrangian L_{eff} describing the rigid 2d skyrmion in a ferromagnet is

$$L_{eff} = \frac{M_s}{2}\dot{\mathbf{R}}^2 - q_s\mathbf{A}.\dot{\mathbf{R}} - \mathcal{V}(R) \tag{44}$$

From this Lagrangian, we learn the equation of the motion of the rigid skyrmion namely $M_s\ddot{X}_j = f_j + 4q_s\hbar S\varepsilon_{zij}\dot{X}^i$; for the limit $M_s = 0$, it reduces to $\dot{X}^i = \frac{1}{4q_s\hbar S}\varepsilon^{zji}f_j$.

5.2 Implementing dissipation

So far we have considered magnetic moment obeying the constraint $\mathbf{n}^2 = 1$ with time evolution given by the LL equation $\dot{\mathbf{n}} = -\gamma\mathbf{f} \wedge \mathbf{n}$ where the force $\mathbf{f} = -\frac{\partial\mathcal{H}}{\partial\mathbf{n}}$. Using this equation, we deduce the typical properties $\mathbf{n}.\dot{\mathbf{n}} = \mathbf{f}.\dot{\mathbf{n}} = 0$ from which we learn that the time variation $\frac{dH}{dt}$ of the Hamiltonian, which reads as $\hbar S\gamma\int d^2\mathbf{r}\frac{\partial\mathcal{H}}{\partial n^a}\dot{n}^a$, vanishes identically as explicitly exhibited below,

$$\frac{dH}{dt} = -\hbar S\gamma\int d^2\mathbf{r}(\mathbf{f}.\dot{\mathbf{n}}) \tag{45}$$

In presence of dissipation, we loose energy; and so one expects that $\frac{dH}{dt} < 0$; indicating that the rigid skyrmion has a damped dynamics. In what follows, we study the effect of dissipation in the ferromagnet and derive the damped skyrmion equation.

5.2.1 Landau-Lifshitz-Gilbert equation

Due to dissipation, the force $\mathbf{F}$ acting on the rigid skyrmion $\mathbf{R}(t)$ has two terms, the old conservative $\mathbf{f} = -\frac{\partial\mathcal{H}}{\partial\mathbf{n}}$; and an extra force $\delta\mathbf{f}$ linearly dependent in magnetisation velocity $\dot{\mathbf{n}}$. Due to this extra force $\delta\mathbf{f} = -\frac{\alpha}{\gamma}\dot{\mathbf{n}}$, the LL equation gets modified; its deformed expression is obtained by shifting the old force $\mathbf{f}$ like $\mathbf{F} = \mathbf{f} - \frac{\alpha}{\gamma}\dot{\mathbf{n}}$ with α a positive damping parameter (Gilbert parameter). As such, the previous LL relation gives the so called Landau-Lifschitz-Gilbert (LLG) equation [68, 69]

$$\dot{\mathbf{n}} = -\gamma \mathbf{f} \wedge \mathbf{n} + \alpha \dot{\mathbf{n}} \wedge \mathbf{n} \tag{46}$$

where its both sides have $\dot{\mathbf{n}}$. From this generalised relation, we still have $\mathbf{n}.\dot{\mathbf{n}} = 0$ (ensuring $\mathbf{n}^2 = 1$); however $\mathbf{f}.\dot{\mathbf{n}} \neq \mathbf{0}$ as it is equal to the Gilbert term namely $-\alpha(\mathbf{f} \wedge \mathbf{n}).\dot{\mathbf{n}}$. Notice that Eq. (46) still describe a rotating magnetic moment in the target space ($d\mathbf{n} = 0$); but with a different angular velocity Ω which, in addition to $\mathbf{f}$, depends moreover on the Gilbert parameter and the magnetisation $\mathbf{n}$. By factorising Eq. (46) like $\dot{\mathbf{n}} = \Omega \wedge \mathbf{n}$, we find

$$\Omega = \frac{-\gamma}{1+\alpha^2}[\mathbf{f} + \alpha(\mathbf{f} \wedge \mathbf{n})] \tag{47}$$

Notice that in presence of dissipation ($\alpha \neq 0$), the variation of the hamiltonian $\frac{dH}{dt}$ given by (45) is no longer non vanishing; by first replacing $\mathbf{f}.\dot{\mathbf{n}} = -\alpha(\mathbf{f} \wedge \mathbf{n}).\dot{\mathbf{n}}$ and putting back in it, we get

$$\frac{dH}{dt} = \alpha\hbar S \int d^2\mathbf{r}\gamma(\mathbf{f} \wedge \mathbf{n}).\dot{\mathbf{n}} \tag{48}$$

then, substituting $\gamma \mathbf{f} \wedge \mathbf{n} = \alpha \dot{\mathbf{n}} \wedge \mathbf{n} - \dot{\mathbf{n}}$; we find that $\frac{dH}{dt}$ is given by $-\alpha\hbar s \int d^2\mathbf{r}.\dot{\mathbf{n}}^2$ indicating that $\frac{dH}{dt} < 0$; and consequently a decreasing energy $H(t)$ (loss of energy) while increasing time.

