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# Gradient Optimal Control of the Bilinear Reaction–Diffusion Equation

*El Hassan Zerrik and Abderrahman Ait Aadi*

## Abstract

In this chapter, we study a problem of gradient optimal control for a bilinear reaction–diffusion equation evolving in a spatial domain  $\Omega \subset \mathbb{R}^n$  using distributed and bounded controls. Then, we minimize a functional constituted of the deviation between the desired gradient and the reached one and the energy term. We prove the existence of an optimal control solution of the minimization problem. Then this control is characterized as solution to an optimality system. Moreover, we discuss two special cases of controls: the ones are time dependent, and the others are space dependent. A numerical approach is given and successfully illustrated by simulations.

**Keywords:** distributed bilinear systems, reaction–diffusion equation, controllability, optimal control

## 1. Introduction

The controllability of distributed bilinear systems governed by partial differential equations has been studied by many authors: in [1], the authors developed the weak controllability of the beam and rod equations in the mono-dimensional case. In [2], the author considered the controllability of semilinear parabolic and hyperbolic systems using distributed controls. In [3], the author studied the exact controllability of the semilinear wave equations in one space dimension. The optimal control problem for a class of infinite dimensional bilinear systems have been considered in many works. In [4], the author proved the existence and characterization of an optimal control of a bilinear convective-diffusive fluid model using bounded controls. In [5], the author developed optimal control problem of a bilinear heat equation with distributed bounded control. In [6], the authors studied optimal control for a class of bilinear systems using unbounded control. In [7], the authors considered the optimal control problem of the wave equation using bounded boundary control. In [8], the authors considered the optimal control problem of the Kirchhoff plate equation with distributed bounded controls. In [9], the author proved the optimal control of the bilinear wave equation using distributed and bounded controls. The regional optimal control problem of a class of infinite dimensional bilinear systems with unbounded controls was developed in [10], then the authors studied the existence and characterization of an optimal control.

In [11], the authors studied the constrained regional optimal control of a bilinear plate equation using distributed and bounded controls. The notion of gradient controllability is very important, since its close to real applications and there exist systems that cannot be controllable but gradient of the state is controllable. Then in [12], the authors proved the regional controllability of parabolic systems using HUM method.

In the present work, we study the gradient optimal control problem of the bilinear reaction–diffusion equation using distributed and bounded controls. Then, we examine the existence and we give characterization of an optimal control. Also, an algorithm and simulations are given. Let  $\Omega$  be an open bounded domain of  $\mathbb{R}^n$ , ( $n \geq 1$ ) with a  $C^2$  boundary  $\partial\Omega$ , we denote by  $Q = \Omega \times (0, T)$  and  $\Sigma = \partial\Omega \times (0, T)$ , and we consider the bilinear reaction–diffusion equation

$$\begin{cases} y_t(x, t) - \Delta y(x, t) = u(x, t)y(x, t) & \text{in } Q \\ y(x, 0) = y_0(x) & \text{in } \Omega \\ y(x, t) = 0 & \text{on } \Sigma, \end{cases} \quad (1)$$

where  $u \in \mathcal{U}_\rho := \{u \in L^\infty(Q) \mid -\rho \leq u \leq \rho \text{ a.e. in } Q\}$  is a scalar control function, and  $\rho$  is a positive constant.

Let us consider the following state space

$$\mathcal{H} := L^2(0, T; H_0^1(\Omega)).$$

For all  $y_0 \in H_0^1(\Omega)$  and  $u \in \mathcal{U}_\rho$ , the system (1) has a unique weak solution  $y \in \mathcal{H}$  (see for example [13, 14]).

Define the operator

$$\begin{aligned} \nabla : H_0^1(\Omega) &\rightarrow (L^2(\Omega))^n \\ y &\rightarrow \nabla y = \left( \frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n} \right), \end{aligned}$$

and  $\nabla^*$  its adjoint.

Let us recall that the system (1) is weakly gradient controllable if for all  $y^d \in (L^2(\Omega))^n$  and  $\varepsilon > 0$ , there exist a control  $u \in \mathcal{U}_\rho$  such that

$$\|\nabla y(\cdot, T) - y^d(\cdot)\|_{(L^2(\Omega))^n} \leq \varepsilon,$$

where  $y^d = (y_1^d, \dots, y_n^d)$  is the gradient of the desired state in  $(L^2(\Omega))^n$ .

