Chapter

Application of Einstein's Methods in a Quantum Theory of Radiation

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Abstract

Einstein showed in his seminal paper on radiation that molecules with a quantum-theoretical distribution of states in thermal equilibrium are in dynamical equilibrium with the Planck radiation. The method he used assigns coordinates fixed with respect to molecules to derive the A and B coefficients, and fixed relative to laboratory coordinates to specify their thermal motion. The resulting dynamical equilibrium between quantum mechanical and classically defined statistics is critically dependent upon considerations of momentum exchange. When Einstein's methods relating classical and quantum mechanical statistical laws are applied to the level of the single quantum oscillator they show that matrix mechanics describes the external appearances of an atom as determined by photon-electron interactions in laboratory coordinates, and wave mechanics describes an atom's internal structure according to the Schrödinger wave equation. Non-commutation is due to the irreversibility of momentum exchange when transforming between atomic and laboratory coordinates. This allows the "rotation" of the wave function to be interpreted as the changing phase of an electromagnetic wave. In order to describe the momentum exchange of a quantum oscillator the Hamiltonian model of atomic structure is replaced by a Lagrangian model that is formulated with equal contributions from electron, photon, and nucleus. The fields of the particles superpose linearly, but otherwise their physical integrity is maintained throughout. The failure of past and present theoretical models to include momentum is attributed to the overwhelming requirement of human visual systems for an explicit stimulus.

Keywords: Einstein's quantum theory, matrix mechanics, wave mechanics, momentum exchange, conservation laws, non-commutation, wave function, Schrödinger wave equation, Lagrangian

1. Introduction

Two possibilities are available in the literature for describing the interaction of matter and radiation, classical theory and nonrelativistic quantum theory. Classical theory explains the continuous aspects of electromagnetic radiation, Maxwell's laws, and the theory of heat. Quantum theory explains the Planck radiation law of black body radiation, the discrete nature of observables, and the statistical properties of matter. A third possibility that has remained relatively obscure as an alternative derives from Einstein's 1917 paper "Quantum theory of radiation" and includes aspects of both theories [1]. He shows there that as a consequence of the conservation of momentum the velocity distribution of molecules emitting black

body radiation *quantum mechanically* are in dynamical equilibrium with the *classically* derived Maxwell-Boltzman velocity distribution due to thermal exchange. The internal energy distribution of the molecules demanded by quantum theory is then in strict conformance with the emission and absorption of radiation. Because the link between quantum mechanical and classical properties of matter is statistically defined and applies to material systems rather than individual atoms Einstein's theory is considered to be unfinished. A description of atomic structure using his methods is sought after here as a way to fulfill these ideas.

The authors of nonrelativistic quantum theory adopted Einstein's ideas for the A and B coefficients, which are determined by the classical field effect resonance, and described the discrete transfer of energy from a radiation beam to an atomic state; but they neglected the effect of momentum exchange required by the conservation of momentum. The momentum of a photon E/c causes an atom or molecule to recoil in the direction of the beam when it is absorbed and in the opposite direction when it is emitted. Nonrelativistic quantum mechanics places primary importance on the observable properties of radiation in the form of energy measurements while ignoring the more subtle effects of momentum which are more difficult to observe. Consequently the Schrödinger wave equation is formulated continuously without provision for transmitting the discontinuous impulses of photons. The relationship between classical and quantum mechanical statistics that Einstein had carefully constructed breaks down so that instead of a gradual evolution of ideas in which classical and quantum concepts develop together a complete break from classical theory occurred. In the absence of an underlying classical foundation different interpretations of quantum mechanics developed which use methods drawn from facts that are supported by experiment specific to that model alone, but show no relationship to each other. No model has emerged that can account for all the facts. In the following discussion we shall see that the reason no single model of quantum mechanics is able to explain all of the experimental facts, discrete and continuous, yet they concern the same topic is that each one addresses a different aspect of the *same* physical phenomenon; the interaction between matter and radiation.

2. Matrix mechanics

2.1 Historical perspectives

The Bohr model of the atom gives the quantum rule for changes in energy state $E_2 - E_1 = h\nu$, but says nothing about the processes of emission and absorption. Improved understanding of radiation came gradually as experimental techniques improved. Einstein's 1917 paper marks the beginning of quantum mechanics since all subsequent research on the absorption, emission, and dispersion of radiation is based upon it [2]. Through the use of thought experiments and results obtained in an earlier paper on Brownian motion he showed how the microscopic structure of matter is able to influence matter macroscopically. The induced absorption of black body radiation occurs continuously due to random inputs of momentum from thermal collisions and radiation, while induced and spontaneous emission occurs discretely according to the Bohr frequency rule for changes in state and is directed along an infinitesimal solid angle consistent with the photon's recoil momentum E/c. A dynamic equilibrium is thereby created between the thermal energy absorbed by molecules and the quantum mechanical emission of radiation.

