Chapter

Multiple Solutions for Some Classes Integro-Dynamic Equations on Time Scales

Svetlin G. Georgiev

Abstract

In this chapter we study a class of second-order integro-dynamic equations on time scales. A new topological approach is applied to prove the existence of at least two non-negative solutions. The arguments are based upon a recent theoretical result.

Keywords: integro-dynamic equations, time scale, BVP, existence, positive solution, fixed point, cone, sum of operators

1. Introduction

Many problems arising in applied mathematics and mathematical physics can be modeled as differential equations, integral equations and integro-differential equations.

Integral and integro-differential equations can be solved using the Adomian decomposition method (ADM) [1, 2], Galerkin method [3], rationalized Haar functions method [4], homotopy perturbation method (HPM) [5, 6] and variational iteration method (VIM) [7]. ADM can be applied for linear and nonlinear problems and it is a method that represents the solution of the considered problems in the form of Adomian polynomials. Rationalized Haar functions and Galerkin methods are numerical methods that can be applied in different ways for the solutions of integral and integro-differential equations. VIM is an analytical method and can be used for different classes linear and nonlinear problems. HPM is a semi-analytical method for solving of linear and nonlinear differential, integral and integro-differential equations.

In recent years, time scales and time scale analogous of some well-known differential equations, integral equations and integro-differential equations have taken prominent attention. The new derivative, proposed by Stefan Hilger in [8], gives the ordinary derivative if the time scale is the set of the real numbers and the forward difference operator if the time scale is the set of the integers. Thus, the need for obtaining separate results for discrete and continuous cases is avoided by using the time scales calculus.

This chapter outlines an application of a new approach for investigations of integro-differential equations and integro-dynamic equations on time scales. The approach is based on a new theoretical result. Let $\mathbb{T}$ be a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. Let also, $a, b \in \mathbb{T}$, $a < b$. In this chapter we will investigate the following second-order integro-dynamic equation

$$x^{\Delta^2}(t) = \int_a^t k(t,s) f\left(s, x(s), x^{\Delta}(s)\right) \Delta s, \quad t \in [a,b], \tag{1}$$

subject to the boundary conditions

$$x(a) = \alpha, \quad x\left(\sigma^2(b)\right) = \beta, \tag{2}$$

where
(H1) $k \in \mathcal{C}_{rd}([a, \sigma^2(b)] \times [a, \sigma^2(b)])$, $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$.
(H2) $f \in \mathcal{C}\left([a, \sigma^2(b)] \times \mathbb{R}^2\right)$ and

$$|f(s,u,v)| \leq a_1(s)|u|^{p_1} + a_2(s)|v|^{p_2} + a_3(s), \quad s \in \left[a, \sigma^2(b)\right], \quad u, v \in \mathbb{R}, \tag{3}$$

$a_j \in \mathcal{C}_{rd}([a, \sigma^2(b)])$, $j \in \{1,2,3\}$, are non-negative functions, $p_1, p_2 \geq 0$.

We will investigate the BVP (1), (2) for existence of non-negative solutions. Our main result in this chapter is as follows.

Theorem 1.1. *Suppose* (*H*1)-(*H*2). *Then the BVP* (1), (2) *has at least two non-negative solutions.*

Linear integro-dynamic equations of arbitrary order on time scales are investigated in [9] using ADM. Nonlinear integro-dynamic equations of second order on time scales are studied in [10] using the series solution method. Asymptotic behavior of non-oscillatory solutions of a class of nonlinear second order integro-dynamic equations on time scales is considered in [11].

The chapter is organized as follows. In the next section, we will give some basic definitions and facts by time scale calculus. In Section 3, we give some auxiliary results which will be used for the proof of our main result. In Section 4, we will prove our main result. In Section 5, we will give an example. Conclusion is given in Section 6.

2. Time scales revisited

Time scales calculus originates from the pioneering work of Hilger [8] in which the author aimed to unify discrete and continuous analysis. Time scales have gained much attention recently. This section is devoted to a brief introduction of some basic notions and concepts on time scales. For detailed introduction to time scale calculus we refer the reader to the books [12, 13].

Definition 2.1. *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of the real numbers.*

Definition 2.2.

1. *The operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *given by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ for } t \in \mathbb{T} \tag{4}$$

will be called the forward jump operator.

2. The operator $\rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for} \quad t \in \mathbb{T} \tag{5}$$

will be called the backward jump operator.

3. The function $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t \quad \text{for} \quad t \in \mathbb{T} \tag{6}$$

will be called the graininess function.

