Chapter

Critical Mach Numbers of Flow around Two-Dimensional and Axisymmetric Bodies

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Abstract

The paper presents the calculated results obtained by the author for critical Mach numbers of the flow around two-dimensional and axisymmetric bodies. Although the previously proposed method was applied by the author for two media, air and water, this chapter is devoted only to air. The main goal of the work is to show the high accuracy of the method. For this purpose, the work presents numerous comparisons with the data of other authors. This method showed acceptable accuracy in comparison with the Dorodnitsyn method of integral relations and other methods. In the method under consideration, the parameters of the compressible flow are calculated from the parameters of the flow of an incompressible fluid up to the Mach number of the incoming flow equal to the critical Mach number. This method does not depend on the means determination parameters of the incompressible flow. The calculation in software Flow Simulation was shown that the viscosity factor does not affect the value critical Mach number. It was found that with an increase in the relative thickness of the body, the value of the critical Mach number decreases. It was also found that the value of the critical Mach number for the two-dimensional case is always less than for the axisymmetric case for bodies with the same cross-section.

Keywords: compressibility, critical Mach number, air, two-dimensional case, axisymmetric case

1. Introduction

This chapter provides an overview of the results obtained by the author by an approximate method for determining the critical Mach numbers for flows in two-dimensional and axisymmetric cases [1–3].

The high subsonic flow velocities in aerodynamics are a common case since most modern passenger and cargo planes fly at Mach numbers exceeding the critical Mach numbers. The compressibility effect increases with an increase in the free-stream Mach number and with an increase in the perturbations created by the flying body at low Mach numbers. Compressibility problems have been considered by many scientists by various methods [4–20]. Knowing the free-stream Mach number, at which the local velocity somewhere on the surface of the body becomes equal to the local sound velocity, makes it possible to correctly choose the basic aerodynamic equations for a body moving at high speed. It is well known that when the local flow velocity becomes equal to the speed of sound, the aerodynamic equations change their own type from elliptic to hyperbolic. The value of the critical Mach number is the transition from one type of equation to another.

2. The approximate method for calculating flow compressibility characteristics

The equations for the existence of the velocity potential and continuity are written for two cases: two-dimensional and axisymmetric for compressible irrotational gas flow

$$\frac{\partial u}{\partial r} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial (r^n \rho u)}{\partial x} + \frac{\partial (r^n \rho v)}{\partial r} = 0, \tag{1}$$

where u, v are components of the velocity along axis x и r accordingly, *m/s*;, *r, x* are coordinates; ρ is local density of gas, *kg/m*^{3}; parameter n equal

$$n = \begin{cases} 0 & -\ 2\text{D body;} \\ 1 & -\ \text{axisymmetric body.} \end{cases}$$

Here x, r are coordinates in the meridional plane for the axisymmetric case ($n = 1$). Let us introduce the special functions proposed by Burago [20]

$$\tau = \frac{2}{\rho_0/\rho + 1}; \quad \sigma = \frac{\rho_0/\rho - 1}{\rho_0/\rho + 1}, \tag{2}$$

ρ_0 is gas density at the stagnation point. Taking into account equation (Eq. (2)), the equation (Eq. (1)) can be rewritten as

$$\frac{\partial}{\partial r}\left(\frac{\tau u}{1-\sigma}\right) - \frac{\partial}{\partial x}\left(\frac{\tau u}{1-\sigma}\right) = 0, \quad \frac{\partial}{\partial x}\left(\frac{r^n \tau u}{1+\sigma}\right) + \frac{\partial}{\partial r}\left(\frac{r^n \tau v}{1+\sigma}\right) = 0, \tag{3}$$

For barotropic model of compressible air we have

$$\frac{\rho}{\rho_\infty} = \left(\frac{p}{p_\infty}\right)^{\frac{1}{\kappa}}, \tag{4}$$

where p is static pressure, *Pa*; κ is ratio of specific heats (for air $\kappa = 1.4$); ∞ subscript indicates flow parameters at infinity. The equation (Eq. (4)) refers to as Poisson's adiabatic curve.

According to the accepted model of a barotropic gas, the enthalpy and pressure function differ only by a constant value $h = P(p) + \text{const}$. The pressure function is

$$P(p) = \int_{p_\infty}^{p} \frac{dp}{\rho}. \tag{5}$$

If we carry out the integration in formula (Eq. (5)) taking into account formula (Eq. (4)), then we get (Eq. (6))

$$h = \frac{\rho^{\kappa-1}}{(\kappa-1)\rho_\infty^{\kappa-1} M_\infty^2}, \tag{6}$$

here M_∞ is Mach number at infinity.

Using the equation for enthalpy (Eq. (6)), the Bernoulli equation for a compressible gas can be written as

$$\frac{|\mathbf{V}|^2}{2} + \frac{\rho^{\kappa-1}}{(\kappa-1)\rho_\infty^{\kappa-1}M_\infty^2} = \frac{1}{2} + \frac{1}{(\kappa-1)M_\infty^2}, \tag{7}$$

where $|\mathbf{V}|$ is the modulus of the vector of the total local velocity of flow.

Based on formula (Eq. (7)), the following equalities can be written:

$$\frac{\rho}{\rho_\infty} = \left[1 + \frac{\kappa-1}{2}M_{\infty 2}\left(1 - \overline{u}^2 - \overline{v}^2\right)\right]^{\frac{1}{\kappa-1}}, \quad \frac{a}{a_\infty} = \left[1 + \frac{\kappa-1}{2}M_{\infty 2}\left(1 - \overline{u}^2 - \overline{v}^2\right)\right]^{\frac{1}{2}}, \tag{8}$$

here $\overline{u} = u/U_\infty$, $\overline{v} = v/U_\infty$ are the dimensionless components of the local velocity, U_∞ is free-stream flow velocity at infinity.

Let us write the well-known isentropic relations for density and sound speed

$$\frac{\rho_0}{\rho} = \left[1 + \frac{\kappa-1}{2}M^2\right]^{\frac{1}{\kappa-1}}, \quad \frac{a_0}{a} = \left[1 + \frac{\kappa-1}{2}M^2\right]^{\frac{1}{2}}, \tag{9}$$

here lower index "0" specifies parameters in stagnation point.

Formulas (9) can be written as

$$\frac{\rho_0}{\rho} = [E(M)]^{\frac{1}{\kappa-1}}, \quad \frac{a_0}{a} = [E(M)]^{\frac{1}{2}}, \quad E(M) = 1 + \frac{\kappa-1}{2}M^2. \tag{10}$$

The speed of sound is defined by formula $a = \sqrt{dp/d\rho}$. The speed of sound in the case of an isentropic air flow is $a = \sqrt{\kappa p/\rho}$. Based on the equation of state in the form (Eq. (4)), we can write

$$a = a_\infty\left(\frac{\rho}{\rho_\infty}\right)^{\frac{\kappa-1}{2}}. \tag{11}$$

Based on the isentropic relationship for density (Eq. (9)), the function σ (Eq. (2)) can be calculated. **Figure 1** shows a graph of the function σ for air versus Mach number.