5.2.2 Damped skyrmion equation

To obtain the damped skyrmion equation due to the Gilbert term, we consider the rigid magnetic moment $\mathbf{n}[\mathbf{r} - \mathbf{R}(t)]$; and compute the expression of the skyrmion velocity $\dot{\mathbf{R}}$ in terms of the conservative force $\mathbf{f}$ and the parameter α. To that purpose we start from Eq. (46) and multiply both equation sides by $\wedge d\mathbf{n}$ while assuming $\mathbf{f}.\mathbf{n} = 0$ (the conservative force transverse to magnetisation), we get $\dot{\mathbf{n}} \wedge d\mathbf{n} = -\gamma(\mathbf{f}.d\mathbf{n})\mathbf{n} + \alpha(d\mathbf{n}.\dot{\mathbf{n}})\mathbf{n}$. Then, multiply scalarly by $\mathbf{n}$, which corresponds to a projection along the magnetisation, we obtain

$$\mathbf{n}.(\dot{\mathbf{n}} \wedge d\mathbf{n}) = -\gamma(\mathbf{f}.d\mathbf{n}) + \alpha(d\mathbf{n}.\dot{\mathbf{n}}) \tag{49}$$

Substituting $d\mathbf{n}$ and $\dot{\mathbf{n}}$ by their expansions $dx^i(\partial_i\mathbf{n})$ and $-\dot{X}^i(\partial_i\mathbf{n})$, then multiplying by $\wedge dx^l$; we end up with a relation involving $dx^j \wedge dx^l$ (which reads as $\varepsilon^{zjl}d^2\mathbf{r}$); so we have

$$\dot{X}^l\left(4\pi J^0 d^2\mathbf{r}\right) = -\gamma\varepsilon^{0lj}\left(\mathbf{f}.\partial_j\mathbf{n}\right)d^2\mathbf{r} + \alpha\varepsilon^{0lj}\dot{X}_j\left(\partial_j\mathbf{n}\right)^2 d^2\mathbf{r} \tag{50}$$

where we have set $J^0 = \frac{1}{2\pi}\varepsilon^{zij}\mathbf{n}.\left[(\partial_i\mathbf{n}) \wedge \left(\partial_j\mathbf{n}\right)\right]$, defining the magnetization density, and where we have replaced $\left(\partial_i\mathbf{n}.\partial_j\mathbf{n}\right)$ by $\delta_{ij}(\partial_k\mathbf{n})^2$. By integrating over the 2d space while using $\int J^0 d^2\mathbf{r} = 4\pi q_s$ and setting $\eta_j = \frac{1}{4\pi}\int d^2\mathbf{r}\left(\partial_j\mathbf{n}\right)^2 \equiv \eta$, we arrive at the relation

$$4\pi q_s \dot{X}^l = \gamma\varepsilon^{0lj}\int\left(\mathbf{f}.\partial_j\mathbf{n}\right)d^2\mathbf{r} - 4\pi\eta\alpha\varepsilon^{0lj}\dot{X}^l \tag{51}$$

with $\varepsilon^{zxy} = -\varepsilon_{zxy} = -1$. The remaining step is to replace the conservative force $\mathbf{f}$ by $-\frac{\partial\mathcal{H}}{\partial\mathbf{n}}$ and proceeds in performing the integral over $\left(\mathbf{f}.\partial_j\mathbf{n}\right)$. Because of the explicit dependence into $\mathbf{r}$, the $\mathbf{f}.\partial_j\mathbf{n}$ can be expressed like $\partial_j^{\exp}\mathcal{H} - \partial_j^{tot}\mathcal{H}$; the explicit derivation term $\partial_j^{\exp}\mathcal{H}$ has been added because the Hamiltonian density has an explicit dependence $\mathcal{H}[\mathbf{n}, \partial_\mu\mathbf{n}, \mathbf{r}]$. Recall that $\partial_j^{tot}\mathcal{H}$ is given by $\partial_j^{\exp}\mathcal{H} + \frac{\partial\mathcal{H}}{\partial\mathbf{n}}.\partial_j\mathbf{n}$ which is equal to $\partial_j^{\exp}\mathcal{H} - \mathbf{f}.\partial_j\mathbf{n}$. Notice also that the term $\partial_j^{\exp}\mathcal{H}$ can be also expressed like $-\frac{\partial\mathcal{H}}{\partial\mathbf{R}}$. Therefore, the integral $(\mathbf{f}.\partial_j\mathbf{n})d^2\mathbf{r}$ has two contributions namely the $\int\left(\partial_j^{tot}\mathcal{H}\right)d^2\mathbf{r}$ which, being a total derivative, vanishes identically; and the term $\int\left(\partial_j^{\exp}\mathcal{H}\right)d^2\mathbf{r}$ that gives $-\frac{\partial\mathcal{V}}{\partial\mathbf{R}}$. Putting this value back into (51), we end up with

$$4\pi q_s\dot{X}^l = \gamma\varepsilon^{zlj}\left(\frac{\partial\mathcal{V}}{\partial X^j} + \frac{4\pi\eta\alpha}{\gamma}\dot{X}^l\right) \tag{52}$$

Implementing the kinetic term of the skyrmion, we obtain the equation with dissipation $M_s\ddot{\mathbf{R}} = -\frac{\partial\mathcal{V}}{\partial\mathbf{R}} + G\left(\mathbf{z}\wedge\dot{\mathbf{R}} - \frac{\eta\alpha}{q_s}\dot{\mathbf{R}}\right)$ where the constant $G = \frac{4\pi\hbar S q_s}{a^2}$ stands for the gyrostropic constant**.**

6. Electron-skyrmion interaction

In this section, we investigate the interacting dynamics between electrons and skyrmions with spin transfer torque (STT) [70]. The electron-skyrmion interaction is given by Hund coupling $J_H\left(\Psi^\dagger\sigma_a\Psi\right).n^a$ which leads to emergent $SU(2)$ gauge potential that mediate the interaction between the spin texture $\mathbf{n}(t,\mathbf{r})$ and the two spin states $(\Psi_\uparrow, \Psi_\downarrow)$ of the electron. We also study other aspects of electron/ skyrmion system like the limit of large Hund coupling; and the derivation of the effective equation of motion of rigid skyrmions with STT.

