Chapter

Spectral Properties of a Non-Self-Adjoint Differential Operator with Block-Triangular Operator Coefficients

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Abstract

In this chapter, the Sturm-Liouville equation with block-triangular, increasing at infinity operator potential is considered. A fundamental system of solutions is constructed, one of which decreases at infinity, and the second increases. The asymptotic behavior at infinity was found out. The Green's function and the resolvent for a non-self-adjoint differential operator are constructed. This allows to obtain sufficient conditions under which the spectrum of this non-self-adjoint differential operator is real and discrete. For a non-self-adjoint Sturm-Liouville operator with a triangular matrix potential growing at infinity, an example of operator having spectral singularities is constructed.

Keywords: differential operators, spectrum, non-self-adjoint, block-triangular operator coefficients, Green's function, resolvent

1. Introduction

The question of the generalization of the oscillatory Sturm theorem for scalar equations of higher orders and for equations with matrix coefficients for a long time remained open. Only in recent joint papers by F. Rofe-Beketov and A. Kholkin (see [1]) a connection was established between spectral and oscillatory properties for self-adjoint operators generated by equations of arbitrary even order with operator coefficients and boundary conditions of general form. Later, a Sturm-type oscillation theorem was proved [2] for a problem on finite and infinite intervals for a second-order equation with block-triangular matrix coefficients. In the case of non-self-adjoint differential operators, oscillation theorems have not been considered earlier.

Results turning out in self-adjoint and non-self-adjoint cases differentiate substantially. The theory of non-self-adjoint singular differential operators, generated by scalar differential expressions, has been well studied. An overview on the theory of non-self-adjoint singular ordinary differential operators is provided in V.E. Lyantse's Appendix I to the monograph [3]. In the study of the connection between spectral and oscillation properties of non-self-adjoint differential operators with block-triangular operator coefficients [2, 4] the question arises of the structure of the spectrum of such operators. For scalar non- self-adjoint differential operators these questions were studied in the papers [5–8]. The theory of singular non-self-adjoint differential operators with matrix and operator coefficients is relatively new. In the context of the inverse scattering problem, for an operator with a triangular matrix potential decreasing at infinity, the first moment of which is bounded, the structure of the spectrum was established in [9, 10]. The theory of equations with block - triangular operator coefficients the first results were published in 2012 in the works of the author [11–13].

In this works we construct the fundamental system of solutions of differential equation with block-triangular operator potential that increases at infinity, one of that is decreasing at infinity, and the second growing. The asymptotics of the fundamental system of solutions of this equation is established. The Green's function is constructed for a non-self-adjoint system with a block-triangular potential, the diagonal blocks of which are self-adjoint operators. We obtained a resolvent for a non-self-adjoint differential operator, using which the structure of the operator spectrum is set. Sufficient conditions at which a spectrum of such non-self-adjoint differential operator is real and discrete are obtained. Here the rate of growth elements, not on the main diagonal, is subordinated to the rate of growth of the diagonal elements. In case of infringement of this condition, the operator can have spectral singularities [14].

2. The fundamental solutions for an non-self-adjoint differential operator with block – triangular operator coefficients.

Let us designate $H_k, k = \overline{1, r}$ as a finite-dimensional or infinite-dimensional separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$. Denote by $\mathbf{H} = H_1 \oplus H_2 \oplus \dots \oplus H_r$. Element $\overline{h} \in \mathbf{H}$ will be written in the form of $\overline{h} = \operatorname{col}(\overline{h}_1, \overline{h}_2, \dots, \overline{h}_r)$, where $\overline{h}_k \in H_k$, $k = 1, 2, \dots, r$, I_k, I- are identity operators in H_k and $\mathbf{H}$ accordingly.

We denote by $L_2(\mathbf{H}, (0, \infty))$ the Hilbert space of vector-valued functions $y(x)$ with values in $\mathbf{H}$ with inner product $\langle y, z \rangle = \int_0^\infty (y(x), z(x))\, dx$ and the norm $\|\cdot\|$.

Now let us consider the equation with block-triangular operator potential in $B(\mathbf{H})$

$$l[\overline{y}] = -\overline{y}'' + V(x)\overline{y} = \lambda\overline{y}, \quad 0 \le x < \infty, \tag{1}$$

where

$$V(x) = v(x) \cdot I + U(x), \quad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \dots & U_{1r}(x) \\ 0 & U_{22}(x) & \dots & U_{2r}(x) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & U_{rr}(x) \end{pmatrix}, \tag{2}$$

$v(x)$ is a real scalar function such that $0 < v(x) \to \infty$ monotonically, as $x \to \infty$, and it has monotone absolutely continuous derivative. Also, $U(x)$ is a relatively small perturbation, e. g. as $x \to \infty$ $|U(x)| \cdot v^{-1}(x) \to 0$ or $|U|v^{-1} \in L^\infty(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x),\ k = \overline{1, r}$ are assumed to be bounded self-adjoint operators in H_k.

In case where

$$v(x) \ge Cx^{2\alpha}, C > 0, \alpha > 1, \tag{3}$$

we suppose that coefficients of the Eq. (1) satisfy relations:

$$\int_0^\infty |U(t)| \cdot v^{-\frac{1}{2}}(t)dt < \infty, \tag{4}$$

$$\int_0^\infty v'^2(t) \cdot v^{-\frac{5}{2}}(t)dt < \infty, \quad \int_0^\infty v''(t) \cdot v^{-\frac{3}{2}}(t)dt < \infty. \tag{5}$$

In case of $v(x) = x^{2\alpha}, 0 < \alpha \le 1$, we suppose that the coefficients of the Eq. (1) satisfy the relation

$$\int_a^\infty |U(t)| \cdot t^{-\alpha}dt < \infty, \ \ a > 0. \tag{6}$$

2.1 Construction of the fundamental system of solutions for an operator differential equation with a rapidly increasing at infinity potential

Consider first the case where $v(x) \ge Cx^{2\alpha}, C > 0, \alpha > 1$.

Condition (3) is performed, for example, quickly increasing functions e^x, $\exp\{e^x\}$ etc.

Rewrite the Eq. (1) in the form

$$-\bar{y}'' + (v(x) + q(x))\bar{y} = ((\lambda + q(x))I - U(x))\bar{y}, \tag{7}$$

where $q(x)$ determined by a formula (cf. with the monograph [15])

$$q(x) = \frac{5}{16}\left(\frac{v'(x)}{v(x)}\right)^2 - \frac{1}{4}\frac{v''(x)}{v(x)}. \tag{8}$$

Now let us denote.

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \ \exp\left(-\int_0^x \sqrt{v(u)}du\right), \gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \ \exp\left(\int_0^x \sqrt{v(u)}du\right). \tag{9}$$

It is easy to see that $\gamma_0(x,\lambda) \to 0, \gamma_\infty(x,\lambda) \to \infty$ as $x \to \infty$. These solutions constitute a fundamental system of solutions of the scalar differential equation

$$-z'' + (v(x) + q(x))z = 0, \tag{10}$$

in such a way that for all $x \in [0,\infty)$ one has.

$$W(\gamma_0, \gamma_\infty) := \gamma_0(x,\lambda) \cdot \gamma'_\infty(x,\lambda) - \gamma'_0(x,\lambda) \cdot \gamma_\infty(x,\lambda) = 1. \tag{11}$$

Theorem 2.1 *Under conditions (3), (4), (5) Eq. (1) has a unique decreasing at infinity operator solution $\Phi(x,\lambda) \in B(\mathbf{H})$, satisfying the conditions*

$$\lim_{x\to\infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I \ \ \text{and} \quad \lim_{x\to\infty} \frac{\Phi'(x,\lambda)}{\gamma'_0(x,\lambda)} = I. \tag{12}$$

Also, there exists increasing at infinity operator solution $\Psi(x,\lambda) \in B(\mathbf{H})$ satisfying the conditions

$$\lim_{x\to\infty}\frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)}=I \ \ and \ \ \lim_{x\to\infty}\frac{\Psi'(x,\lambda)}{\gamma'_\infty(x,\lambda)}=I. \tag{13}$$