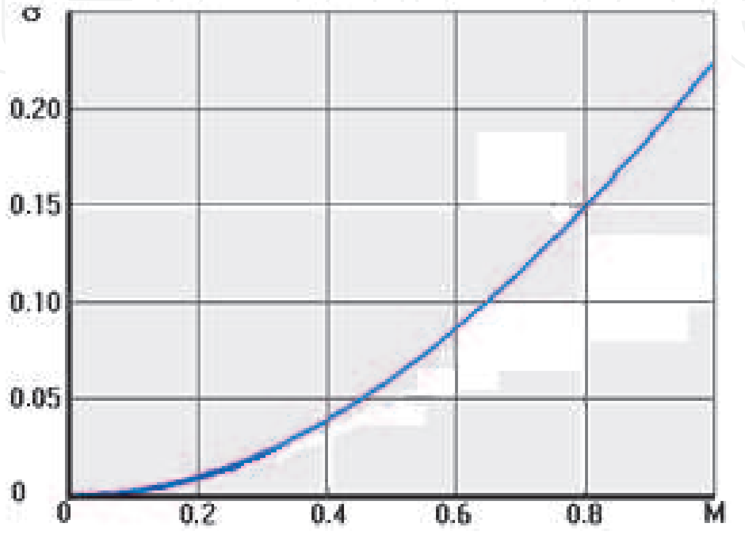

Figure 1.
The function σ vs. Mach number.

In **Figure 1** it is seen that the values σ lie in a narrow range [0; 0.22] for air when changing the Mach number from 0 to 1.0. This narrow range allows us to take $\sigma \approx$ const, and then using formulas (Eq. (2)), we can replace (Eq. (3)) with an approximate

$$\frac{\partial}{\partial x}\left(\frac{r^n \tau u}{\rho_0+\rho}\right)+\frac{\partial}{\partial r}\left(\frac{r^n \tau v}{\rho_0+\rho}\right)=0, \quad \frac{\partial}{\partial r}\left(\frac{\tau u}{\rho_0+\rho}\right)-\frac{\partial}{\partial x}\left(\frac{\tau u}{\rho_0+\rho}\right)=0. \tag{12}$$

If we introduce new variables u° and v° associated with u and v as follows

$$u=\frac{\rho_\infty}{\rho}\frac{(\rho+\rho_0)}{(\rho_\infty+\rho_0)}u^0=\frac{1+\frac{\rho_0}{\rho}}{1+\frac{\rho_0}{\rho_\infty}}u^0=\eta_c u^0; \quad v=\frac{\rho_\infty}{\rho}\frac{(\rho+\rho_0)}{(\rho_\infty+\rho_0)}v^0=\frac{1+\frac{\rho_0}{\rho}}{1+\frac{\rho_0}{\rho_\infty}}v^0=\eta_c v^0. \tag{13}$$

where η_c is introduced

$$\eta_c=\left(1+\frac{\rho_0}{\rho}\right)/\left(1+\frac{\rho_0}{\rho_\infty}\right). \tag{14}$$

The coefficient η_c will be called the coefficient of compressibility. The new speeds (Eq. (13)) allow equations (Eq. (12)) to be written in the form

$$\frac{\partial u^0}{\partial r}-\frac{\partial v^0}{\partial x}=0, \quad \frac{\partial(r^n u^0)}{\partial x}+\frac{\partial(r^n v^0)}{\partial r}=0. \tag{15}$$

Equations (Eq. (15)) repeat equations (Eq. (1)) describing potential flow of an incompressible liquid ($\rho \approx const$) with a velocity having components (u^0, v^0). Equations (Eq. (15)) allow us to assert that the boundary conditions at infinity for a compressible gas flow with velocity components (u, v) will be identical to the corresponding conditions for an incompressible fluid flow with velocity components (u^0, v^0). The boundary conditions on the body surface will also be the same. Indeed, in a compressible flow on the surface of a body v_n = 0. On the body surface in an incompressible fluid flow, based on (Eq. (13)), we can write

$$v_n^0=\frac{\rho}{\rho_\infty}\frac{(\rho_\infty+\rho_0)}{(\rho+\rho_0)}v_n=0. \tag{16}$$

Thus, the equations (Eq. (13)) determine the approximate relationship between the components of the velocities in compressible and incompressible flows around the same body under the same conditions at infinity and on the surface of the body. So, to determine velocities of compressible gas flow, it is necessary to use the known isentropic relationships for the local velocity of a flow and density

$$\frac{|\mathbf{V}|}{a_0}=\frac{M}{\sqrt{1+\frac{\kappa-1}{2}M^2}}, \quad \frac{\rho_0}{\rho_\infty}=\left[1+\frac{\kappa-1}{2}M_\infty^2\right]^{\frac{1}{\kappa-1}}, \tag{17}$$

here M is the local Mach number at an arbitrary point. The left side of the first formula (Eq. 17) can be written as

$$\frac{|\mathbf{V}|}{a_0}=\frac{U_\infty a_\infty\sqrt{\overline{u}^2+\overline{v}^2}}{a_\infty a_0}=\frac{M_\infty\sqrt{\overline{u}^2+\overline{v}^2}}{\sqrt{1+\frac{\kappa-1}{2}M_\infty^2}}. \tag{18}$$

From (Eq. (17)) and (Eq. (18)) it follows

$$M = M_\infty \sqrt{\frac{\overline{u}^2 + \overline{v}^2}{1 + \frac{\kappa - 1}{2} M_\infty^2 \left(1 - \overline{u}^2 - \overline{v}^2\right)}}. \tag{19}$$

Equation (Eq. 19) shows that the local Mach number M is determined by the local dimensionless velocities $\overline{u}$ and $\overline{v}$, which, according equations (Eq. 13) and (Eq. 10) depends on local Mach number. To account for this, we will use successive approximations. We will use the velocities calculated for an incompressible flow, which we will substitute in (Eq. (19)) at the first stage of approximation

$$\overline{u}^{(1)} = u^0 / U_\infty; \quad \overline{v}^{(1)} = v^0 / U_\infty. \tag{20}$$

Thus, based on these values, the local Mach number $M^{(1)}$ for the first approximation is determined. Thereafter, the computed first approximation Mach number $M^{(1)}$ is used to calculate the second approximation velocity components from $\overline{u}^{(2)}$, $\overline{v}^{(2)}$ from (Eqs. (13) and (14)), and isentropic formula for density (Eq. (10)). The obtained values of the velocity components make it possible to calculate on the basis (Eq. (19)), the local Mach number for the second approximation $M^{(2)}$, which, in turn, is used to calculate the local velocity components at the stage of the third approximation $\overline{u}^{(3)}$, $\overline{v}^{(3)}$, and so on. The approximation process must be continued until the following inequality is satisfied $\left|\eta_c^{(n)} - \eta_c^{(n-1)}\right| - \left|\eta_c^{(n-1)} - \eta_c^{(n-2)}\right| \leq \varepsilon$, where ε is a given calculation error. Preliminary calculations using the described algorithm showed that the number of approximation steps increases as $M \to 1.0$, but usually the maximum number of approximations does not exceed 30.

