Chapter

The Uniformly Parabolic Equations of Higher Order with Discontinuous Data in Generalized Morrey Spaces and Elliptic Equations in Unbounded Domains

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Abstract

We study the regularity of the solutions of the Cauchy-Dirichlet problem for linear uniformly parabolic equations of higher order with vanishing mean oscillation (VMO) coefficients. We prove continuity in generalized parabolic Morrey spaces $M_{p,\varphi}$ of sublinear operators generated by the parabolic Calderon-Zygmund operator and by the commutator of this operator with bounded mean oscillation (BMO) functions. We obtain strong solution belongs to the generalized Sobolev-Morrey space $\overset{\circ}{W}{}^{m,1}_{p,\varphi}(Q)$. Also we consider elliptic equation in unbounded domains.

Keywords: higher order parabolic equations, generalized Morrey spaces, sublinear operators, Calderon-Zygmund integrals, VMO, Cauchy-Dirichlet problem, elliptic equations, unbounded domain

1. Introduction

We consider the higher order linear Cauchy-Dirichlet problem in $Q = \Omega \times (0, T)$, being a cylinder in $\mathbb{R}^{n+1}$, $\Omega \subset R^n$ be a bounded domain $0 < T < \infty$

$$u_t - \sum_{\substack{|\alpha| \le m, \\ |\beta| \le m}} a_{\alpha\beta}(x,t) D^{\alpha\beta} u(x,t) = f(x,t), \ a.e. \ \text{in } Q \tag{1}$$

$$u(x,t) = 0 \ \text{ on } \partial_p Q, \tag{2}$$

where $\partial_p Q = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t = 0\})$ stands for the parabolic boundary of Q and $D^{\alpha\beta} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}, \cdots, \partial x_n^{\alpha_n}} \cdots \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1}, \cdots, \partial y_n^{\beta_n}}$, $|\alpha| = \sum_{k=1}^n \alpha_k$, $\beta = \sum_{k=1}^n \beta_k$.

The unique strong solvability of this type problem was proved in [1]. In [2] the regularity of the solution in the Morrey spaces $L_{p,\lambda}(\mathbb{R}^{n+1})$ with $p \in (1, \infty)$,

$\lambda \in (0, n+2)$ and also its Hölder regularity was studied. In [3] Nakai extend these studies on generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^{n+1})$ with a weight φ satisfying the integral condition

$$\int_r^\infty \frac{\varphi(a,s)}{s} ds \le c\varphi(a,r), \forall a \in \mathbb{R}^{n+1}, r > 0.$$

The generalized Morrey space is then defined to be the set of all $f \in L_{p,loc}(\mathbb{R}^{n+1})$ such that

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} = \sup_{\mathcal{E}} \frac{1}{\varphi(\mathcal{E})} \left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(x)|^p dx \right)^{\frac{1}{p}},$$

where the supremum is taken over all parabolic balls $\mathcal{E}$ with respect to the parabolic distance.

The main results connected with these spaces is the following celebrated lemma: let $|Df| \in L_{p,n-\lambda}$ even locally, with $n - \lambda < p$, then u is Holder continuous of exponent $\alpha = 1 - \frac{n-\lambda}{p}$. This result has found many applications in theory elliptic and parabolic equations. In [2] showed boundedness of the maximal operator in $L_{p,\lambda}(\mathbb{R}^{n+1})$ that allows them to prove continuity in these spaces of some classical integral operators. So was put the beginning of the study of the generalized Morrey spaces $M_{p,\varphi}, p > 1$ with φ belonging to various classes of weight functions. In [3] proved boundedness of maximal and Calderon-Zygmund operators in $M_{p,\varphi}$ imposing suitable integral and doubling conditions on φ. These results allow to study the regularity of the solutions of various linear elliptic and parabolic value problems in $M_{p,\varphi}$ (see [4–6]). Here we consider a supremum condition on the weight which is optimal and ensure the boundedness of the maximal operator in $M_{p,\varphi}$. We use maximal inequality to obtain the Calderon-Zygmund type estimate for the gradient of the solution of the problem (1) and (2) in the $M_{p,\varphi}$.

The results presented here are a natural extension of the previous paper [7] to parabolic equations. Here we study the boundedness of the sublinear operators, generated by Calderon-Zygmund operators in generalized Morrey spaces and the regularity of the solutions of higher order uniformly elliptic boundary value problem in local generalized Morrey spaces where domain is bounded. Also hear we study higher order uniformly elliptic boundary value problem where domain is unbounded.

In paper [8] Byun, Palagachev and Wang is study the regularity problem for parabolic equation in classical Lebesgue classes and of Byun, Palagchev and Softova [9, 10] where the problem studied in weighted Lebesgue and Orlicz spaces with a Muckenhoupt weight and the classical Morrey spaces $L_{p,\lambda}(Q)$ with $\lambda \in (0, n+2)$.

In papers [11, 12] the authors studied second order linear elliptic and parabolic equations with VMO coefficients.

Denote by a the coefficient $a(x,t) = \{a_{\alpha\beta}(x,t)\} : Q \to M^{n\times n}$ and by $f(x,t)$ nonhomogeneous term. Suppose that the operator is uniformly parabolic.

The paper is organized as follows. In section 2 we introduce some notations and give the definition of the generalized Morrey spaces $M_{p,\varphi}(Q)$. In section 3 we study sublinear operators generated by parabolic singular integrals in generalized Morrey spaces. In section 4 we is consider sublinear operators generated by non-singular integrals, in section 5 singular and non-singular integrals in generalized Morrey spaces. In section 6 we consider uniformly parabolic equations of higher order with VMO coefficients and proved regularity of solutions. In section 7 we study uniformly elliptic equations in unbounded domains.

2. Some notation and definition

The following notations are used in this paper:

$$x = (x', t), y = (y', \tau) \in \mathbb{R}^{n+1} = R^n \times R^n, \mathbb{R}^{n+1}_+ = R^n \times R_+;$$

$$x = (x'', x_n, t) \in D^{n+1}_+ = R^{n-1} \times R_+ \times R_+, D^{n+1}_- = R^{n-1} \times R_- \times R_+;$$

$|\cdot|$ is the Euclidean metric, $|x| = \left(\sum_{i=1}^n x_i^2 + t^2\right)^{\frac{1}{2}}$; $B_r(x') = \{y' \in R^n : |x' - y'| < r\}$, $|B_r| = c \cdot r^n$; $\mathcal{I}_r(x', t) = \{y \in \mathbb{R}^{n+1} : |x' - y'| < r, |t - \tau| < r^2\}$, $|\mathcal{I}_r(x', t)| = c \cdot r^{n+2}$; $Q_r = \mathcal{I}_r(x, \tau) \cap Q$ for each $(x, \tau) \in Q$, $2\mathcal{I}_r(x, \tau) = \mathcal{I}_{2r}(x, \tau)$.

