Chapter

On the Generalized Simplest Equations: Toward the Solution of Nonlinear Differential Equations with Variable Coefficients

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Abstract

The simplest equations with variable coefficients are considered in this research. The purpose of this study is to extend the procedure for solving the nonlinear differential equation with variable coefficients. In this case, the generalized Riccati equation is solved and becomes a basis to tackle the nonlinear differential equations with variable coefficients. The method shows that Jacobi and Weierstrass equations can be rearranged to become Riccati equation. It is also important to highlight that the solving procedure also involves the reduction of higher order polynomials with examples of Korteweg de Vries and elliptic-like equations. The generalization of the method is also explained for the case of first order polynomial differential equation.

Keywords: the simplest equation, Riccati equation, nonlinear differential equations, reduction of polynomial

1. Introduction

Despite the advent of supercomputers in numerical methods, increasing activities are devoted to solving nonlinear differential equations by analytical method in recent years [1–3]. Analytical solutions have their own importance concerning the physical phenomena as they are often pave the way to the construction of right theory [4]. Many methods have been proposed concerning this important problem and generally, for the problems with constant coefficients. One of the useful methods is the method of simplest Equation [5] or for some authors, the auxiliary Equations [6]. The method is built by the utilization of the first integral of simplest nonlinear differential equations, such as Bernoulli and Riccati Equations [7]. The method had produced many new solutions of the considered nonlinear differential equations, generally with constant coefficients [8, 9].

For the more general cases, we have found that the method can be extended such as involving the solution of the nonlinear differential equations with variable coefficients. The nature of variable coefficients often arises in the equation describing the heterogenous media and composites [10] or in other cases are produced by the coordinate transformation of the partial differential Equations [11]. Those two categories are developed rapidly in recent years with the capacity of high-speed super computers which sufficient for computing nonlinear problem with complex geometries [12, 13], as sometimes desired by engineering design activities. The role of analytical solutions is as a benchmark to validate the computer algorithm with simpler geometries as it is usually performed [14].

In this chapter, the solution method of the simplest equations is different from the cases of constant coefficients except, on Bernoulli equation. Hence, we will start from Riccati equation instead of Bernoulli equation as the simplest equation to highlight the novelty of the procedure. The method is then followed by examples and conclusion.

2. The first integral of the simplest equations with variable coefficients

2.1 Riccati equation

Consider the Riccati equation with variable coefficients as follows,

$$A_\xi = a_1(\xi)A^2 + a_2(\xi)A + a_3(\xi) \tag{1}$$

Let $A = \beta_1\beta_2$ and the above equation can be rearranged as,

$$\beta_2\beta_{1\xi} + \beta_1\beta_{2\xi} = a_1\beta_1^2\beta_2^2 + a_2\beta_1\beta_2 + a_3 \text{ or}$$

$$\beta_2\beta_{1\xi} - a_1\beta_1^2\beta_2^2 - a_2\beta_1\beta_2 = -\beta_1\beta_{2\xi} + a_3 = \gamma\beta_1\beta_2.$$

and is separated as

$$\beta_{1\xi} - a_1\beta_1^2\beta_2 - (a_2+\gamma)\beta_1 = 0 \text{ and } \beta_{2\xi} + \gamma\beta_2 - \frac{a_3}{\beta_1} = 0 \tag{2}$$

The solutions for β_1 and β_2 are

$$\beta_2 = e^{\int_\xi (a_2+\gamma)d\xi}\left[\int_\xi e^{\int_\xi (a_2+\gamma)d\xi} a_1\beta_2 d\xi + C_1\right]^{-1}$$

and

$$\beta_3 = e^{-\int_\xi \gamma d\xi}\left(\int_\xi e^{\int_\xi \gamma d\xi}\frac{a_3}{\beta_1}d\xi + C_2\right) \tag{3}$$

The relation for $A = \beta_1\beta_2$ is thus,

$$A = \beta_1\beta_2 = e^{\int_\xi a_2 d\xi}\left[\int_\xi e^{\int_\xi (a_2+\gamma)d\xi} a_1\beta_2 d\xi + C_1\right]^{-1}\left(\int_\xi e^{\int_\xi \gamma d\xi}\frac{a_3}{\beta_1}d\xi + C_2\right) \tag{4}$$