Proof

a. Eq. (7) equivalently to integral equation

$$\Phi(x,\lambda)=\gamma_0(x,\lambda)I+\int\limits_x^\infty K(x,t,\lambda)\cdot\Phi(t,\lambda)dt, \tag{14}$$

where

$$K(x,t,\lambda)=C(x,t,\lambda)\cdot[(\lambda+q(t))I-U(t)], \tag{15}$$

$$C(x,t,\lambda)=\gamma_\infty(x,\lambda)\cdot\gamma_0(t,\lambda)-\gamma_\infty(t,\lambda)\cdot\gamma_0(x,\lambda), \tag{16}$$

with $C(x,t,\lambda)$ being the Cauchy function that in each variable satisfies Eq. (10) and the initial conditions $C(x,t,\lambda)|_{x=t}=0, \quad C'_x(x,t,\lambda)|_{x=t}=1, \quad C'_t(x,t,\lambda)|_{x=t}=-1.$ Set $\chi(x,\lambda)=\frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)}$ to rewrite Eq. (14) in form

$$\chi(x,\lambda)=I+\int\limits_x^\infty R(x,t,\lambda)\chi(t,\lambda)dt, \tag{17}$$

where $R(x,t,\lambda)=K(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}$. Thus

$$\begin{aligned}\left|C(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right| &= \left|\gamma_0^2(t,\gamma)\cdot\frac{\gamma_\infty(x,\lambda)}{\gamma_0(x,\lambda)}-\gamma_0(t,\lambda)\cdot\gamma_\infty(t,\lambda)\right| = \\ &= \left|\frac{1}{2\sqrt{v(t)}}\cdot\exp\left(-2\int\limits_0^t\sqrt{v(u)}du\right)\cdot\exp\left(2\int\limits_0^x\sqrt{v(u)}du\right)-\frac{1}{2\sqrt{v(t)}}\right| = \\ &= \frac{1}{2\sqrt{v(t)}}\cdot\left|\exp\left(-2\int\limits_x^t\sqrt{v(u)}du\right)-1\right|\end{aligned} \tag{18}$$

and since with $x\le t$ one has $\exp\left(-2\int\limits_x^t\sqrt{v(u)}du\right)\le 1$, we deduce that

$$\left|C(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right|\le\frac{1}{\sqrt{v(t)}}. \tag{19}$$

Hence.

$$|R(x,t,\lambda)|=\left|C(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\gamma)}\cdot[(\lambda+q(t))I-U(t)]\right|\le\frac{1}{\sqrt{v(t)}}(|\lambda|+|q(t)|+|U(t)|). \tag{20}$$

By virtue of (3)–(5), (8),

$$\frac{1}{\sqrt{v(t)}}(|\lambda|+|q(t)|+|U(t)|)\in L(0,\infty), \tag{21}$$

and therefore integral equation has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)| \leq$ const. By (17), one has that $\lim_{x\to\infty}\chi(x,\lambda) = I$, where the first part of formula (12) follows from.

Differentiable (14) to get $\frac{\Phi'(x,\lambda)}{\gamma_0'(x)} = I + \int_x^\infty S(x,t,\lambda)\chi(t,\lambda)dt$, where $S(x,t,\lambda) = K_x'(x,t,\lambda)\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)} = C_x'(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)}\cdot[(\lambda+q(t))I - U(t)]$. We have similarly (18), that $\left|C_x'(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)}\right| \leq \frac{1}{\sqrt{v(t)}}$, and therefore $|S(x,t,\lambda)| \leq \frac{1}{\sqrt{v(t)}}\cdot[|\lambda| + |q(t)| + |U(t)|] \in L(0,\infty)$, where the second part of formula (12) follows from.

b. Denote by $\hat{\Psi}(x,\lambda) \in B(\mathbf{H})$ block-triangular operator solution of Eq. (1) that increases at infinity, $\Psi_{kk}(x,\lambda) \in B(H_k, H_k)$, $k = \overline{1,r}$ -its diagonal blocks. Now Eq. (7) is equivalent to the integral equation

$$\hat{\Psi}(x,\lambda) = \gamma_\infty(x,\lambda)\cdot I - \int_0^x K(x,t,\lambda)\cdot\hat{\Psi}(t,\lambda)dt, \tag{22}$$

where, just as in (14), the kernel $K(x,t,\lambda)$ is given by (15). Now set $\chi(x,\lambda) = \frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)}$ to rewrite Eq. (22) in form

$$\chi(x,\lambda) = I - \int_0^x R(x,t,\lambda)\cdot\chi(t,\lambda)dt, \tag{23}$$

where $R(x,t,\lambda) = C(x,t,\lambda)\cdot\frac{\gamma_\infty(t,\lambda)}{\gamma_\infty(x,\lambda)}\cdot[(q(t)+\lambda)\cdot I - U(t)]$. Similarly we can prove that the integral Eq. (23) has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)| \leq$ const. Pass in (23) to a limit as $x \to \infty$ to get $\lim_{x\to\infty}\chi(x,\lambda) = I + \tilde{C}(\lambda)$ where $\tilde{C}(\lambda)$ is block-triangular operator in $\mathbf{H}$, that is

$$\lim_{x\to\infty}\frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)} = I + \tilde{C}(\lambda). \tag{24}$$

Now consider another block-triangular operator solution $\tilde{\Psi}(x,\lambda)$ that increases at infinity diagonal blocks which are defined by.

$$\tilde{\Psi}_{kk}(x,\lambda) = \Phi_{kk}(x,\lambda)\int_a^x \Phi_{kk}^{-1}(t,\lambda)\left(\Phi_{kk}^*(t,\lambda)\right)^{-1}dt, k = \overline{1,r}, (a \geq 0), \tag{25}$$

$\Phi_{kk}(x,\lambda)$ are the diagonal blocks of operator solution $\Phi(x,\lambda)$ as in Section a). In view (16) and the definition of the functions $\gamma_0(x), \gamma_\infty(x)$ can be proved that

$$\lim_{x\to\infty}\frac{\tilde{\Psi}_{kk}(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_k, \quad k = \overline{1,r}. \tag{26}$$

Since $\hat{\Psi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$ are the operator solutions of Eq. (1) that increase at infinity,

$$\hat{\Psi}(x,\lambda) = \tilde{\Psi}(x,\lambda) + \Phi(x,\lambda)\cdot C_0(\lambda), \tag{27}$$

where $C_0(\lambda)$ is some block-triangular operator. Thus $\lim\limits_{x\to\infty}\frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x)} = \lim\limits_{x\to\infty}\frac{\tilde{\Psi}(x,\lambda)}{\gamma_\infty(x)}$, hence, by virtue (26), $\lim\limits_{x\to\infty}\frac{\Psi_{kk}(x,\lambda)}{\gamma_\infty(x)} = I_k,\ k=\overline{1,r}$ and in (24) has

$$\tilde{C}(\lambda) = \begin{pmatrix} 0 & C_{12}(\lambda) & \dots & C_{1r}(\lambda) \\ 0 & 0 & \dots & C_{2r}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{28}$$

The solution $\Psi(x,\lambda)$ given by $\Psi(x,\lambda) = \hat{\Psi}(x,\lambda)\big(I+\tilde{C}(\lambda)\big)^{-1}$ is subject to first from condition (13). Use (12) to differentiate (27), then find the asymptotes of $\tilde{\Psi}'(x,\lambda)$ as $x\to\infty$ similarly to (21) to obtain the second part of formula (13). Theorem is proved. □

In this section, the fundamental system of solution is constructed for an operator differential equation with a rapidly increasing at infinity potential.

2.2 Asymptotic of the fundamental system solutions of equation with block-triangular potential

Now consider the case when $v(x) = x^{2\alpha}$, $0<\alpha\le 1$ and coefficients of Eq. (1) satisfy the condition (6). Rewrite Eq. (1) in the form

$$-\bar{y}'' + \big(x^{2\alpha} - \lambda + q(x,\lambda)\big)\bar{y} = (q(x,\lambda)\cdot I - U(x))\bar{y}, \tag{29}$$

where $q(x,\lambda)$ determined by a formula

$$q(x,\lambda) = \frac{5\alpha^2}{4}\left(\frac{x^{2\alpha-1}}{x^{2\alpha}-\lambda}\right)^2 - \frac{\alpha(2\alpha-1)x^{2\alpha-2}}{2(x^{2\alpha}-\lambda)}. \tag{30}$$

Denote

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha}-\lambda)}}\cdot \exp\left(-\int_a^x \sqrt{u^{2\alpha}-\lambda}\,du\right), \tag{31}$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha}-\lambda)}}\cdot \exp\left(\int_a^x \sqrt{u^{2\alpha}-\lambda}\,du\right). \tag{32}$$

There solutions constitute a fundamental system of solutions of the scalar differential equation $-z'' + (x^{2\alpha} - \lambda + q(x,\lambda))z = 0$, in such a way that for all $x\in[0,\infty)$ one has $W(\gamma_0,\gamma_\infty) := \gamma_0(x,\lambda)\cdot\gamma'_\infty(x,\lambda) - \gamma'_0(x,\lambda)\cdot\gamma_\infty(x,\lambda) = 1$.

We are about to establish the asymptotics[1] of $\gamma_0(x,\lambda)$ as $x\to\infty$:

$$\gamma_0(x,\lambda) = (2x^\alpha)^{-\frac{1}{2}}\cdot\left(1-\frac{\lambda}{x^{2\alpha}}\right)^{-\frac{1}{4}}\cdot \exp\left(-\int_a^x u^\alpha\left(1-\frac{\lambda}{u^{2\alpha}}\right)^{\frac{1}{2}}du\right). \tag{33}$$

[1] For $\alpha=1$ and $\alpha=\frac{1}{2}$, i.e., for $v(x)=x^2$ and $v(x)=x$, the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ is known.