S^n is the unit sphere in $\mathbb{R}^{n+1}$;

$$D_i u = \frac{\partial u}{\partial x_i}, Du = (D_1 u, \ldots, D_n u), u_t = \frac{\partial u}{\partial t};$$

$$D^{\alpha\beta} u = \frac{\partial^{|\alpha|} \partial^{|\beta|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \cdot \partial y_1^{\beta_1} \cdot \partial y_n^{\beta_n}}$$

the letter C is used for various positive constants.

In the following, besides the standard parabolic metric $\rho(x, t) = \max\left(|x'|, |t|^{\frac{1}{2}}\right)$. We use the equivalent one

$$\rho(x, t) = \left(\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}\right)^{\frac{1}{2}}$$

considered by Fabes and Riviere in [13]. The topology induced by $\rho(x, t)$ consists of the ellipsoids

$$\mathcal{E}_r(x) = \left\{ y \in \mathbb{R}^{n+1} : \frac{|x' - y'|^2}{r^2} + \frac{|t - \tau|^2}{r^4} < 1 \right\}, |\mathcal{E}_r| = C \cdot r^{n+2},$$

$$\mathcal{E}_1(x) \equiv B_1(x).$$

It is easy to see that the this metrics ore equivalent. In fact, for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{I}}$ and $\overline{\mathcal{I}}$ with measure comparable to r^{n+2} such that $\underline{\mathcal{I}} \subset \mathcal{E}_r \subset \overline{\mathcal{I}}$.

Let $Q = \Omega \times (0, T), T > 0,$ be a cylinder in R^{n+1}_+. We give the definitions of the functional spaces that we are going to use. Let $a \in L_{1,loc}\left(\mathbb{R}^{n+1}\right.$ and let $a_{\mathcal{E}_r} = |\mathcal{E}_r|^{-1} \int_{\mathcal{E}_r} a(y) dy$ be the mean value of the integral of a. Denote

$$\eta_a(R) = \sup_{r \le R} \frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} \left| f(y) - f_{\mathcal{E}_r} \right| dy \text{ for every } R > 0,$$

where $\mathcal{E}_r$ ranges over all ellipsoids in $\mathbb{R}^{n+1}$. We say $a \in BMO$ (bounded mean oscillation [14]) if

$$\|a\|_* = \sup_{R > 0} \eta_\alpha(R)$$

is finite. $\|\cdot\|_*$ is a norm in a BMO constant functions.

We say $a \in VMO$ (vanishing mean oscillation) [14] if $a \in BMO$ and

$$\lim_{R \to 0} \eta_a(R) = 0$$

$\eta_a(R)$ is called the VMO-modulus of a. For any bounded cylinder Q we define $BMO(Q)$ and $VMO(Q)$ taking $a \in L_1(Q)$ and $Q_r = Q \cap \mathcal{E}_r(x),\ x \in Q,$ instead of $\mathcal{E}_r$ in the definition above. If a function $a \in BMO$ or $VMO,$ it is possible to extend the function in the whole of $\mathbb{R}^{n+1}$ preserving its BMO-norm or VMO-modulus, respectively (see [15]). Any bounded uniformly continuous (BUC) function f with modulus of continuity $\omega_f(R)$ belongs to VMO with $\eta_f(R) = \omega_f(R)$. Besides, BMO and VMO also contain discontinuous functions, and the following example shows the inclusion $W^1_{n+2}(\mathbb{R}^{n+1}) \subset VMO \subset BMO$.

Example 2.1. We have that $f(x) = |\log \rho(x,t)| \in BMO \setminus VMO$; $\sin f(x) \in BMO \cap L_\infty(\mathbb{R}^{n+1})$; $f_\alpha(x) = |\log \rho(x,t)|^\alpha \in VMO$ for any $\alpha \in (0,1)$; $f_\alpha \in W^1_{n+2}(\mathbb{R}^{n+1})$ for $\alpha \in \left(0, 1 - \frac{1}{n+2}\right)$; $f_\alpha \notin W^1_{n+2}(\mathbb{R}^{n+1})$ for $\alpha \in \left[1 - \frac{1}{n+2}, 1\right)$.

Let $\varphi : \mathbb{R}^{n+1} \times R_+ \to R_+$ be a measurable function and $p \in [1, \infty)$. The generalized parabolic Morrey space $M_{p,\varphi}(\mathbb{R}^{n+1})$ consists of all $f \in L_{p,loc}(\mathbb{R}^{n+1})$ such that

$$\|f\|_{p,\varphi;\, \mathbb{R}^{n+1}} = \sup_{(x,r) \in \mathbb{R}^{n+1} \times R_+} \varphi^{-1}(x,r) \left(r^{-n-2} \int_{\mathcal{E}_r(x)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

The space $M_{p,\varphi}(Q)$ consists of $L_p(Q)$ functions provided the following norm is finite

$$\|f\|_{p,\varphi;\, Q} = \sup_{(x,r) \in \mathbb{R}^{n+1} \times R_+} \varphi^{-1}(x,r) \left(r^{-n-2} \int_{Q_r(x)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

The generalized weak parabolic Morrey space $WM_{1,\alpha}(R_{n+1})$ consists of all measurable functions such that

$$\|f\|_{WM_{1,\alpha}(\mathbb{R}^{n+1})} = \sup_{(x,r) \in \mathbb{R}^{n+1} \times R_+} \varphi^{-1}(x,r) r^{-n-2} \|f\|_{WL_1(\mathcal{E}_r(x))},$$

where WL_1 denotes the weak L_1 space. The generalized Sobolev-Morrey space $W^{2m,1}_{p,\varphi}(Q),\ p \in [1, \infty),$ consists of all Sobolev functions $U \in W^{2m,1}_p(Q)$ with distributional derivatives $D^l_t D^s_x u \in M_{p,\varphi}(Q),\ 0 \le 2l + |s| \le 2m,$ endowed by the norm

$$\|u\|_{W^{2m,1}_{p,\varphi}(Q)} = \|u_t\|_{p,\varphi;\, Q} + \sum_{|\delta| \le 2m} \|D^s u\|_{p,\varphi;\, Q}.$$

We also define the space

$$\overset{0}{W}{}^{2m,1}_{p,\varphi}(Q) = \left\{ u \in W^{2m,1}_{p,\varphi}(Q) : u(x) = 0,\ x \in \partial Q \right\},$$
$$\|u\|_{\overset{0}{W}{}^{2m,1}_{p,\varphi}(Q)} = \left\{ \|u\|_{W^{m,1}_{p,\varphi}(Q)} \right\},$$

where ∂Q means the parabolic boundary $\Omega \cup (\partial\Omega \times (0,T))$. In problem (1) and (2) the coefficient matrix $a(x,t) = \left(a_{\alpha,\beta}(x,t)\right)^n_{i,j=1}$, $|\alpha|, |\beta| = m$ satisfies

$$\exists\gamma > 0 \ \gamma \sum_{|\alpha|=m} \xi_\alpha^2 \le \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha,\beta}(x,t)\xi_\alpha\xi_\beta, \tag{3}$$

for a.e. $(x,t) \in Q, \forall \xi \in R^n, \xi = \{\xi_\alpha, |\alpha = m| \in R^N\}$, N–number different multiindeks with length equal to m, $a_{\alpha,\beta}(x,t) = a_{\beta,\alpha}(x,t)$, which implies $a_{\alpha,\beta}(x,t) \in L_\infty(Q)$.