6.1 Hund coupling

We start by recalling that a magnetic atom (like iron, manganese, ...) can be modeled by a localized magnetic moment $\mathbf{n}(t,\mathbf{r})$ and mobile carriers represented by a two spin component field $\Psi(t,\mathbf{r})$; the components of the fields $\mathbf{n}$ and Ψ are respectively given by $n^a(t,\mathbf{r})$ with a=1,2,3; and by $\Psi_\alpha(t,\mathbf{r})$ with $\alpha = \uparrow\downarrow$. Using the electronic vector density $\boldsymbol{j}_{(e)} = \Psi^\dagger\sigma\Psi$, the interaction between localised and itinerant electrons of the magnetic atom are bound by the Hund coupling reading as $H_{e-n} = -J_H\mathbf{n}.\boldsymbol{j}_{(e)}$ with Hund parameter $J_H > 0$ promoting alignment of $\mathbf{n}$ and $\boldsymbol{j}_{(e)}$. So, the dynamics of the interacting electron with the backround $\mathbf{n}$ is given by the Lagrangian density $\mathcal{L}_e = \hbar\Psi^\dagger\frac{i\partial}{\partial t}\Psi - H_{e-n}$ expanding as follows

$$\mathcal{L}_e[\Psi,\mathbf{n}] = \hbar\Psi^\dagger\frac{i\partial}{\partial t}\Psi - \Psi^\dagger\left(\frac{\mathbf{P}^2}{2m} - J_H\sigma.\mathbf{n}\right)\Psi \tag{53}$$

where $\mathbf{P} = \frac{\hbar}{i}\nabla$ and $\sigma.\mathbf{n} = \sigma^x n_x + \sigma^y n_y + \sigma^z n_z$.

6.1.1 Emergent gauge potential

Because of the ferromagnetic Hund coupling $(J_H > 0)$, the spin observable $\hat{S}_e^z = \frac{\hbar}{2}\sigma^z$ of the conduction electron tends to align with the orientation $\sigma^{\mathbf{n}} = \sigma.\mathbf{n}$ of the magnetisation $\mathbf{n}$ — with angle $\theta = (\widehat{\mathbf{e}_z, \mathbf{n}})$—; this alinement is accompanied by a local phase change of the electronic wave function Ψ which becomes $\psi = U\Psi$ where $U(t,\mathbf{r}) = e^{i\Theta(t,\mathbf{r})}$ is a unitary $SU(2)$ transformation mapping σ^z into $\sigma^{\mathbf{n}}$; that is $\sigma^{\mathbf{n}} = U^\dagger\sigma^z U$. For later use, we refer to the new two components of the electronic field like $\psi_{+\mathbf{n}}, \psi_{-\mathbf{n}}$ (for short $\psi_{\dot\alpha}$ with label $\dot\alpha = \pm$) such that the gauge transformation reads as $\psi_{\dot\alpha} = U^\alpha_{\dot\alpha}\Psi_\alpha$; that is $\psi_\pm = U_{\pm\downarrow}\Psi_\uparrow + U_{\pm\uparrow}\Psi_\downarrow$. This local rotation of the electronic spin wave induces a non abelian gauge potential with components $\mathcal{A}_\mu = -iU\partial_\mu U^\dagger$ mediating the interaction between the electron and the magnetic texture. Indeed, putting the unitary change into $\mathcal{L}_e[\Psi,\mathbf{n}]$, we end up with an equivalent Lagrangian density; but now with new field variables as follows

$$\mathcal{L}_e[\psi,\mathcal{A}_\mu] = \hbar\psi^\dagger(i\partial_0 - A^a_0\sigma_a)\psi - \psi^\dagger\left(\frac{(\mathbf{P}+\hbar\mathbf{A}^a\sigma_a)^2}{2m} - J_H\sigma^z\right)\psi \tag{54}$$

Here, the vector potential matrix $\mathcal{A}_\mu$ is valued in the $SU(2)$ Lie algebra generated by the Pauli matrices σ^a; so it can be expanded as $A^x_\mu\sigma^x + A^y_\mu\sigma^y + A^z_\mu\sigma^z$ with components $A^a_\mu = \frac{1}{2}Tr(\sigma^a\mathcal{A}_\mu)$. Notice that in going from the old $\mathcal{L}_e[\Psi,\mathbf{n}]$ to the new $\tilde{\mathcal{L}}_e[\psi,\mathcal{A}_\mu]$, the spin texture $\mathbf{n}$ has disappeared; but not completely as it is manifested by an emergent non abelian gauge potential $\mathcal{A}_\mu$; so everything is as if we have an electron interacting with an external field $\mathcal{A}_\mu$. To get the explicit relation between the gauge potential and the magnetisation, we use the isomorphism $SU(2) \sim \mathbb{S}^3$ and the Hopf fibration $\mathbb{S}^1 \times \mathbb{S}^2$ to write the unitary matrix U as follows

$$U = e^{i\gamma}\begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{+i\varphi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix} \quad , \quad \mathcal{A}_\mu = \begin{pmatrix} \mathfrak{Z}_\mu & W^-_\mu \\ W^+_\mu & -\mathfrak{Z}_\mu\end{pmatrix} \tag{55}$$

where the factor $e^{i\gamma}$ describes $\mathbb{S}^1$ and where, for later use, we have set $W^\pm_\mu = A^1_\mu \pm iA^2_\mu$ and $\mathfrak{Z}_\mu = A^3_\mu$. So, a specific realisation of the gauge transformation is given by fixing $\gamma = cst$ (say $\gamma = 0$); it corresponds to restricting $\mathbb{S}^3$ down to $\mathbb{S}^2$ and SU(2) reduces down to SU(2)/U(1). In this parametrisation, we can also express the unitary matrix U like $\mathbf{m}.\sigma$ with magnetic vector $\mathbf{m} = (\sin\frac{\theta}{2}\cos\varphi, \sin\frac{\theta}{2}\sin\varphi, \cos\frac{\theta}{2})$ obeying the property $\mathbf{m}^2 = 1$; the same constraint as before. By putting back into $U\sigma.\mathbf{n}U^\dagger$, and using some algebraic relations like $\varepsilon_{abd}\varepsilon_{dce} = \delta_{ac}\delta_{be} - \delta_{bc}\delta_{ae}$, we obtain $[2(\mathbf{m}.\mathbf{n})\mathbf{m} - \mathbf{n}].\sigma$. Then, substituting $\mathbf{n}$ by its expression $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, we end up with the desired direction σ^z appearing in Eq. (54). On the other hand, by putting $U = \mathbf{m}.\sigma$ back into $-iU\partial_\mu U^\dagger$, we obtain an explicit relation between the gauge potential and the magnetic texture namely $A^a_\mu = \varepsilon^{abc}m_b\partial_\mu m_c$. From this expression, we learn the entries of the potential matrix $\mathcal{A}_\mu$ of Eq. (55); the relation with the texture $\mathbf{n}$ is given in what follows seen that $\mathbf{m}(\theta) = \mathbf{n}(\theta/2)$.

