Chapter

Continuous One Step Linear Multi-Step Hybrid Block Method for the Solution of First Order Linear and Nonlinear Initial Value Problem of Ordinary Differential Equations

Kamoh Nathaniel, Kumleng Geoffrey and Sunday Joshua

Abstract

In this paper, a collocation approach for solving initial value problem of ordinary differential equations (ODEs) of the first order is presented. This approach consists of reducing the problem to a set of linear multi-step algebraic equations by approximating the ODE with a shifted Legendre polynomial basis function to determine the unknown constants. The proposed method is simple and efficient; it approximates the solutions very closely to the closed form solutions. Some problems were considered using Maple Software to illustrate the simplicity, efficiency and accuracy of the method. The results obtained revealed that the hybrid method can be suitable candidate for all forms of first order initial value problems of ordinary differential equations.

Keywords: collocation, hybrid block method, consistent, zero stable, convergent

1. Introduction

The development of mathematics parallels the human endeavor to understand our physical environment. Differential equations were discovered when the need to understand the behavior of nearly all systems undergoing change became more demanding. They are found in science and engineering as well as economics, social science, biology, business and health care. Many systems described by differential equations are so large and complex that a purely analytic solution is sometimes not traceable [1–5]. However mathematicians have studied the nature of these equations for decades of years and there are many well-developed numerical methods for the solution of different order initial value problems of ordinary differential equations. Unfortunately, in trying to achieve efficient and accurate solution, the choice of the numerical method to be adopted becomes very essential [4, 6–8].

The main goal of this paper is to derive a one step continuous hybrid block method using shifted Legendre polynomials basis function with the expectation that the numerical (proposed) method will give a solution that is close to the close form solution of the initial value problems of first order nonlinear ordinary differential equations. The paper is structured as follows. In Section 2, we derived and analyze the obtained schemes for consistency, zero stability and convergence. Some first order nonlinear problems of ordinary differential equations were solved using the derived schemes and the main results are presented in Section 3. Finally, we end with some concluding remarks in Section 4, where we compared our results with some earlier results contained in the literature.

1.1 Linear multistep methods (LMMs)

Linear Multi-step Methods (LMMs) are very popular for solving Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs). They are also applied in solving higher order ODEs. LMMs are not self-starting and therefore, need starting values from single-step methods like Euler's method and Runge–Kutta family of methods [1, 9, 10].

The general $k-step$ LMM of the discrete form as given in [11–14] is;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{1}$$

where α_j and β_j are uniquely determined and $\alpha_j + \beta_j \neq 0$. The LMM (1) generates discrete schemes which are used to solve first order ODEs. However, the continuous Linear Multi-step Methods (CLMMs) which is now being used was introduced by [15] and used by so many researchers such as [6, 7, 9, 16, 17] leading to the development of what is now called continuous Linear Multi-step Methods (CLMMs) given by;

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j}, \tag{2}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are expressed as continuous functions of x and are continuously differentiable at least m times $(m \geq 1)$. According to [1, 2, 11, 12, 18], the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion. While the collocation and interpolation technique is widely used for the derivation of CLMMs, this method is derived using different techniques and approaches.

The introduction of CLMMs have numerous advantages which is of great importance; better global error is estimated, it can be used to recover some standard schemes, it provides a simplified form of coefficients for further analytical work at different points and guarantee easy approximation of solutions at all interior points of the integration interval [1, 7, 16, 19, 20].

In this work, the CLMM is developed for the solution of (linear and nonlinear) first-order initial value problems of ordinary differential equations using the shifted Legendre polynomials basis function. The corresponding discrete schemes are obtained from the evaluation of the continuous scheme at some selected grid points.