From (Eq. (18) and (19)) we can write

$$\frac{M_\infty \sqrt{\overline{u}^2 + \overline{v}^2}}{\sqrt{1 + \frac{\kappa - 1}{2} M_\infty^2}} = \frac{M}{\sqrt{1 + \frac{\kappa - 1}{2} M^2}}. \tag{21}$$

Putting in the formula (Eq. (21)) $M = 1$ and $M_\infty = M_*$, we can get the formula for calculating the critical Mach number

$$M_* = \sqrt{\frac{1}{\overline{u}^2 + \overline{v}^2 + \frac{\kappa - 1}{2}\left(\overline{u}^2 + \overline{v}^2 - 1\right)}} \equiv \sqrt{\frac{1}{\overline{U}^2 + \frac{\kappa - 1}{2}\left(\overline{U}^2 - 1\right)}}, \tag{22}$$

where the notation $\overline{U}$ is introduced for the local total relative velocity. From (Eq. (22)) it can be seen that the minimal value M_* corresponds to the maximal value $\overline{U}$. Therefore, to calculate the critical Mach number for a flow near a body, it is necessary to determine the maximum local velocity on the body surface. The dimensionless velocities $\overline{u}$ and $\overline{v}$ (Eq. (22)) also should correspond to the critical free-stream Mach number. For this, it is necessary to apply a method of successive approximations, similar to that used to calculate the local Mach number (Eq. (19)). As a first approximation, we will use the velocities $\overline{u}^{(1)}$ and $\overline{v}^{(1)}$ calculated for an incompressible flow. The critical Mach number $M_*^{(1)}$ attained in the first approximation should be considered as the free-stream Mach number for calculating the local relative velocities for the second approximation $\overline{u}^{(2)}$ and $\overline{v}^{(2)}$. The approximation process must be continued until the calculated critical Mach number for subsequent approximations changes by a given error ε, i.e., until the condition is satisfied $\left|M_*^{(i)} - M_*^{(i-1)}\right| \leq \varepsilon$. It turns out that the critical Mach number can be calculated using only the values of the incompressible flow speeds. It should be noted that the method described above is applicable for compressible flows for which the inequality $M_\infty \leq M_*$ is valid. This condition determines the absence of transonic and supersonic zones in the flow field. To calculate the critical Mach number M_* and the velocity field (u, v) in a compressible flow, it is sufficient to calculate the velocity field (u^0, v^0) around the body for an incompressible flow.

It follows from the described method that the critical Mach number for a given 2-D body can be calculated using only the values of the incompressible flow speeds. The method is applicable for compressible flows for which the free-stream Mach numbers are less than the critical Mach number. So, to calculate the critical Mach number, for an incompressible flow, it is sufficient to calculate the velocity field near the streamlined body, but in practical aerodynamics, the pressure distribution on the body surface is obtained experimentally more often than the velocity field. To compare the results of calculations and experimental data, it is convenient to use the pressure-drop coefficient. This circumstance leads to the necessity of obtaining a connection between the pressure field in a compressible flow and the pressure field in an incompressible flow. To establish this connection we write down the pressure-drop coefficient for an incompressible flow

$$c_p^0 = \frac{p^0 - p_\infty}{0,5\rho V_\infty^2} = 1 - \left(\frac{V^0}{V_\infty}\right)^2. \tag{23}$$

Based on formulas (Eq. (13)), the pressure-drop coefficient for an incompressible flow can be represented through the gas flow velocity

$$c_p^0 = 1 - \left(\frac{V}{\eta_c V_\infty}\right)^2. \tag{24}$$

The pressure-drop coefficient for the compressible gas flow can be written

$$c_p = \frac{p - p_\infty}{0.5\rho_\infty V_\infty^2} = 2\frac{p_\infty}{\rho_\infty V_\infty^2}\left(\frac{p}{p_\infty} - 1\right) \tag{25}$$

Let us rewrite the last formula in terms of the Mach number of the free-stream flow. Using the barotropic model of compressed air, we can write the following equalities

$$\frac{p_\infty}{\rho_\infty V_\infty^2} = \frac{1}{\kappa}\frac{a_\infty^2}{V_\infty^2} = \frac{1}{\kappa M_\infty^2},$$

then we rewrite formula (Eq. (25)) in the form

$$c_p = \frac{2}{\kappa M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) = \frac{2}{\kappa M_\infty^2}\left(\frac{p}{p_0}\frac{p_0}{p_\infty} - 1\right). \tag{26}$$

In formula (Eq. (24)), the ratio of the squares of the velocities can be represented as the identity

$$\frac{V^2}{V_\infty^2} = \frac{V^2}{a^2}\frac{a^2}{a_0^2}\frac{a_0^2}{a_\infty^2}\frac{a_\infty^2}{V_\infty^2} = \frac{M^2}{M_\infty^2}\frac{a^2}{a_0^2}\frac{a_0^2}{a_\infty^2}.$$

The use of isentropic formula (Eq. (13)) allows us to write the last equation in the form

$$\frac{V^2}{V_\infty^2} = \frac{M^2}{M_\infty^2} \cdot \frac{E(M_\infty)}{E(M)}.$$

Substitution of the found ratio of the squares of the velocities into the formula (Eq. (24)) for the pressure coefficient of the incompressible flow gives

$$c_p^0 = 1 - \eta_c^2 \frac{M^2}{M_\infty^2} \cdot \frac{E(M_\infty)}{E(M)}. \tag{27}$$

Using isentropic formulas for pressure ratios in the form

$$p_0/p = [E(M)]^{\frac{\kappa}{\kappa-1}}, \quad p_0/p_\infty = [E(M_\infty)]^{\frac{\kappa}{\kappa-1}},$$

we write down the pressure-drop coefficient of the compressible gas (Eq. (26)) as follows

$$c_p = \frac{2}{\kappa M_\infty^2} \left\{ \left[\frac{E(M_\infty)}{E(M)} \right]^{\frac{\kappa}{\kappa-1}} - 1 \right\}. \tag{28}$$

In order to calculate the pressure coefficient for a compressible gas from the known value of the incompressible flow coefficient, we introduce auxiliary functions proposed by G. Burago [20].

$$F(M) = [E(M)]^{\frac{\kappa}{\kappa-1}}, \quad G(M) = \frac{M^2}{E(M)} \cdot \left\{ 1 + [E(M)]^{\frac{1}{\kappa-1}} \right\}^{-2}. \tag{29}$$

Using the new functions, the formulas for the pressure-drop coefficients of an incompressible flow and a compressible gas can be represented in the form

$$G(M) - G(M_\infty)\left(1 - c_p^0\right) = 0, \tag{30}$$

$$c_p = \frac{2}{\kappa M_\infty^2} \left[\frac{F(M_\infty)}{F(M)} - 1 \right]. \tag{31}$$

In order to recalculate the values of the pressure-drop coefficient of the incompressible flow to the pressure-drop coefficient of the compressible gas for a given value of the Mach number of the free-stream flow, it is necessary to solve the transcendental Eq. (30) with respect to the local Mach number, and then use the formulas (31) and (29) to calculate the pressure-drop coefficient of the compressible gas.

The dependence of the critical Mach number on the pressure-drop coefficient of the incompressible flow can be obtained. For this, it is necessary to solve Eq. (30) with respect to the condition $M = 1.0$, i.e. solve an equation of the form

$$G(1.0) - G(M_*)\left(1 - c_p^0\right) = 0. \tag{32}$$

3. Results and discussions

The presented method is approximate; therefore, it is necessary to demonstrate the consistency of this method with other calculation methods in order to analyze the accuracy of the calculations.

The calculation of the dependence of the critical Mach number on the pressure-drop coefficient of an incompressible flow using Eq. (30) is shown in **Figure 2**. **Figure 2** also shows the Khristianovich's curve [13].

Comparison of the two calculations shows good agreement. After the publication of the work of Burago [20] in 1949, a number of approximate methods based on the Chaplygin gas model were developed. Burago's method refers to the approximate mathematical models for accounting for the compressibility of the flow. The effectiveness of approximate mathematical models is always shown by comparison with experimental or calculated data for various bodies. Calculations carried out according to the method described above showed that this method is not inferior in accuracy to more rigorous methods, at the same time the advantage of this method over other methods of accounting for the flow compressibility in terms of speeds is undeniable.