Although the A and B coefficients of Einstein's radiation theory have been incorporated into nonrelativistic quantum mechanics the transition of energy from a classical thermal origin to the discrete energy states of atoms and molecules

creates discontinuities that are not accounted for by the Schrödinger wave equation. A vastly improved knowledge of the mechanical properties of photons due to momentum in the astronomical sciences, molecular manipulation, optical tweezers, and laser cooling; technological advances has not translated into an understanding of how to incorporate momentum into the equations of quantum mechanics. The momentum of light is treated separately from energy, and Einstein's theory of radiation is the only one that makes explicit use of it when describing absorption and emission. To see why this is true it will be necessary to examine the historical origins of quantum mechanics.

The dynamic equilibrium between classical and quantum mechanical statistical laws that exists for black body radiation is closely related to the phenomenon of dispersion. Dispersion is the continuous change in the angle of refraction of different frequencies of light by a prism or other medium. Although light disperses continuously across the entire spectrum, at certain specific frequencies characteristic of the medium, it is completely absorbed forming lines. When Bohr introduced his theory of electron orbitals he immediately recognized the possibility that the discrete lines of atomic spectra are related to the discrete lines in dispersion phenomena [3]. Other researchers, in particular Debye and Sommerfeld, were also inspired by that possibility and a series of papers appeared that tried to explain the discrete and continuous properties of dispersion by introducing classically inspired modifications of the electron orbitals [2, 4, 5]. However, when experiments revealed that the characteristic frequencies of anomalous dispersion coincide with the frequencies of the spectral lines it was evident that orbiting electrons could not account for both and a complete break from classical theory was necessary. Ladenburg was the first to suggest how the new quantum theory would appear by following Einstein's reasoning leading to the A and B coefficients [5–7]. This enabled him to equate two theoretical expressions, the energy absorbed/emitted by N classical resonators and the energy absorbed/emitted by N′ quantum atoms. By obtaining a statistical balance between classical and quantum mechanical energy exchange he satisfied the conservation of energy, but not that of momentum. Four years later Kramers reinterpreted Ladenburg's results by using the Bohr model of the atom as a multiply periodic system of virtual oscillators [2, 5, 8, 9]. In that model a quantum mechanical variable X is described with a classical Fourier series, where A(n, n- τ) is the quantum analog of the classical amplitude, n indicates the electron orbital number, and τ assumes integral values to denote positive or negative transitions [9].

$$X = \sum_{\tau} A(n, n-\tau) \exp\left[2i\pi\nu(n, n-\tau)t\right], \tau = \mp 1, \mp 2, \ldots \quad (1)$$

The Bohr-Kramer method distanced itself from that of Einstein in an important way. Einstein argued that momentum conservation is what distinguishes classical properties observed in laboratory coordinates from quantum mechanical properties observed in atomic coordinates. The discrete and continuous properties of matter are thereby separated from each other physically. In the interpretation by (1), on the other hand, matter-radiation interactions are described exclusively in laboratory coordinates. Fields are described classically by means of Fourier series while quantization is imposed on the field energy. Quantization is thereby understood to be a localization of energy even though the fields extend to infinity and are therefore diffuse. The concept of photon momentum, a property whose displacement in time is *directional*, is replaced by a wave model that is *isotropic* and treats emission as a spherically symmetric process with no net momentum transfer, and processes that are *reversible* in time.

Once Kramer had reinterpreted Einstein's quantum theory of radiation with fictitious harmonic oscillators Heisenberg was able to use it to formulate a theory of quantum mechanics that reconciles the continuity of radiation fields with the discrete energy states of an atom [2, 7, 9]. The complex sets of mathematical rules that he used to describe the frequencies and intensities of spectral lines, may be expressed in the form of a matrix.

$$\sum_k (p_{nk}q_{km} - q_{nk}p_{km}) = \begin{matrix} i\hbar \, for \, n = m \\ 0 \, for \, n \neq m \end{matrix} \tag{2}$$

Each matrix element represents a pair of energy states of the type (1) with the observable properties, frequency and intensity, of an electromagnetic wave. The complete matrix has an infinite number of components and corresponds in its entirety to one of the dynamic variables; the coordinates, the momenta, or the velocities of the particles. The matrix products do not commute as they do in classical theory. When $n = m$ the elements are diagonal and the value of the equation is equal to iħ. For non-diagonal elements, $n \neq m$, and its value is zero.