6.1.2 Large Hund coupling limit

We start by noticing that the non abelian gauge potential A^a_μ obtained above can be expressed in a condensed form like $\varepsilon^{abc}m_a\partial_\mu m_b$ (for short $\mathbf{m}\wedge\partial_\mu\mathbf{m}$); so it is normal to $\mathbf{m}$; and then it can be expanded as follows

$$A^a_\mu = \frac{1}{2}\mathrm{e}^a\partial_\mu\theta - \mathrm{f}^a\sin\frac{\theta}{2}\partial_\mu\varphi \tag{56}$$

where we have used the local basis vectors $\mathbf{m}(\theta), \mathbf{e}(\theta)$ and $\mathbf{f}(\theta)$. This is an orthogonal triad which turn out to be intimately related with the triad vectors given by Eq. (5); the relationships read respectively like $\mathbf{n}(\theta/2), \mathbf{u}(\theta/2)$ and $\mathbf{v}(\theta/2)$ involving $\theta/2$ angle instead of θ. Substituting these basis vectors by their angular values, we obtain

$$\begin{pmatrix} A_\mu^1 \\ A_\mu^2 \\ A_\mu^3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}\partial_\mu\theta - \frac{1}{2}\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta - 1 \end{pmatrix}\partial_\mu\varphi \tag{57}$$

from which we learn that the two first components combine in a complex gauge field $W_\mu^\pm = A_\mu^1 \pm iA_\mu^2$ which is equal to $\frac{i}{2}e^{i\varphi}\mathbf{w}^\pm\partial_\mu\mathbf{n}$ with $\mathbf{w}^\pm = \mathbf{u} \pm i\mathbf{v}$; and the third component A_μ^3 has the remarkable form $\frac{1}{2}(1 - \cos\theta)\partial_\mu\varphi$ whose structure recalls the geometric Berry term (7). Below, we set $A_\mu^3 = \mathfrak{Z}_\mu$ as in Eq. (55); it contains the temporal component $\mathfrak{Z}_0$ and the three spatial ones $\mathfrak{Z}_i$ — denoted in Section 2 respectively as a_0 and a_i—.

In the large Hund coupling ($J_H >> 1$), the spin of the electron is quasi- aligned with the magnetisation $\mathbf{n}$; so the electronic dynamics is mainly described by the chiral wave function $(\psi_+, 0)$ denoted below as $\chi = (\chi, 0)$. Thus, the effective properties of the interaction between the electron and the skyrmion can be obtained by restricting the above relations to the polarised electronic spin wave χ. By setting $\psi_- = 0$ into Eq. (54) and using $\chi^\dagger\sigma^x\chi = \chi^\dagger\sigma^y\chi = 0$ and $\chi^\dagger\sigma^z\chi = \bar{\chi}\chi$ as well as replacing $\left(A_\mu^x\sigma_x\right)^2 + \left(A_\mu^y\sigma_y\right)^2$ by $\frac{1}{4}\left(\partial_\mu\mathbf{n}\right)^2$, the Lagrangian (54) reduces to the polarised $\mathcal{L}_e^{(\mathrm{pol})} = \mathcal{L}_e\left[\chi, \mathbf{n}, \mathcal{Z}_\mu\right]$ given by

$$\mathcal{L}_e^{(\mathrm{pol})} = \hbar\chi^\dagger(i\partial_0 - \mathfrak{Z}_0\sigma^z)\chi - \chi^\dagger\left(\frac{\left(P_i + \hbar\mathfrak{Z}_i^a\sigma_a\right)^2}{2m} + \frac{\hbar^2}{8m}\left(\partial_\mu\mathbf{n}\right)^2 - J_H\sigma^z\right)\chi \tag{58}$$

where $(\mathfrak{Z}_0, \mathfrak{Z}_i)$ define the four components of the emergent abelian gauge prepotential $\mathfrak{Z}_\mu$ associated with the Pauli matrix σ^z; their explicit expressions are given by $\mathfrak{Z}_0 = \frac{1}{2}(1 - \cos\theta)\dot{\varphi}$ and $\mathfrak{Z}_i = \frac{1}{2}(1 - \cos\theta)\partial_i\varphi$; their variation with respect to the magnetic texture are related to the magnetisation field like $\frac{\delta\mathfrak{Z}_\mu}{\delta\mathbf{n}} = \frac{1}{2}\partial_\mu\mathbf{n}\wedge\mathbf{n}$.