After expanding here the integral, we obtain the exponential as follows

$$\exp\left(-\int_a^x u^\alpha\cdot\left(1-\frac{1}{2}\cdot\frac{\lambda}{u^{2\alpha}}-\sum_{k=2}^{\infty}\frac{1\cdot 3\cdot\ \ldots\ \cdot(2k-3)}{k!\cdot 2^k}\cdot\left(\frac{\lambda}{u^{2\alpha}}\right)^k\right)du\right). \tag{34}$$

In case $\frac{\alpha+1}{2\alpha}=n\in N$, i.e. $\alpha=\frac{1}{2n-1}$, this expression after integration acquires the form:

$$\begin{gathered}c\cdot\exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \ldots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\\ \cdot\exp\left(\frac{1\cdot 3\cdot\ \ldots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\cdot\ln x+o(1)\right)=\\ =c\cdot\exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \ldots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot\\ \cdot x^{\frac{1\cdot 3\cdot\ \ldots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n}\cdot(1+o(1)).\end{gathered} \tag{35}$$

The asymptotics of $\gamma_0(x,\lambda)$ as $x\to\infty$ is as follows:

$$\begin{gathered}\gamma_0(x,\lambda)=c\cdot\exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \ldots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\\ \cdot x^{\frac{1\cdot 3\cdot\ \ldots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n-\frac{\alpha}{2}}\cdot(1+o(1)).\end{gathered} \tag{36}$$

In particular, for $\alpha=1\ \ (n=1)$, $\gamma_0(x,\lambda)$ has the following asymptotics at infinity:

$$\gamma_0(x,\lambda)=c\cdot x^{\frac{\lambda-1}{2}}\cdot\exp\left(-\frac{x^2}{2}\right)(1+o(1)). \tag{37}$$

In case $\frac{\alpha+1}{2\alpha}\notin N$ we set $n=\left[\frac{\alpha+1}{2\alpha}\right]+1$, with $[\beta]$ being the integral part of β, to obtain the following asymptotics for $\gamma_0(x,\lambda)$ at infinity:

$$\begin{gathered}\gamma_0(x,\lambda)=c\cdot x^{-\frac{\alpha}{2}}\exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \ldots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot\\ \cdot\exp\left(-\frac{1\cdot 3\cdot\ \ldots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\cdot\frac{x^{-\alpha}}{\alpha}\right)\cdot(1+o(x^{-\alpha}))\end{gathered} \tag{38}$$

In particular, with $\alpha=\frac{1}{2}\ \ (n=2)$ one has

$$\gamma_0(x,\lambda)=cx^{-\frac{1}{4}}\cdot\exp\left(-\frac{2}{3}x^{\frac{3}{2}}+\lambda x^{\frac{1}{2}}-\left(\frac{\lambda}{2}\right)^2x^{-\frac{1}{2}}\right)\cdot\left(1+o\left(x^{-\frac{1}{2}}\right)\right). \tag{39}$$

A similar procedure allows to establish the asymptotics of $\gamma_\infty(x)$ as $x\to\infty$. If $\frac{\alpha+1}{2\alpha}=n\in N$, i.e. $\alpha=\frac{1}{2n-1}$, then

$$\gamma_\infty(x,\lambda) = c\cdot\ \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha} - \sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot \\ \cdot x^{-\left(\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n+\frac{\alpha}{2}\right)}\cdot(1+o(1)). \tag{40}$$

With $\alpha=1\ (n=1)$, this becomes

$$\gamma_\infty(x,\lambda) = c\cdot x^{-\frac{\lambda+1}{2}}\cdot\ \exp\left(\frac{x^2}{2}\right)(1+o(1)). \tag{41}$$

In case $\frac{\alpha+1}{2\alpha}\notin N$, we set $n=\left[\frac{\alpha+1}{2\alpha}\right]+1$ to get the asymptotics.

$$\gamma_\infty(x,\lambda) = c\cdot x^{-\frac{\alpha}{2}}\exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot \\ \cdot\exp\left(\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\cdot\frac{x^{-\alpha}}{\alpha}\right)\cdot(1+o(x^{-\alpha})). \tag{42}$$

In case $\alpha=\frac{1}{2}\ (n=2)$, one has

$$\gamma_\infty(x,\lambda) = cx^{-\frac{1}{4}}\cdot\ \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}} + \left(\frac{\lambda}{2}\right)^2 x^{-\frac{1}{2}}\right)\cdot\left(1+o\left(x^{-\frac{1}{2}}\right)\right). \tag{43}$$

Theorem 2.2 *Under* $0<\alpha\le 1$ *and condition (6), the statement of Theorem 2.1 is also valid for Eq. (1).*

Proof is similar to Theorem 2.1. Moreover, note that

$$\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right| = \left|\gamma_0^2(t,\lambda)\cdot\frac{\gamma_\infty(x,\lambda)}{\gamma_0(x,\lambda)} - \gamma_0(t,\lambda)\cdot\gamma_\infty(t,\lambda)\right| = \\ = \left|\frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\cdot\ \exp\left(-2\int_a^t\sqrt{u^{2\alpha}-\lambda}du\right)\cdot\ \exp\left(2\int_a^x\sqrt{u^{2\alpha}-\lambda}du\right) - \frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\right| \\ = \frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\cdot\left|\exp\left(-2\int_x^t\sqrt{u^{2\alpha}-\lambda}du-1\right)\right|. \tag{44}$$

As $x\le t$, one has $\exp\left(-2\int_x^t\sqrt{u^{2\alpha}-\lambda}du\right)\le 1$, and that is why

$$\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right| \le \frac{1}{\sqrt{t^{2\alpha}-\lambda}}. \tag{45}$$

Hence

$$|R(x,t,\lambda)| = \left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\cdot[q(t,\lambda)\cdot I - U(t)]\right| \le \frac{1}{\sqrt{t^{2\alpha}-\lambda}}(|q(t,\lambda)|+|U(t)|). \tag{46}$$

By virtue of (6) and (30), $\frac{1}{\sqrt{t^{2\alpha}-\lambda}}(|q(t,\lambda)|+|U(t)|)\in L(a,\infty)$ and therefore integral equation has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)|\le$ const. By (17), one has that $\lim_{x\to\infty}\chi(x,\lambda)=I$, where the first part of formula (12) follows from.

The remaining statements of Theorem 2.1 are proved similarly. □

From Theorem 2.2 and the asymptotic formulas (37), (39), (41), (43) follows.

Corollary 2.1 *If $\alpha = 1$, i.e. $v(x) = x^2$, then, under condition (6), the solutions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ have common (known) asymptotics, as in the quality $\gamma_0(x, \lambda)$ and $\gamma_\infty(x, \lambda)$ you can take the following functions.*

$$\gamma_0(x, \lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right), ..\gamma_\infty(x, \lambda) = x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right). \tag{47}$$

If $\alpha = \frac{1}{2}$, i.e. the coefficient $v(x) = x$, and the condition (6) holds, then.

$$\gamma_0(x, \lambda) = x^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3}x^{\frac{3}{2}} + \lambda x^{\frac{1}{2}}\right), \gamma_\infty(x, \lambda) = x^{-\frac{1}{4}} \cdot \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}}\right). \tag{48}$$

Remark 2.1 It is known that scalar equation

$$-\varphi'' + x^2 \cdot \varphi = \lambda\varphi \tag{49}$$

for $\lambda = 2n + 1$ has the solution $\varphi_n(x) = H_n(x) \cdot exp\left(-\frac{x^2}{2}\right)$, where $H_n(x)$ is the Chebyshev – Hermitre polynomial, that at $x \to \infty$ has next asymptotics $H_n(x) = (2x)^n(1 + o(1))$. Hence the solution $\varphi_n(x)$ of the Eq. (49) at $x \to \infty$ will have the following asymptotics at infinity: $\varphi_n(x) = (2x)^n \cdot exp\left(-\frac{x^2}{2}\right) \cdot (1 + o(1))$.

In the case of $U(x) = 0, v(x) = x^2$ in (2), the Eq. (1) is splitting into infinity system scalar equations of the form (49). The operator solution $\Phi(x, \lambda)$ will be diagonal in this case. Denote by $\varphi(x, \lambda)$ the diagonal elements of the operator $\Phi(x, \lambda)$. Then, by Corollary 2.1, the solution $\varphi(x, \lambda)$ will have the following asymptotics at infinity: $\varphi(x, \lambda) = (x)^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)(1 + o(1))$. In particular, for $\lambda = 2n + 1$, this yields the solution proportional to $\varphi_n(x)$.