Theorem 2.1. *(Main results) Let $a(x,t) \in VMO(Q)$ with $\eta_{\alpha,\beta} = \sum_{i,j=1}^n \eta_{\alpha\beta ij}$ satisfy* (3), *and, for each $p \in (1,\infty)$, let $u(x,t) \in \overset{0}{W}{}_p^{2m,1}(Q)$ be a strong solution* (1) and (2). *If $f \in M_{p,\varphi}(Q)$ with $\varphi(x,r)$ being a measurable positive function satisfying*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) \frac{\operatorname*{ess\ inf}_{s<\xi<\infty} \varphi(x,\xi)\xi^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}} + 1} ds \le C\varphi(x,r) \tag{4}$$

$(x,r) \in Q \times R_+$, *then $u(x,t) \in \overset{0}{W}{}_{p,\varphi}^{2m,1}(Q)$ and*

$$\|u\|_{\overset{0}{W}{}_{p,\varphi}^{2m,1}(Q)} \le C\|f\|_{p,\varphi;Q} \tag{5}$$

with $C = C(n,p,\gamma,\partial\Omega,T,\eta_\alpha,\|a\|_{\infty;Q})$.

3. Sublinear operators generated by parabolic singular integrals in generalized Morrey spaces

Let $f \in L_1(\mathbb{R}^{n+1})$ be a function with a compact support and $a \in BMO$. For any $x \notin suppf$ define the sublinear operators T and T_a such that

$$|Tf(x)| \le c\int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho^{n+2}(x-y)} dy, \tag{6}$$

$$|T_a f(x)| \le c\int_{\mathbb{R}^{n+1}} |a(x) - a(y)| \frac{|f(y)|}{\rho^{n+2}(x-y)} dy, \tag{7}$$

This operators are bounded in $L_p(\mathbb{R}^{n+1})$ satisfy the estimates

$$\|Tf\|_{L_p} \le C\|f\|_{L_p}, \|T_a f\|_{L_p} \le C\|a\|_*\|f\|_{L_p}, \tag{8}$$

where constants independent of a and f. Let we have the Hardy operator $Hg(r) = \frac{1}{r}\int_0^r g(s)ds, r > 0$.

Theorem 3.1. *(see [12]) The inequality*

$$\operatorname*{ess\ sup}_{r>0} \omega(r)Hg(r) \le A \operatorname*{ess\ sup}_{r>0} \vartheta(r)g(r) \tag{9}$$

holds for all non-increasing functions $g : R_+ \to R_+$ if and only if

$$A = C \sup_{r>0} \frac{\omega(r)}{r} \int_0^r \frac{ds}{\underset{0<\xi<s}{\text{ess} \ \sup} \vartheta(\xi)} < \infty \tag{10}$$

Lemma 3.1. *(see [12]) Let* $f \in L_{p,loc}(\mathbb{R}^{n+1}), p \in [1, \infty)$, *be such that*

$$\int_r^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p(\mathcal{E}_s(\gamma_0))} ds < \infty \ \ \forall (x_0, r) \in \mathbb{R}^{n+1} \times R_+ \tag{11}$$

and let T be a sublinear operator satisfying (6).
i. *If* $p > 1$ *and T is bounded on* $L_p(\mathbb{R}^{n+1})$, *then*

$$\|Tf\|_{L_p(\mathcal{E}_r)(x_0)} \le c r^{\frac{n+2}{p}} \int_{2r}^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p(\mathcal{E}_s(\gamma_0))} ds \tag{12}$$

ii. *If* $p = 1$ *and T is bounded from* $L_1(\mathbb{R}^{n+1})$ *on* $WL_1(\mathbb{R}^{n+1})$, *then*

$$\|Tf\|_{WL_1(\mathcal{E}_r)(x_0)} \le c r^{n+2} \int_{2r}^\infty s^{-(n+3)} \|f\|_{L_1(\mathcal{E}_s(x_0))} ds, \tag{13}$$

where the constants are independent of r, x_0 and f.

Theorem 3.2. *(see [12]) Let* $p \in [1, \infty)$ *and* $\varphi(x, r)$ *be a measurable positive function satisfying*

$$\int_r^\infty \frac{\underset{s<\xi<\infty}{\text{ess} \ \inf} \varphi(x, \xi) \xi^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}+1}} ds \le C \varphi(x, r), \ \ \forall (x, r) \in \mathbb{R}^{n+1} \times R_+ \tag{14}$$

and let T be a sublinear operator satisfying (6).

i. *If* $p > 1$ *and T is bounded on* $L_p(\mathbb{R}^{n+1})$, *then T is bounded on* $M_{p,\varphi}(\mathbb{R}^{n+1})$, *and*

$$\|Tf\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} \le C \|f\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} \tag{15}$$

ii. *If* $p = 1$ *and T is bounded from* $L_1(\mathbb{R}^{n+1})$ *to* $WL_1(\mathbb{R}^{n+1})$, *then it is bounded from* $M_{1,\varphi}(\mathbb{R}^{n+1})$ *to* $WM_{1,\varphi}(\mathbb{R}^{n+1})$, *and*

$$\|Tf\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} \le C \|f\|_{M_{1,\varphi}(\mathbb{R}^{n+1})} \tag{16}$$

with constants independent of f.

Our next step is to show boundedness of T_a in $M_{p,\varphi}(\mathbb{R}^{n+1})$. For this we recall some properties of the *BMO* functions.

Lemma 3.2. *John-Nirenberg lemma [[12], Lemma 2.8]. Let* $a \in BMO$ *and* $p \in [1, \infty)$. *Then, for any* $\mathcal{E}_r$,

$$\left(\frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} |a(y) - a_{\mathcal{E}_r}|^p dy \right)^{\frac{1}{p}} \le c(p) \|a\|_*.$$

As an immediate consequence of (7) we get the following property.

Corollary 3.1. *Let* $a \in BMO$. *Then, for all* $0 < 2r < s$,

$$|a_{\mathcal{E}_r} - a_{\mathcal{E}_s}| \le C(n)\left(1 + \ln \frac{s}{r}\right) \cdot \|a\|_* \tag{17}$$

Now we estimate the norm of T_a.