The quantum mechanical reformulation of the classical Fourier series (1) and (2) is further simplified into its more familiar form by replacing the summed elements with single terms.

$$\mathbf{pq} - \mathbf{qp} = \mathrm{i}\hbar \tag{3}$$

The momentum **p** and position **q** are not numbers; but rather arrays of quantities, or matrices. Each component of the matrix is a Fourier series associated with any two of an infinite number of orbits. Because the orbits may extend to infinity both in space and in time exchanges of momentum are delocalized.

2.2 Classical interpretation of matrix mechanics

After three successive modifications from Ladenburger to Kramers to Heisenberg, Einstein's theory is scarcely recognizable. Mathematical modifications that dilute its physical content are given by the Eqs. (1)–(3). Very little remains of Einstein's carefully crafted relationship between classical and quantum mechanical variables despite the fact that all three reinterpretations and the Eqs. (1) through (3) claim to describe the same physical phenomenon, the interaction between matter and radiation. The theories differ dramatically because the directional properties of emitted radiation due to recoil momentum have been replaced by virtual harmonic oscillators which emit energy isotropically as spherical waves and are reversible in time. The balance between thermal energy and radiative energy maintained by momentum exchange depends on oscillators that absorb thermal energy classically and emit energy quantum mechanically directed along an infinitesimal solid angle with momentum E/c. Virtual oscillators that emit isotropically disrupt the delicate balance between classical and quantum mechanical statistical principles which Einstein had so carefully constructed.

The advantage of using energy rather than momentum in a theory of radiation is its ease of use. Energy is defined as a magnitude, which is easier to describe mathematically, to measure, and to calculate. The advantage of momentum, on the other hand, is that its description provides a more accurate picture of a system's time evolution. Position coordinates are assigned to particles relative to a system of reference in order to specify the direction and magnitude of momentum. The conservation of momentum may then be applied and used to interpret observable

phenomena. The Ptolemaic planetary system, for example, introduced fictitious epicycles in violation of the conservation of momentum, but continued to be used for a thousand years because it successfully reproduced what was observed. If astronomers had understood the universal properties of momentum they would have immediately rejected a theory that suggests massive objects could reverse motion in empty space.

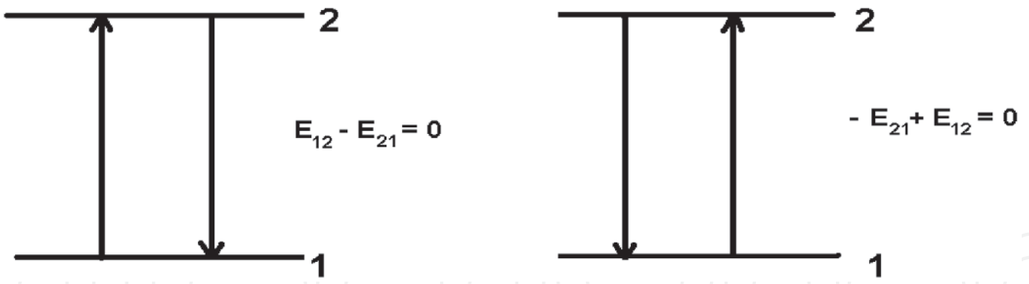
Einstein used atomic coordinates fixed with respect to a molecule to derive his A and B coefficients describing momentum exchange during the absorption and emission of energy. The linear momentum of molecules due to thermal impulses is described by introducing a second coordinate system defined with respect to the black body container, that is, in laboratory coordinates. The momentum exchange between the opposing external and internal forces of molecules creates a dynamic equilibrium and allows a clear separation between classical and quantum observables respectively. In contrast, the Bohr-Kramers method describes all observables, discrete and continuous, externally with respect to laboratory coordinates. From Heisenberg's perspective there was no need to treat the discrete spectral lines due to atomic orbitals and the continuous observables due to dispersion phenomena differently, concluding that [10], "Quantum mechanics [is] founded exclusively upon relationships between quantities which are in principle observable."