In this section, the asymptotics of the fundamental system of solutions for the Sturm-Liouville equation with block-triangular operator potential, increasing at infinity is established. One of the solutions is found decreasing at infinity, the other one increasing.

3. Green's function for an operator differential equation with block – triangular coefficients

Let us suppose that at the $x = 0$ given boundary conditions

$$\cos A \cdot \bar{y}'(0) - \sin A \cdot \bar{y}(0) = 0, \tag{50}$$

where A- the block-triangular operator of the same structure as the coefficients of the differential equation, $A_{kk},\ k = \overline{1, r}$- bounded self-adjoint operators in H_k, which satisfy the conditions

$$-\frac{\pi}{2}I_k < < A_{kk} \le \frac{\pi}{2}I_k. \tag{51}$$

Together with the problem (1), (50) we consider the separated system

$$l_k\left[y_k\right] = -y_k'' + (v(x)I_k + U_{kk}(x))y_k = \lambda y_k,\ \ k = \overline{1, r} \tag{52}$$

with the boundary conditions

$$\cos A_{kk} \cdot {y_k}'(0) - \sin A_{kk} \cdot y_k(0) = 0, \quad k = \overline{1, r}. \tag{53}$$

Let L' denote the minimal differential operator generated by differential expression $l[\overline{y}]$ (1) and the boundary condition (50), and let ${L_k}'$, $k = \overline{1, r}$ denote the minimal differential operator on $L_2(H_k, (0, \infty))$ generated by differential expression $l_k\left[y_k\right]$ and the boundary conditions (53). Taking into account the conditions on coefficients, as well as sufficient smallness of perturbations $U_{kk}(x)$ and conditions (51), we conclude that, for every symmetric operator ${L_k}'$, there is a case of limit point at infinity. Hence their self-adjoint extensions L_k are the closures of operators ${L_k}'$ respectively. The operators L_k are semi-bounded below, and their spectra are discrete.

Let L denote the operator extensions L', by requiring that $L_2(\mathbf{H}, (0, \infty))$ be the domain of operator L.

The following theorem is proved in [4].

Theorem 3.1 *Suppose that, for Eq. (1) conditions (3)-(5) are satisfied for* $\alpha > 1$ *or condition (6) for* $0 < \alpha \leq 1$. *Then the discrete spectrum of the operator L is real and coincides with the union of spectra of the self-adjoint operators* L_k, $k = \overline{1, r}$, *i.e.,* $\sigma_d(L) = \cup_{k=1}^{r} \sigma(L_k)$.

Comment 3.1 Note that this theorem contains a statement of the discrete spectrum of the non-self-adjoint operator L only and no allegations of its continuous and residual spectrum.

Along with the Eq. (1) we consider the equation

$$l_1[\overline{y}] = -\overline{y}'' + V^*(x)\overline{y} = \lambda \overline{y} \tag{54}$$

($V^*(x)$ is adjoint to the operator $V(x)$). If the space $\mathbf{H}$ is finite-dimensional, then the Eq. (54) can be rewritten as

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda \tilde{y}, \tag{55}$$

where $\tilde{y} = \left(\tilde{y}_1\ \tilde{y}_2 \dots \tilde{y}_r\right)$ and the equation is called the left.

For operator -functions $Y(x, \lambda), Z(x, \lambda) \in B(\mathbf{H})$ let

$$W\{Z^*, Y\} = Z^{*\prime}\left(x, \overline{\lambda}\right) Y(x, \lambda) - Z^*\left(x, \overline{\lambda}\right) Y'(x, \lambda). \tag{56}$$

If $Y(x, \lambda)$ - operator solution of the Eq. (1), and $Z(x, \lambda)$ - operator solution of Eq. (54), the Wronskian does not depend on x.

Now we denote $Y(x, \lambda)$ and $Y_1(x, \lambda)$ the solutions of the Eqs. (1) and (54), respectively, satisfying the initial conditions

$$Y(0, \lambda) = \cos A, Y'(0, \lambda) = \sin A, Y_1(0, \lambda) = (\cos A)^*, {Y_1}'(0, \lambda) = (\sin A)^*, \ \lambda \in \mathbb{C}. \tag{57}$$

Because the operator function $Y_1^*\left(x, \overline{\lambda}\right)$ satisfies equation

$$-{Y_1^*}''\left(x, \overline{\lambda}\right) + Y_1^*\left(x, \overline{\lambda}\right) \cdot V(x) = \lambda Y_1^*\left(x, \overline{\lambda}\right), \tag{58}$$

the operator function $\tilde{Y}(x, \lambda) =: Y_1^*\left(x, \overline{\lambda}\right)$ is a solution to the left of the equation

$$-\tilde{Y}''(x, \lambda) + \tilde{Y}(x, \lambda) \cdot V(x) = \lambda \tilde{Y}(x, \lambda) \tag{59}$$

and satisfies the initial conditions $\tilde{Y}(0, \lambda) = \cos A$, $\tilde{Y}'(0, \lambda) = \sin A$, $\lambda \in \mathbb{C}$.

Operator solutions of Eq. (54) decreasing and increasing at infinity will be denoted by $\Phi_1(x,\lambda)$, $\Psi_1(x,\lambda)$, and the corresponding solutions of the Eq. (59) denote by $\tilde{\Phi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$. For the system operator solutions $Y(x,\lambda), \tilde{\Phi}(x,\lambda) \in B(\mathbf{H})$ of the Eqs. (1) and (59), respectively, will take the form of Wronskian $W\{\tilde{\Phi}, Y\} = \tilde{\Phi}'(x,\lambda)Y(x,\lambda) - \tilde{\Phi}(x,\lambda)Y'(x,\lambda)$ and do not depend on x.

Let us designate

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\{\tilde{\Phi}, Y\}\right)^{-1}\tilde{\Phi}(t,\lambda) & 0 \le x \le t \\ -\Phi(x,\lambda)\left(W\{\tilde{Y}, \Phi\}\right)^{-1}\tilde{Y}(t,\lambda) & x \ge t \end{cases}. \tag{60}$$

In the following theorem it is proved that the operator function $G(x,t,\lambda)$ possesses all the classical properties of the Green's function.

Theorem 3.2 *The operator function $G(x,t,\lambda)$ is the Green's function of the differential operator L, i.e.:*

1. *The function $G(x,t,\lambda)$ is continuous for all values $x,t \in [0,\infty)$;*

2. *For any fixed t, the function $G(x,t,\lambda)$ has a continuous derivative with respect to x on each of the intervals $[0,t)$ and (t,∞), and at $x = t$ it has the jump*

$$G_x{}'(x+0,x,\lambda) - G_x{}'(x-0,x,\lambda) = -I. \tag{61}$$

3. *For a fixed t, the function $G(x,t,\lambda)$ of the variable x is an operator solution of Eq. (1) on each of the intervals $[0,t)$, (t,∞), and it satisfies the boundary condition (50), and at a fixed x function $G(x,t,\lambda)$ of the variable t is an operator solution of the Eq. (59) on each of the intervals $[0,x)$, (x,∞), and it satisfies the boundary condition $\tilde{y}'(0)\cdot\cos A - \tilde{y}(0)\cdot\sin A = 0$.*

Proof The function $G(x,t,\lambda)$ is continuous with respect to x at each of the intervals $[0,t)$ and (t,∞). Similarly to the variable t. To prove the continuity of the function $G(x,t,\lambda)$ for all $x,t \ge 0$, it is sufficient that the identity shown as

$$Y(x,\lambda)\left(W\{\tilde{\Phi}, Y\}\right)^{-1}\tilde{\Phi}(x,\lambda) + \Phi(x,\lambda)\left(W\{\tilde{Y}, \Phi\}\right)^{-1}\tilde{Y}(x,\lambda) \equiv 0. \tag{62}$$

is satisfied for all $x \ge 0$. This identity shown as

$$\begin{gathered} Y(x,\lambda)\left(\tilde{\Phi}(x,\lambda)Y'(x,\lambda) - \tilde{\Phi}'(x,\lambda)Y(x,\lambda)\right)^{-1}\tilde{\Phi}(x,\lambda) - \\ -\Phi(x,\lambda)\left(\tilde{Y}'(x,\lambda)\Phi(x,\lambda) - \tilde{Y}(x,\lambda)\Phi'(x,\lambda)\right)^{-1}\tilde{Y}(x,\lambda) \equiv 0 \end{gathered} \tag{63}$$

or

$$\left(Y'(x,\lambda)Y^{-1}(x,\lambda) - \tilde{\Phi}^{-1}(x,\lambda)\tilde{\Phi}'(x,\lambda)\right)^{-1} \equiv \left(\tilde{Y}^{-1}(x,\lambda)\tilde{Y}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda)\right)^{-1},$$

$$Y'(x,\lambda)Y^{-1}(x,\lambda) - \tilde{\Phi}^{-1}(x,\lambda)\tilde{\Phi}'(x,\lambda) \equiv \tilde{Y}^{-1}(x,\lambda)\tilde{Y}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda), \tag{64}$$

which is equivalent to

$$Y'(x,\lambda)Y^{-1}(x,\lambda) - \tilde{Y}^{-1}(x,\lambda)\tilde{Y}'(x,\lambda) \equiv \tilde{\Phi}^{-1}(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda) \tag{65}$$

or to.

$$\tilde{Y}^{-1}(x,\lambda)\left(\tilde{Y}(x,\lambda)Y'(x,\lambda)-\tilde{Y}'(x,\lambda)Y(x,\lambda)\right)Y^{-1}(x,\lambda)\equiv$$
$$\equiv-\tilde{\Phi}^{-1}(x,\lambda)\left(\tilde{\Phi}(x,\lambda)\Phi'(x,\lambda)-\tilde{\Phi}'(x,\lambda)\Phi^{-1}(x,\lambda)\right)\Phi^{-1}(x,\lambda). \tag{66}$$

This follows from the fact that $W\{\tilde{Y},Y\}=W\{\tilde{\Phi},\Phi\}=0$.