Our problem consists in finding a control  $u$  that steers the gradient of state close to  $y^d$ , over the time interval  $[0, T]$  with a reasonable amount of energy. This may be stated as the following minimization problem

$$\min_{u \in \mathcal{U}_\rho} J(u), \quad (2)$$

where

$$J(u) = \frac{1}{2} \int_0^T \|\nabla y(\cdot, t) - y^d(\cdot)\|_{(L^2(\Omega))^n}^2 dt + \frac{\beta}{2} \int_Q u^2(x, t) dQ, \quad (3)$$

with  $\beta > 0$ .

The rest of the paper is organized as follows: in section 2, we study the existence of an optimal control solution of (2). In section 3, we give a characterization of an optimal control solution of the problem (2), and we discuss two special cases of an optimal control solution of such problem. Finally, in section 4, we present an algorithm and simulations.

## 2. Existence of an optimal control

The main result of the existence of an optimal control solution of (2) is given by the following theorem.

**Theorem 1.** There exists an optimal control  $u^* \in \mathcal{U}_\rho$ , solution of (2).

**Proof:** Let  $u^n$  be a minimizing sequence in  $\mathcal{U}_\rho$ , such that

$$\lim_{n \rightarrow +\infty} \inf_{u \in \mathcal{U}_\rho} J(u^n) = \inf_{u \in \mathcal{U}_\rho} J(u). \quad (4)$$

Then, according to the nature of the cost function  $J$ , we can deduce that  $u^n$  is uniformly bounded in  $\mathcal{U}_\rho$ .

So, we can extract from  $u^n$  a subsequence also denoted by  $u^n$  such that  $u^n \rightharpoonup u$  weakly in  $\mathcal{U}_\rho$ .

In other hand, using the weak form of system (1), we deduce that

$$\frac{1}{2} \frac{d}{dt} \|y^n\|_{L^2(\Omega)}^2 + \int_{\Omega} \nabla y^n \nabla y^n dx = \int_{\Omega} u^n |y^n|^2 dx. \quad (5)$$

By integration with respect to time and using the function  $u^n$  is uniformly bounded in  $L^\infty(Q)$ , we have

$$\|y^n\|_{L^2(\Omega)}^2 + \int_0^t \|y^n\|_{H_0^1(\Omega)}^2 ds \leq c_1 \int_0^t \|y^n\|_{L^2(\Omega)}^2 ds, \quad (6)$$

where  $c_1$  is a positive constant.

Using Gronwall's Lemma, we deduce that  $y^n$  uniformly bounded in  $L^\infty(0, T; L^2(\Omega))$ , and then  $y^n$  uniformly bounded in  $L^2(0, T; H_0^1(\Omega))$ .

Using the previous result and system (1), we obtain that  $y_t^n$  is uniformly bounded in  $L^2(0, T; H^{-1}(\Omega))$ , and then  $y^n$  is uniformly bounded in  $\mathcal{H}$ .

Using the above bounds, we can extract a subsequence satisfying the following convergence properties

$$y^n \rightharpoonup y^* \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)) \quad (7)$$

$$y^n \rightarrow y^* \quad \text{strongly in } L^2(Q) \quad (8)$$

$$u^n \rightharpoonup u^* \quad \text{weakly in } L^2(Q). \quad (9)$$

Since  $\mathcal{U}_\rho$  is a closed and convex subset of  $L^\infty(Q) \subset L^2(Q)$ ,  $\mathcal{U}_\rho$  is weakly closed in  $L^2(Q)$ . Then  $u^* \in \mathcal{U}_\rho \subset L^2(Q)$ . On the other hand, since  $-\rho \leq u^n \leq \rho$  for all  $n$ ,  $u^n \rightharpoonup u^{**}$  weakly\* in  $L^\infty(Q)$ , and hence  $u^n \rightharpoonup u^{**}$  weakly in  $L^2(Q)$ . By the uniqueness of the weak limit, we obtain  $u^* = u^{**}$  and  $u^* \in \mathcal{U}_\rho \subset L^\infty(Q)$ .