We set

$$\inf \emptyset = \sup\mathbb{T}, \quad \sup\emptyset = \inf \mathbb{T}. \tag{7}$$

Observe that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$ and $\rho(t) \leq t$ for any $t \in \mathbb{T}$. Below, suppose that $\mathbb{T}$ is a time scale with forward jump operator and backward jump operator σ and ρ, respectively.

Definition 2.3. *We define the set*

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if} \quad \sup\mathbb{T} < \infty \\ \mathbb{T} & \text{otherwise.} \end{cases} \tag{8}$$

Using the forward and backward jump operators, one can classify the elements of a time scale.

Definition 2.4. *The point $t \in \mathbb{T}$ is said to be*

1. right-scattered if $\sigma(t) > t$.

2. right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$.

3. left-scattered if $\rho(t) < t$.

4. left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

5. isolated if it is left-scattered and right-scattered at the same time.

6. dense if it is left-dense and right-dense at the same time.

Definition 2.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a given function and $t \in \mathbb{T}^{\kappa}$. The delta or Hilger derivative of f at t will be called the number $f^{\Delta}(t)$, provided that it exists, if for any $\varepsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon \ |\sigma(t) - s| \quad \text{for} \quad \text{all} \quad s \in U. \tag{9}$$

If $f^{\Delta}(t)$ exists for any $t \in \mathbb{T}^{\kappa}$, then we say that f is delta or Hilger differentiable in $\mathbb{T}^{\kappa}$. The function $f^{\Delta} : \mathbb{T} \to \mathbb{R}$ will be called the delta derivative or Hilger derivative, shortly derivative, of f in $\mathbb{T}^{\kappa}$.

Remark 2.6. *The delta derivative coincides with the classical derivative in the case when $\mathbb{T} = \mathbb{R}$.*

Note that the delta derivative is well defined.

Theorem 2.7. *Let $f : \mathbb{T} \to \mathbb{R}$ be a given function and $t \in \mathbb{T}^{\kappa}$.*

1. The function f is continuous at t, if it is differentiable at t.

2. The function f is differentiable at t and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \tag{10}$$

if f is continuous at t and t is tight-scattered.

3. Let t is right-dense. Then the function f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{11}$$

exists as a finite number. In this case, we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{12}$$

4. We have

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t), \tag{13}$$

if f is differentiable at t.

Definition 2.8. *Let $f : \mathbb{T}|to\mathbb{R}$ is a given function.*

1. *We say that f is regulated if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

2. *We say that f is pre-differentiable with region of differentiation D if*

   a. *it is continuous,*

   b. $D \subset \mathbb{T}^{\kappa}$,

   c. $\mathbb{T}^{\kappa} \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$,

   d. f *is differentiable at each* $t \in D$.

To define indefinite integral and Cauchy integral on time scale we have a need of the following basic result.

Theorem 2.9. *Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}$, $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ be a given regulated function. Then there exists unique function F that is pre-differentiable and*

$$F^{\Delta}(t) = f(t) \text{ for any } t \in D, \quad F(t_0) = x_0. \tag{14}$$

Definition 2.10.

1. *Let $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Then any function F in Theorem 2.9. is said to be a pre-antiderivative of the function f and the indefinite integral of the regulated function f is defined by*

$$\int f(t) \Delta t = F(t) + c. \tag{15}$$

Here c is an arbitrary constant. Define the Cauchy integral as follows

$$\int_{\tau}^{s} f(t) \Delta t = F(s) - F(\tau) \text{ for all } \tau, s \in \mathbb{T}. \tag{16}$$

2. A function $F : \mathbb{T} \to \mathbb{R}$ is said to be an antiderivative of the function $f : \mathbb{T} \to \mathbb{R}$ if

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^{\kappa}. \tag{17}$$

Definition 2.11. *Let $f : \mathbb{T} \to \mathbb{R}$ be a given function. If it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$, then we say that f is rd-continuous. With $C_{rd}(\mathbb{T})$ we will denote the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ and with $C^1_{rd}(\mathbb{T})$ we will denote the set of all functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative are rd-continuous.*

We will note that if f is rd-continuous, then it is regulated Below, we will list some of the properties of the Cauchy integral.