To make sure that the jump in the first derivative at $t=x$ is equal to $(-I)$, i.e., that the equality (61) holds, it is sufficient to prove the identity

$$Y'(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(x,\lambda)+\Phi'(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(x,\lambda)\equiv I. \tag{67}$$

Now we consider the function

$$C(x,t,\lambda)=Y(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(t,\lambda)+\Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(t,\lambda), \tag{68}$$

which is an analogue of the Cauchy function. This function is the solution of Eq. (1) of the variable x, and it is the solution of Eq. (59) of the variable t. By (62), we have $C(x,x,\lambda)\equiv 0$. But in this case $C_{xx}''|_{t=x}=(V(x)-\lambda I)C|_{t=x}\equiv 0$, and, therefore, $C_x'(x,t,\lambda)|_{t=x}\equiv\Omega_1(\lambda)$, i.e.,

$$Y'(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(x,\lambda)+\Phi'(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(x,\lambda)\equiv\Omega_1(\lambda). \tag{69}$$

It shows that $\Omega_1(\lambda)=I$, we obtain (61).

Since operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ form a fundamental system of solutions of Eq. (1), the operator solution $Y(x,\lambda)$ of Eq. (1) satisfying the initial conditions (57), can be written as $Y(x,\lambda)=\Phi(x,\lambda)A(\lambda)+\Psi(x,\lambda)B(\lambda)$, where $A(\lambda)=-W\{\tilde{\Psi},Y\}, B(\lambda)=W\{\tilde{\Phi},Y\}$,

$$Y(x,\lambda)=\Psi(x,\lambda)W\{\tilde{\Phi},Y\}-\Phi(x,\lambda)W\{\tilde{\Psi},Y\}. \tag{70}$$

Similarly, operator solution $\tilde{Y}(x,\lambda)$ of Eq. (59) can be represented in the form

$$\tilde{Y}(x,\lambda)=\tilde{W}\{\tilde{\Phi},Y\}\tilde{\Psi}(x,\lambda)-\tilde{W}\{\tilde{\Psi},Y\}\tilde{\Phi}(x,\lambda), \tag{71}$$

where

$$\tilde{W}\{\tilde{\Phi},Y\}=\sin A\cdot\Phi(0,\lambda)-\cos A\cdot\Phi'(0,\lambda)=-\Omega(0,\lambda)=-W\{\tilde{Y},\Phi\}. \tag{72}$$

Similarly we get $\tilde{W}\{\tilde{\Psi},Y\}=-W\{\tilde{Y},\Psi\}$. Thus,

$$\tilde{Y}(x,\lambda)=W\{\tilde{Y},\Psi\}\tilde{\Phi}(x,\lambda)-W\{\tilde{Y},\Phi\}\tilde{\Psi}(x,\lambda). \tag{73}$$

Substituting (70) and (73) into the formula (69), using the fact that the equality (69) is performed on x identically, we obtain

$$\Omega_1(\lambda)=\lim_{x\to\infty}\left[\Psi'(x,\lambda)\tilde{\Phi}(x,\lambda)-\Phi'(x,\lambda)\tilde{\Psi}(x,\lambda)\right]. \tag{74}$$

By Theorem 2.1, on the asymptotic behavior of functions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ at infinity, we have

$$\Omega_1(\lambda) = \lim_{x\to\infty} [\gamma_0(x,\lambda)\gamma_\infty{}'(x,\lambda) - \gamma_0{}'(x,\lambda)\gamma_\infty(x,\lambda)] \cdot I = W\{\gamma_0, \gamma_\infty\} \cdot I = I. \qquad (75)$$

This completes the proof of the formula (61), and with it the theorem 3.1. □

Corollary. *By the definition (60), function $G(x,t,\lambda)$ is meromorphic of the parameter λ with the poles coincide with the eigenvalues of the operator L.*

We constructed Green's function for the non-self-adjoint differential operator.

4. Resolvent for an non-self-adjoint operator differential equation with block – triangular coefficients

We consider the operator R_λ defined in $L_2(\mathbf{H},(0,\infty))$ by the relation

$$\left(R_\lambda \bar{f}\right)(x) = \int_0^\infty G(x,t,\lambda)\bar{f}(t)dt = \qquad (76)$$

$$= -\int_0^x \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(t,\lambda)\bar{f}(t)\,dt + \int_x^\infty Y(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(t,\lambda)\bar{f}(t)dt.$$

Theorem 4.1 *The operator R_λ is the resolvent of the operator$L$.*

Proof One can directly verify that, for any function $\bar{f}(x)\in L_2(\mathbf{H},(0,\infty))$, the vector-function $\bar{y}(x,\lambda) = \left(R_\lambda \bar{f}\right)(x)$ is a solution of the equation $l[\bar{y}] - \lambda\bar{y} = \bar{f}$ whenever $\lambda \notin \sigma(L)$. We will prove that $\bar{y}(x,\lambda)\in L_2(\mathbf{H},(0,\infty))$.

Since operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ form a fundamental system of solutions of Eq. (1), the operator solution $Y(x,\lambda)$ of Eq. (1) satisfying the initial conditions (57), can be written as $Y(x,\lambda) = \Phi(x,\lambda)A(\lambda) + \Psi(x,\lambda)B(\lambda)$, where $A(\lambda) = W\{\tilde{\Psi},Y\}, B(\lambda) = -W\{\tilde{\Phi},Y\}$,

$$Y(x,\lambda) = \Phi(x,\lambda)W\{\tilde{\Psi},Y\} - \Psi(x,\lambda)W\{\tilde{\Phi},Y\}. \qquad (77)$$

Similarly, the operator solution $\tilde{Y}(x,\lambda)$ of Eq. (59) can be represented in the following form

$$\tilde{Y}(x,\lambda) = W\{\tilde{Y},\Phi\}\tilde{\Psi}(x,\lambda) - W\{\tilde{Y},\Psi\}\tilde{\Phi}(x,\lambda). \qquad (78)$$

By using formulas (77) and (78), we can rewrite the relation (76) as follows:

$$(R_\lambda f)(x) = -\int_0^a \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(t,\lambda)f(t)\,dt + \bar{\chi}_1(x,\lambda) - \bar{\chi}_2(x,\lambda) + \bar{\chi}_3(x,\lambda) - \bar{\chi}_4(x,\lambda), \qquad (79)$$

where $a > 0$ and

$$\bar{\chi}_1(x,\lambda) = \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}W\{\tilde{Y},\Psi\}\int_a^x \tilde{\Phi}(t,\lambda)\bar{f}(t)\,dt, \qquad (80)$$

$$\bar{\chi}_2(x,\lambda) = \Phi(x,\lambda)\int_a^x \tilde{\Psi}(t,\lambda)\bar{f}(t)\,dt\ , \qquad (81)$$

$$\overline{\chi}_3(x,\lambda) = \Phi(x,\lambda)\, W\{\tilde{\Psi}, Y\}\left(W\{\tilde{\Phi}, Y\}\right)^{-1} \int_x^{\infty} \tilde{\Phi}\,(t,\lambda)\overline{f}(t)\,dt, \tag{82}$$

$$\overline{\chi}_4(x,\lambda) = \Psi(x,\lambda) \int_x^{\infty} \tilde{\Phi}\,(t,\lambda)\overline{f}(t)\,dt. \tag{83}$$