Here are some comparisons. In **Figure 3** shows a comparison of the calculation by the method described above with the calculated data of L. Sedov [5] and by the Glauert formulas ([4], p. 311) and Kárman-Tsien ([4], p. 311). Glauert formula is

$$C_p = \frac{C_p^0}{\sqrt{1 - M_\infty^2}}, \tag{33}$$

and the Kárman-Tsien formula is

$$C_p = \frac{C_p^0}{\sqrt{1 - M_\infty^2} + 0,5\left[1 - \sqrt{1 - M_\infty^2}\right] C_p^0}. \tag{34}$$

In **Figure 3**, for the pressure coefficient, the experimental data are presented as points ([4], Figure 121) and a curve that is an approximation of these points ([5], p.388). Comparison analysis indicates that the calculation by the Glauert formula (34) is approximate and for the Mach number $M_\infty \geq 0,4$ it can be argued that the Glauert formula should not be used. The Kárman-Tsien formula (35) gives a better agreement with the experimental data than the Glauert formula, but for the considered pressure-drop coefficients ($C_p^0 = -0.6$ and $C_p^0 = -0.73$) it is inferior in accuracy to the calculated data of L. Sedov [5] and the data obtained by the author

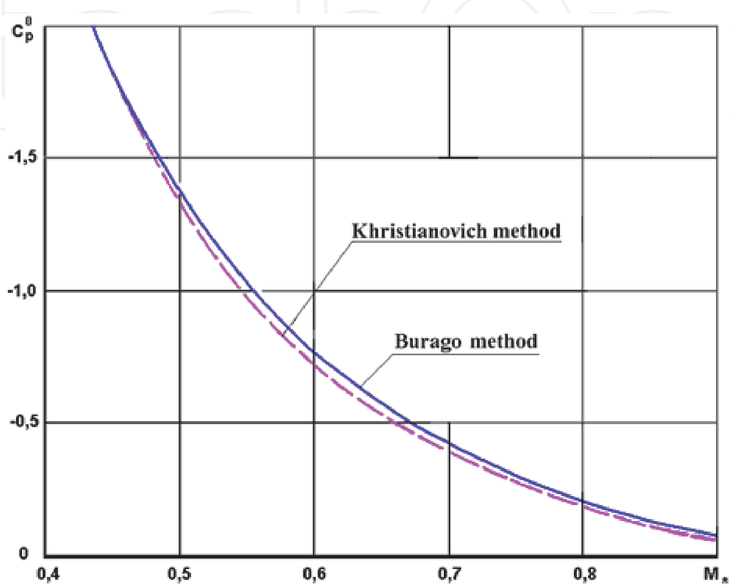

Figure 2.
Comparison of calculation results for critical Mach number based on Burago and Khristianovich methods.

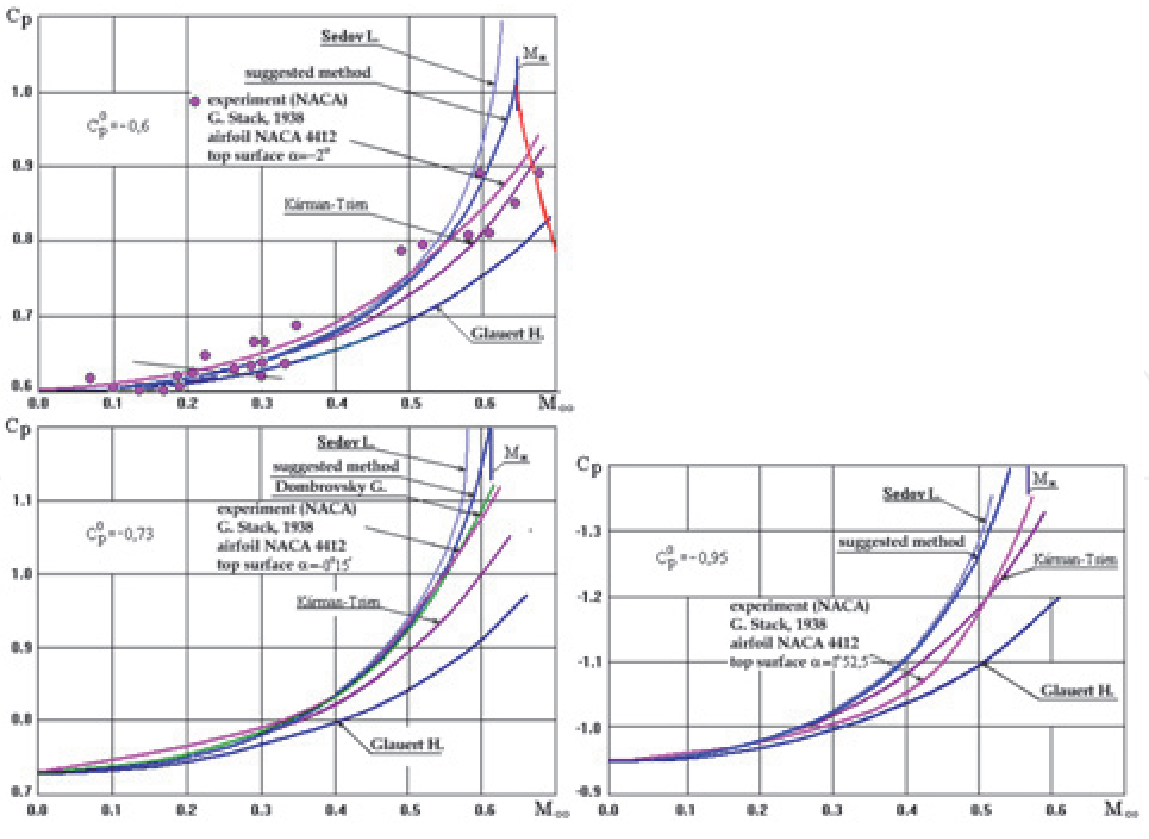

Figure 3.
Comparison of different methods account compressibility.

of the chapter by the method outlined above [1]. It should be noted that the calculated data by Burago's method for all three considered cases are closer to the experimental data of G. Stack [19] than the results of the calculation by the Sedov's method [5].

Accounting for gas compressibility according to the method described above and the Kárman-Tsien method [see. Formula (35)] is based on recalculating the pressure-drop coefficients of an incompressible liquid C_p^0 by the values of the pressure-drop coefficients of a compressible gas C_p for the given free-stream Mach numbers and ratio of specific heatsκ. An interesting question is: how much differ these two methods in a wide range of variation of Mach numbers and pressure coefficients for an incompressible fluid? **Table 1** compares these two methods based on the data in **Figure 3**, from which it follows that the maximum discrepancy increases with decreasing value C_p^0 and at $M_\infty \to M_*$. The maximum relative error of the two methods in the considered range of variation of Mach numbers and pressure coefficients reaches 15–16%.

The advantage of the Burago method over the Kárman-Tsien method is demonstrated in **Figure 4**, which presents the calculated data and experimental data of various authors for two bodies: an ellipse (**Figure 4a**) and a biconvex airfoil (**Figure 4b**) with relative thicknesses of 20% and 6% respectively. **Figure 4a** shows a comparison of the results of calculating the pressure drop coefficient on the surface of an ellipse of relative thickness δ = 0.2 in a compressible flow. The calculations are performed for the critical Mach number M_* = 0.7 (**Table 2**).