1.2 Shifted Legendre polynomials

The shifted Legendre polynomials are well known family of orthogonal polynomials on the interval $[0, A]$ and are denoted by $P_i(t)$, the $P_i(t)$ can be obtained by the recurrence formula:

$$p_i(t) = \sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!t^k}{(i-k)!(k!)^2 A^k}, i = 1, 2, \dots,$$

where $P_0(t) = 1$ and $P_1(t) = 2t - 1$

The first few terms of the shifted Legendre polynomials on the interval [0, A] with $A = 1$ are:

$$P_0(t) = 1,$$
$$P_1(t) = 2t - 1,$$
$$P_2(t) = 6t^2 - 6t + 1,$$
$$P_3(t) = 20t^3 - 30t^2 + 12t - 1,$$
$$P_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t + 1,$$
$$P_5(t) = 252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1,$$
$$P_6(t) = 924t^6 - 2772t^5 + 3150t^4 - 1680t^3 + 420\,t^2 - 42t + 1.$$

1.3 Collocation method

A collocation is a method which involves the determination of an approximate solution of a functional equation in a suitable set of functions called trial or basis functions. The approximate solution is required to satisfy the equation and its supplementary conditions at certain points in the range of interest called collocation points.

2. Derivation of one step hybrid block methods with shifted Legendre polynomials

We consider the first order ordinary differential equation of the form

$$y'(x) = f(x, y(x)), y(x_0) = y_0, \tag{3}$$

where $y(x)$ is the unknown function to be determined. The idea here is to approximate the exact solution $y(x)$ of (3) in the partition $I_n = [a = x_0 < x_1 < x_2 < \dots < x_n = b]$ of the integration interval $[a, b]$ with a constant step size $h = x_i - x_{i-1}, i = 1, \dots, n$ by a shifted Legendre polynomial basis function of degree $s + r - 1$ of the form;

$$y(x) = \sum_{i=0}^{s+r-1} c_i P_i(t), \tag{4}$$

where $c_i \in \mathbb{R}$, $y \epsilon C^1(a, b)$ and $t = (x - x_n)$. The first derivative of (4), is substituted into (3), to obtain a differential system of the form

$$y'(x) = \sum_{i=0}^{s+r-1} c_i p_i'(t) = f(x, y(x)), \tag{5}$$

Now interpolating (4) at $x_{n+s}, s = \frac{1}{2}, \frac{3}{4}$ and collocating (5) at $x_{n+r}, r = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, k$ where s and r represents the interpolation and collocation points respectively, and k is the step number, after some substitutions and manipulations the continuous scheme of the form;

$$y(x) = \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(x)y_{n+\frac{3}{4}} + h\left(\sum_{\tau=0}^{k}\beta_\tau(x)f\left(x_{n+\tau}, y_{n+\tau}\right) + \beta_\mu(x)f\left(x_{n+\mu}, y_{n+\mu}\right)\right),$$
$$\mu = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}. \tag{6}$$

is obtained with the following continuous coefficients

$$\left.\begin{aligned}
\alpha_{\frac{1}{2}}(x) &= \frac{27}{11} + \frac{24576}{11h^5}t^5 - \frac{26880}{11h^4}t^4 + \frac{12800}{11h^3}t^3 - \frac{2304}{11h^2}t^2 - \frac{8192}{11h^6}t^6 \\
\alpha_{\frac{3}{4}}(x) &= -\frac{16}{11} + \frac{2304}{11h^2}t^2 - \frac{12800}{11h^3}t^3 + \frac{26880}{11h^4}t^4 - \frac{24576}{11h^5}t^5 + \frac{8192}{11h^6}t^6 \\
\beta_0(x) &= t - \frac{3}{40}h - \frac{149}{30h}t^2 + \frac{110}{9h^2}t^3 - \frac{16}{h^3}t^4 + \frac{32}{3h^4}t^5 - \frac{128}{45h^5}t^6 \\
\beta_{\frac{1}{4}}(x) &= -\frac{21}{55}h + \frac{736}{55h}t^2 - \frac{5248}{99h^2}t^3 + \frac{2864}{33h^3}t^4 - \frac{2176}{33h^4}t^5 + \frac{9472}{495h^5}t^6 \\
\beta_{\frac{1}{2}}(x) &= \frac{9}{55}h - \frac{2154}{55h}t^2 + \frac{6916}{33h^2}t^3 - \frac{4608}{11h^3}t^4 + \frac{4032}{11h^4}t^5 - \frac{19456}{165h^5}t^6 \\
\beta_{\frac{3}{4}}(x) &= \frac{9}{55}h - \frac{3712}{165h}t^2 + \frac{12608}{99h^2}t^3 - \frac{3024}{11h^3}t^4 + \frac{8576}{33h^4}t^5 - \frac{44288}{495h^5}t^6 \\
\beta_1(x) &= -\frac{3}{440}h + \frac{97}{110h}\ t^2 - \frac{518}{99h^2}t^3 + \frac{400}{33h^3}t^4 - \frac{416}{33h^4}t^5 + \frac{2432}{495h^5}t^6
\end{aligned}\right\}. \tag{7}$$