Now, we show that  $u^n y^n \rightharpoonup u^* y^*$  weakly in  $L^2(Q)$ .

Since  $u^n y^n - u^* y^* = u^n (y^n - y^*) + (u^n - u^*) y^*$ , and using (7), (8) and (9), we obtain  $u^n y^n \rightharpoonup u^* y^*$  weakly in  $L^2(Q)$ .

Thus  $y^* = y(u^*)$  is the solution of system (1) with control  $u^*$ .

Since the functional  $J$  is lower semi-continuous with respect to weak convergence (basically Fatou's lemma), we obtain

$$\begin{aligned} J(u^*) &\leq \frac{1}{2} \liminf_{n \rightarrow +\infty} \int_0^T \|\nabla y^n(\cdot, t) - y^d(\cdot)\|_{(L^2(\Omega))^n}^2 dt + \frac{\beta}{2} \liminf_{n \rightarrow +\infty} \int_Q (u^n)^2(x, t) dx dt \\ &\leq \liminf_{n \rightarrow +\infty} J(u^n) \\ &= \inf_{u \in \mathcal{U}_\rho} J(u). \end{aligned}$$

Finally, we conclude that  $u^*$  is an optimal control.

### 3. Characterization of an optimal control

This section is devoted to characterization of an optimal control solution of the problem (2).

#### 3.1 Time and space control dependent

In this part, we give characterization of an optimal control that depend on time and space.

The following result give the differentiability of the mapping  $u \rightarrow y(u)$ .

**Lemma 1** The mapping  $u \in \mathcal{U}_\rho \rightarrow y(u) \in \mathcal{H}$  is differentiable in the following sense

$$\frac{y(u + \varepsilon h) - y(u)}{\varepsilon} \rightharpoonup \phi \text{ weakly in } \mathcal{H} \text{ as } \varepsilon \rightarrow 0, \text{ for any } u, u + \varepsilon h \in \mathcal{U}_\rho$$

Moreover,  $\phi = \phi(y, h)$  satisfies the following system

$$\begin{cases} \phi_t(x, t) - \Delta \phi(x, t) = u(x, t)\phi(x, t) + h(x, t)y(x, t) & \text{on } Q \\ \phi(x, 0) = 0 & \text{in } \Omega \\ \phi(x, t) = 0 & \text{in } \Sigma. \end{cases} \quad (10)$$

**Proof:** Consider  $y^\varepsilon = y(u + \varepsilon h)$  and  $y = y(u)$ . Then  $\left(\frac{y^\varepsilon - y}{\varepsilon}\right)$  is a weak solution of

$$\begin{cases} \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t - \Delta \left(\frac{y^\varepsilon - y}{\varepsilon}\right) = u \left(\frac{y^\varepsilon - y}{\varepsilon}\right) + h y^\varepsilon & \text{on } Q \\ \left(\frac{y^\varepsilon - y}{\varepsilon}\right)(x, 0) = 0 & \text{in } \Omega \\ \left(\frac{y^\varepsilon - y}{\varepsilon}\right)(x, t) = 0 & \text{in } \Sigma. \end{cases}$$

Using the result (6), it follows that

$$\left\| \frac{y^\varepsilon - y}{\varepsilon} \right\|_{\mathcal{H}} \leq C,$$

where  $C$  depends on the  $L^\infty$  bound on  $h$ , but is independent of  $\varepsilon$ . Hence on a subsequence, by weak compactness, we have

$$\begin{aligned}\frac{y^\varepsilon - y}{\varepsilon} &\rightharpoonup \phi \quad \text{weakly in } L^\infty([0, T]; H_0^1(\Omega)) \\ \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t &\rightharpoonup \phi_t \quad \text{weakly in } L^\infty([0, T]; H^{-1}(\Omega)).\end{aligned}$$

By the definition of weak solution, we have

$$\left\langle \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t, \psi \right\rangle - \int_\Omega \nabla \left(\frac{y^\varepsilon - y}{\varepsilon}\right) \nabla \psi dx = \int_\Omega u \left(\frac{y^\varepsilon - y}{\varepsilon}\right) \psi dx + \int_\Omega h y^\varepsilon \psi dx, \quad (11)$$

for any  $\psi \in H_0^1(\Omega)$ , and a.e  $0 \leq t \leq T$ .