The calculated data using the Sells finite difference method (1968) are taken from [16]. It can be noted that the results of the Burago method and the Kárman-Tsien (34) are in good agreement to within approximate values $x \le 0.1$. For values of relative coordinate along the big axis of an ellipse $0.1 \le x \le 0.5$ the significant divergence between the results of the Kárman-Tsien formula (34) and Burago's

C_p^0	Mach number M_∞	C_p Kárman-Tsien	C_p Burago	δ %
−0.1	0.3	−0.1051	−0.1048	0.29
	0.6	−0.1266	−0.1269	0.24
	$M_* = 0.886$	−0.2289	−0.2165	6
−0.5	0.3	−0.5305	−0.5315	0.18
	0.6	−0.6667	−0.7042	5.0
	$M_* = 0.679$	−0.7489	−0.8696	14
−1.0	0.3	−1.0742	−1.0796	0.5
	0.4	−1.1432	−1.1659	2.0
	$M_* = 0.558$	−1.3427	−1.5874	15
−1.5	0.3	−1.6315	−1.6474	1.0
	0.4	−1.7566	−1.8320	4
	$M_* = 0.486$	−1.9245	−2.2790	16
−2.0	0.3	−2.2029	−2.2389	1.6
	0.4	−2.4009	−2.6067	8
	$M_* = 0.437$	−2.5034	−2.9683	16
−2.5	0.3	−2.7890	−2.8618	2.5
	$M_* = 0.400$	−3.0782	−3.6432	16
−3.0	0.3	−3.3904	−3.5221	3.7
	$M_* = 0.371$	−3.6515	−4.3127	15
−3.5	0.3	−4.0076	−4.2352	5
	$M_* = 0.348$	−4.2265	−4.9949	15
−4.0	0.3	−4.6414	−5.0245	8
	$M_* = 0.329$	−4.8020	−5.6830	15
−4.5	0.3	−5.2922	−5.9404	11
	$M_* = 0.312$	−5.3714	−6.3339	15

Table 1.
Comparison Burago's and Kárman-Tsien methods.

method is observed. For the values of the relative coordinate along the major axis of the ellipse $0.1 \le x \le 0.5$, there is a significant discrepancy between the Kárman-Tsien results and the Burago method. However, it can be noted that the calculations of other authors are in good agreement with the calculations by the Kárman-Tsien formula (34). The calculation according to this method approaches the value of the pressure coefficient, indicated in **Figure 4** as C_p^*, which corresponds to the critical Mach number. As can be seen from **Figure 4**, the values determined in the article [16] and by the author practically coincide. **Figure 4** also shows the distribution of the pressure-drop coefficient C_p^0 for an incompressible fluid. **Figure 4** shows the strong influence of the compressibility factor. Of greatest interest to researchers is the value of the critical Mach number for a cylinder in a transverse flow. In **Figure 5 (a, b)**, the results of calculations of the relative velocity on the surface of the cylinder, obtained using the proposed approximate technique, are compared with the calculations of other authors performed by other methods.

Integral relationship method [17], method of Legendre transformations [10], method approximation of adiabat [7], Rayleigh-Janzen method (Simisaki [17]; Shih-I

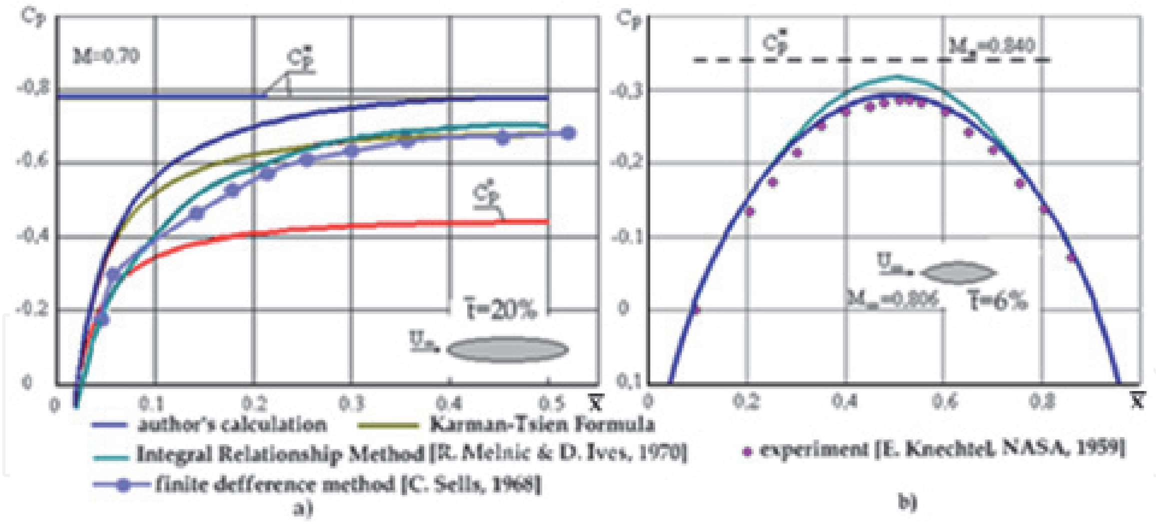

Figure 4.
Comparison of the calculated values of the pressure drop coefficient on the surface of the ellipse (relative thickness 20%) (a); for a biconvex airfoil (relative thickness 6%) (b).

M_*	Method and authors of calculation results	δ %
0.415	C. Jacob's calculation [7]	10
0.409	Poggi method. Calculations by S. Kaplan [12]	9
0.404	I. Imei's calculation	8
0.400	Rayleigh-Janzen method. T. Simisaki's calculations	7
0.399	The method of integral relations by A.A. Dorodnitsyn. calculations by P.I. Chushkin	7
0.39853¤0.00002	The multi-layer method of integrated relations by A.A. Dorodnitsyn. calculations by R. Melnik and D. C. Ives	7
0.3983¤0.0002	The Rayleigh-Janzen method. Hoffman's calculations	7
0.396	The method of approximation of adiabat. Approximation A4. G. A. Dombrovsky	6
0.390	The method of approximation of adiabat. A3 approximation. G. A. Dombrovsky	5
0.390	The method of integral relations by A.A. Dorodnitsyn. Calculations by M. Holt and B. Messon	5
0.390	Calculations by I. Imei. Z. Hashimoto	5
0.37170¤0.00001	author's calculation	—

Table 2.
Comparison of the calculated values for the critical Mach number on the surface of the cylinder.

Pai [6]) are compared in **Figure 5**. The results of Simisaki's calculations were taken from [17]. The calculations were carried out for the Mach number almost equal to the critical Mach number $M_\infty \approx 0.372$ (**Table 2**). It can be noted that in **Figure 5**, the author's calculation is in good agreement with the data of other authors for velocities on the cylinder surface, taking into account the compressibility factor.

Let us consider the numerical values of the critical Mach numbers M_* for a flow around a circular cylinder obtained by different authors. **Table 1** shows a comparison of the results of calculating the critical Mach numbers by different methods. The value δ denotes the relative error of comparing the values of the critical Mach numbers calculated by different methods and by this method.