Lemma 3.3. *(see [12]) Let $a \in BMO$. and T_a be a bounded operator in $L_p(\mathbb{R}^{n+1}), p \in (1, \infty)$, satisfying (7) and (8). Suppose that, for any $f \in L_{p,loc}(\mathbb{R}^{n+1})$,*

$$\int_r^\infty \left(1 + \ln \frac{s}{r}\right) \|f\|_{L_p(\mathcal{E}_s(x_0))} \frac{ds}{s^{\frac{n+2}{p}+1}} < \infty \ \forall (x_0, r) \in \mathbb{R}^{n+1} \times R^+ \tag{18}$$

Then,

$$\|T_a f\|_{L_p(\mathcal{E}_s(x_0))} \le c \cdot \|a\|_* r^{\frac{n+2}{p}} \int_{2r}^\infty \left(1 + \ln \frac{s}{r}\right) \|f\|_{L_p(\mathcal{E}_s(x_0))} \frac{ds}{s^{\frac{n+2}{p}+1}} \tag{19}$$

where C is independent of a, f, x_0 and r.

Theorem 3.3. *Let $p \in (1, \infty)$ and $\varphi(x, r)$ be measurable positive functions such that*

$$\int_r^\infty \left(1 + \ln \frac{s}{r}\right) \frac{\operatorname*{ess\ inf}_{s<\xi<\infty} \varphi(x, \xi) \xi^{\frac{n+2}{p}}}{s^{\frac{(n+2+p)}{p}}} ds \le C \varphi(x, r) \tag{20}$$

for $\forall (x, r) \in \mathbb{R}^{n+1} \times R_+$, where C is independent of x and r. Suppose that $a \in BMO$ and let T_a be a sublinear operator satisfying (7). If T_a is bounded in $L_p(\mathbb{R}^{n+1})$, then bounded in $M_{p,\varphi}(\mathbb{R}^{n+1})$, and

$$\|T_a f\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} \le C \|a\|_* \cdot \|f\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} \tag{21}$$

constant C independent of a and f.

Then basic results of the theorem follows by Lemma 3.3 and Theorem 3.1 in the same manner as for Theorem 3.2. For example the functions $\varphi(x, r) = r^{\beta - \frac{n+2}{p}}$, $\varphi(x, r) = r^{\beta - \frac{n+2}{p}} \cdot \log^m(l + r)$ with $0 < \beta < \frac{n+2}{p}$ and $m \ge 1$, are weight functions satisfying the condition (20).

4. Non-singular integrals in generalized Morrey spaces

Let $x \in D_+^{n+1}$, define $\overline{x} = (x'', -x_n, t) \in D_-^{n+1}$ and $x^0 = (x'', 0, 0) \in R^{n-1}$. Consider the semi-ellipsoids $\mathcal{E}_r^+(x^0) = \mathcal{E}_r^+(x^0) \cap D_-^{n+1}$. Let $f \in L_1(D_+^{n+1})$, $a \in BMO(D_+^{n+1})$, and $\overline{T}, \overline{T}_a$ be sublinear operators such that

$$|\overline{T} f(x)| \le C \int_{D_+^{n+1}} \frac{|f(y)|}{\rho(\overline{x} - y)^{n+2}} dy \tag{22}$$

$$|\overline{T}_a f(x)| \le C \int_{D_+^{n+1}} |a(x) - a(y)| \frac{|f(y)|}{\rho(\overline{x} - y)^{n+2}} dy \tag{23}$$

Let both the operators be bounded in $L_p(D_+^{n+1})$, satisfy the estimates

$$\|\overline{T} f\|_{L_p(D_+^{n+1})} \le C \|f\|_{L_p(D_+^{n+1})}, \ \|\overline{T}_a f\|_{L_p(D_+^{n+1})} \le C \|a\|_* \|f\|_{L_p(D_+^{n+1})} \tag{24}$$

constants C independent of a and f.

The following results hold, which can be proved in the some manner as in Section 3 (see [12]).

Lemma 4.1. *Let* $f \in L_{p,loc}\left(D_+^{n+1}\right)$, $p \in (1,\infty)$ *and for all* $(x^0, r) \in R^{n-1} \times R_+$

$$\int_r^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} ds < \infty. \tag{25}$$

If $\overline{T}$ *is bounded on* $L_p\left(D_+^{n+1}\right)$, *then*

$$\|\overline{T}f\|_{L_p\left(\mathcal{E}_r^+(x^0)\right)} \le c r^{\frac{n+2}{p}} \int_{2r}^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} ds, \tag{26}$$

where the constant c is independent of r, x^0 and f.

Theorem 4.1. *Suppose* φ *be a weight function satisfying* (14), *and let* $\overline{T}$ *be a sublinear operator satisfying* (22) *and* (24). *Then* $\overline{T}$ *is bounded in* $M_{p,\varphi}\left(D_+^{n+1}\right)$, $p \in (1,\infty)$ *and*

$$\|\overline{T}f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} \le C \|f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} ds, \tag{27}$$

with a constant c independent of f.

Lemma 4.2. *Let* $p \in (1,\infty)$, $a \in BMO\left(D_+^{n+1}\right)$ *and* $\overline{T}_a$ *satisfy* (23) *and* (24). *Suppose that, for all* $f \in L_{p,loc}\left(D_+^{n+1}\right)$,

$$\int_r^\infty \left(1 + \ln \frac{s}{r}\right) \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} s^{-\frac{n+2}{p}-1} ds < \infty, \ \forall (x^0, r) \in \mathbb{R}^{n+1} \times R_+. \tag{28}$$

Then

$$\|\overline{T}_a f\|_{L_p\left(\mathcal{E}_r^+(x^0)\right)} \le C \|a\|_* r^{\frac{n+2}{p}} \int_{2r}^\infty \left(1 + \ln \frac{s}{r}\right) \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} \frac{ds}{s^{\frac{n+2}{p}+1}}$$

with a constant c independent of a, f, x^0 and r.

Theorem 4.2. *Let* $p \in (1,\infty)$, $a \in BMO\left(D_+^{n+1}\right)$, *let* $\varphi(x^0, r)$ *be a weight function satisfying* (20) *and* $\overline{T}_a$ *be a sublinear operator satisfying* (7), (8). *Then sublinear operator* $\overline{T}_a$ *is bounded in* $M_{p,\varphi}\left(D_+^{n+1}\right)$ *and*

$$\|\overline{T}_a f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} \le C \|a\|_* \|f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} \tag{29}$$

constant c independent of a and f.

5. Singular and non-singular integrals in generalized Morrey spaces

We apply the above results to Calderon-Zygmund-type operators with parabolic kernel. Since these operators are sublinear and bounded in $L_p\left(\mathbb{R}^{n+1}\right)$, their continuity in $M_{p,\varphi}$ follows immediately. We are called a parabolic Calderon-Zygmund kernel if the following a measurable function $K(x,\xi) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \backslash \{0\} \to R$.

1. $K(x,\cdot)$ is a parabolic Calderon-Zygmund kernel for a.e. $x \in \mathbb{R}^{n+1}$:

   1_a. $K(x,\cdot) \in C^\infty\left(\mathbb{R}^{n+1}\right) \backslash \{0\}$,

   1_b. $K(x, (\mu\xi', \mu^2 s)) = \mu^{-n-2} K(x,\xi)$ for all $\mu > 0$, $\xi = (\xi', s)$,

   1_c. $\int_{S^n} K(x,\xi) d\sigma_\xi = 0$, $\int_{S^n} |K(x,\xi)| d\sigma_\xi < +\infty$.