Theorem 2.12. *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$. Then we have the following.*

i. $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,

ii. $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,

iii. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,

iv. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

v. $\int_a^b f(\sigma(t)) g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(t)\Delta t$,

vi. $\int_a^b f(t) g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(\sigma(t))\Delta t$,

vii. $\int_a^a f(t)\Delta t = 0$,

viii. If $|f(t)| \le g(t)$ on $[a, b)$, then

$$\left| \int_a^b f(t)\Delta t \right| \le \int_a^b g(t)\Delta t, \tag{18}$$

ix. If $f(t) \ge 0$ for all $a \le t < b$, then $\int_a^b f(t)\Delta t \ge 0$.

Let

$$G(t,s) = \begin{cases} -\dfrac{(\sigma(s) - a)(\sigma^2(b) - t)}{\sigma^2(b) - a}, & \sigma(s) \le t, \\ -\dfrac{(t - a)(\sigma^2(b) - \sigma(s))}{\sigma^2(b) - a}, & t \le s, \quad t \in [a, \sigma^2(b)], \quad s \in [a, \sigma(b)]. \end{cases} \tag{19}$$

We have

$$|G(t,s)| \le \sigma^2(b) - a, \quad t \in [a, \sigma^2(b)], \quad s \in [a, \sigma(b)]. \tag{20}$$

In [12], it is proved that G is the Green function for the BVP

$$x^{\Delta^2} = 0, \quad x(a) = x(\sigma^2(b)) = 0. \tag{21}$$

3. Auxiliary results

Let X be a real Banach space.

Definition 3.1. *A mapping $K : X \to X$ that is continuous and maps bounded sets into relatively compact sets will be called completely continuous.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 3.2. *Suppose that Ω_X is the class of all bounded sets of X. The function $\alpha : \Omega_X \to [0, \infty)$ that is defined in the following manner*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam(Y_j) \le \delta, \quad j \in \{1, \dots, m\}\right\}, \tag{22}$$

where $\operatorname{diam}(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of $Y_j, j \in \{1, \dots, m\}$, is said to be Kuratowski measure of noncompactness.

For the main properties of measure of noncompactness we refer the reader to [14].

Definition 3.3. *If the mapping $K : X \to X$ is continuous and bounded and there exists a nonnegative constant k such that*

$$\alpha(K(Y)) \le k\alpha(Y), \tag{23}$$

for any bounded set $Y \subset X$, then we say that it is a k-set contraction.

Note that any completely continuous mapping $K : X \to X$ is a 0-set contraction (see [15]).

Definition 3.4. *Suppose that X and Y are real Banach spaces. Then the map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which*

$$\|Kx - Ky\|_Y \ge h\|x - y\|_X \tag{24}$$

for any $x, y \in X$.

Definition 3.5. *A closed, convex set $\mathcal{P}$ in X is said to be cone if.*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,*

2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P} \setminus \{0\}$,

$$\begin{aligned} \mathcal{P}_{r_1} &= \{u \in \mathcal{P} : \|u\| < r_1\}, \\ \mathcal{P}_{r_1, r_2} &= \{u \in \mathcal{P} : r_1 < \|u\| < r_2\} \end{aligned} \tag{25}$$

for positive constants r_1, r_2 such that $0 < r_1 \le r_2$. The following result will be used to prove our main result. We refer the reader to [16, 17] for more details.

Theorem 3.6. *Let $\mathcal{P}$ be a cone in a Banach space $(E, \|\cdot\|)$. Let Ω be a subset of $\mathcal{P}$, $0 \in \Omega$ and $0 < r < L < R$ are real constants. Let also, $T : \Omega \to E$ is an expansive operator with a constant $h > 1$, $S : \overline{\mathcal{P}_R} \to E$ is a k-set contraction with $0 \le k < h - 1$ and $S(\overline{\mathcal{P}_R}) \subset (I - T)(\Omega)$. Assume that $\mathcal{P}_{r,L} \cap \Omega \ne \emptyset$, $\mathcal{P}_{L,R} \cap \Omega \ne \emptyset$ and there exist an $u_0 \in \mathcal{P}^*$ such that $T(x - \lambda u_0) \in \mathcal{P}$ for all $\lambda \ge 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ and the following conditions hold:*

a. $Sx \ne x - \lambda u_0$, $x \in \partial\mathcal{P}_r$, $\lambda \ge 0$,

b. $\|Sx + T0\| \le (h - 1)\|x\|$ and $Tx + Fx \ne x$, $x \in \partial\mathcal{P}_L \cap \Omega$,

c. $Sx \ne x - \lambda u_0$, $x \in \mathcal{P}_R$, $\lambda \ge 0$.