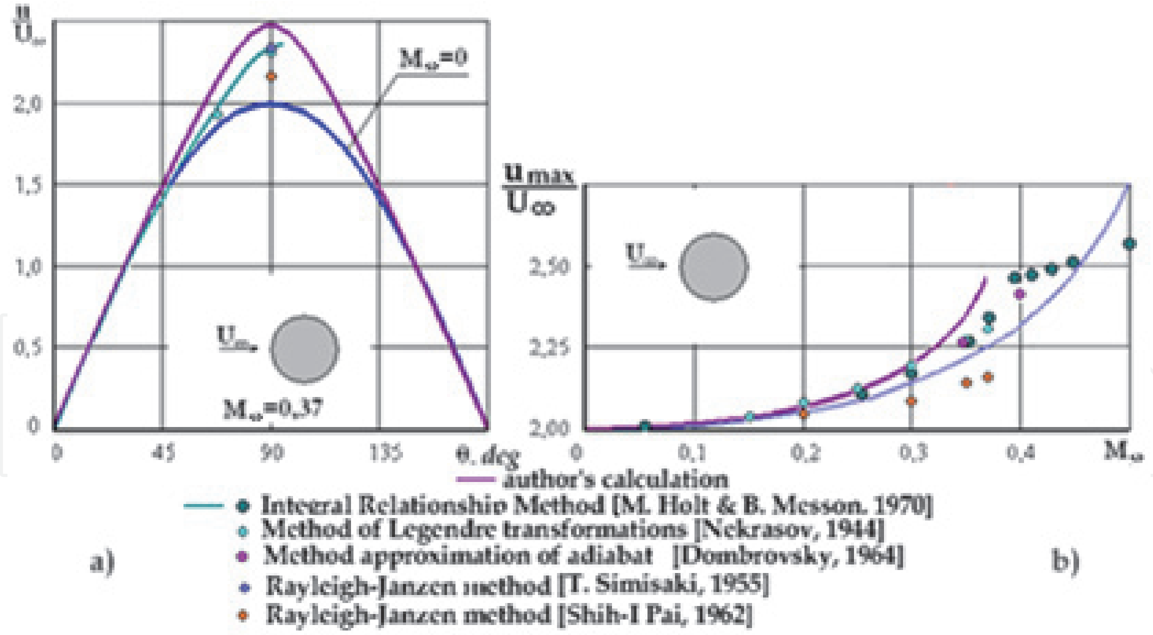

Figure 5.
The velocity on the cylinder surface vs. Mach number.

If we exclude from the comparison the early work of Kaplan [12] in which the adiabatic exponent for air was taken equal κ = 1.408 which affected the value of the calculated critical Mach number and the work of Jacob (see [7]) then based on the data in **Table 1** it can be argued that the accuracy of calculating the critical Mach number on the surface of the circular cylinder by the Burago's method in comparison with other methods on average approximately corresponds to a relative error of 5%. From **Table 1** it can be seen that the critical Mach number on the surface of the circular cylinder calculated by Burago's method is the smallest of the data presented in the **Table 1**. This does not mean that this value is the most inaccurate compared

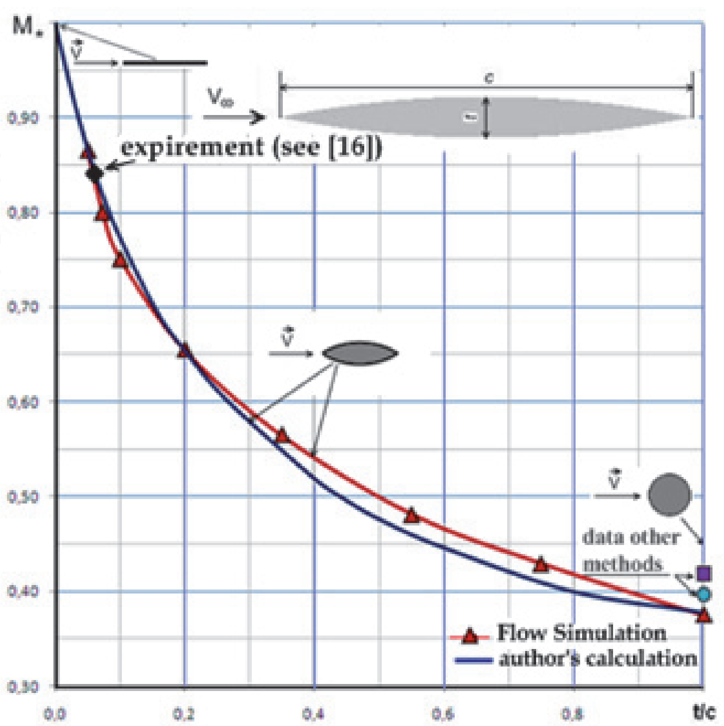

Figure 6.
Critical Mach number vs. relative thickness of the biconvex airfoil.

to other methods. Holt and Messon [17] point out that "symmetric flows (in which a shock does not occur) were calculated only up to the Mach number of the free-stream flow M_∞=0.37". This circumstance indirectly confirms that the exact value of the critical Mach number is closer to that determined by the Burago method than by other methods.

An important question is the question: how does the viscosity of the medium affect the value of the critical Mach number? **Figure 6** shows a comparison of the calculated data for the flow around the biconvex airfoil using the ideal gas and viscous models. **Figure 6** also shows the experimental results that can be observed also in **Figure 4(b)**.

Calculations for viscous gas are performed in a computational package Flow Simulation (SolidWorks). The calculations in the Flow Simulation software are based on solving the Navier–Stokes equations. The calculation domain and the grid used in the calculations Flow Simulation are shown in **Figure 7(a)** and **(b)**. The flow velocity was calculated at a distance equal to the thickness of the boundary

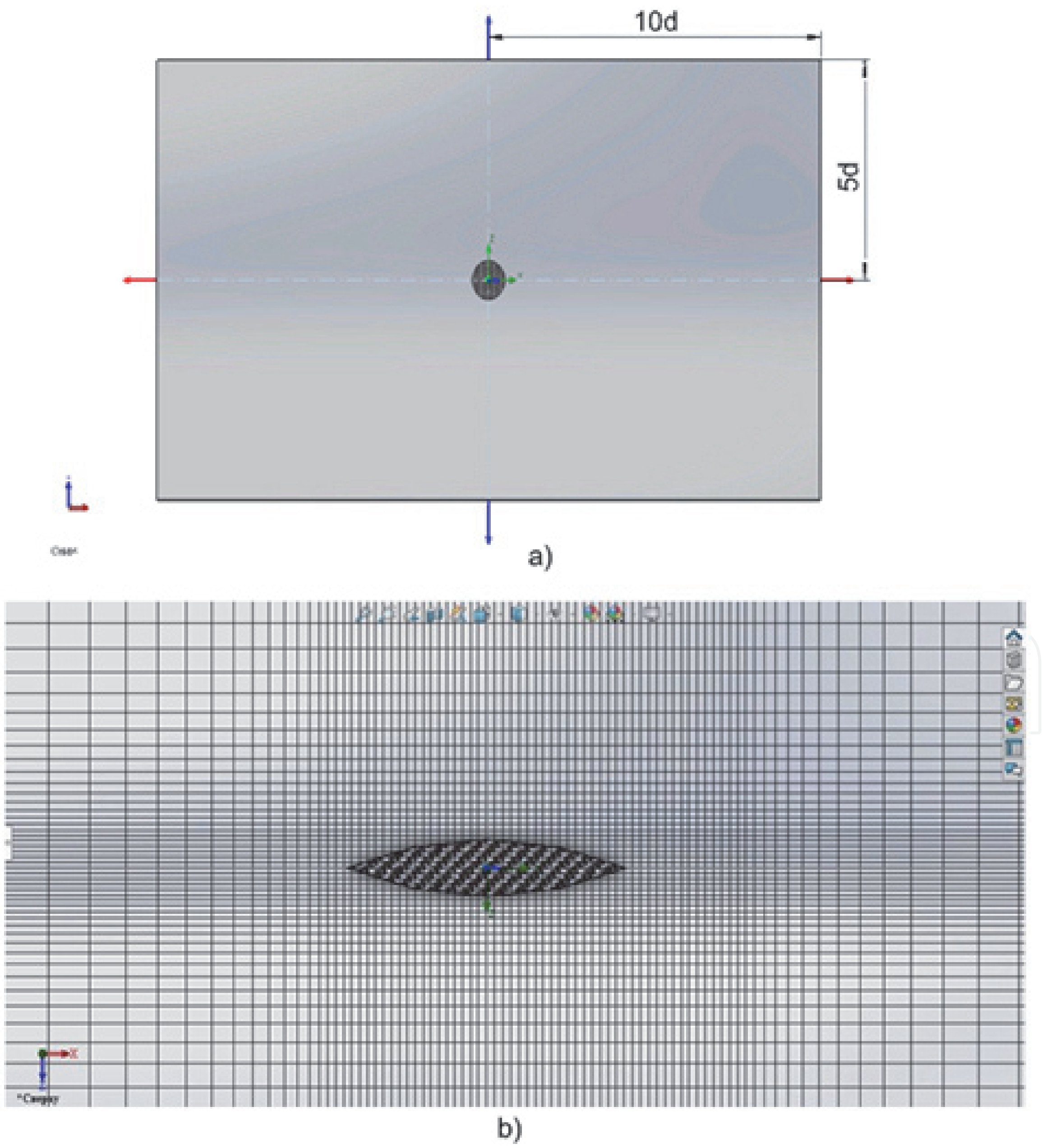

Figure 7.
The calculation domain and grid used in package flow simulation.