Letting  $\varepsilon \rightarrow 0$  in (11), we conclude that  $\phi$  is the weak solution of system (10).

Now, we give characterization of an optimal control that depend on time and space.

**Theorem 2** An optimal control solution of problem (2) is given by the formula

$$u^*(x, t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t), \rho \right) \right), \quad (12)$$

where  $p \in \mathcal{C}([0, T]; \mathcal{H})$  is the weak solution of the adjoint system

$$\begin{cases} p_{it}(x, t) - \Delta p_i(x, t) = -u^*(x, t) p_i(x, t) & \text{on } Q \\ p_i(x, T) = \left( \frac{\partial y(T)}{\partial x_i} - y_i^d \right) & \text{in } \Omega \\ p_i(x, t) = 0 & \text{in } \Sigma. \end{cases} \quad (13)$$

**Proof:** Let  $u^* \in \mathcal{U}_\rho$  and  $y = y(u^*)$  be the corresponding weak solution, and let  $u^* + \varepsilon h \in \mathcal{U}_\rho$ , for  $\varepsilon > 0$  and  $y^\varepsilon = y(u^* + \varepsilon h)$ .

Since  $J$  reaches its minimum at  $u^*$ , then

$$\begin{aligned}0 \leq \lim_{\varepsilon \rightarrow 0^+} \frac{J(u^* + \varepsilon h) - J(u^*)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \sum_{i=1}^n \frac{1}{2} \int_\Omega \left( \int_0^T \frac{\partial \phi}{\partial x_i} \frac{\partial p_i}{\partial t} dt \right. \\ &\quad \left. + \int_0^T \left( -\Delta \frac{\partial \phi}{\partial x_i} + u \frac{\partial \phi}{\partial x_i} + h \frac{\partial y}{\partial x_i} p_i \right) dt \right) dx \\ &\quad + \lim_{\varepsilon \rightarrow 0^+} \frac{\beta}{2} \int_Q (2hu^* + \varepsilon h^2) dQ.\end{aligned}$$

Then

$$0 \leq \int_Q \beta h u dQ + \sum_{i=1}^n \int_Q h \frac{\partial y}{\partial x_i} p_i dQ = \int_Q h \left( \beta u + \sum_{i=1}^n \int_Q h \frac{\partial y}{\partial x_i} p_i \right) dQ.$$

Using a standard control argument based on the choices for the variation  $h(x, t)$ , an optimal control is given by

$$u^*(x, t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t), \rho \right) \right).$$

### 3.2 Time or space control dependent

In this subsection, we study two cases of controls: the first ones are time dependent  $u(t)$ , and the others are space dependent  $u(x)$ .

- **Case 1:**  $u = u(t)$ .

Here, we consider the admissible controls set

$$\mathcal{U}_\rho = \{u \in L^\infty(0, T) : -\rho \leq u \leq \rho \quad \text{a.e in } (0, T)\} \quad (14)$$

and we take the functional cost

$$J(u) = \frac{1}{2} \int_0^T \|\nabla y(\cdot, t) - y^d(\cdot)\|_{(L^2(\Omega))^n}^2 dt + \frac{\beta}{2} \int_0^T u^2(t) dt. \quad (15)$$

**Corollary 1** Under conditions (14) and (15), an optimal control is given by the formula

$$u(t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \int_\Omega \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dx, \rho \right) \right), \quad (16)$$

where  $y$  is the weak solution of the equation

$$\begin{cases} y_t(x, t) - \Delta y(x, t) = u(t)y(x, t) & \text{on } Q \\ y(x, 0) = y_0(x), & \text{in } \Omega \\ y(x, t) = 0 & \text{in } \Sigma, \end{cases}$$

and  $p_i$  is the weak solution of the adjoint equation

$$\begin{cases} p_{it}(x, t) - \Delta p_i(x, t) = -u^*(t)p_i(x, t) & \text{on } Q \\ p_i(x, T) = \left( \frac{\partial y(T)}{\partial x_i} - y_i^d \right) & \text{in } \Omega \\ p_i(x, t) = 0 & \text{in } \Sigma. \end{cases}$$

**Proof:** Using the same steps as in the proof of Theorem 2, let  $h = h(t)$  be an arbitrary function with  $u + \varepsilon h \in \mathcal{U}_\rho$  for small  $\varepsilon$ .