In order to obtain the block discrete scheme for ($K = 1$), Eq. (7) is evaluated at $x = x_n, x_{n+\frac{1}{8}}, x_{n+\frac{1}{4}}, x_{n+1}$ and its first derivative at $x_{n+\frac{1}{8}}$ to give the following discrete schemes;

$$\left.\begin{aligned}
y_{n+\frac{1}{8}} &= \frac{325}{352}y_{n+\frac{1}{2}} - \frac{15}{2048}hf_n + \frac{27}{352}y_{n+\frac{3}{4}} - \frac{15}{22528}hf_{n+1} - \frac{735}{5632}hf_{n+\frac{1}{2}} - \frac{2895}{11264}hf_{n+\frac{1}{4}} + \frac{15}{11264}hf_{n+\frac{3}{4}} \\
y_{n+\frac{1}{4}} &= \frac{1}{360}hf_n + y_{n+\frac{3}{4}} + \frac{1}{360}hf_{n+1} - \frac{19}{60}hf_{n+\frac{1}{2}} - \frac{17}{180}hf_{n+\frac{1}{4}} - \frac{17}{180}hf_{n+\frac{3}{4}} \\
y_{n+\frac{1}{2}} &= -\frac{11}{27}y_n + \frac{11}{360}hf_n + \frac{16}{27}y_{n+\frac{3}{4}} + \frac{1}{360}hf_{n+1} - \frac{1}{15}hf_{n+\frac{1}{2}} + \frac{7}{45}hf_{n+\frac{1}{4}} - \frac{1}{15}hf_{n+\frac{3}{4}} \\
y_{n+\frac{3}{4}} &= y_{n+\frac{1}{2}} - \frac{11}{720}hf_n - \frac{13}{3360}hf_{n+1} + \frac{283}{1440}hf_{n+\frac{1}{2}} - \frac{49}{480}hf_{n+\frac{1}{4}} + \frac{151}{1440}hf_{n+\frac{3}{4}} + \frac{22}{315}hf_{n+\frac{1}{8}} \\
y_{n+1} &= \frac{27}{11}y_{n+\frac{1}{2}} + \frac{1}{360}hf_n - \frac{16}{11}y_{n+\frac{3}{4}} + \frac{281}{3960}hf_{n+1} + \frac{49}{165}hf_{n+\frac{1}{2}} - \frac{13}{495}hf_{n+\frac{1}{4}} + \frac{257}{495}hf_{n+\frac{3}{4}}
\end{aligned}\right\} \tag{8}$$

Eq. (7) is the continuous scheme while (8) is the block discrete schemes for step number $K = 1$.

2.1 Order and error constant

Expanding (8), in Taylor's series gives;