layer. In the calculations, the free-stream velocity varied, and as soon as the local flow velocity at the boundary of the layer boundary became equal to the sound velocity, the critical Mach number was calculated.

Based on **Figure 6**, it can be concluded that viscosity has practically no effect on the value of the critical Mach number.

Another two-dimensional body, the flow around which is well studied is an ellipse. **Figure 8** shows the calculated results for the critical Mach number depending on the degree of compression (relative thickness, $p = a/b$) of the ellipse. **Figure 8** also shows the value of the critical Mach number for a two-dimensional body formed by two contacting cylinders. Calculations of the critical Mach number for such a 2-D body were first performed by the author [1].

We have looked at comparing calculations for two-dimensional bodies. The high accuracy of the calculation of the critical Mach numbers by the Burago method is shown.

Let us now consider the applicability of the Burago method for axisymmetric flows. It is also of interest to compare the critical Mach numbers for two-dimensional and axisymmetric flows. Ellipses and ellipsoids of revolution (spheroids) with various factor of compression δ are chosen. For calculations the well-known potential models for ellipses and spheroids [21] were used. It is obvious that the maximal error will correspond to flow around thick bodies ($\delta \to 1.0$) and for the case $M_\infty = M_*$. In **Table 3** and in **Figure 9(a)** results of calculation of critical Mach number for ellipses and spheroids with an offered method and with Dorodnitsyn method of integral relations executed by Chushkin [14, 15] are compared.

Table 3 and **Figure 9(a)** shows a comparison of the results of calculating the critical Mach numbers by two different methods. **Table 3** and **Figure 9(a)** shows a good agreement. It is necessary to notice that calculations in a range of $(0.4 \le \delta \le 0.8)$ were executed by Chushkin only for the second approximation and each subsequent approximation resulted in reduction the value of critical Mach number. Therefore, some additional error can be explained by this fact. It can be noted that the relative difference in values of the critical Mach numbers increases with increasing δ for both ellipses and spheroids.

However, the maximum relative difference in values of the critical Mach numbers for the elliptic cylinder does not exceed 7% and for the sphere 8%. **Figure 9(b)** shows a comparison of the ratio of the maximum velocity at the surface of an

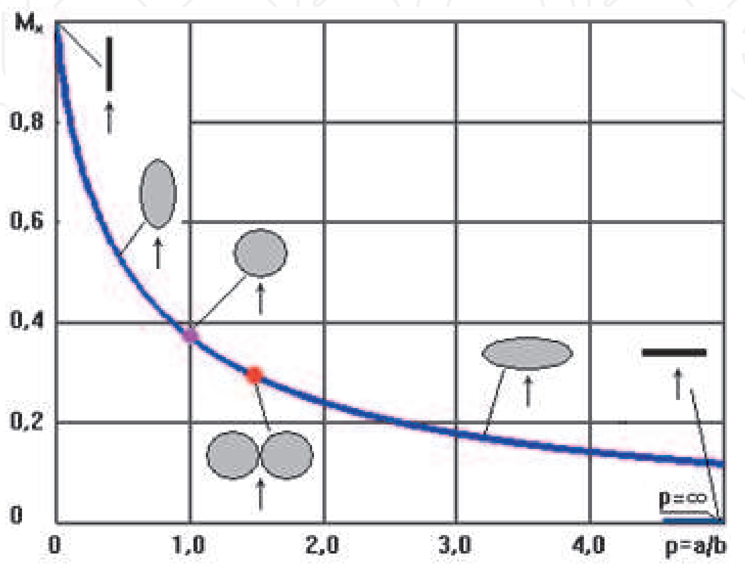

Figure 8.
Critical Mach number vs. relative thickness of the ellipse.

δ	Ellipses				Ellipsoids			
	$C^0_{p\ \min}$	**Chushkin [14]** M_*	**Calculation** M_*	**Relative difference %**	$C^0_{p\ \min}$	**Chushkin [14]** M_*	**Calculation** M_*	**Relative difference %**
0.05	−0.103	0.869	0.884	1.8	−0.014	0.984	0.980	0.4
0.10	−0.210	0.803	0.807	0.5	−0.042	0.957	0.945	1.3
0.15	−0.323	0.752	0.748	0.5	−0.080	0.929	0.905	2.6
0.20	−0.440	0.709	0.700	1.3	−0.122	0.899	0.868	4.0
0.40	−0.960	0.588	0.566	3.7	−0.337	0.783	0.742	5
0.60	−1.560	0.506	0.480	5	−0.602	0.692	0.648	6
0.80	−2.240	0.447	0.418	6	−0.908	0.620	0.576	7
1.00	−3.000	0.399	0.372	7	−1.250	0.563	0.519	8

Table 3.
Critical Mach number at a flow around ellipses and spheroids.

elliptical contour to the velocity at infinity as a function of the Mach number (symmetric flow). **Figure 9(b)** uses the relative ellipse thickness $\delta = 0.1$. Here, along with the calculations by the proposed method, the results of calculations obtained by the theory of small perturbations and by the Hantz and Wendt second approximation (see the Book [7]), as well as the results of calculations obtained by the Dombrovsky method [7] are shown. It can be noted that the results of calculations using the proposed method practically coincided with the results of Dombrovsky. The critical Mach number equal to $M_* \approx 0.807$ (**Figure 9(b)**) was obtained by the Dombrovsky method of approximation of the adiabat. This is in very good agreement with Lighthill's data (see Book [7]) and Dombrovsky's result [7] $M_* \approx 0.81$ and Chushkin's result $M_* \approx 0.803$ (**Table 2**). Kaplan's result [12] $M_* \approx 0.857$ should be considered less accurate, but the relative error does not exceed 7%.

Also, it follows from **Figure 9(a)** that critical Mach number for the two-dimensional case is always less than for axisymmetric case for bodies with the same cross-section.