We have

$$\int_0^T h(t) \left( \int_\Omega \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dx + \beta u(t) \right) dt \geq 0.$$

By using a standard control argument concerning the sign of the variation  $h$ , we obtain

$$u(t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \int_\Omega \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dx, \rho \right) \right).$$

- **Case 2:**  $u = u(x)$ .

We consider the admissible controls set

$$\mathcal{U}_\rho = \{u \in L^\infty(\Omega) : -\rho \leq u \leq \rho \quad \text{a.e in } \Omega\} \quad (17)$$



and we take the functional cost

$$J(u) = \frac{1}{2} \int_0^T \|\nabla y(\cdot, t) - y^d(\cdot)\|_{(L^2(\Omega))^n}^2 dt + \frac{\beta}{2} \int_{\Omega} u^2(x) dx. \quad (18)$$

**Corollary 2** Under conditions (17) and (18), an optimal control satisfies

$$u(x) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \int_0^T \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dt, \rho \right) \right), \quad (19)$$

where  $y$  is the solution of system

$$\begin{cases} y_t(x, t) - \Delta y(x, t) = u(x)y(x, t) & \text{on } Q \\ y(x, 0) = y_0(x), & \text{in } \Omega \\ y(x, t) = 0 & \text{in } \Sigma, \end{cases}$$

and  $p_i$  is the solution of system

$$\begin{cases} p_{i_t}(x, t) - \Delta p_i(x, t) = -u^*(x, t)p_i(x, t) & \text{on } Q \\ p_i(x, T) = \left( \frac{\partial y(T)}{\partial x_i} - y_i^d \right) & \text{in } \Omega \\ p_i(x, t) = 0 & \text{in } \Sigma. \end{cases}$$

**Proof:** Using the same notations as in the proof of Theorem 2, let  $h = h(x)$  be an arbitrary function with  $u + \varepsilon h \in \mathcal{U}_\rho$  for small  $\varepsilon$ .

We have

$$\int_{\Omega} h(x) \left( \int_0^T \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dt + \beta u(x) \right) dx \geq 0.$$

A standard control argument gives

$$u(x) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \int_0^T \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dt, \rho \right) \right).$$

#### 4. Algorithm and simulations

We have the solution of the problem (2) is given by the formula

$$u^*(x, t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t), \rho \right) \right),$$

where  $y^*$  is the weak solution of the Eq. (1) and  $p_i$  is the weak solution of the adjoint Eq. (13).

The computation of an optimal control solution the problem (2) can be realized by

$$\begin{cases} u_{n+1}^*(x, t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \sum_{i=1}^n \frac{\partial y^n(x, t)}{\partial x_i} p_i^n(x, t), \rho \right) \right), \\ u_0^* = 0, \end{cases} \quad (20)$$



where  $y^n$  is the solution of the Eq. (1) associated to  $u_n^*$  and  $p^n$  is the solution of the adjoint Eq. (13). Then, we consider the following algorithm

**Step 1 : Initialization**

⊙ Initial state  $y_0, u_0^*$  and  $y^d$ .

⊙ Threshold accuracy  $\varepsilon$  and the final time  $T$ .

**Step 2 :**

⊙ Solving the system (1) gives  $y^n$ .

⊙ Solving the system (13) gives  $p^n$ .

⊙ Calculate  $u_{n+1}^*$  by the formula (20).

Until  $\|u_{n+1}^* - u_n^*\|_{L^\infty(Q)} \leq \varepsilon$  stop, else  $n = n + 1$  go to step 2.