Dispersion phenomena are observed and measured in laboratory coordinates, and not in the coordinates of an atom. They are given by off-diagonal elements of matrices $n \neq m$ where elements above the diagonal refer to changes in frequency due to energy absorption and elements below the diagonal refer to frequency changes due to energy emission. The elements represent the continuously variable resonances of radiation with an atom's valence electrons. The energy of an absorption offsets the energy of an emission except for a difference in phase so a value of zero is obtained for Eq. (2). On the other hand, the diagonal elements of matrices for $n = m$ are real eigenvalues representing ground state energy levels. Absorption results in stimulation to a higher orbital and the subsequent emission of a photon upon decay according to the Bohr frequency condition. The off-diagonal interactions due to continuous momentum exchanges are governed by the Compton equation $p\lambda = h$. Each matrix element is a photon-electron interaction obtained by resolving the Fourier series (1) into its individual components. It is hypothesized that the complete matrix array expresses the conservation of momentum. Heisenberg mistakenly believed that matrices describe atomic structure, but as Einstein showed atomic structure must be described by internally defined coordinates in the unobservable space–time of an atom. To compare atomic and laboratory coordinates a transformation of coordinates must be performed. Transformations may be visualized with the assistance of the electron oscillator shown in the figure.

2.3 Non-commutation

To see how non-diagonal and diagonal elements differ we introduce the idea of an electron oscillator in the figure below. If an electron is raised from the ground state $|1\rangle$ to an excited state $|2\rangle$ and then returns a photon is irreversibly emitted. This is shown schematically in the figure below, where 1 and 2 denote the states and arrows refer to transitions. On the left the energy of an electron increases and then decreases, while on the right the reverse occurs. Each arrow represents one-half cycle of the electron oscillator. If the arrows are used to describe off-diagonal matrix elements, they refer to different atoms. If the elements are diagonal they refer to the same atom. It is a simple way of comparing the laboratory coordinates of matrix elements, as determined by photons, to coordinates of atomic structure determined by electron shells during the absorption and emission of radiation. Although the

final state of the quantum system differs the two processes are identical when described in terms of energy differences.



Now consider what happens when the same two energy exchanges are analyzed in terms of the momentum. Using Compton's equation for the momentum of a photon, $p = h/\lambda$, the first exchange may be expressed:

$$p_{12}\lambda_{12} - p_{21}\lambda_{21} = 0 \tag{4}$$

Angular momentum increases by an amount ħ when the electron is excited and is then reduced by the same amount when the atom returns to its ground state $|1\rangle$. Thus this type of photon emission ends up with the atomic system in its ground state.

However, when the order of the electron transitions is reversed on the right of the figure we see by the following expression that a description of momentum exchange gives a different result.

$$p_{21}\lambda_{21} - p_{12}\lambda_{12} = \hbar \tag{5}$$

The electron begins in an excited state $|2\rangle$, reverts to the ground state $|1\rangle$ by emitting a photon, and is excited once again. Thus the final state of the atomic system has an angular momentum that is greater than the ground state by an amount ħ. In both cases (4) and (5) a photon is emitted, but because the order of the physical variables changed the angular momentum of the atomic system described by (5) is greater than (4) and the physical variables do not commute. Non-commutation is interpreted as the irreversibility of momentum when transforming between atomic and laboratory coordinates.

3. Wave mechanics

3.1 Historical perspectives

Einstein introduced the founding principles of wave mechanics with concepts from his 1905 papers on special relativity and the photoelectric effect which de Broglie extended to material particles. He also provided the stimulus which led to completion of these ideas in a series of papers on the quantum theory of gases by showing that the same statistics Bose had applied to light quanta could also be used to describe emission from a monatomic ideal gas [11]. This led directly to the further development of wave mechanics by Schrödinger and the introduction of the wave function who openly acknowledged his indebtedness to Einstein in a letter [12]. "By the way, the whole thing would not have started at present or at any other time (I mean as far as I am concerned) had not your second paper on the degenerate gas directed my attention to the importance of de Broglie's ideas." His papers also

stimulated Dirac to write the first paper on quantum electrodynamics introducing the concept of second quantization [13]. Despite the implicit dependence of their theories upon his ideas none of them heeded his advice about momentum [1]. "Most important appears to me the result about the momentum transferred to the molecule by incoming and outgoing radiation." If they had followed Einstein's logic a more coherent description of quantum mechanics would have emerged.