2. $\|D_\xi^\beta K\|_{L_\infty\left(\mathbb{R}^{n+1}\times S^n\right)} \le M(\beta) < \infty$ for every multi-index β.

Moreover,

$$|K(x, x-y)| \le \rho(x-y)^{-n-2} |K\left(x, \left(\frac{x'-y'}{\rho(x-y)}, \frac{t-\tau}{\rho^2(x-y)}\right)\right)| \le \frac{M}{\rho(x-y)^{n+2}},$$

which means the singular integrals

$$\mathcal{B}f(x) = PV\int_{\mathbb{R}^{n+1}} K(x, x-y) f(y) dy, \tag{30}$$

$$\mathcal{C}[a,f](x) = PV\int_{\mathbb{R}^{n+1}} K(x, x-y)[a(y) - a(x)] f(y) dy$$

are sublinear and bounded in $L_p\left(\mathbb{R}^{n+1}\right)$ according to the results in [1, 13].

Theorem 5.1. *Let $f \in M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)m$ then there exist constants c depending on n, p and the kernel such that*

$$\|\mathcal{B}f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)}, \tag{31}$$

$$\|\mathcal{C}[a,f]\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|a\|_* \|f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)}.$$

Corollary 5.1. *For any cylinder Q in $\mathbb{R}_+^{n+1}$, $f \in M_{p,\varphi}(Q)$, $a \in BMO(Q)$ and $K(x,\xi) : Q \times \mathbb{R}_+^{n+1} \backslash \{0\} \to R$. Then the operators (30) are bounded in $M_{p,\varphi}(Q)$ and*

$$\|\mathcal{B}f\|_{M_{p,\varphi}(Q)} \le C\|f\|_{M_{p,\varphi}(Q)}, \;\; \|\mathcal{C}[a,f]\|_{M_{p,\varphi}(Q)} \le C\|a\|_* \|f\|_{M_{p,\varphi}(Q)}. \tag{32}$$

constant c independent of a and f.

We define the extensions

$$\overline{K}(x,\xi) = \begin{cases} K(x,\xi), & (x,\xi) \in Q \times R_+^{n+1} \backslash \{0\} \\ 0, & \text{elsewhere} \end{cases}, \; \bar{f}(x) = \begin{cases} f(x), & x \in Q \\ 0, & x \notin Q \end{cases}$$

and then the singular integral satisfying inequalities

$$|\mathcal{B}f(x)| \le |\overline{\mathcal{B}}f(x)| \le C\int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho(x-y)^{n+2}} dy$$

and

$$\|\mathcal{B}f\|_{M_{p,\varphi}(Q)} \le \|\overline{\mathcal{B}}f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|\bar{f}\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} = C\|f\|_{M_{p,\varphi}(Q)}.$$

Corollary 5.2. *Let $a \in VMO$. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon, \eta_a)$ such that for any $\mathcal{E}_r(x_0)$ with a radius $r \in (0, r_0)$ and all $f \in M_{p,\varphi}(\mathcal{E}_r(x_0))$*

$$\|\mathcal{C}[a,f]\|_{M_{p,\varphi}(\mathcal{E}_r(x_0))} \le C\varepsilon\|f\|_{M_{p,\varphi}(\mathcal{E}_r(x_0))}, \tag{33}$$

where c is independent of $\mathcal{E}, f, r,$ and x_0.

For the proof of corollary see [12].

For any $x' \in R_+^n$ and any fixed $t > 0$, define the generalized reflexion

$$\tau(x) = (\tau'(x), t), \ \tau'(x) = x' - 2x_n \frac{a^n_{\alpha\beta(x',t)}}{a^{nn}_{\alpha\beta(x',t)}}, \tag{34}$$

where $|\alpha| \le m, \ |\beta| \le m, \ a^n_{\alpha\beta}(x)$ is the last row of the coefficients matrix $a(x) = \left(a_{\alpha\beta}(x)\right)$ of (1). The function $\tau'(x)$ maps R^n_+ into R^n_-, and the kernel $K(x, \tau(x) - y) = K(x, \tau'(x) - y', t - \tau)$ is non-singular for any $x, y \in D^{n+1}_+$. Taking $\overline{x} \in D^{n+1}_+$, there exists positive constants K_1 and K_2 such that

$$K_1 \rho(\overline{x} - y) \le \rho(\tau(x) - y) \le K_2 \rho(\overline{x} - y). \tag{35}$$

Let $f \in M_{p,\varphi}\left(D^{n+1}_+\right)$, $a \in BMO\left(D^{n+1}_+\right)$ define the non-singular integral operators

$$\begin{aligned} \overline{\mathcal{B}}f(x) &= \int_{D^{n+1}_+} K(x, \tau(x) - y) f(y) dy, \\ \overline{\mathcal{C}}[a,f](x) &= \int_{D^{n+1}_+} K(x, \tau(x) - y)[a(y) - a(x)] f(y) dy. \end{aligned} \tag{36}$$

Since $K(x, \tau(x) - y)$ is still homogeneous and satisfies 1_b, we have

$$|K(x, \tau(x) - y)| \le \frac{M}{\rho(\tau(x) - y)^{n+2}} \le \frac{C}{\rho(\overline{x} - y)^{n+2}}.$$

Hence, the operators (36) are sublinear and bounded in $L_p\left(D^{n+1}_+\right)$, $p \in (1, \infty)$. From section 4 the following results are obtained.

Theorem 5.2. *Let* $a \in BMO\left(D^{n+1}_+\right)$ *and* $f \in M_{p,\varphi}\left(D^{n+1}_+\right)$ *with* (p, φ) *as in* (8) *Then the non-singular operators are continuous in* $M_{p,\varphi}\left(D^{n+1}_+\right)$ *and*

$$\begin{aligned} \|\overline{\mathcal{B}}f(x)\|_{M_{p,\varphi}(D^{n+1}} &\le C\|f\|_{D^{n+1}_+}, \\ \|\overline{\mathcal{C}}[a,f](x)\|_{M_{p,\varphi}(D^{n+1}} &\le C\|a\|_* \|f\|_{D^{n+1}_+} \end{aligned} \tag{37}$$

constant C independent of a and f.

Corollary 5.3. *For any* $a \in VMO$. *Then there exists a positive number* $r_0 = r_0(\varepsilon, \varphi_a)$ *such that for any* $\mathcal{E}_r(x_0)$ *with a radius* $r \in (0, r_0)$ *and all* $\|f\|_{M_{p,\varphi}\left(\mathcal{E}^+_r(x^0)\right)}$

$$\|\mathcal{C}[a,f]\|_{M_{p,\varphi}\left(\mathcal{E}^+_r(x^0)\right)} \le C\varepsilon \|f\|_{M_{p,\varphi}\left(\mathcal{E}^+_r(x^0)\right)} \tag{38}$$

where C is independent of $\mathcal{E}, f, r$ and x^0, $\varepsilon > 0$.