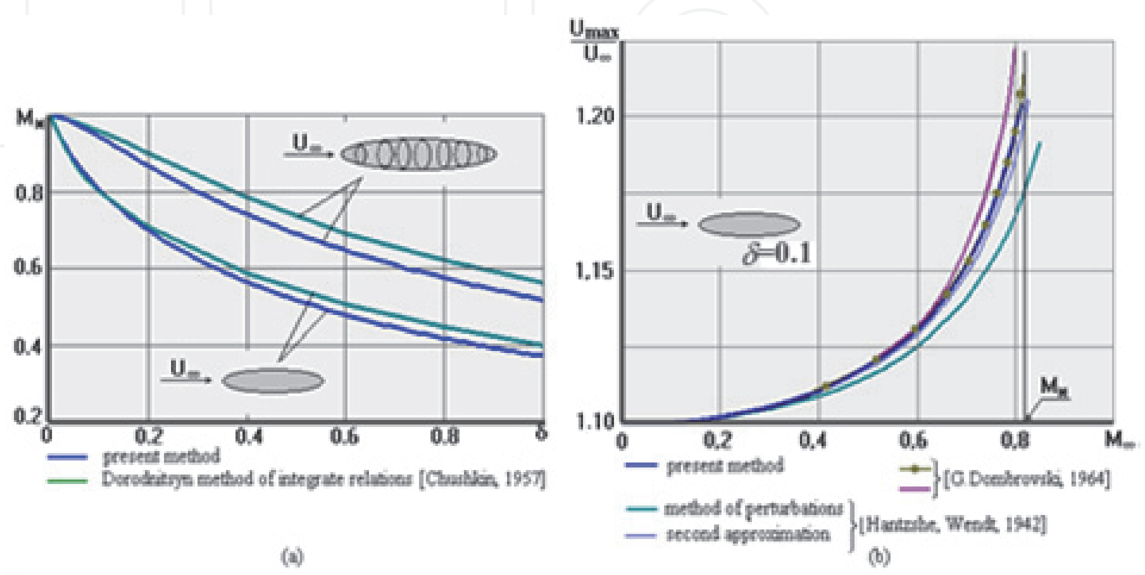

Figure 9.
(a) Critical Mach number vs. relative thickness of an ellipse and ellipsoid; (b) relative maximal velocity of the flow around ellipse vs. Mach number.

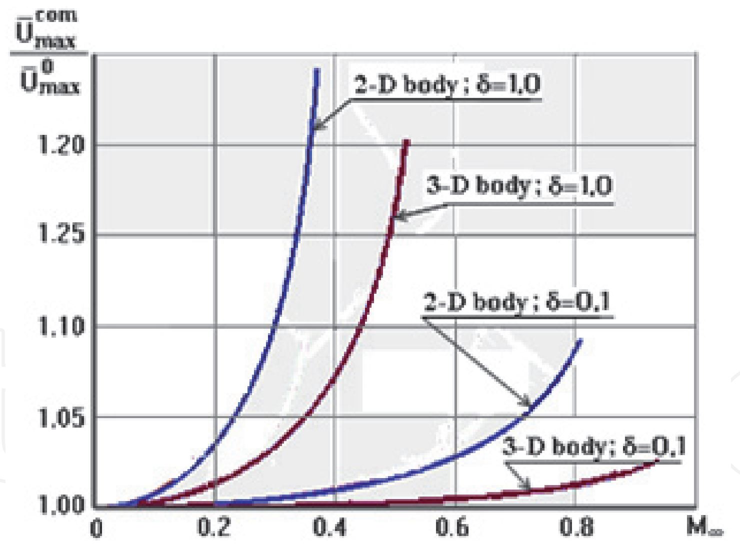

Figure 10.
Maximal velocities on surface of the ellipse and spheroid vs. Mach number.

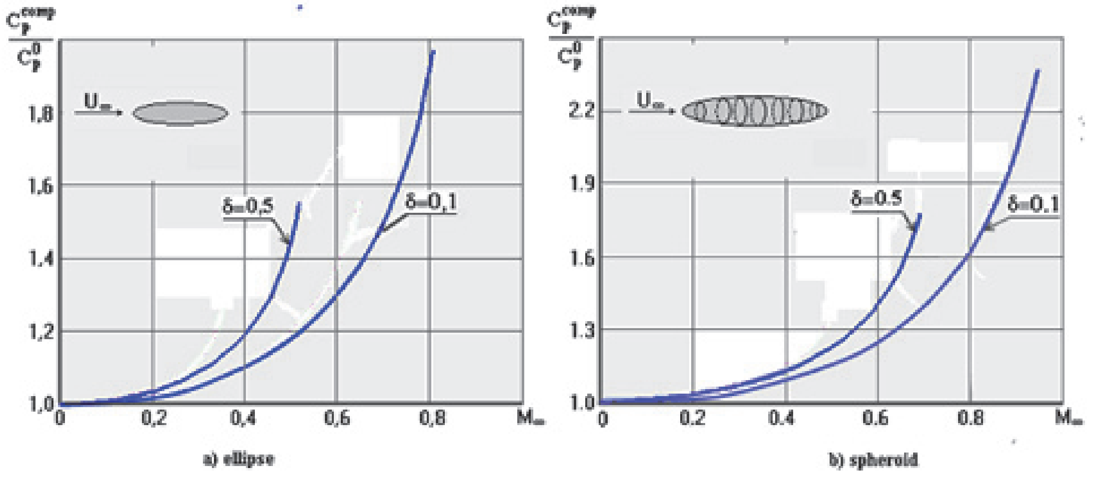

Figure 11.
Influence of Mach number of compressible flow on pressure-drop coefficient on the surface of the an ellipse and spheroid.

The effect of compressibility is shown in **Figure 10**, which shows the ratio of the relative maximum velocities $\left(\overline{U}_{\max} = U_{\max}/U_\infty\right)$ for compressible and incompressible flows on the surface of 2D ellipses and 3D spheroids.

It follows from **Figure 10** that the effect of compressibility for two-dimensional ellipses at the same Mach number is greater than for 3D-spheroids. **Figure 11** shows the results of calculating the pressure-drop coefficient for compressible and incompressible flows on the surfaces of two- and three-dimensional bodies.

As seen in **Figure 11**, the compressibility effect has a stronger effect on 2D bodies than on axisymmetric bodies.

4. Conclusions

This paper presents the results of calculating the critical Mach numbers of the flow around two-dimensional and axisymmetric bodies. A sufficiently high accuracy of calculating the critical Mach numbers for engineering calculations using the proposed method is shown. The method allows one to determine the parameters of a compressible flow from the values of the flow of an incompressible fluid up to a speed corresponding to the critical Mach number. This method does not depend on the means determination parameters of the incompressible flow. The calculation in software Flow Simulation was shown that the viscosity factor does not affect the value critical Mach number. It was found that with an increase in the relative thickness of the body, the value of the critical Mach number decreases. It was also found that the value of the critical Mach number for the two-dimensional case is always less than for the axisymmetric case for bodies with the same cross-section.

Conflict of interest

The author declares no conflict of interest.

Acronyms and abbreviations

AMM	Applied Mechanics and Materials
NACA	National Advisory Committee for Aeronautics
TsAGI	Central Aerohydrodynamic Institute
TM	Technical Memorandum
TR	Technical Report
HSH	High Speed Hydrodynamics

Appendices and nomenclature

u, v	velocity components along axis x и r;
ρ	local density of gas;
τ, σ	special functions (Eq. (2));
p	the static pressure;
∞	subscript indicates flow parameters at infinity;
0	lower index, specifies parameters of the flow in the stagnation point or upper index and specifies parameters of the flow of the incompressible liquid;
κ	the ratio of specific heats;
$P(p)$	the pressure function;
h	the enthalpy;
$\mathbf{V}$	the vector of full local velocity of flow;
M_∞	the Mach number at infinity;
M	the local Mach number;
$\overline{u} = u/U_\infty,\ \overline{v} = v/U_\infty$	dimensionless velocities;
$E(M)$	the function (Eq. (11));
a	the sound speed;
η_c	the compressibility factor of the flow (Eq. (15));
ε	given accuracy of calculation;
$\overline{U}$	local total relative velocity (Eq. (23));
c_p^0	the pressure-drop coefficient for an incompressible flow (Eq. (28));
$G(M), F(M)$	the functions (Eq. (30)).

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