**Step 3 :** The control  $u_n^*$  is optimal.

#### 4.1 Simulations

On  $\Omega = ]0, 1[$ , we consider the following equation

$$\begin{cases} y_t(x, t) - \Delta y(x, t) = u(t)y(x, t) & \text{on } Q \\ y(x, 0) = x(1 - x)(1 + x), & \text{in } \Omega \\ y(x, t) = 0 & \text{in } \Sigma, \end{cases} \quad (21)$$

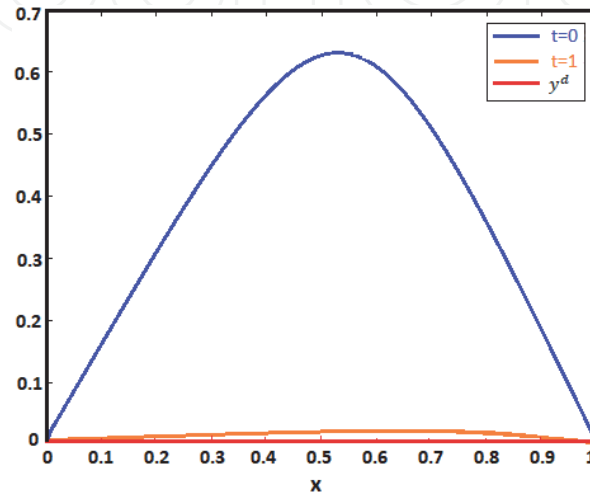
and consider problem (2) with the control set

$$\mathcal{U}_\rho = \{u \in L^\infty(0, T) : -\rho \leq u \leq \rho \text{ a.e in } (0, T)\}.$$

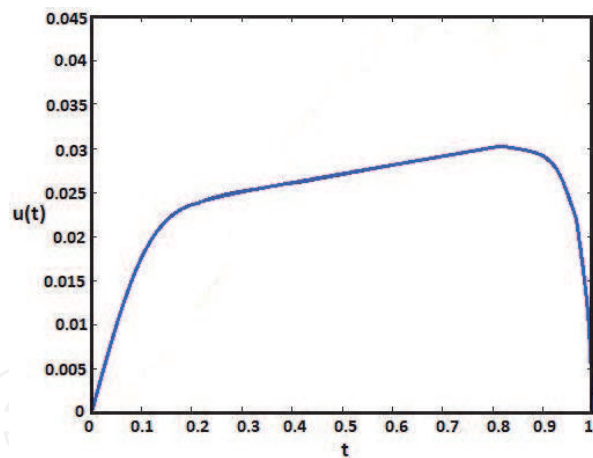
An optimal control solution of problem (2) is given by the following formula

$$u^*(t) = \max \left( -\rho, \min \left( -\frac{1}{\beta} \int_0^1 \sum_{i=1}^n \frac{\partial y(x, t)}{\partial x_i} p_i(x, t) dx, \rho \right) \right),$$

where  $y^*$  is solution of the Eq. (21) associated to the control  $u^*$  and  $p$  is the solution of the following adjoint system



**Figure 1.**  
The gradient of the state on  $]0, 1[$ .



**Figure 2.**  
 Evolution of the control function.

$$\begin{cases} p_i(x,t) - \Delta p_i(x,t) = u^*(t)p_i(x,t) & \text{on } Q \\ p_i(x,T) = \left(\frac{\partial y(T)}{\partial x_i} - y_i^d\right) & \text{in } \Omega \\ p_i(x,t) = 0 & \text{in } \Sigma. \end{cases}$$

We take  $T = 1, \rho = 1, \beta = 0.1, y_0(x) = x(1 - x)(1 + x)$ , and  $y^d(x) = 0$ . Applying the previous algorithm, with  $\varepsilon = 10^{-4}$  we obtain.

**Figure 1** shows that the gradient state is very close to the desired one with error  $\|\nabla y(T)\| = 5.33 \times 10^{-5}$  and the evolution of control is given by **Figure 2**.

### 5. Conclusion

Gradient optimal control problem of the bilinear diffusion equation with distributed and bounded controls is considered. The existence and characterized of an optimal control are proved. The obtained results are tested by numerical examples. Questions are still open, as is the case of gradient optimal control problem of the semilinear reaction–diffusion equation.

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