3.2 Physical interpretation of the wave function

The concept of electron oscillator may be used to describe the rotation of the wave function of half-integer spin particles [14]. Excitation consists of the rotation of an electron's wave function through 2π radians during the *absorption* of one complete cycle of an electromagnetic wave. Decay corresponds to a second rotation of 2π radians during the *emission* of a complete wave cycle. In other words, a complete electron cycle, excitation and decay, consists of two wave function rotations, or 4π radians, and two cycles of an electromagnetic wave, where rotation refers to a change in phase of the electromagnetic field rather than a change in physical space. The electron begins its cycle during energy absorption by entering into a superposition state with a photon's sinusoidal electromagnetic fields and it exits the superposition state when the photon is released. The completed rotation consists of one cycle of an electron oscillator and two cycles of a wave. Thus changes in state can be viewed variously as the excitation and decay of an electron, photon creation and annihilation, superposition of fields, or cycling of a wave; depending upon which physical aspect of the phenomenon one chooses to describe. We use imaginary numbers to describe the transformation of coordinates from the atom to ordinary space so that it is possible to describe the rotation of a wave function mathematically.

The transfer of photon momenta to molecules in induced absorption and emission was predicted theoretically by Einstein and has been verified macroscopically by experiments of many types. It has also been verified microscopically by recent experiments with ultracold three-level artificial atoms which support the idea that momentum is a necessary parameter for the description of emission processes [15]. In the quantum Zeno effect frequent measurements arrest the progress of a "quantum jump". The measurements are equivalent to momentum exchange thereby confirming the earlier hypothesis that photon momenta need to be included in theories of the stimulated absorption and emission of radiation. An incoming photon transfers a momentum $+E/c$ to an atom in the ground state and superposes its fields with an electron's fields. When it exits the superposition state it transfers recoil momentum $-E/c$ to the atom and is expelled. The induced absorption and emission momenta are applied at different locations, the ground state electron shell and the excited state electron shell; and they are directed in opposite directions. Taking momentum into account during the time evolution of absorption and emission processes suggests that the electron oscillator cycles at discrete points in space due to momentum exchange and discontinuously in time.

In the wave mechanical view emission occurs by discrete energy exchange, but momentum exchange is either undetectable or does not occur; a situation that is refuted by the Einstein theory of radiation and cannot be sustained by experiment. The Schrödinger wave equation must be reformulated to reflect the discontinuous spatial coordinates and asymmetry of time necessary for momentum exchange.

3.3 Lagrangian model

The matrix mechanical observables of matter-radiation interactions are described in laboratory coordinates, while wave mechanical properties of matter are

described in atomic coordinates. Both describe the same characteristic, the steady states of an atom, but they approach them from different points of view; external and internal. The Einstein theoretical model of matter-radiation interactions adopts both points of view simultaneously within a single material system, as the dynamic equilibrium between external and internal forces. To describe the radiative processes of a single atom a wave equation is needed that includes photons, describes the time evolution of the wave function, and explains how discrete exchanges of momentum can occur during stimulated absorption and emission. Finally, in order to be in agreement with special relativity theory it must be symmetric in the space and time coordinates.

It is possible to formulate a relativistic wave equation by taking the action integral of a Lagrangian $S = \int Ldt$. Dirac has previously advised on the proper use of the Lagrangian in quantum mechanics [16], "we ought to consider the classical Lagrangian not as a function of the coordinates and velocities but rather as a function of the coordinates at time t and the coordinates at time t + dt." Following Dirac's initiative we let the coordinates at time t and at time t + dt denote electron shells corresponding to the states $|1\rangle$ and $|2\rangle$ respectively. Next, "We introduce at each point of space-time a Lagrangian density, which must be a function of the coordinates and their first derivatives with respect to x,y,z, and t corresponding to the Lagrangian in particle theory being a function of coordinates and velocities. The integral of the Lagrangian density over any (four-dimensional) region of space-time must then be stationary for all small variations of the coordinates inside the region, provided the coordinates on the boundary remain invariant"; where the "four-dimensional region of space-time" refers to the area between electron shells and "the coordinates on the boundary" refers to the electron shells. Absorption initiates from the steady state $|1\rangle$ with coordinates $r_1 = (x_1,y_1,z_1)$ and time t_1, and it finalizes at $|2\rangle$ with coordinates $r_2 = (x_2,y_2,z_2)$ and time t_2; where r_1 and r_2 denote electron shells. The Lagrangian density within the four-dimensional space–time region bounded by the electron shells is a function of the coordinates and their first derivatives $L\left(\phi_i, \phi_{i,\mu}\right)$. The conditions are satisfied by an action integral of the Lagrangian density.