6. Proof of the first main result

Now using boundedness of singular integral of Calderon-Zygmund operators in generalized Morrey spaces we will get interval estimates for solutions of problem (1), (2) with coefficients from VMO spaces.

Let Ω to be open bounded domain in $R^n, n \ge 3$ and we suppose that its boundary is sufficiently smoothness.

Let coefficients $a_{\alpha\beta}(x)$, $|\alpha|, |\beta| \le m$ are symmetric and satisfying to the condition uniform ellipticity, essential boundedness of the coefficient $a_{\alpha\beta}(x) \in L_\infty(Q)$ and regularity $a_{\alpha\beta}(x) \in VMO(Q)$. Let $f \in M_{p,\varphi}(Q)$, (p, φ) as in (8) Since $M_{p,\varphi}(Q)$ is a proper subset of $L_p(Q)$, (1) and (2) is uniquely solvable and the solution $u(x)$ belongs at least to $W_p^{2m,1}(Q)$. Our aim is to show that this solution also belong to $\overset{0}{W}{}_{p,\varphi}^{2m,1}(Q)$. For this we need an a priori estimate of u, which we prove in two steps. Before we give interior estimate. For any $x_0 \in R_+^{n+1}$ define the parabolic semi-cylinders $C_r(x_0) = B_r(x_0') \times (t_0 - r^2, t_0)$. Let $\vartheta \in C_0^\infty(C_r)$ and suppose that $\vartheta(x,t) = 0$, for $t \le 0$. According to [1, 7, 16], for any $x \in supp\,\vartheta$ the following representation formula for the higher derivatives of ϑ holds true if $u \in \overset{0}{W}{}_p^{2m}(Q)$

$$\begin{aligned} D^{|\alpha|}u(x) &= P.V.\int_{\mathbb{R}^{n+1}} D^{|\alpha|}\Gamma(x, x-y)\left[\sum_{|\alpha|,\,|\beta| \le 2m} \left(a_{\alpha\beta}(x) - a_{\alpha\beta}(y)\right)D^{\alpha,\beta}\vartheta(y)\right]dy \\ &+P.V.\int_{\mathbb{R}^{n+1}} D^{|\alpha|}\Gamma(x, x-y)L\vartheta(y)dy + L\vartheta(x)\int_{S^n} D^{|\beta|}\Gamma(x,y)\nu_i d\sigma_y \end{aligned} \tag{39}$$

where $\nu = (\nu_1, \ldots, \nu_{n+1})$ is the outward normal to S^n. Here, $\Gamma(x,\xi)$ is the fundamental solution of the operator L. $\Gamma(x,t)$ can be represented in form

$$\Gamma(x,\xi) = \frac{1}{(n-2)\omega_n\left(deta_{\alpha\beta}\right)^{\frac{1}{2}}}\left(\sum_{i,j=1}^{n} A_{\alpha\beta}(x)\xi_i\xi_j\right)^{\frac{2-n}{2}}$$

for a.e. $x \in \mathbb{R}^{n+1}$ and $\forall \xi \in R^n \backslash \{0\}$, where $\left(A_{\alpha\beta}\right)_{n\times n}$ is inverse matrix for $\left(a_{\alpha\beta}\right)_{n\times n}$. Since any function $\vartheta \in W_p^{2m,1}(Q)$ can be approximated by C_0^∞ functions, the representation formula (39) still holds for any $\vartheta \in W_p^{2m,1}(C_r(x_0))$. The properties of the fundamental solution (see [7, 17]) imply that $D^{|\alpha|}\Gamma(x,y)$ are variable Calderon-Zygmund kernels in the sense of our definition above. By notation above, we can write

$$D^{\alpha,\beta}\vartheta(x) = D^{\alpha,\beta}C\left[a_{\alpha,\beta}, \vartheta\right](x) + D^{\alpha,\beta}B(L\vartheta)(x) + L\vartheta(x)\int_{S^n} D^{\alpha}\Gamma(x,y)\nu_i d\sigma y. \qquad |\alpha|, |\beta| \le m. \tag{40}$$

The operators $D^{\alpha,\beta}B$ and $D^{\alpha,\beta}C$ are defined by (30) with $K(x, x-y) = D^{\alpha,\beta}\Gamma(x, x-y)$. Due to (30) and (31) and the equivalence of the metrics we obtain for $\mathcal{E} > 0$ there exists $r_0(\mathcal{E})$ such that for any $r < r_0(\mathcal{E})$

$$\|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_r(x_0))} \le C\left(\|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_r(x_0))} + \|L\vartheta\|_{M_{p,\varphi}(C_r(x_0))}\right) \tag{41}$$

for some r small enough. From (41) we get that

$$\|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_r(x_0))} \le C(n,p,\varphi_\alpha)\cdot\|D^{\alpha,\beta}\Gamma\|_{L_\infty(Q)})\|L\vartheta\|_{M_{p,\varphi}(C_r(x_0))}.$$

Define a cut-off function $\psi(x) = \psi_1(x')\psi_2(t)$, with $\psi_1 \in C_0^\infty\left(B_r(x_0')\right), \psi_2 \in C_0^\infty(R)$ such that

$$\psi_1(x') = \begin{matrix} 1, & x' \in B_{\theta_r}(x_0') \\ 0, & x' \notin B_{\theta_r'}(x_0'), \end{matrix}$$

$$\psi_2(t) = \begin{matrix} 1, & t \in \left(t_0 - (\theta r)^2, t_0\right] \\ 0, & t < (t_0 - (\theta\prime r)^2 \end{matrix}$$

with $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > 0$ and $|D^\alpha \psi| \le C[\theta(1-\theta)r]^{-\alpha}$, $|\alpha| \le 2m$, $|\psi_t| \sim |D^\alpha \psi|$. For any solution $u \in W_p^{2m,1}(Q)$ of (1) and (2) define $\vartheta(x) = \varphi(x)u(x) \in W_p^{2m,1}(C_r)$. Hence,

$$\begin{aligned} &\|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_{\theta' r}(x_0))} \\ &\le C\|L\vartheta\|_{M_{p,\varphi}(C_{\theta' r}(x_0))} \le C\|f\|_{M_{p,\varphi}(C_{\theta' r}(x_0))} + \frac{\|D^\alpha u\|_{M_{p,\varphi}(C_{\theta' r}(x_0))}}{\theta(1-\theta)r} + \frac{\|u\|_{M_{p,\varphi}(C_{\theta' r}(x_0))}}{[\theta(1-\theta)r]^2}. \end{aligned}$$