Then $T + S$ has at least two fixed points $x_1 \in \mathcal{P}_{r,L} \cap \Omega$, $x_2 \in \mathcal{P}_{L,R} \cap \Omega$, i.e.,

$$r < \|x_1\| < L < \|x_2\| < R. \tag{26}$$

Let

$$\begin{aligned} A_1 &= \frac{\alpha|\sigma^2(b)| + (\alpha+2|\beta|)\max\left\{|a|,|\sigma^2(b)|\right\}}{\sigma^2(b)-a}, \\ A_2 &= \max_{(t,s)\in[a,\sigma^2(b)]\times[a,\sigma^2(b)]} |k(t,s)|, \\ A_3 &= \max\left\{\max_{s\in[a,\sigma^2(b)]} a_j(s), \quad j=1,2,3\right\}, \\ A_4 &= \max\left\{\left(\sigma^2(b)-a\right)^2, \sigma^2(b)-a\right\}, \end{aligned} \tag{27}$$

and

$$\phi(t) = \frac{\alpha\sigma^2(b) - \beta a + (\beta-\alpha)t}{\sigma^2(b)-a}, \quad t\in\left[a,\sigma^2(b)\right]. \tag{28}$$

Then

$$\begin{aligned} |\phi(t)| &\le \frac{\alpha|\sigma^2(b)| + |\beta|\max\left\{|a|,|\sigma^2(b)|\right\} + (|\beta|+\alpha)\max\left\{|a|,|\sigma^2(b)|\right\}}{\sigma^2(b)-a} \\ &= A_1, \quad t\in\left[a,\sigma^2(b)\right]. \end{aligned} \tag{29}$$

Suppose that $E = C^1_{rd}([a,\sigma^2(b)])$ is endowed with the norm

$$\|x\| = \max\left\{\max_{t\in[a,\sigma^2(b)]}|x(t)|, \quad \max_{t\in[a,\sigma^2(b)]}\left|x^\Delta(t)\right|\right\}, \tag{30}$$

provided it exists. Next two lemmas give integral representations of the solutions of the BVP (1), (2).

Lemma 3.7. *If $x\in E$ is a solution to the integral equation*

$$x(t) = \int_a^{\sigma(b)} G(t,s)\int_a^s k(s,s_1)f\left(s_1,x(s_1),x^\Delta(s_1)\right)\Delta s_1\Delta s + \phi(t), \quad t\in\left[a,\sigma^2(b)\right], \tag{31}$$

then x is a solution to the BVP (1), (2).

Proof. Since G is the Green function of the BVP (3) and $\phi^{\Delta^2}(t)=0$, $t\in[a,\sigma^2(b)]$, we get

$$x^{\Delta^2}(t) = \int_a^t k(t,s)f\left(s,x(s),x^\Delta(s)\right)\Delta s, \quad t\in\left[a,\sigma^2(b)\right], \tag{32}$$

and

$$\begin{aligned} x(a) = \phi(a) &= \frac{\alpha\sigma^2(b) - \beta a + (\beta-\alpha)a}{\sigma^2(b)-a} = \alpha, \\ x\left(\sigma^2(b)\right) = \phi\left(\sigma^2(b)\right) &= \frac{\alpha\sigma^2(b) - \beta a + (\beta-\alpha)\sigma^2(b)}{\sigma^2(b)-a} = \beta. \end{aligned} \tag{33}$$

Thus, x is a solution to the BVP (1), (2). This completes the proof. □

For $x \in E$, define the operator

$$F_1 x(t) = \int_a^t (t - \sigma(s))(x(s) \\ - \int_a^{\sigma(b)} G(s, s_1) \int_a^{s_1} k(s_1, s_2) f\left(s_2, x(s_2), x^\Delta(s_2)\right) \Delta s_2 \Delta s_1 - \phi(s)) \Delta s, \tag{34}$$

$t \in \left[a, \sigma^2(b)\right]$.