6.2 Skyrmion with spin transfer torque

Here, we investigate the full dynamics of the electron/skyrmion system $\{e^-, \mathbf{n}\}$ described by the Lagrangian density $\mathcal{L}_{tot}$ containing the parts $\mathcal{L}_\mathbf{n} + \mathcal{L}_{e-\mathbf{n}}$; the electronic Lagrangian $\mathcal{L}_{e-\mathbf{n}}$ is given by Eq. (54). The Lagrangian $\mathcal{L}_\mathbf{n}$, describing the skyrmion dynamics, is as in eqs (5)–(7) namely $-\hbar S\mathfrak{Z}_0 - \mathcal{H}_\mathbf{n}$ with $\mathfrak{Z}_0 = \frac{1}{2}(1 - \cos\theta)\dot{\varphi}$. By setting $\tilde{\mathcal{H}}_\mathbf{n} = \mathcal{H}_\mathbf{n} + \frac{\hbar^2}{8m}\left(\partial_\mu\mathbf{n}\right)^2\psi^\dagger\psi$, the full Lagrangian density $\mathcal{L}_{tot}$ with can be then presented like $\tilde{\mathcal{L}}\left[\psi, \mathfrak{Z}_\mu\right] - \tilde{H}_\mathbf{n}$ like

$$\tilde{\mathcal{L}}\left[\psi, \mathfrak{Z}_\mu\right] = -\hbar S\mathfrak{Z}_0 + \hbar\psi^\dagger\left(i\frac{\partial}{\partial t} - \mathfrak{Z}_0\sigma^z\right)\psi - \psi^\dagger\left(\frac{\left(P_i + \hbar\mathfrak{Z}_i\sigma^z\right)^2}{2m}\right)\psi \tag{59}$$

with $(P_i + \hbar\mathfrak{Z}_i\sigma^z)^2$ expanding as $P_i^2 + \hbar_i^2\mathfrak{Z}^2 + \hbar\left(P_i\mathfrak{Z}^i + \mathfrak{Z}^iP_i\right)\sigma^z$. The equations of motion of ψ and $\mathbf{n}$ are obtained as usual by computing the extremisation of this

Lagrangian density with respect to the corresponding field variables. In general, we have $\delta\mathcal{L}_{tot} = (\delta\mathcal{L}_{tot}/\delta\mathbf{n}).\delta\mathbf{n} + (\delta\mathcal{L}_{tot}/\delta\psi).\delta\psi + hc$ which vanishes for $\delta\mathcal{L}_{tot}/\delta\mathbf{n} = 0$ and $\delta\mathcal{L}_{tot}/\delta\psi^\dagger = 0$.

6.2.1 Modified Landau-Lifshitz equation

Regarding the spin texture $\mathbf{n}$, the associated field equation of motion is given by $\delta\mathcal{L}_{tot}/\delta\mathbf{n} = 0$; the contributions to this equation of motion come from the variations $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{H}}_{\mathbf{n}}$ with respect to $\delta\mathbf{n}$ namely

$$\frac{\delta\tilde{\mathcal{H}}}{\delta\mathbf{n}} - \frac{\delta\tilde{\mathcal{L}}}{\delta\mathfrak{Z}_\mu}\frac{\delta\mathfrak{Z}_\mu}{\delta\mathbf{n}} = 0 \tag{60}$$

The variation $\frac{\delta\mathcal{H}_{\mathbf{n}}}{\delta\mathbf{n}}$ depends on the structure of the skyrmion Hamiltonian density $\tilde{\mathcal{H}}$; its contribution to the equation of motion can be presented like $\lambda\partial^\mu\partial_\mu\mathbf{n} = \mathbf{F}$ with some factor λ. However, the variation $\frac{\delta\mathcal{L}}{\delta\mathfrak{Z}_\mu}\frac{\delta\mathfrak{Z}_\mu}{\delta\mathbf{n}}$ describes skyrmion-electron interaction; and can be done explicitly into two steps; the first step concerns the calculation of the time like component $\frac{\delta\mathcal{L}}{\delta\mathfrak{Z}_0}\frac{\delta\mathfrak{Z}_0}{\delta\mathbf{n}}$; it gives $-\frac{\hbar}{2}[2S + \psi^\dagger\sigma^z\psi](\dot{\mathbf{n}}\wedge\mathbf{n})$; it is normal to $\mathbf{n}$ and to velocity $\dot{\mathbf{n}}$ and involves the eletron spin density $\rho_e^z = \psi^\dagger\sigma^z\psi$.

The second step deals with the calculation of the space like component $-\frac{\delta\mathcal{L}}{\delta\mathfrak{Z}_i}\frac{\delta\mathfrak{Z}_i}{\delta\mathbf{n}}$; the factor $\frac{\delta\mathcal{L}}{\delta\mathfrak{Z}_i}$ gives $-\hbar\mathcal{J}^i$ with a 3-component current vector density reading as follows

$$\mathcal{J}_i = \frac{1}{2m}\left(\psi^\dagger\sigma^z P_i\psi - P_i\psi^\dagger\sigma^z\psi\right) + \frac{\hbar}{m}\left(\psi^\dagger\psi\right)\mathfrak{Z}_i \tag{61}$$

This vector two remarkable properties: (1) it is given by the sum of two contributions as it it reads like $\mathcal{J}_i^{(+\mathbf{n})} + \mathcal{J}_i^{(-\mathbf{n})}$ with

$$\begin{aligned}\mathcal{J}_l^{(+\mathbf{n})} &= \frac{\hbar}{m}\left(\overline{\psi}_{+\mathbf{n}}\psi_{+\mathbf{n}}\right)\mathfrak{Z}_l + \frac{1}{2m}\left(\overline{\psi}_{+\mathbf{n}}\frac{\hbar}{i}\partial_l\psi_{+\mathbf{n}} - \frac{\hbar}{i}\partial_l\overline{\psi}_{+\mathbf{n}}\psi_{+\mathbf{n}}\right)\\ \mathcal{J}_l^{(-\mathbf{n})} &= \frac{\hbar}{m}\left(\overline{\psi}_{-\mathbf{n}}\psi_{-\mathbf{n}}\right)\mathfrak{Z}_l - \frac{\hbar}{2im}\left(\overline{\psi}_{-\mathbf{n}}\frac{\hbar}{i}\partial_l\psi_{-\mathbf{n}} - \frac{\hbar}{i}\partial_l\overline{\psi}_{-\mathbf{n}}\psi_{-\mathbf{n}}\right)\end{aligned} \tag{62}$$