As so,

$$\begin{aligned} &[\theta(1-\theta)r]^2 \|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \\ &\le C\left(r^2\|f\|_{M_{p,\varphi}(Q)}\right) + \theta'(1-\theta')r\|D^\alpha u\|_{M_{p,\varphi}(C_{\theta'\cdot r}(x_0))} + \|u\|_{M_{p,\varphi}(C_{\theta'\cdot r}(x_0))}. \end{aligned}$$

We introduce

$$\theta_\alpha = \sup_{0<\theta<1} [\theta(1-\theta)r]^\alpha \|D^\alpha u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))}, \quad |\alpha| \le 2m,$$

the above inequality becomes

$$[\theta(1-\theta)r]^2 \cdot \|D^\alpha u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \theta_{2m} \le C\left(r^2\|f\|_{M_{p,\varphi}(Q)+\theta_m+\theta_0}\right) \tag{42}$$

Now we use following interpolation inequality (see [5])

$$\theta_m \le \varepsilon \cdot \theta_{2m} + \frac{c}{\varepsilon}\theta_0 \ \ for \ \ any \ \ \varepsilon \in (0, 2m).$$

where there exists a positive constant C independent of r. Thus (42) becomes

$$[\theta(1-\theta)r]^2\|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \theta_{2m} \le C\left(r^2 + \theta_0\right), \ \forall\theta \in (0,1).$$

Taking $\theta = \frac{1}{2}$ we obtain the Caccioppoli-type estimate

$$\|D^{\alpha,\beta}u\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} \le C\left(\|f\|_{M_{p,\varphi}(Q)} + \frac{1}{r^2}\|u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))}\right)$$

We get the boundedness of the coefficients

$$\begin{aligned} &\|u_t\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} \le \|a\|_{L_\infty(Q)} \cdot \|D^{\alpha,\beta}u\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} + \\ &+ \|f\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} \le C\left(\|f\|_{M_{p,\varphi}(Q)+\frac{1}{r_2}}\|u\|_{M_{p,\varphi}(C_r(x_0))}\right). \end{aligned}$$

Let $Q' = \Omega' \times (0,T)$ and $Q'' = \Omega'' \times (0,T)$ the cylinders with $\Omega' \Subset \Omega'' \Subset \Omega$. By the standard covering procedure and partition of the unity we obtain that

$$\|u\|_{W^{2m,1}_{p,\varphi}(Q')} \le C\left(\|f\|_{M_{p,\varphi}(Q)}\right) + \|u\|_{M_{p,\varphi}(Q''))} \tag{43}$$

where C depends on $n, p, \Lambda, T, \|D\Gamma\|_{L_\infty(Q)}, \eta_\alpha, \|a\|_{L_\infty(Q)}$ and $dist(\Omega', \partial\Omega'')$. Now we give boundary estimates. For any fixed $(x^0, r) \in \mathbb{R}^{n+1} \times R_+$ define the semi-cylinders

$$C_r^+(x^0) = B_r^+(x^{0\prime}) \times (0, r^2) = |x^0 - x'| < r, \; x_n > 0, \; 0 < t < r^2$$

with $S_r^+ = (x'', 0, t) : |x^0 - x''| < r, \; 0 < t < r^2$. For any solution $u \in W_p^{2m,1}(C_r^+(x^0))$ with $supp\, u \in C_r^+(x^0)$, the following boundary representation formula holds (see [1, 7, 16]).

$$D^{\alpha,\beta}u(x) = \mathcal{C}_{ij}\left[a_{\alpha,\beta}, D^{\alpha,\beta}u\right](x) + \mathcal{B}_{ij}(Lu)(x) + Lu(x)\int_{S^n} D^\alpha \Gamma \nu_i d\sigma_y - J_{ij}(x),$$

where

$$J_{ij}(x) = \mathcal{B}_{ij}(Lu)(x) + \tilde{\mathcal{C}}_{ij}\left[a_{\alpha,\beta,D^{\alpha,\beta}}u\right](x), \; i,j = 1, \dots, n-1,$$

$$J_{in}(x) = J_{ni}(x) = \sum_{i=1}^{n}\left(\frac{\partial \tau(x)}{\partial x_n}\right)^l \left[\overline{\mathcal{C}}_{il}\left[a_{\alpha,\beta,D^{\alpha,\beta}u}\right](x) + \overline{\mathcal{B}}_{il}(Lu)(x)\right], \; i = 1, \dots, n$$

$$J_{nn}(x) = \sum_{r,l=1}^{n}\left(\frac{\partial \tau(x)}{\partial x_n}\right)^r \left(\frac{\partial \tau(x)}{\partial x_n}\right)^l \left[\overline{\mathcal{C}}_{il}\left[c, D^{\alpha,\beta}u\right](x) + \overline{\mathcal{B}}_{il}(Lu)(x)\right],$$

$$\left(\frac{\partial \tau(x)}{\partial x_n}\right) = \left(-2\frac{a_{\alpha,\beta}^{n1}(x)}{a_{\alpha,\beta}^{nn}(x)}, \dots, -2\frac{a_{\alpha,\beta}^{nn-1}(x)}{a_{\alpha,\beta}^{nn}(x)}, -1, 0\right).$$

Here $\overline{B}_{ij}$ and $\overline{C}_{ij}$ are non-singular operators defined by (36) with a kernel $K(x, \tau(x) - y) = D^{\alpha,\beta}\Gamma(x, \tau(x) - y)$. Applying the estimates (37) and (38) and having in mind that the components of the vector $\left(\frac{\partial \tau(x)}{\partial x_n}\right)$ are bounded, we obtain that

$$\|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_r(x_0))} \le C\left(\|f\|_{M_{p,\varphi}(Q)} + r^2\|u\|_{M_{p,\varphi}(C_r(x_0))}\right)$$

Taking r small enough we can move the norm of u on the left-hand side, obtaining that

$$\|u\|_{M_{p,\varphi}(C_r(x_0))} \le C\|f\|_{M_{p,\varphi}(Q)}$$

with a constant C depending on $n, p, \Lambda, T, \eta_\alpha, \|a\|_{L_\infty(Q)}$. By covering the boundary with small cylinders, using a partition of the unit subordinated by that covering and local flattening of $\partial\Omega$ we get that

$$\|u\|_{W_{p,\varphi}^{2m,1}(Q \setminus Q')} \le C\|f\|_{M_{p,\varphi}(Q)} \tag{44}$$

Using (43) and (44), we obtain (5).

7. The higher order elliptic equations in unbounded domains

Now we are consider boundary value the Dirichlet problem for higher order nondivergence uniformly elliptic equations with coefficients in modified Morrey spaces in unbounded domains Ω

$$\begin{aligned} Lu &= \sum_{|\alpha| \le |\beta| \le m} a_{\alpha,\beta}D^{\alpha,\beta}u = f(x) \text{ in } \Omega \\ D^\alpha u &= g(x) \; |\alpha| \le m - 1 \text{ on } \partial\Omega \end{aligned} \tag{45}$$

where the coefficients matrix $a(x) = \left\{a^{ij}_{\alpha,\beta}(x)\right\}^{n}_{i,j=1}$ satisfies

$$\exists\Lambda > 0 \ \Lambda \sum_{|\alpha|=m} \xi^2_\alpha \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}\xi_\alpha\xi_\beta, \tag{46}$$

for a.e. $x \in \Omega$, $\forall \xi \in R^n$, $a_{\alpha,\beta} = a_{\beta,\alpha}, \xi = \left\{\xi_\alpha, \||\alpha| = m \in R^N\right\}$, N–number different multiindeks with length equal to m.