Let us show that each of these vector-functions $\overline{\chi}_1(x,\lambda), \overline{\chi}_2(x,\lambda), \overline{\chi}_3(x,\lambda), \overline{\chi}_4(x,\lambda)$ belongs to $L_2(\mathbf{H}, (0,\infty))$. Since the operator solution $\Phi(x,\lambda)$ decays fairly quickly as $x \to \infty$, then $|\Phi(x,\lambda)| \in L_2(0,\infty)$. It follows that

$$|\overline{\chi}_1(x,\lambda)| \le c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\int_a^x \left|\tilde{\Phi}(t,\lambda)\right|\cdot\left|\overline{f}(t)\right|dt \le$$

$$\le c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\left(\int_a^x\left|\tilde{\Phi}(t,\lambda)\right|dt\right)^{\frac{1}{2}}\cdot\left(\int_a^x\left|\overline{f}(t)\right|dt\right)^{\frac{1}{2}} <$$

$$< c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\left(\int_a^{\infty}\left|\tilde{\Phi}(t,\lambda)\right|dt\right)^{\frac{1}{2}}\cdot\left(\int_a^{\infty}\left|\overline{f}(t)\right|dt\right)^{\frac{1}{2}} \le c_1(\lambda)\cdot|\Phi(x,\lambda)|, \tag{84}$$

and therefore $\overline{\chi}_1(x,\lambda) \in L_2(\mathbf{H}, (0,\infty))$. Similarly we get that $\overline{\chi}_3(x,\lambda) \in L_2(\mathbf{H}, (0,\infty))$. First we prove the assertion for the function $\overline{\chi}_2(x,\lambda)$, when $\alpha > 1$ and the coefficients of the Eq. (1) satisfy the conditions (3)-(5). In this case, we have $|\overline{\chi}_2(x,\lambda)| \le |\Phi(x,\lambda)| \int_a^x \left|\tilde{\Psi}\,(t,\lambda)\right|\left|\overline{f}(t)\right|dt$.

By virtue of the asymptotic formulas for the operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ we obtain that

$$|\overline{\chi}_2(x,\lambda)| \le c_1(\lambda)\gamma_0(x,\lambda)\int_a^x \gamma_{\infty}(t,\lambda)\left|\overline{f}(t)\right|dt. \tag{85}$$

Let us rewrite this relation in the following form

$$|\overline{\chi}_2(x,\lambda)| \le c_1(\lambda)\gamma_0(x,\lambda)\gamma_{\infty}(x,\lambda)\int_a^x \frac{\gamma_{\infty}(t,\lambda)}{\gamma_{\infty}(x,\lambda)}\left|\overline{f}(t)\right|dt. \tag{86}$$

By using the definition of the functions $\gamma_0(x,\lambda)$ and $\gamma_{\infty}(x,\lambda)$ (see (9)) and by applying the Cauchy-Bunyakovskii inequality we obtain

$$|\overline{\chi}_2(x,\lambda)| \le \frac{1}{2}c_1(\lambda)\frac{1}{\sqrt{v(x)}}\left(\int_a^x\sqrt{\frac{v(x)}{v(t)}}\exp\left(-2\int_t^x\sqrt{v(u)}du\right)dt\right)^{\frac{1}{2}}\left(\int_0^{\infty}\left|\overline{f}(t)\right|^2dt\right)^{\frac{1}{2}}. \tag{87}$$

Since $t \le x$, we get $\exp\left(-2\int_t^x\sqrt{v(u)du}\right) \le 1$, and then the latter estimate for $\chi_2(x,\lambda)$ can be rewritten as follows

$$|\overline{\chi}_2(x,\lambda)| \le c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^x \frac{1}{\sqrt{v(t)}}dt\right)^{\frac{1}{2}} \le c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^\infty \frac{1}{\sqrt{v(t)}}dt\right)^{\frac{1}{2}}. \tag{88}$$

By formula (3), we get $|\overline{\chi}_2(x,\lambda)| \le \frac{c_3(\lambda)}{\sqrt[4]{v(x)}}$, and hence, if $\alpha > 1$ and the coefficients of the Eq. (1) satisfy the conditions (3)-(5), we have $\overline{\chi}_2(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. In the case of $v(x) = x^{2\alpha}, 0 < \alpha \le 1$, the assertion can be proved similarly.

For the function $\overline{\chi}_4(x,\lambda)$ we will conduct the proof for the case when $v(x) = x^{2\alpha}, 0 < \alpha \le 1$ and the coefficients of the Eq. (1) satisfy the condition (6). As in (85) we have $|\overline{\chi}_4(x,\lambda)| \le c_1(\lambda)\gamma_\infty(x,\lambda)\int_x^\infty \gamma_0(t,\lambda)\left|\overline{f}(t)\right|dt$, which can be rewritten as follows $|\overline{\chi}_4(x,\lambda)| \le c_1(\lambda)\gamma_0(x,\lambda)\gamma_\infty(x,\lambda)\int_x^\infty \frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|\overline{f}(t)\right|dt$.

Let us use the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$, for example, in the case $\frac{\alpha+1}{2\alpha} = n \in N$, i.e. $\alpha = \frac{1}{2n-1}$ (see (36) and (40)). Setting $a(\alpha,\lambda) = \frac{1\cdot 3\cdot \ldots \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n$, we obtain

$$|\overline{\chi}_4(x,\lambda)| \le c_2(\lambda)x^{-\alpha}\int_x^\infty \frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|\overline{f}(t)\right|dt \le c_2(\lambda)x^{-\alpha}\left(\int_a^x \left(\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right)^2 dt\right)^{\frac{1}{2}}\left(\int_0^\infty \left|\overline{f}(t)\right|^2 dt\right)^{\frac{1}{2}}, \tag{89}$$

$$|\overline{\chi}_4(x,\lambda)| \le c_3(\lambda)x^{-\alpha}\left(\int_x^\infty \left(\frac{t}{x}\right)^{2a(\alpha,\lambda)-\alpha}\exp\frac{-2x^{\alpha+1}\left(\left(\frac{t}{x}\right)^{\alpha+1}-1\right)}{1+\alpha}dt\right)^{\frac{1}{2}}. \tag{90}$$

Replacing variables $t = xu$, we get

$$|\overline{\chi}_4(x,\lambda)| \le c_3(\lambda)x^{-\alpha+\frac{1}{2}}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\exp\frac{-2x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha}du\right)^{\frac{1}{2}}. \tag{91}$$

Since the inequality $\exp\frac{-x^{\alpha+1}\left(u^{\alpha+1}-1\right)}{1+\alpha} \le x^{-2}$ holds for all $\alpha \in (0,1]$ and $u \in [1,\infty)$ with sufficiently large $u \in [1,\infty)$, we have

$$|\overline{\chi}_4(x,\lambda)| \le c_3(\lambda)x^{-\alpha-\frac{1}{2}}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\exp\frac{-x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha}du\right)^{\frac{1}{2}}. \tag{92}$$

Hence it follows that $|\overline{\chi}_4(x,\lambda)| \le c_4(\alpha,\lambda)x^{-\alpha-\frac{1}{2}}$, therefore $\overline{\chi}_4(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. In case, where $0 < \alpha \le 1$ and $\frac{\alpha+1}{2\alpha} \notin N$ and where $\alpha > 1$, the proof is similar.

Thus, $R_\lambda\overline{f} \in L_2(\mathbf{H},(0,\infty))$ for any function $\overline{f} \in L_2(\mathbf{H},(0,\infty))$. □

Since the resolvent R_λ is a meromorphic function of λ, the poles of which coincide with the eigenvalues of the operator L, the statement of Theorem 3.1 can be refined.

Theorem 4.2 *If the conditions (3)-(5)) where*$\alpha > 1$ *or condition (6) where* $0 < \alpha \le 1$ *are satisfied for the Eq. (1), then the spectrum of the operator L is real, discrete and coincides with the union of spectra of self-adjoint operators* L_k, $k = \overline{1,m}$, *i.e.* $\sigma(L) = \cup_{k=1}^r \sigma(L_k)$.

In this section, a resolvent for a non-self-adjoint differential operator with a block-triangular operator potential, increasing at infinity, is constructed. Sufficient conditions under which the spectrum is real and discrete are obtained.

5. Spectral singularities of differential operator with triangular matrix coefficients

Remark 5.1 *If the perturbation $U(x)$ in Eq. (1) does not satisfy conditions (3)-(5) or condition (6), then the statement of Theorem 4.2 ceases to be true, which is shown by the following example.*

Example 5.1 Consider the equation:

$$l[\bar{y}] = -\bar{y}'' + \begin{pmatrix} x^2 & q(x) \\ 0 & \pi^2 x^2 \end{pmatrix}\bar{y} = \lambda\bar{y}, \quad 0 \le x < \infty, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{93}$$

with the boundary condition

$$\bar{y}(0) = 0. \tag{94}$$

Together with the problem (93), (94), consider the separated system

$$l_1[y_1] = -{y_1}'' + x^2 y_1 = \lambda y_1, \tag{95}$$

$$l_2[y_2] = -y_2'' + \pi^2 x^2 y_2 = \lambda y_2 \tag{96}$$

with the boundary conditions.

$$y_1(0) = 0, y_2(0) = 0. \tag{97}$$

As above, denote by L_0 the differential operator generated by the differential expression $l[\bar{y}]$ (93) and the boundary condition (94), and by L_1, L_2 denote the minimal symmetric operators on $L_2(0;\infty)$, generated by the differential expressions $l_1[y_1], l_2[y_2]$ and the boundary conditions (97). Their self-adjoint extensions $\tilde{L_1}, \tilde{L_2}$ are the closures of the operators L_1, L_2, respectively. The operators $\tilde{L_1}, \tilde{L_2}$ are semi-bounded; let us denote their spectra by $\sigma_1 = \sigma(\tilde{L_1}), \quad \sigma_2 = \sigma(\tilde{L_2})$.