$$S[\phi_i(t)] = \int_{r_1}^{r_2}\int_{t_1}^{t_2} L\left(\phi_i, \phi_{\mathrm{i},\mu}\right) d^3\mathrm{x}\mathrm{d}\mathrm{t} = \mathrm{h} \tag{6}$$

The action is a functional, a function of the values of coordinates on the *discrete* boundaries of the space–time surfaces r_1 and r_2 which are in turn functions of the *continuous* space–time variables of the fields within the surface. The discrete space–time variables assigned to the limits of integration describe electron shells and the continuous space–time variables of the Lagrangian density describe electromagnetic fields. Thus the photon is represented as a four-dimensional localization of field within the electron shells. Momentum exchange occurs when a photon makes contact with a point on the electron shell whether by absorption or emission. Even though complementarity denies the simultaneous presence of wave and particle properties in free space, they are present in atomic space when a photon's sinusoidal fields are localized within electron shells.

3.4 Physical model of the atom

If the photon is created as an independent entity when energy is absorbed; then quantum mechanics refers to not two, but three bodies. It presumes that the three

field sources are loosely bound within a conservative, or frictionless system, that they are free to interact with each other, and that each of the three particles contributes to the atomic system independently. For the related case of three particles with gravitational fields no general closed form solution is possible [17]. Gravitationally bound three-body systems result in chaos for nearly all initial conditions. It should not be surprising therefore that a physical system consisting of three electromagnetic field sources; electron, photon, and nucleus; also has an indeterminate outcome. To obtain the equations of motion for an electromagnetic three-body problem when the only knowledge available about the particles is their field properties, we need to obtain a series of partial solutions, which are the different mathematical models. Because an exact solution is not possible for the dynamic evolution of a three body system all solutions are considered approximations.

The three-body model of atomic structure may be described formally by introducing a wave-like, physically independent field source ε, the localized photon, into our description of excited atomic states. The modified Hamiltonian is now given by,

$$H = T + \varepsilon + V \tag{7}$$

where T refers to an electron, ε represents a "captured" photon, and V represents the nucleus. Each of the three field sources (or particles), possesses a unique vector field; that is, a well-defined field geometry, while the plus and minus signs indicate that the superposition of fields is linear. The Eq. (7) contains the essence of quantum mechanics as a three-body conservative system in real space, as opposed to nonrelativistic descriptions in abstract space. The equations revert to their classical two-body form when the influence of ε is negligible.

4. Conclusion

If momentum is not taken into account the structure of an atom and its observable properties may be described in the same space. In other words, we can plot the motion of a hydrogen atom's electron in the same space as the motion of the nucleus. If momentum is included a single space–time no longer suffices. When a photon interacts with an atom its linear momentum is transformed into angular momentum and an electron is excited. The angular momentum can no longer be described in laboratory coordinates and instead is expressed in atomic coordinates. All matter has internal and external aspects that are described in distinct coordinate systems. The idea of internal and external properties of matter is as old as science itself having first been expressed by Socrates and Aristotle; however, by introducing Eq. (6) it is proposed as a universal property of matter. Only Einstein fully grasped the need for distinct coordinate systems to describe matter through his theories of the photoelectric effect and Brownian motion. He concluded his quantum theory of radiation by stating [1], "For a *theoretical* discussion such small effects [due to momentum] should be considered on a completely equal footing with the more conspicuous effects of a radiative *energy* transfer, since energy and momentum are linked in the closest possible way." His advice was not fully appreciated due to an inability to visualize the time evolution of a radiating atom.

The conscious mind requires mental images to be able to understand and describe natural phenomena. "For Plato says that we would be engaging in futile labor if we tried to explain these phenomena without images that speak to the eyes." [18]. The need for visual images forms the foundation of classical physics and is the source and origin of science itself. All stages of formulating a theory; whether

observation, analysis, or experiment; is intimately connected to the visual system. In fact the visual cortex is so dominant an area of the brain that when blindness occurs it processes tactile and auditory sensory data instead. Visualization was an important factor during the derivation of quantum mechanics and as well of scientific theory in the past. The need to visualize explains why Heisenberg insisted on a theory of "observables", and it also explains why wave mechanics quickly became more popular. It also accounts for the fact that none of the mathematical models explicitly includes the photon.

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