Lemma 3.8. *If $x \in E$ is a solution to the integral equation*

$$F_1 x(t) = 0, \quad t \in \left[a, \sigma^2(b)\right], \tag{35}$$

then x is a solution to the BVP (1), (2).
Proof. We have

$$0 = (F_1 x)^\Delta(t) \\ = \int_a^t \left(x(s) - \int_a^{\sigma(b)} G(s, s_1) \int_a^{s_1} k(s_1, s_2) f\left(s_2, x(s_2), x^\Delta(s_2)\right) \Delta s_2 \Delta s_1 - \phi(s)\right) \Delta s, \; t \in \left[a, \sigma^2(b)\right], \tag{36}$$

whereupon

$$0 = (F_1 x)^{\Delta^2}(t) \\ = x(t) - \int_a^{\sigma(b)} G(t, s) \int_a^s k(s, s_1) f\left(s_1, x(s_1), x^\Delta(s_1)\right) \Delta s_1 - \phi(t), \tag{37}$$

$t \in [a, \sigma^2(b)]$. Hence and Lemma 3.7, we conclude that x is a solution to the BVP (1), (2). This completes the proof. □

Now, we will give an estimate of the norm of the operator F_1.
Lemma 3.9. *If $x \in E$ and $\|x\| \le c$ for some positive constant c, then*

$$\|F_1 x\| \le A_4 \left(c + \left(\sigma^2(b) - a\right)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1 \right). \tag{38}$$

Proof. We have

$$|F_1 x(t)| \le \int_a^t (t - \sigma(s))(|x(s)| \\ + \int_a^{\sigma(b)} |G(s, s_1)| \int_a^{s_1} |k(s_1, s_2)| \left|f\left(s_2, x(s_2), x^\Delta(s_2)\right)\right| \Delta s_2 \Delta s_1 \\ + |\phi(s)|) \Delta s \\ \le \int_a^t (t - \sigma(s))(c \\ + \left(\sigma^2(b) - a\right) \int_a^{\sigma(b)} \int_a^{s_1} A_2 \left(a_1(s_2) |x(s_2)|^{p_1} + a_2(s_2) \left|x^\Delta(s_2)\right|^{p_2} \right. \\ \left. + a_3(s_2)\right) \Delta s_2 \Delta s_1 + A_1) \Delta s \\ \le A_4 \left(c + \left(\sigma^2(b) - a\right)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1 \right), \tag{39}$$

$t \in [a, \sigma^2(b)]$, and

$$
\begin{aligned}
\left|(F_1x)^{\Delta}(t)\right| \leq & \int_a^t (|x(s)| \\
& + \int_a^{\sigma(b)} |G(s,s_1)| \int_a^{s_1} |k(s_1,s_2)| |f(s_2, x(s_2), x^{\Delta}(s_2))| \Delta s_2 \Delta s_1 \\
& + |\phi(s)|) \Delta s \\
\leq & \int_a^t (c \\
& + (\sigma^2(b) - a) \int_a^{\sigma(b)} \int_a^{s_1} A_2 \left(a_1(s_2)|x(s_2)|^{p_1} + a_2(s_2)\left|x^{\Delta}(s_2)\right|^{p_2}\right. \\
& \left. + a_3(s_2)\right) \Delta s_2 \Delta s_1 + A_1) \Delta s \\
\leq & A_4 \left(c + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1\right),
\end{aligned}
\tag{40}
$$

$t \in [a, \sigma^2(b)]$. Thus,

$$\|F_1x\| \leq A_4 \left(c + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1\right). \tag{41}$$

This completes the proof. □

Below, suppose

(H3) Suppose that the positive constants A, m, ε, r_1, L_1, R_1 and R satisfy the following conditions

$$r_1 < \frac{L_1}{20} < L_1 < R_1, \quad m \in \left(0, \frac{4}{5}\right), \quad \varepsilon > 1, \quad R_1 < \varepsilon \frac{L_1}{20}, \tag{42}$$

$$AA_4 \left(R_1 + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 \left(R_1^{p_1} + R_1^{p_2} + 1\right) + A_1\right) < \frac{L_1}{20}, \tag{43}$$

$$AA_4 \left(L_1 + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 \left(L_1^{p_1} + L_1^{p_2} + 1\right) + A_1\right) < \left(\frac{4}{5} - m\right) L_1. \tag{44}$$

In the next section, we will give an example for constants A, m, ε, r_1, L_1, R_1 and R that satisfy $(H3)$. For $x \in E$, define the operator

$$Fx(t) = AF_1x(t), \quad t \in \left[a, \sigma^2(b)\right]. \tag{45}$$

By Lemma 3.9, we get the following result.

Lemma 3.10. *If $x \in E$ and $\|x\| \leq c$ for some positive constant c, then*

$$\|Fx\| \leq AA_4 \left(c + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1\right). \tag{46}$$

Lemma 3.11. *If $x \in E$ is a solution to the integral equation*

$$0 = \frac{L_1}{5} + Fx(t), \quad t \in \left[a, \sigma^2(b)\right], \tag{47}$$

then it is a solution to the BVP (1), (2).