The Eq. (95) (cf. (49)) has the solution $y_{1,n}(x) = H_n(x) \cdot exp\left(-\frac{x^2}{2}\right)$ for $\lambda = 2n + 1$. Since $H_{2n+1}(0) = 0$, the eigenvalues of the operator $\tilde{L_1}$ are $\lambda_n = 4n + 3$. The sets σ_1 and σ_2 do not intersect.

Denote by L the extension of the operator L_0 generated by the requirement on the functions from the domain of the operator L to belong to $L_2(H_2, (0;\infty))$, and by $\sigma(L)$ its spectrum.

Denote by $Y(x,\lambda) = \begin{pmatrix} y_{11}(x,\lambda) & y_{12}(x,\lambda) \\ 0 & y_{22}(x,\lambda) \end{pmatrix}$ the matrix solution of the Eq. (93), satisfying the initial conditions $Y(0,\lambda) = 0, Y'(0,\lambda) = I$.

If some $\lambda_0 \in \sigma(\tilde{L}_1)$, and $y(x,\lambda_0)$- is the corresponding eigenfunction of the operator $\tilde{L}_1$, then the vector function $\bar{y}(x,\lambda_0) = \begin{pmatrix} y(x,\lambda_0) \\ 0 \end{pmatrix}$ is the eigenfunction of the operator L, corresponding to the eigenvalue λ_0, i.e. $\lambda_0 \in \sigma(L)$. Moreover, $\lambda_0 \in \sigma(\tilde{L}_2)$ is the eigenvalue of the operator L if and only if the solution $y_{12}(x,\lambda_0)$ of the equation

$$-y_{12}{}'' + x^2 y_{12} + q(x) y_{22} = \lambda_0 y_{12}, \tag{98}$$

satisfying the initial conditions $y_{12}(0,\lambda) = y_{12}{}'(0,\lambda) = 0$, belongs to $L_2(0;\infty)$. Let $u(x,\lambda), v(x,\lambda)$ be the solutions of the Eq. (95), satisfying the initial conditions $u(0,\lambda) = 0, u'(0,\lambda) = 1, v(0,\lambda) = -1, v'(0,\lambda) = 0$, and let $C(x,t,\lambda) = u(x,\lambda)v(t,\lambda) - v(x,\lambda)u(t,\lambda)$- be the Cauchy function of the Eq. (95). Then the solution $y_{12}(x,\lambda_0)$ is given by

$$y_{12}(x,\lambda_0) = \int_0^x q(t) \cdot C(x,t,\lambda_0) \cdot y_{22}(t,\lambda_0)\, dt. \tag{99}$$

Choose the coefficient $q(x) = y_{22}(x,\lambda_0)e^{x^\mu}$, where $\mu > 2$ (for instance, $\mu = 4$), and show that the integral $\int_0^\infty y_{12}^2(x,\lambda_0)\,dx$ diverges and, consequently, $\lambda_0 \notin \sigma(L)$. Indeed, since the solution $y_{22}(x,\lambda_0)$ has finitely many zeros, we conclude that, for any $x \geq N_1 > 0$,

$$y_{22}(x,\lambda_0) \geq c_1 e^{-\alpha x^2}, \alpha > 0, \tag{100}$$

and the Cauchy function decays no faster than $e^{-(x-t)^2}$. Hence, if $|x-t| > N_2$, we have

$$C(x,t,\lambda_0) \geq c_2 e^{-(x-t)^2}. \tag{101}$$

In the case of $\frac{x}{4} \leq t \leq \frac{x}{2}$ and $x \geq \max(4N_1, 2N_2)$, the inequalities (100) and (101) are fulfilled simultaneously, therefore, $y_{12}(x,\lambda_0) > c_3 \int_{\frac{x}{4}}^{\frac{x}{2}} e^{t^4} \cdot e^{-2\alpha t^2} \cdot e^{-(x-t)^2}\, dt$. Since $e^{-(x-t)^2} \geq e^{-\frac{x^2}{4}}$ for $t \leq \frac{x}{2}$, we get $y_{12}(x,\lambda_0) > c_3 e^{-\frac{x^2}{4}} \int_{\frac{x}{4}}^{\frac{x}{2}} e^{t^4} \cdot e^{-2\alpha t^2}\, dt$. If x is sufficiently large and $t \in \left[\frac{x}{4}, \frac{x}{2}\right]$, we have $e^{t^4 - 2\alpha t^2} > e^{\frac{1}{2}t^4} \geq e^{\frac{x^4}{32}}$, hence for $x \to \infty y_{12}(x,\lambda_0) > c_3 \frac{x}{4} e^{-\frac{x^2}{4} + \frac{x^4}{32}} \to \infty$. It follows that $y_{12}(x,\lambda_0) \notin L_2(0;\infty)$ and $\lambda_0 \notin \sigma(L)$.

There arises the question on the nature of such values λ.

Consider the equation with a triangular matrix potential:

$$l[\bar{y}] = -\bar{y}'' + \begin{pmatrix} p(x) & q(x) \\ 0 & r(x) \end{pmatrix} \bar{y} = \lambda \bar{y}, \quad 0 \leq x < \infty, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{102}$$

where $p(x), q(x), r(x)$ are scalar functions, $p(x), r(x)$ are real functions and $p(x), r(x) \to \infty$ monotonically as $x \to \infty$.

Let the boundary condition is given at $x = 0$:

$$\cos A \cdot \bar{y}'(0) - \sin A \cdot \bar{y}(0) = 0, \tag{103}$$

where A is a triangular matrix, $cosA = \begin{pmatrix} \cos\alpha_{11} & cos\,\alpha_{12} \\ 0 & \cos\alpha_{22} \end{pmatrix}$.

Consider the separated system

$$l_1[y_1] = -y_1{}'' + p(x)y_1 = \lambda y_1, \tag{104}$$

$$l_2[y_2] = -y_2'' + r(x)y_2 = \lambda y_2. \tag{105}$$

with the boundary conditions

$$\cos\alpha_{11}y_1'(0) - \sin\alpha_{11}y_1(0) = 0, \tag{106}$$

$$\cos\alpha_{22}y_2'(0) - \sin\alpha_{22}y_2(0) = 0. \tag{107}$$

Let L_0 be the differential operator generated by the differential expression $l[\overline{y}]$ (103) and the boundary condition (104), and let L_1, L_2 be minimal symmetric operators on $L_2(0,\infty)$ generated by the differential expressions $l_1[y_1], l_2[y_2]$ and the boundary conditions (106), (108) respectively. Denote by $\tilde{L_1}, \tilde{L_2}$ the self-adjoint extensions of the operators L_1, L_2 respectively. The operators $\tilde{L_1}, \tilde{L_2}$ are semi-bounded; let us denote their spectra by σ_1 and σ_2 respectively. Denote by L the extension of the operator L_0 and by $\sigma(L)$ its spectrum.

Let $u(x,\lambda), v(x,\lambda)$ be the solutions of the Eq. (104) with the boundary conditions $u(0,\lambda) = 0,\ u'(0,\lambda) = 1,\ v(0,\lambda) = -1,\ v'(0,\lambda) = 0$. The general solution of the Eq. (104) has the form $\varphi(x,\lambda) = u(x,\lambda) + lv(x,\lambda)$ up to a constant. Choose an l such that the condition $\varphi(b,\lambda) = 0$ holds true. This equality is valid for $l = l(b,\lambda) = -\frac{u(b,\lambda)}{v(b,\lambda)}$ (the solution $v(x,\lambda)$ has finitely many zeros for a fixed λ, hence $v(b,\lambda) \neq 0$ whenever b is sufficiently large). Put $\varphi_{11}^{(b)}(x,\lambda) = u(x,\lambda) + l(b,\lambda)v(x,\lambda)$. Since for the operator L_1 there is the case of a limit point, then, as is known, $l(b,\lambda)$ has a unique limit $m(\lambda)$ as $b \to \infty$, and the solution of the Eq. (104) satisfies $\varphi_{11}(x,\lambda) = u(x,\lambda) + m(\lambda)v(x,\lambda) \in L_2(0,\infty)$. Similarly we obtain that the solution of the Eq. (105) satisfies $\varphi_{22}(x,\lambda) \in L_2(0,\infty)$.