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$$C_q = \begin{bmatrix} \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left((1)^j - \frac{27}{11}\left(\frac{1}{2}\right)^j + \frac{16}{11}\left(\frac{3}{4}\right)^j\right) y_n^j - h\left(\frac{1}{360}(0)^j - \frac{13}{495}\left(\frac{1}{4}\right)^j + \frac{49}{165}\left(\frac{1}{2}\right)^j + \frac{257}{495}\left(\frac{3}{4}\right)^j + \frac{281}{3960}(1)^j\right) y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!} y_n^j \left(\left(\left(\frac{3}{4}\right)^j - \left(\frac{1}{2}\right)^j\right) y_n^j - h\left(-\frac{11}{720}(0)^j - \frac{49}{480}\left(\frac{1}{4}\right)^j + \frac{283}{1440}\left(\frac{1}{2}\right)^j + \frac{151}{1440}\left(\frac{3}{4}\right)^j + \frac{22}{315}\left(\frac{1}{8}\right)^j - \frac{13}{3360}(1)^j\right) y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left(\left(\frac{1}{2}\right)^j - \frac{16}{27}\left(\frac{1}{2}\right)^j + \frac{11}{27}(0)^j\right) y_n^j - h\left(\frac{11}{360}(0)^j + \frac{7}{45}\left(\frac{1}{4}\right)^j - \frac{1}{15}\left(\frac{1}{2}\right)^j - \frac{7}{15}\left(\frac{3}{4}\right)^j + \frac{1}{360}(1)^j\right) y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left(\left(\frac{1}{4}\right)^j - \left(\frac{3}{4}\right)^j\right) y_n^j - h\left(\frac{1}{360}(0)^j - \frac{17}{180}\left(\frac{1}{4}\right)^j - \frac{19}{60}\left(\frac{1}{2}\right)^j - \frac{17}{180}\left(\frac{3}{4}\right)^j + \frac{1}{360}(1)^j\right) y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left(\left(\frac{1}{8}\right)^j - \frac{325}{352}\left(\frac{1}{2}\right)^j - \frac{27}{352}\left(\frac{3}{4}\right)^j\right) y_n^j - h\left(-\frac{15}{2048}(0)^j - \frac{2895}{5632}\left(\frac{1}{4}\right)^j - \frac{735}{5632}\left(\frac{1}{2}\right)^j + \frac{15}{11264}\left(\frac{3}{4}\right)^j - \frac{15}{22528}(1)^j\right) y_n^{(j+1)}\right) = 0 \end{bmatrix}$$

and collecting like terms in powers of h, gives $c_0 = c_1 = c_2 = \ldots = c_6 = (0, 0, 0, 0, 0)^T$ and $c_7 = \left(-\frac{17}{85155840}, \frac{311}{1981808640}, -\frac{1}{7741440}, -\frac{1}{12386304}, \frac{135}{2583691264}\right)^T$. Hence, the method has order $p = (6, 6, 6, 6, 6)^T$ and with error constants of $c_7 = \left(-\frac{17}{85155840}, \frac{311}{1981808640}, -\frac{1}{7741440}, -\frac{1}{12386304}, \frac{135}{2583691264}\right)^T$.

2.2 Consistency

The linear multi-step method (8) is said to be consistent if the following conditions hold:

i. it has order $\breve{p} \geq 1$,

ii. $\sum_{j=0}^{k} \breve{\alpha}_j = 0$,

iii. $\sum_{j=0}^{k} j\breve{\alpha}_j = \sum_{j=0}^{k} \breve{\beta}_j$,

iv. $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$,

where $\rho(r)$ and $\sigma(r)$ are the first and the second characteristic polynomials of (8) respectively, [21]. Following [8, 14], (i) is sufficient condition for the block method (8) to be consistent since $p = (6, 6, 6, 6, 6)^T > 1$. Hence, the method is consistent.

2.3 Zero stability

The block solution (8), is said to be zero stable if the roots $z_r; r = 1, \ldots, n$ of the first characteristic polynomial $p(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfies $|z_r| \leq 1$ and every root with $|z_r| = 1$ has multiplicity not exceeding the order of the differential equation in the limit as $h \to 0$.

Calculations from all available information revealed that the block method (8) has the following roots

$$z^4(z - 1) = 0 \Rightarrow z = (0, 0, 0, 0, 1)$$

Hence the block method is zero stable, since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

2.4 Convergence

According to [8, 14, 22], we can safely assert the convergence of the block method (8) since the method is consistent and zero stable.

2.5 Region of absolute stability of the block method

Reformulating the block (8) as a General Linear Method (GLM) containing a partition of matrices A and B using the stability polynomial $(Ar - B)$, where

$$A = \begin{bmatrix} 1 & \frac{2895}{11264}z & -\frac{325}{352} + \frac{735}{5632}z & -\frac{27}{352} - \frac{15}{11264}z & \frac{15}{22528}z \\ 0 & 1 + \frac{17}{180}z & \frac{19}{60}z & -1 + \frac{17}{180}z & -\frac{1}{360}z \\ 0 & -\frac{7}{45}z & 1 + \frac{1}{15}z & -\frac{16}{27} + \frac{1}{15}z & -\frac{1}{360}z \\ -\frac{22}{315}z & -\frac{49}{480} & 1 + \frac{283}{1440}z & 1 + \frac{151}{1440}z & \frac{13}{3360}z \\ 0 & \frac{13}{495}z & -\frac{27}{11} - \frac{49}{165}z & \frac{16}{11} - \frac{257}{495}z & 1 - \frac{281}{3960}z \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{15}{2048}z \\ 0 & 0 & 0 & 0 & \frac{1}{360}z \\ 0 & 0 & 0 & 0 & \frac{11}{27} + \frac{11}{360}z \\ 0 & 0 & 0 & 0 & \frac{11}{720}z \\ 0 & 0 & 0 & 0 & \frac{13}{3360}z \end{bmatrix}$$