Proof. We have

$$0 = (Fx)^{\Delta^2}(t) = A(F_1x)^{\Delta^2}(t), \quad t \in \left[a, \sigma^2(b)\right], \tag{48}$$

whereupon

$$0 = A\left(x(t) - \int_a^{\sigma(b)} G(t,s) \int_a^s k(s,s_1) f\left(s_1, x(s_1), x^{\Delta}(s_1)\right) \Delta s_1 - \phi(t)\right), \tag{49}$$

$t \in [a, \sigma^2(b)]$, and

$$x(t) = \int_a^{\sigma(b)} G(t,s) \int_a^s k(s,s_1) f\left(s_1, x(s_1), x^{\Delta}(s_1)\right) \Delta s_1 + \phi(t), \tag{50}$$

$t \in [a, \sigma^2(b)]$. Now, the assertion follows from Lemma 3.7. This completes the proof. □

4. Proof of the Main result

Let

$$\tilde{P} = \{u \in E : u \geq 0 \text{ on } [t_0, \infty)\}. \tag{51}$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\tilde{P}$. For $v \in E$, define the operators

$$\begin{aligned} Tv(t) &= (1 + m\varepsilon)v(t) - \varepsilon \frac{L_1}{10}, \\ Sv(t) &= -\varepsilon Fv(t) - m\varepsilon v(t) - \varepsilon \frac{L_1}{10}, \end{aligned} \tag{52}$$

$t \in [t_0, \infty)$. Note that any fixed point $v \in E$ of the operator $T + S$ is a solution to the IVP (1). Define

$$\begin{aligned} \mathcal{P}_{r_1} &= \{v \in \mathcal{P} : \|v\| < r_1\}, \\ \mathcal{P}_{L_1} &= \{v \in \mathcal{P} : \|v\| < L_1\}, \\ \mathcal{P}_{R_1} &= \{v \in \mathcal{P} : \|v\| < R_1\}, \\ \mathcal{P}_{r_1,L_1} &= \{v \in \mathcal{P} : r_1 < \|v\| < L_1\}, \\ \mathcal{P}_{L_1,R_1} &= \{v \in \mathcal{P} : L_1 < \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m} A_4 \left(R_1 + \left(\sigma^2(b) - a\right)(\sigma(b) - a)^2 A_2 A_3 \left(R_1^{p_1} + R_1^{p_2} + 1\right) + A_1\right) \\ &\quad + \frac{L_1}{5m}, \\ \Omega &= \mathcal{P}_{R_2} = \{v \in \mathcal{P} : \|v\| < R_2\}. \end{aligned} \tag{53}$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|Tv_1 - Tv_2\| = (1 + m\varepsilon)\|v_1 - v_2\|, \tag{54}$$

whereupon $T : \Omega \to E$ is an expansive operator with a constant $1 + m\varepsilon > 1$.

2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned}\|Sv\| &\le \varepsilon\|Fv\| + m\varepsilon\|v\| + \varepsilon\frac{L_1}{10}\\ &\le \varepsilon\Big(AA_4\Big(R_1 + \big(\sigma^2(b) - a\big)(\sigma(b) - a)^2 A_2A_3\big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1\Big)\\ &\quad + mR_1 + \frac{L_1}{10}\Big).\end{aligned} \tag{55}$$

Therefore $S(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S : \overline{\mathcal{P}}_{R_1} \to E$ is continuous, we have that $S(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S : \overline{\mathcal{P}}_{R_1} \to E$ is a 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m}Fv_1 + \frac{L_1}{5m}. \tag{56}$$

Note that by the second inequality of $(H3)$ and by Lemma 3.10, it follows that $\varepsilon Fv_1 + \varepsilon\frac{L_1}{5} \ge 0$ on $[t_0, \infty)$. We have $v_2 \ge 0$ on $[t_0, \infty)$ and

$$\begin{aligned}\|v_2\| &\le \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m}\\ &\le R_1 + \frac{A}{m}A_4\Big(R_1 + \big(\sigma^2(b) - a\big)(\sigma(b) - a)^2 A_2A_3\big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1\Big)\\ &\quad + \frac{L_1}{5m} = R_2.\end{aligned} \tag{57}$$

Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon F v_1 - \varepsilon\frac{L_1}{10} - \varepsilon\frac{L_1}{10} \tag{58}$$

or

$$(I - T)v_2 = -\varepsilon m v_2 + \varepsilon\frac{L_1}{10} = Sv_1. \tag{59}$$

Consequently $S(\overline{\mathcal{P}}_{R_1}) \subset (I - T)(\Omega)$.

4. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v - \lambda v_0) \in \mathcal{P}$ for all $\lambda \ge 0$, $v \in \partial\mathcal{P}_{r_1} \cap (\Omega + \lambda v_0)$ and $Sv = v - \lambda v_0$ for some $\lambda \ge 0$ and for some $v \in \mathcal{P}_{r_1}$. Then

$$\begin{aligned}r_1 &\ge \|v - \lambda v_0\| = \|Sv\| \ge -Sv(t)\\ &= \varepsilon Fv(t) + \varepsilon m v(t) + \varepsilon\frac{L_1}{10} \ge \varepsilon\frac{L_1}{20}, \quad t \in [t_0, \infty),\end{aligned} \tag{60}$$

because by the second inequality of $(H3)$ and by Lemma 3.10, it follows that $\varepsilon Fv + \varepsilon\frac{L_1}{20} \ge 0$ on $[t_0, \infty)$.

5. Let $v \in \partial\mathcal{P}_{L_1}$. Then

$$\|Sv + T0\| = \|\varepsilon Fv + m\varepsilon v + \varepsilon \frac{L_1}{5}\| \le \varepsilon\left(\|Fv\| + m\|v\| + \frac{L_1}{5}\right)$$
$$\le \varepsilon\big(AA_4\big(L_1 + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3\big(L_1^{p_1} + L_1^{p_2} + 1\big) \tag{61}$$
$$+A_1) + \left(m + \frac{1}{5}\right)L_1\right) \le \varepsilon L_1 = \varepsilon\|v\|.$$

Note that in the last inequality we have used the third inequality of $(H3)$.

6. Now, assume that $v \in \partial\mathcal{P}_{L_1} \cap \Omega$ is such that

$$v = Tv + Sv, \tag{62}$$

whereupon

$$Fv + \frac{L_1}{5} \equiv 0 \quad \text{on} \quad [t_0, \infty). \tag{63}$$

Since $v \in \partial\mathcal{P}_L$, we have that $v \not\equiv 0$ on $[t_0, \infty)$ and by the second inequality of $(H3)$ and by Lemma 3.10, it follows that $Fv + \frac{L_1}{5} > Fv + \frac{L_1}{20} \ge 0$ on $[t_0, \infty)$. This is a contradiction.

7. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v - \lambda v_0) \in \mathcal{P}$ for all $\lambda \ge 0$, $v \in \partial\mathcal{P}_{R_1}$, $v \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ and $Sv = v - \lambda v_0$ for some $\lambda \ge 0$ and for some $v \in \mathcal{P}_{R_1}$. Then

$$\begin{aligned} R_1 &\ge \|v - \lambda v_0\| = \|Sv\| \ge -Sv(t) \\ &= \varepsilon Fv(t) + \varepsilon m v(t) + \varepsilon\frac{L_1}{10} \ge \varepsilon\frac{L_1}{20}, \quad t \in [t_0, \infty), \end{aligned} \tag{64}$$

which is a contradiction.

Therefore all conditions of Theorem 3.6 hold. Hence, the IVP (1) has at least two solutions u_1 and u_2 so that

$$r_1 < \|u_1\| < L_1 < \|u_2\| < R_1. \tag{65}$$

5. An example

In this section we will illustrate our main result with an example. Firstly, we will give an example for the constants A, m, ε, r_1, L_1 and R_1 that satisfy the hypothesis $(H3)$. Let $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = 10$, $\alpha = \beta = 1$,

$$a_1(s) = a_2(s) = a_3(s) = \frac{1}{3}, \quad f(s, u, v) = \frac{1}{1 + v^4}, \quad k(s, s_1) = s_1^2, \tag{66}$$

$s \in [0, 12]$, $s_1 \in [0, 11]$, and

$$\begin{aligned} &r_1 = 1, \quad L_1 = 100, \quad R_1 = 200, \quad \varepsilon = 10^{10}, \\ &p_1 = p_2 = 0, \quad m = \frac{1}{2}, \quad A = \frac{1}{10^{50}}. \end{aligned} \tag{67}$$