Denote by $\Phi_b(x,\lambda) = \begin{pmatrix} \varphi_{11}^{(b)}(x,\lambda) & \varphi_{12}^{(b)}(x,\lambda) \\ 0 & \varphi_{22}^{(b)}(x,\lambda) \end{pmatrix}$ the matrix solution of the Eq. (103) satisfying the initial conditions $\Phi_b(b,\lambda) = 0,\ \Phi_b{}'(b,\lambda) = I$. We have $\varphi_{11}^{(b)}(x,\lambda) \to \varphi_{11}(x,\lambda) \in L_2(0,\infty)$; $\varphi_{22}^{(b)}(x,\lambda) \to \varphi_{22}(x,\lambda) \in L_2(0,\infty)$ as $b \to \infty$.

The solution $\varphi_{12}^{(b)}(x,\lambda)$ is given by $\varphi_{12}^{(b)}(x,\lambda) = \int_0^x q(t)\cdot C(x,t,\lambda)\cdot \varphi_{22}^{(b)}(t,\lambda)\,dt$, where $C(x,t,\lambda) = u(x,\lambda)v(t,\lambda) - v(x,\lambda)u(t,\lambda)$ is the Cauchy function of the Eq. (104).

Further, we have $\varphi_{12}^{(b)}(x,\lambda) \to \int_0^x q(t)\cdot C(x,t,\lambda)\cdot\varphi_{22}(t,\lambda)\,dt \coloneqq \varphi_{12}(x,\lambda)$ as $b \to \infty$.

Put $\Phi(x,\lambda) = \begin{pmatrix} \varphi_{11}(x,\lambda) & \varphi_{12}(x,\lambda) \\ 0 & \varphi_{22}(x,\lambda) \end{pmatrix}$.

Together with the Eq. (102), we consider the left equation.

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda\tilde{y},\quad \tilde{y} = (y_1\ y_2). \tag{108}$$

The matrix solutions of the Eq. (108) will be denoted by $\tilde{\Phi}_b(x,\lambda)$ and $\tilde{\Phi}(x,\lambda)$.

Denote by $Y(x,\lambda)$ and $\tilde{Y}(x,\lambda)$ the solutions of the Eqs. (102) and (108) respectively satisfying the initial conditions

$$Y(0,\lambda) = \cos A,\ Y'(0,\lambda) = \sin A,\ \tilde{Y}(0,\lambda) = \cos A,\ \tilde{Y}'(0,\lambda) = \sin A,\ \ \lambda\in\mathbb{C}. \tag{109}$$

Put

$$G_b(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\left(\tilde{\Phi}_b, Y\right)\right)^{-1}\tilde{\Phi}_b(t,\lambda) & 0 \le x \le t \\ -\Phi_b(x,\lambda)\left(W\left(\tilde{Y},\Phi_b\right)\right)^{-1}\tilde{Y}(t,\lambda) & t \le x \le b \end{cases}. \tag{110}$$

The function $G_b(x,t,\lambda)$ is the Green function of the operator L_b^0 generated by the problem (102), (103), $y(b) = 0$, which spectrum coincides with the union of spectra of the operators $L_{b,1}^0, L_{b,2}^0$ generated by the problems (104), (106), $y_1(b) = 0$ and (105), (107), $y_2(b) = 0$ respectively. Eigenvalues of the operators $L_{b,1}^0$ and $L_{b,2}^0$ tend to ones of the operators $\tilde{L}_1$ and $\tilde{L}_2$ respectively as $b \to \infty$, $\Phi_b(x,\lambda) \to \Phi(x,\lambda)$, $\tilde{\Phi}_b(x,\lambda) \to \tilde{\Phi}(x,\lambda)$, and

$$W(\tilde{Y},\Phi_b) = \cos A \cdot {\Phi_b}'(0,\lambda) - \sin A \cdot \Phi_b(0,\lambda) \to \cos A \cdot \Phi'(0,\lambda) - \sin A \cdot \Phi(0,\lambda) = \\ = W(\tilde{Y},\Phi), W(\tilde{\Phi}_b, Y) \to W(\tilde{\Phi}, Y), \tag{111}$$

$$G_b(x,t,\lambda) \to G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W(\tilde{\Phi},Y)\right)^{-1}\tilde{\Phi}(t,\lambda) & 0 \le x \le t \\ -\Phi(x,\lambda)\left(W(\tilde{Y},\Phi)\right)^{-1}\tilde{Y}(t,\lambda) & t \le x \end{cases}. \tag{112}$$

Poles of the Green function $G(x,t,\lambda)$ of the operator L coincide with the zero set of the determinant $\Delta(\lambda) := \det\Omega(\lambda)$, where.

$$\Omega(\lambda) = W(\tilde{Y},\Phi)\big|_{x=0} = \cos A \cdot \Phi'(0,\lambda) - \sin A \cdot \Phi(0,\lambda). \tag{113}$$

Since the matrices $\cos A, \sin A, \Phi(0,\lambda), \Phi'(0,\lambda)$ are triangle, we have $\Delta(\lambda) = \Delta_1(\lambda)\cdot\Delta_2(\lambda)$, where $\Delta_k(\lambda) = \cos\alpha_{kk}\cdot\varphi'_{kk}(0,\lambda) - \sin\alpha_{kk}\cdot\varphi_{kk}(0,\lambda), k = 1,2$. On the other hand, zeros of the function $\Delta_k(\lambda)$ are eigenvalues of the self-adjoint operator $\tilde{L}_k$. Hence the poles of the Green function $G(x,t,\lambda)$ of the operator L are situated on the real axis, and their set coincides with the union of spectra of the operators $\tilde{L}_1$ and $\tilde{L}_2$.

Consider the operator $R_{\lambda,b}$ defined on $L_2(H_2,(0;b))$ by.

$$\left(R_{\lambda,b}\overline{f}\right)(x) = \int_0^b G_b(x,t,\lambda)\overline{f}(t)dt = -\int_0^x \Phi_b(x,\lambda)\left(W(\tilde{Y},\Phi_b)\right)^{-1}\tilde{Y}(t,\lambda)\overline{f}(t)dt + \\ + \int_x^b Y(x,\lambda)\left(W(\tilde{\Phi}_b,Y)\right)^{-1}\tilde{\Phi}(t,\lambda)\overline{f}(t)dt. \tag{114}$$

One can directly verify that the operator $R_{\lambda,b}$ is the resolvent of the operator L_b^0.

Let $\overline{f}(x)$ be an arbitrary vector function square integrable on $[0,\infty)$. Choose a sequence of finite continuous vector functions $\left\{\overline{f}_n(x)\right\}$ $(n = 1,2,\dots)$ converging in mean square to $\overline{f}(x)$. Substituting $\overline{f}_n$ for $\overline{f}$ in (114) and letting first $b \to \infty$ and then $n \to \infty$, we obtain the following formula for the resolvent R_λ of the operator L: $\left(R_\lambda\overline{f}\right)(x) = \int_0^\infty G(x,t,\lambda)\overline{f}(t)dt$, where the Green function of the operator L is defined by the formula (112).

Theorem 5.1 *The operator R_λ is the resolvent of the operator L. The resolvent's poles coincide with the union of the spectra of the self-adjoint operators $\tilde{L}_1$ and $\tilde{L}_2$.*

Remark 5.2 *As in Example 5.1, if $\lambda_0 \in \sigma(\tilde{L}_2)$ and $\varphi_{12}(x,\lambda_0) \notin L_2(0,\infty)$, then λ_0 is the pole of the resolvent R_λ of the operator L but it is not the eigenvalue of this operator, i.e., λ_0 is the* ***point of the spectral singularity of the operator*** *L.*

Theorem 5.1 implies that, if the rate of the coefficient's growth $q(x)$ of the Eq. (102) is subordinated to one of $p(x)$ and $r(x)$, then the operator L has no spectral singularities, and its spectrum is real and coincides with the union of the spectra of the operators $\tilde{L}_1$ and $\tilde{L}_2$.

For a non-self-adjoint Sturm-Liouville operator with a triangular matrix potential growing at infinity, an example of operator having spectral singularities is constructed. A special role of these points was found first by M.A. Naimark in [16]. The notion "spectral singularity" was introduced later due to J. Schwartz [17] (see also Supplement I in the monograph [3]).

6. Conclusion

We consider the Sturm-Liouville equation with block-triangular, increasing at infinity operator potential. For him, built a fundamental system of solutions, one of which is decreasing at infinity, and the second is growing. The asymptotics of these solutions at infinity is defined. For non-self-adjoint operator generated by such differential expression obtained the Green's function. A resolvent of such an operator is constructed. Sufficient conditions at which a spectrum of such non-self-adjoint differential operator is real and discrete are obtained.

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