we obtain the region of absolute stability shown in **Figure 1** below

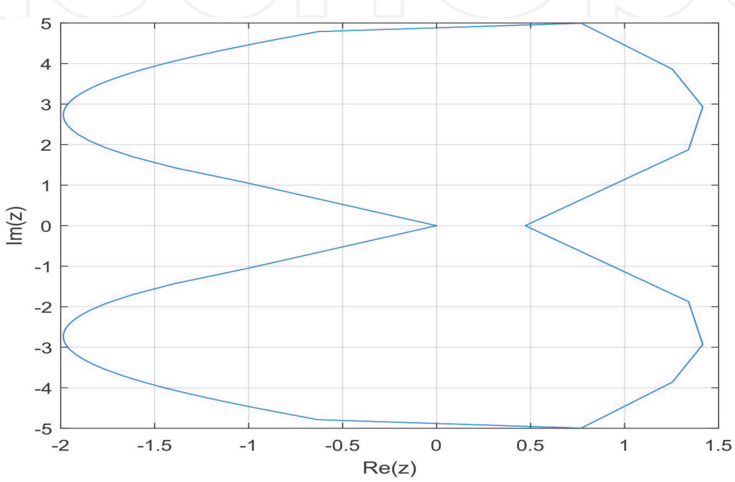

Figure 1.
Region of absolute stability.

3. Numerical experiments

This section discusses the implementation of the derived method by solving some first order nonlinear initial value problems of ordinary differential equations.

Problem 1

We consider a nonlinear first order initial value problem of ordinary differential problem which was solved by [23]. $y'(x) = -10(y-1)^2; y(0) = 2,$ $h = 0.01$ With exact solution given as $y(x) = 1 + \frac{1}{1+10x}$, the result is shown in **Table 1**, while the theoretical and numerical results are presented graphically in **Figure 2**.

Problem 2

Given a nonlinear first order ordinary differential problem solved by [24] (**Table 2**). $y'(x) = 2xy, y(0) = 1,\ h = 0.1\ x \in [0, 1]$ with exact solution given by $y(x) = e^{x^2}$, the result is shown in **Table 2**, **Figure 3** shows the solution curve for problem 2.

x	Exact solution	Result of Proposed Method	Error in Proposed Method	Error in [23]
0.01	1.90909090909091	1.90909090891558	1.7533×10^{-10}	2.829001×10^{-7}
0.02	1.83333333333333	1.83333333310133	2.3200×10^{-10}	4.045782×10^{-7}
0.03	1.76923076923077	1.76923076898962	2.4115×10^{-10}	4.472541×10^{-7}
0.04	1.71428571428571	1.71428571405431	2.3140×10^{-10}	4.509027×10^{-7}
0.05	1.66666666666667	1.66666666645183	2.1484×10^{-10}	4.356251×10^{-7}
0.06	1.62500000000000	1.62499999980340	1.9660×10^{-10}	4.117637×10^{-7}
0.07	1.58823529411765	1.58823529393878	1.7887×10^{-10}	3.846989×10^{-7}
0.08	1.55555555555556	1.55555555539306	1.6250×10^{-10}	3.572176×10^{-7}
0.09	1.52631578947368	1.52631578932595	1.4773×10^{-10}	3.307245×10^{-7}
0.10	1.50000000000000	1.49999999986543	1.3457×10^{-10}	3.058785×10^{-7}

Table 1.
(Problem 1): Comparing results of proposed method with [23].