These vectors are respectively interpreted as two spin polarised currents; the $\mathcal{J}_i^{(+\mathbf{n})}$ is associated with the $\psi_{+\mathbf{n}}$ wave function as it points in the same direction as $\mathbf{n}$; the $\mathcal{J}_i^{(-\mathbf{n})}$ is however associated with $\psi_{-\mathbf{n}}$ pointing in the opposite direction of $\mathbf{n}$. (2) Each one of the two $\mathcal{J}^{(+\mathbf{n})}$ and $\mathcal{J}^{(-\mathbf{n})}$ are in turn given by the sum of two contributions as they can be respectively split like $\frac{\hbar}{m}\left(\overline{\psi}_{+\mathbf{n}}\psi_{+\mathbf{n}}\right)\mathfrak{Z} + \mathbf{j}_{\psi_{+\mathbf{n}}}$ and $\frac{\hbar}{m}\left(\overline{\psi}_{-\mathbf{n}}\psi_{-\mathbf{n}}\right)\mathfrak{Z} + \mathbf{j}_{\psi_{-\mathbf{n}}}$ with vector density $\mathbf{j}_\psi$ standing for the usual current vector $\mathbf{j}_\psi = \frac{1}{2m}\overline{\psi}\overleftrightarrow{\mathbf{P}}\psi$. The contribution $\frac{\hbar}{m}\left(\overline{\psi}\psi\right)\mathfrak{Z}$ is proportional to the emergent gauge field $\mathfrak{Z}$; it defines a spin torque transfert to the vector current density $\mathcal{J}_i$.

Regarding the factor $\frac{\delta\mathfrak{Z}_i}{\delta\mathbf{n}}$, it gives $\frac{1}{2}(\partial_i\mathbf{n}\wedge\mathbf{n})$; by substituting, the total contribution of $\frac{\delta\mathcal{L}}{\delta\mathfrak{Z}_i}\frac{\delta\mathfrak{Z}_i}{\delta\mathbf{n}}$ leads to $-\frac{\hbar}{2}\left(\mathcal{J}^i\partial_i\mathbf{n}\right)\wedge\mathbf{n}$ that reads in a condensed form like $-\frac{\hbar}{2}(\mathcal{J}.\nabla\mathbf{n})\wedge\mathbf{n}$. Putting back into Eq. (60), we end up with the following modified LL equation

$$-\frac{\hbar}{2}\left[2S + \psi^\dagger\sigma^z\psi\right](\dot{\mathbf{n}}\wedge\mathbf{n}) + \frac{\hbar}{2}(\mathcal{J}.\nabla\mathbf{n})\wedge\mathbf{n} - \frac{\delta H_{\mathbf{n}}}{\delta\mathbf{n}} = 0 \tag{63}$$

To compare this equation with the usual LL equation ($\hbar S\dot{\mathbf{n}} = \frac{\delta H_{\mathbf{n}}}{\delta \mathbf{n}} \wedge \mathbf{n}$) in absence of Hund coupling (which corresponds to putting ψ to zero), we multiply Eq. (63) by $\wedge\,\mathbf{n}$ in order to bring it to a comparable relation with LL equation. By setting $\rho_e^z = \psi^\dagger\sigma^z\psi$, describing the electronic spin density $|\psi_{+\mathbf{n}}|^2 - |\psi_{-\mathbf{n}}|^2$; we find

$$-\hbar\left[\frac{S}{\mathrm{a}^{-d}} + \frac{\rho_e^z}{2}\right]\dot{\mathbf{n}} = \frac{\delta H_{\mathbf{n}}}{\delta \mathbf{n}} \wedge \mathbf{n} - \hbar[(\mathcal{J}.\nabla)\mathbf{n}] \tag{64}$$

where, due to $\mathbf{n}^2 = 1$, the space gradient $\mathcal{J}.\nabla\mathbf{n}$ is normal to $\mathbf{n}$; and so it can be set as $\Omega^{(e)} \wedge \mathbf{n}$ with $\Omega^{(e)} = \mathcal{J}^i\omega_i^{(e)}$. The above equation is a modified LL equation; it describes the dynamics of the spin texture interacting with electrons through Hund coupling. Notice that for $\psi \to 0$, this equation reduces to $\hbar\frac{S}{\mathrm{a}^{-d}}\dot{\mathbf{n}} = \omega^{(n)} \wedge \mathbf{n}$ showing that the vector $\mathbf{n}$ rotates with $\omega^{(n)} = -\frac{\delta H_{\mathbf{n}}}{\delta \mathbf{n}}$. By turning on ψ, we have $\dot{\mathbf{n}} \sim (\omega^{(n)} + \Omega^{(e)}) \wedge \mathbf{n}$ indicating that the LL rotation is drifted by $\Omega^{(e)}$ coming from two sources: (i) the term $\hbar[(\mathcal{J}.\nabla)\mathbf{n}]$ which deforms LL vector $\omega^{(n)}$ drifted by the $\mathbf{n} \wedge (\mathcal{J}.\nabla\mathbf{n})$; and (ii) the electronic spin density $\rho_e^z = \frac{N_e}{\mathrm{a}^{-d}}$; this term adds to the density $\frac{S}{\mathrm{a}^{-d}}$ of the magnetic texture per unit volume; it involves the number $N_e = N_e^{+\mathbf{n}} - N_e^{-\mathbf{n}}$ with $N_e^{\pm\mathbf{n}}$ standing for the filling factor of polarized conduction electrons. Moreover, if assuming $\mathbf{n}(t, \mathbf{r}) = \mathbf{n}(\mathbf{r} - \mathbf{V}_s t)$ with a uniform $\mathbf{V}_s$, then the drift velocity $\dot{n}^a = -(\partial_i n^a)V_s^i$ and $(J_e^i\partial_i)n^a = J_e^a$. Putting back into the modified LLG equation, we end up with the following relation between the $\mathbf{V}_s$ and $\mathbf{v}_e$ velocities $(S + \frac{n_e}{2})v_s^a = n_e v_e^a$ where we have set $(\partial_i n^a)V_s^i = v_s^a$ and $J_e^a = n_e v_e^a$.