Under these assumptions we prove that the maximal operator M are bounded from the modified Morrey space $\tilde{L}_{p,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$ if and only if,

$$\frac{\alpha}{n} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n-\lambda}.$$

For $x \in R^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t) = R^n \backslash B(x,t)$. One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} \Big| B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \ \ 0 \leq \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball B(x,t). The fractional maximal function $M_\alpha f$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$.

Let $1 \leq p < \infty, \ 0 \leq \lambda \leq n, \ [t]_1 = \min\{1,t\}$. We denote by $\tilde{L}_{p,\lambda}(R^n)$ the modified Morrey space, as the set of locally integrable functions $f(x), \ x \in R^n,$ with the finite norm

$$\|f\|_{\tilde{L}_{p,\lambda}} = \sup_{x \in R^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dy\right)^{\frac{1}{p}}$$

Note that

$$\tilde{L}_{p,0}(R^n) = L_{p,0}(R^n) = L_p(R^n),$$
$$\tilde{L}_{p,\lambda}(R^n) \hookrightarrow L_{p,\lambda}(R^n) \cap L_p(R^n) \ \text{and} \ \max\left\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\right\} \leq \|f\|_{\tilde{L}_{p,\lambda}},$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(R^n) = \tilde{L}_{p,\lambda}(R^n) = \theta$, where θ is the set of all functions equivalent to 0 on R^n. $W\tilde{L}_{p,\lambda}(R^n)$-the modified weak Morrey space as the set of locally integrable functions $f(x), \ x \in R^n$ with finite norm

$$\|f\|_{W\tilde{L}_{p,\lambda}} = \sup_{r>0} r \sup_{x \in R^n, t>0} \left([t]_1^{-\lambda} |\{y \in B(x,t) : |f(y)| > r\}|\right)^{\frac{1}{p}}.$$

Note that

$$W\tilde{L}_{p,0}(R^n) = WL_{p,0}(R^n) = WL_p(R^n),$$
$$\tilde{L}_{p,\lambda}(R^n) \subset W\tilde{L}_{p,\lambda}(R^n) \ \text{and} \ \|f\|_{W\tilde{L}_{p,\lambda}} \leq \|f\|_{\tilde{L}_{p,\lambda}}.$$

We study the $\tilde{L}_{p,\lambda}$-boundedness of the maximal operator M.

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then the Riesz potential I_α is bounded from $L_p(R^n)$ to $L_q(R^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(R^n)$ to $WL_q(R^n)$ if and only if $\alpha = n\left(1 - \frac{1}{q}\right)$. D.R. Adams studied the boundedness of the I_α in Morrey spaces and proved the follows statement.

Theorem **(Adams)** Let $0 < \alpha < n$ and $0 \le \lambda < n - \alpha$, $1 \le p < \frac{n-\lambda}{\alpha}$.

1. If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{p,\lambda}(R^n)$ to $L_{q,\lambda}(R^n)$.

2. If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{1,\lambda}(R^n)$ to $WL_{q,\lambda}(R^n)$.

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of Theorem reduced to the aforementioned result by Hardy-Littlewood-Sobolev Theorem also implies the boundedness of the fractional maximal operator M_α.

In this section we study the fractional maximal integral and the Riesz potential in the modified Morrey space. In the case $p = 1$ we prove that the operator I_α is bounded from $\tilde{L}_{1,\lambda}(R^n)$ to $W\tilde{L}_{q,\lambda}(R^n)$ if and only if, $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. In the case $1 < p < \frac{n-\lambda}{\alpha}$ we prove that the operator I_α is bounded from $\tilde{L}_{p,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$ if and only if, $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$.

Theorem 7.1. *If* $f \in \tilde{L}_{p,\lambda}(R^n),\ 1 < p < \infty,\ 0 \le \lambda < n,$ *then* $Mf \in \tilde{L}_{p,\lambda}(R^n)$ *and*

$$\|Mf\|_{\tilde{L}_{p,\lambda}} \le C_{p,\lambda} \|f\|_{\tilde{L}_{p,\lambda}},$$

where $C_{p,\lambda}$ depends only on p, λ and n.

Proof. We use Fefferman-Stein inequality and get

$$\int_{B(x,t)} (Mf(y))^p dy \le C \int_{R^n} |f(y)|^p M_{\chi_{B(x,t)}}(y) dy.$$

Later from some estimates for $M_{\chi_{B(x,t)}}$ we have the following inequalities

$$\int_{B(x,t)} (Mf(y))^p dy \le C \left(\int_{B(x,t)} |f(y)|^p dy + \right.$$
$$\left. + \sum_{j=0}^{\infty} \int_{B\left(x,2^{j+1}t\right) \setminus B\left(x,2^j t\right)} \frac{t^n |f(y)|^p dy}{(|x-y|+t)^n} \right) \le C\,[t]_1^\lambda \cdot \|f\|_{\tilde{L}_{p,\lambda}}^p.$$

□

Theorem 7.2. *(see [18]) Let* $0 < \alpha < n,\ 0 \le \lambda < n - \alpha$ *and* $1 \le p < \frac{n-\lambda}{\alpha}$.

1. If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the Riesz potential operator I_α from $\tilde{L}_{p,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$.

2. If $p = 1 < \frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $\tilde{L}_{1,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$.

Recall that, for $0 < \alpha < n$

$$M_\alpha f(x) \le \nu_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x)$$

where ν_n is the volume of the unit ball in R^n. From [7] for unbounded domains $\Omega \subset R^n$ we have following result.

Theorem 7.3. *Let $\Omega \subset R^n$ be an unbounded domains with noncompact boundary $\partial\Omega$, and $0 < \alpha < n$, $0 \le \lambda < n - \alpha$ and $1 < p < \frac{n-\lambda}{\alpha}$. Also let satisfies conditions $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$, $f \in \tilde{L}_{q,\lambda}(\Omega)$, function $U(x)$ is a solution of problem* (45). *Then there is exist constant C which dependent only at n, λ, p, q, Ω such that*

$$\|U\|_{\tilde{W}^{2m}_{p,\lambda}}(\Omega) \le C\|f\|_{\tilde{L}_{q,\lambda}(\Omega)}, \tag{47}$$

where $\tilde{W}^{2m}_{p,\lambda}$-is correspondingly modified Sobolev-Morrey space.

The proved Theorem 7.3 consequence from methods of [7] and Theorem 7.1 and 7.2.

Additional classifications

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