x	Exact solution	Result of Proposed Method	Error in Proposed Method	Error in [24]
0.1	1.01005016708417	1.01005016708855	4.3800×10^{-12}	1.899500×10^{-1}
0.2	1.04081077419239	1.04081077421089	1.8500×10^{-11}	1.714527×10^{-1}
0.3	1.09417428370521	1.09417428375087	4.5660×10^{-11}	1.556419×10^{-1}
0.4	1.17351087099181	1.17351087108422	9.2410×10^{-11}	1.415053×10^{-1}
0.5	1.28402541668774	1.28402541685820	1.7046×10^{-10}	1.280382×10^{-1}
0.6	1.43332941456034	1.43332941486021	2.9987×10^{-10}	1.141249×10^{-1}
0.7	1.63231621995538	1.63231622047036	5.1498×10^{-10}	9.839200×10^{-2}
0.8	1.89648087930495	1.89648088017992	8.7497×10^{-10}	7.9005900×10^{-2}
0.9	2.24790798667647	2.24790798815910	1.48263×10^{-9}	5.3376500×10^{-2}
1.0	2.71828182845905	2.71828183097715	2.51810×10^{-9}	1.7703800×10^{-2}

Table 2.
(Problem 2): Comparing results of proposed method with [24].

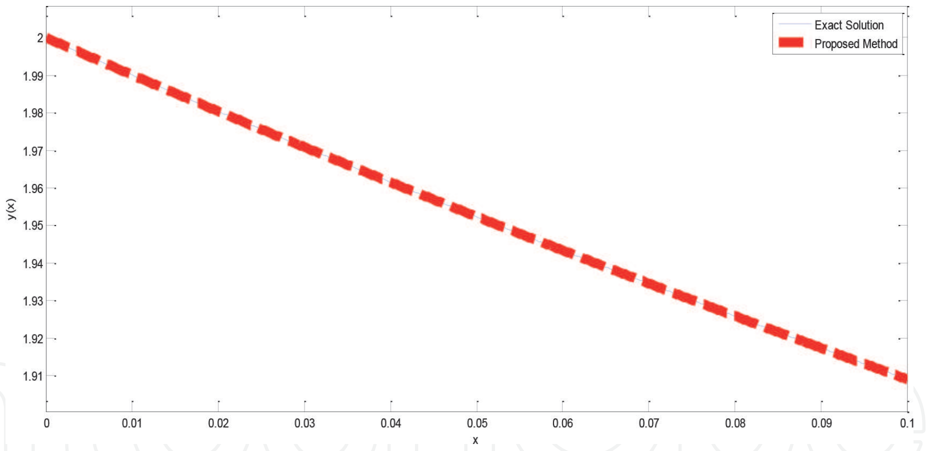

Figure 2.
Solution curve for problem 1.

Problem 3

Considering the first order initial value problem of ordinary differential problem solved by [25] (**Table 3**). $y'(x) = -y^2, y(0) = 1,\ h = 0.01\ x \in [0, 1]$ with exact

x	Exact solution	Result of Proposed Method	Error in Proposed Method	Error in [25]
0.01	0.990099009900990	0.990099009900989	1.000000×10^{-15}	2.91799×10^{-11}
0.02	0.980392156862745	0.980392156862744	1.000000×10^{-15}	3.71577×10^{-11}
0.03	0.970873786407767	0.970873786407766	1.000000×10^{-15}	3.93663×10^{-11}
0.04	0.961538461538462	0.961538461538460	2.000000×10^{-15}	3.39936×10^{-11}
0.05	0.952380952380952	0.952380952380951	1.000000×10^{-15}	2.94922×10^{-11}
0.06	0.943396226415094	0.943396226415093	1.000000×10^{-15}	2.61278×10^{-11}
0.07	0.934579439252336	0.934579439252335	1.000000×10^{-15}	2.31487×10^{-11}
0.08	0.925925925925926	0.925925925925925	1.000000×10^{-15}	6.80704×10^{-11}
0.09	0.917431192660550	0.917431192660549	1.000000×10^{-15}	8.31745×10^{-11}
0.10	0.909090909090909	0.909090909090908	1.000000×10^{-15}	7.50649×10^{-11}

Table 3.
(Problem 3): Comparing results of proposed method with [25].