6.2.2 Rigid skyrmion under spin transfer torque

Here, we investigate the dynamics of a 2D rigid skyrmion $[\mathbf{n} = \mathbf{n}(\mathbf{r} - \mathbf{R})]$ under a spin transfer torque (STT) induced by itinerant electrons. For that, we apply the method, used in sub-subSection 5.1.2 to derive L_s from the computation space integral of $\int d^2\mathbf{r}\mathcal{L}_s$ and Eq. (36). To begin, recall that in absence of the STT effect, the Lagrangian L_s of the 2D skyrmion's point- particle, with position $\mathbf{R} = (X, Y)$ and velocity $\dot{\mathbf{R}} = (\dot{X}, \dot{Y})$, is given by $\frac{M_s}{2}\dot{\mathbf{R}}^2 - \frac{G}{2}\mathbf{z}.(\mathbf{R} \wedge \dot{\mathbf{R}}) - V(\mathbf{R})$ with effective scalar energy potential $V(\mathbf{R}) = \int d^2\mathbf{r}\mathcal{H}(\mathbf{r}, \mathbf{R})$ and a constant $G = \frac{4\pi\hbar}{\mathrm{a}^2}q_s S$. Under STT induced by Hund coupling, the Lagrangian L_s gets deformed into $\tilde{\mathrm{L}}_s = L_s + \Delta L_s$, that is

$$\tilde{\mathrm{L}}_s = \frac{M_s}{2}\dot{\mathbf{R}}^2 - \frac{G}{2}\mathbf{z}.(\mathbf{R} \wedge \dot{\mathbf{R}}) - V(\mathbf{R}) + \Delta L_s \tag{65}$$

To determine ΔL_s, we start from $\tilde{\mathrm{L}}_s = \int d^2\mathbf{r}\tilde{\mathcal{L}}_{tot}$ with Lagrangian density as $\tilde{\mathcal{L}}_{tot} = \tilde{\mathcal{L}} - \tilde{\mathcal{H}}_{\mathbf{n}}$ with $\tilde{\mathcal{L}}$ given by Eq. (59). For convenience, we set $\tilde{\mathcal{L}} = -\hbar S\mathfrak{Z}_0 + \tilde{\mathcal{L}}_{e-\mathbf{n}}$ and set

$$\tilde{\mathcal{L}}_{e-\mathbf{n}} = \hbar\psi^\dagger\left(i\frac{\partial}{\partial t} - \mathfrak{Z}_0\sigma^z\right)\psi - \psi^\dagger\left(\frac{(P_i + \hbar\mathfrak{Z}_i\sigma^z)^2}{2m}\right)\psi \tag{66}$$

The deviation ΔL_s with respect to L_s in (65) comes from those terms in Eq. (66). Notice that this expression involves the wave function ψ coupled to the emergent gauge potential field $\mathfrak{Z}_\mu = (\mathfrak{Z}_0, \mathfrak{Z}_i)$; that is $-\hbar\int d^2\mathbf{r}\psi^\dagger\sigma^z\psi\mathfrak{Z}_0$ and

$-\frac{1}{2m}\int d^2\mathbf{r}\psi^\dagger\left[(P_i+\hbar\mathfrak{Z}_i\sigma^z)^2\right]\psi$. Thus, to obtain ΔL_s, we first calculate the variation $\frac{\delta(\Delta L_s)}{\delta\mathfrak{Z}_\mu}\delta\mathfrak{Z}_\mu$ and put $\delta\mathfrak{Z}_\mu=\frac{\delta\mathfrak{Z}_\mu}{\delta\mathbf{R}}.\delta\mathbf{R}$. Once, we have the explicit expression of this variation, we turn backward to deduce the value of ΔL_s. To that purpose, we proceed in two steps as follows: (i) We calculate the temporal contribution $\frac{\delta(\Delta L_s)}{\delta\mathfrak{Z}_0}\frac{\delta\mathfrak{Z}_0}{\delta\mathbf{R}}.\delta\mathbf{R}$; and (ii) we compute the spatial $\frac{\delta(\Delta L_s)}{\delta\mathfrak{Z}_i}\frac{\delta\mathfrak{Z}_i}{\delta\mathbf{R}}.\delta\mathbf{R}$. Using the variation $\delta\mathfrak{Z}_0=\frac{1}{2}\delta\mathbf{n}.(\mathbf{n}\wedge\partial_j\mathbf{n})\dot{X}^j$, the contribution of the first term can be put as follows

$$\frac{\delta(\Delta L_s)}{\delta\mathfrak{Z}_0}\frac{\delta\mathfrak{Z}_0}{\delta X^l}\delta X^l=-\frac{\hbar}{2}\mathbf{J}_0^z\varepsilon_{zij}\left[\dot{X}^i\delta X^j\right] \tag{67}$$

where we have set $\rho^z=\psi^\dagger\sigma^z\psi$ and $\mathbf{J}_0^z=\int d^2\mathbf{r}\frac{\rho^z}{2}\varepsilon^{zkl}\mathbf{n}.(\partial_k\mathbf{n}\wedge\partial_l\mathbf{n})$. Notice that the right hand side in above relation can be also put into the form $\frac{\hbar}{2}\boldsymbol{J}_0^z\varepsilon_{zij}\left[\delta\dot{X}^iX^j\right]-\delta\left[\frac{\hbar}{2}\varepsilon_{zij}\boldsymbol{J}_0^z\dot{X}^iX^j\right]$ indicating that ΔL_s must contain the term $\frac{\hbar}{2}\varepsilon_{zij}\boldsymbol{J}_0^z\dot{X}^iX^j$ which reads as well like $\frac{\hbar}{2}J_0\mathbf{z}.(\dot{\mathbf{R}}\wedge\mathbf{R})$. Regarding the spatial part $\frac{\delta(\Delta L_s)}{\delta\mathfrak{Z}_i}.\frac{\delta\mathfrak{Z}_i}{\delta X^l}\delta X^l$, we have quite similar calculations allowing to put it in the following form