Then

$$A_1 = \frac{12 + 12 \cdot 3}{12} = 4, \quad A_2 = 144, \quad A_3 = \frac{1}{3}, \quad A_4 = 144 \tag{68}$$

and

$$\begin{aligned}
&AA_4\Big(R_1 + \big(\sigma^2(b) - a\big)(\sigma(b) - a)^2 A_2 A_3 \big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1\Big) \\
&= \frac{1}{10^{50}} \cdot 144 \left(200 + 12^3 \cdot 144 \cdot \frac{1}{3} \cdot 3 + 4\right) \\
&< 5 = \frac{L_1}{20}, \\
&AA_4\Big(L_1 + \big(\sigma^2(b) - a\big)(\sigma(b) - a)^2 A_2 A_3 \big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1\Big) \\
&= \frac{1}{10^{50}} \cdot 144 \left(100 + 12^3 \cdot 144 \cdot \frac{1}{3} \cdot 3 + 4\right) \\
&< \frac{3}{10} \cdot 5 = \left(\frac{4}{5} - m\right) \frac{L_1}{20},
\end{aligned} \tag{69}$$

i.e., $(H1)$-$(H3)$ hold. Consequently the BVP

$$\begin{aligned}
&x^{\Delta^2}(t) = \int_0^t s^2 \frac{1}{1 + \left(x^{\Delta}(s)\right)^4} \Delta s, \quad t \in [0, 10], \\
&x(0) = x(12) = 1,
\end{aligned} \tag{70}$$

has at least two non-negative solutions.

6. Conclusion

In this chapter we introduce a class of BVPs for a class second-order integro-dynamic equations on time scales. We give some integral representations of the solutions of the considered BVP. We apply a new multiple fixed point theorem and we prove that the considered BVP has at least two nontrivial solutions. The approach in this chapter can be applied for investigations of IVPs and BVPs for dynamic equations and integro-dynamic equations of arbitrary order on time scales.

Additional classifications

AMS Subject Classification: 39 A 10, 39 A 99

Author details

Svetlin G. Georgiev
Sorbonne University, Paris, France

*Address all correspondence to: svetlingeorgiev1@gmail.com

References

[1] Biazar J, Babolian E, Islam R. Solution of a system of Volterra integral equations of the first kind by Adomian method. Appl. Math. Comput. 2003;**139**: 249-258

[2] Goghary H, Javadi S, Babolian E. Restarted Adomian method for system of nonlinear Volterra integral equations. Appl. Math. Comput. 2005;**161**:745-751

[3] K. Maleknejad and M. Kajani. Solving linear integro-differential equation system by Galerkin methods with hybrid functions, Appl. Math. Comput. 159 (2004), pp. 603612.

[4] K. Maleknejad, F. Mirzaee and S. Abbasbandy. Solving linear integro-differential equations system by using rationalized Haar functions method, Appl. Math. Comput. 155 (2004), pp. 317328.

[5] Biazar J, Ghazvini H, Eslami M. He's homotopy perturbation method for systems of integro-differential equations. Chaos Solitons. Fractals. 2007. DOI: 10.1016/j.chaos.2007.06.001.

[6] Yusufoglu E. An efficient algorithm for solving integro-differential equations system. Appl. Math. Comput. 2007;**192**:51-55

[7] Wang S, He J. Variational iteration method for solving integro-differential equations. Phys. Lett. A. 2007;**367**: 188-191

[8] Hilger S. Analysis on measure chains - a unified approach to continuous and discrete calculus. Results Math. 1990;**18**: 18-56

[9] Georgiev S. Integral Equations on Time Scales. Atlantis Press; 2016

[10] Georgiev S, Erhan I. Nonlinear Integral Equations on Time Scales, Nova Science. Publishers; 2019

[11] Grace S, El-Bellagy M, Deif S. Asymptotic Behavior of Non-Oscillatory Solutions of Second Order Integro-Dynamic Equations on Time Scales. J. Appl. Computat. Math. 2013:2-4

[12] Bohner M, Petrson A. Dynamic Equations on Time Scales. Birkhauser; 2001

[13] Bohner M, Georgiev S. Multivariable Dynamic Calculus on Time Scales. Springer; 2017

[14] Banas J, Goebel K. Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics. Vol. **60**. New York: Marcel Dekker, Inc.; 1980

[15] Drabek P, Milota J. Methods in Nonlinear Analysis. Birkhäuser: Applications to Differential Equations; 2007

[16] Benzenati L, Mebarki K. Multiple Positive Fixed Points for the Sum of Expansive Mappings and k-Set Contractions. Math. Meth. Appl. Sci. 1919;**42**(13):4412-4426

[17] Georgiev S, Khaled Z. *Multiple Fixed-Point Theorems and Applications in the Theory of ODEs, FDEs and PDEs*. CRC Press; 2020