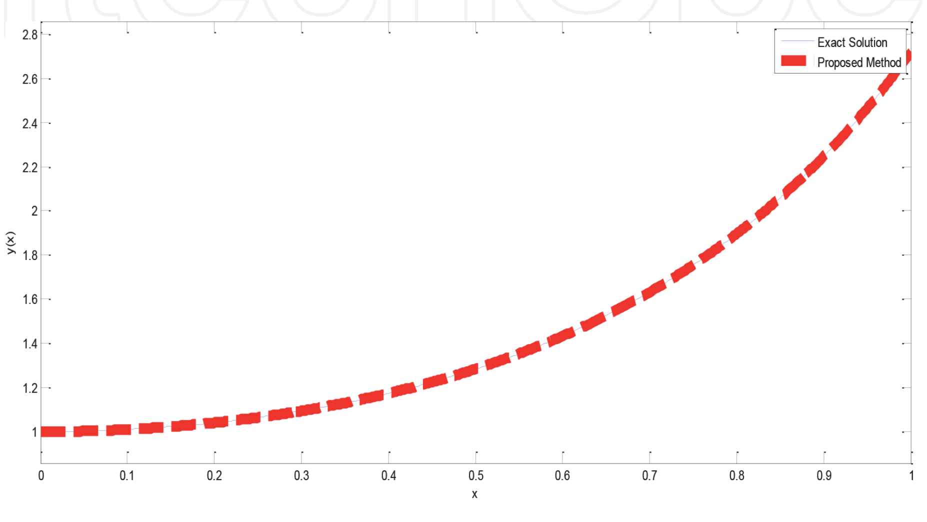

Figure 3.
Solution curve for problem 2.

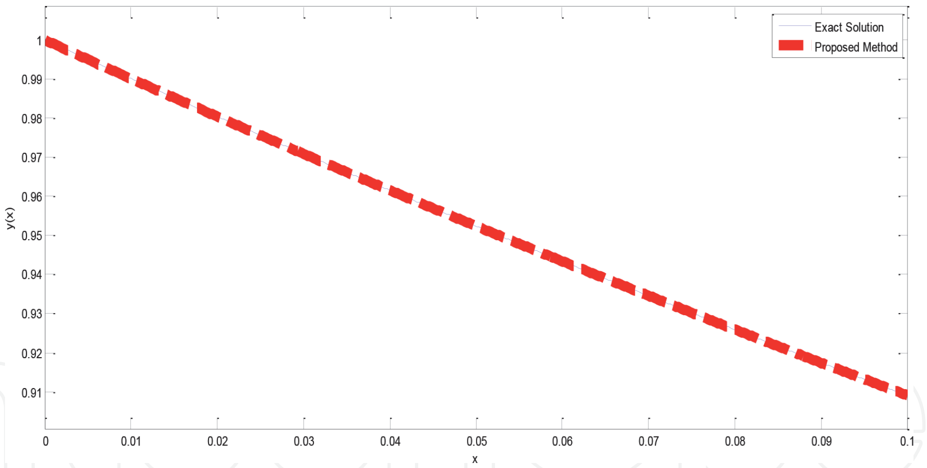

Figure 4.
Solution curve for problem 3.

solution given by $y(x) = \frac{1}{1+x}$ with the result is shown in **Table 3**, **Figure 4** compare the two results (theoretical and numerical graphically).

4. Conclusion

In this paper, we derived one step block hybrid continuous linear multi-step method for solving first order initial value problems of ordinary differential equations, the method was found to be consistent, zero stable and convergent. The method was implemented on some nonlinear initial value problems of ordinary differential equations and the numerical results were found to be accurate when compared with the exact solutions and other numerical methods as contained in **Tables 1–3** and their respective solution curves. The new hybrid block method can be suitable candidate for all forms (linear and nonlinear) of first order initial value problems of ordinary differential equations.

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Authors contributions

This work was carried out in collaboration among the authors. Author Kamoh, N.M. proposed, derived and implemented the method. Author Kumleng, G.M. analyzed the method while Author Sunday; J. presented the numerical results graphically. All the authors managed the literature searches, read and approved the final manuscript.

Author details

Kamoh Nathaniel*, Kumleng Geoffrey and Sunday Joshua
Department of Mathematics, University of Jos, Jos, Nigeria

*Address all correspondence to: mahwash1477@gmail.com

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