$$\frac{\delta(\Delta L_s)}{\delta\mathfrak{Z}_i}.\frac{\delta\mathfrak{Z}_i}{\delta X^l}\delta X^l=-\hbar\varepsilon_{zij}\boldsymbol{J}^{zi}\delta X^j \tag{68}$$

where we have set $\boldsymbol{J}^{zi}(t)=\int d^2\mathbf{r}\mathcal{J}^{zi}(t,\mathbf{r})$ with $\mathcal{J}^{zi}(t,\mathbf{r})$ given by Eq. (61). Here also notice that the right hand of above equation can be put as well like $\delta\left[-\hbar\varepsilon_{zij}\boldsymbol{J}^{zi}X^j\right]$ indicating that ΔL_s contains in addition to $\frac{\hbar}{2}J_0\mathbf{z}.(\dot{\mathbf{R}}\wedge\mathbf{R})$, the term $-\hbar\varepsilon_{zij}\boldsymbol{J}^{zi}X^j$ which reads also as $-\hbar\mathbf{z}.\boldsymbol{J}\wedge\mathbf{R}$ with two component vector $\boldsymbol{J}=(\boldsymbol{J}^{zx},\boldsymbol{J}^{zy})$. Thus, we have the following modified skyrmion equation

$$\tilde{\mathrm{L}}_s=\frac{M_s}{2}\dot{\mathbf{R}}^2-\frac{1}{2}(G+\hbar J_0)\mathbf{z}.(\dot{\mathbf{R}}\wedge\mathbf{R})+\hbar\mathbf{z}.(\boldsymbol{J}\wedge\mathbf{R})-V(\mathbf{R}) \tag{69}$$

from which we determine the modified equation of motion of the rigid skyrmion in presence of spin transfer torque.

7. Comments and perspectives

In this bookchapter, we have studied the basic aspects of the solitons dynamics in various $(1+d)$ spacetime dimensions with $d=1,2,3$; while emphasizing the analysis of their topological properties and their interaction with the environment. After having introduced the quantum SU(2) spins, their coherent vector representation $\mathbf{S}=\mathcal{R}(\alpha,\beta,\gamma)\mathbf{S}_0$ with $\mathbf{S}_0$ standing for the highest weight spin state; and their link with the magnetic moments $\mu\propto S\mathbf{n}$, we have revisited the time evolution of coherent spin states; and proceeded by investigating their spatial distribution while focusing on kinks, 2d and 3d skyrmions. We have also considered the rigid skyrmions dissolved in the magnetic texture without and with dissipation. Moreover, we explored the interaction between electrons and skyrmions and analyzed the effect of the spin transfer torque. In this regard, we have refined the results concerning the modified LL equation for the rigid skyrmion in connection with emergent non abelian SU(2) gauge fields. It is found that the magnetic skyrmions, existing in a ferromagnetic (FM) medium, show interesting behaviors such as emergent electrodynamics [71] and current-driven motion at low current densities

[72, 73]. Consequently, the attractive properties of ferromagnetic skyrmions make them promising candidates for high-density and low-power spintronic technology. Besides, ferromagnetic skyrmions have the potential to encode bits in low-power magnetic storage devices. Therefore, alternative technology of forming and controlling skyrmions is necessary for their use in device engineering. This investigation was performed by using the field theory method based on coherent spin states described by a constrained spacetime field captured by $f(\mathbf{n}) = 1$. Such condition supports the topological symmetry of magnetic solitons which is found to be characterised by integral topological charges Q that are interpreted in terms of magnetic skyrmions and antiskyrmion; these topological states can be imagined as (winding) quasiparticle excitations with $Q > 0$ and $Q < 0$ respectively.

Regarding these two skyrmionic configurations, it is interesting to notice that, unlike magnetic skyrmions, the missing rotational symmetry of antiskyrmions leads to anisotropic DMI, which is highly relevant for racetrack applications. It follows that antiskyrmions exist in certain Heusler materials having a particular type of DMI, including MnPtPdSn [36] and MnRhIrSn [74]. It is then deduced that stabilized antiskyrmions can be observed in materials exhibiting D_{2d} symmetry such as layered systems with heavy metal atoms. Furthermore, the antiskyrmion show some interesting features, namely long lifetimes at room temperature and a parallel motion to the applied current [75]. Thus, antiskyrmions are easy to detect using conventional experimental techniques and can be considered as the carriers of information in racetrack devices.

To lift the limitations associated with ferromagnetic skyrmions for low-power spintronic devices, recent trends combine multiple subparticles in different magnetic surroundings. Stable room-temperature antiferromagnetic skyrmions in synthetic Pt/Co/Ru antiferromagnets result from the combination of two FM nano-objects coupled antiferromagnetically [76]. Compared to their ferromagnetic analogs, antiferromagnetic skyrmions exhibit different dynamics and are driven with several kilometers per second by currents. Coupling two subsystems with mutually reversed spins, gives rise to ferrimagnetic skyrmions as detected in GdFeCo films using scanning transmission X-ray microscopy [77]. At ambient temperature, these skyrmions move at a speed of $50 m/s$ with a reduced skyrmion Hall angle of 20°. Characterized by uncompensated magnetization, the vanishing angular momentum line can be utilized as a self-focusing racetrack for skyrmions. Another technologically promising object is generated by the coexistence of skyrmions and antiskyrmions in materials with D_{2d} symmetry. The resulting spin textures constitute information bits ‘0’ and ‘1’ generalizing the concept of racetrack device. Insensitive to the repulsive interaction between the two distinct nano-objects, such emergent devices are promising solution for racetrack storage applications.

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