Without loss of generality, suppose that $\beta_1 = e^{\int_\xi \gamma d\xi}$ and thus the above relation is performed as,

$$\beta_1\beta_2 = e^{\int_\xi a_2 d\xi}\left[\int_\xi e^{\int_\xi a_2 d\xi} a_1\beta_1\beta_2 d\xi + C_1\right]^{-1}\left(\int_\xi a_3 d\xi + C_2\right) \tag{5}$$

Rearrange Eq. (5) and integrate once,

$$a_1 e^{\int_\xi a_2 d\xi} \beta_1 \beta_2 \left[\int_\xi e^{\int_\xi a_2 d\xi} a_1 \beta_1 \beta_2 d\xi + C_1 \right] = a_1 e^{2\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right)$$

or

$$\left[\int_\xi e^{\int_\xi a_2 d\xi} a_1 \beta_1 \beta_2 d\xi + C_1 \right]^2 = 2\int_\xi a_1 e^{2\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right) d\xi + C_3$$

The solution for A is then,

$$A = \beta_1 \beta_2 = \frac{\sqrt{2}}{2} e^{\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right) \left[\int_\xi a_1 e^{2\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}} \quad (6)$$

where the coefficients a_i will be determined later from the substitution into the considered nonlinear differential equations.

2.2 The Jacobi and Weierstrass equations

It is interesting to know that other simplest equations can also be rearranged into the Riccati-type equations. The famous examples are Jacobi [15] and Weierstrass Equations [16], which can solve a large class of nonlinear differential equations. Let us consider Jacobi type equation with variable coefficients,

$$\phi_\xi^2 = b_1(\xi)\phi^4 + b_2(\xi)\phi^3 + b_3(\xi)\phi^2 + b_4(\xi)\phi + b_5(\xi) \quad (7)$$

and the Weierstrass equation as follows,

$$\phi_\xi^2 = b_1(\xi)\phi^3 + b_2(\xi)\phi^2 + b_3(\xi)\phi + b_4(\xi) \quad (8)$$

Here, the reader should not be confused by the coefficients which represent different functions with the same index. Take $\phi = \frac{1}{v} + a(\xi)$ and the Weierstrass equation becomes Jacobi equation which admits the similar method of solution.

Concerning the search for obtaining solution of (7) and (8), the balancing principle suggests the substitution of the first order series $\phi = b_6 + b_7 A$ as in the following,

$$\begin{aligned} (b_{6\xi} + b_{7\xi}A + b_7 A_\xi)^2 &= b_1 b_7^4 A^4 + \left(4b_1 b_6 b_7^3 + b_2 b_7^3\right)A^3 + \left(6b_1 b_6^2 b_7^2 + 3b_2 b_6 b_7^2 + b_3 b_7^2\right)A^2 \\ &+ \left(4b_1 b_6^3 b_7 + 3b_2 b_6^2 b_7 + 2b_3 b_6 b_7 + b_4 b_7\right)A + b_1 b_6^4 + b_2 b_6^3 + b_3 b_6^2 + b_4 b_6 + b_5 \end{aligned} \quad (9)$$

Performing the Riccati equation $A_\xi = a_1 b_7 A^2 + a_2 A + a_3$ into (9) and we generate the following expression,

$$\begin{aligned} (b_{6\xi} + b_{7\xi}A + b_7 A_\xi)^2 &= \left[a_1 b_7^2 A^2 + (a_2 b_7 + b_{7\xi})A + a_3 b_7 + b_{6\xi}\right]^2 = a_1^2 b_7^4 A^4 + 2a_1 b_7 (a_2 b_7 + b_{7\xi})A^3 \\ &+ \left[2a_1 b_7 (a_3 b_7 + b_{6\xi}) + (a_2 b_7 + b_{7\xi})^2\right]A^2 + 2(a_2 b_7 + b_{7\xi})(a_3 b_7 + b_{6\xi})A + (a_3 b_7 + b_{6\xi})^2 \end{aligned}$$

The coefficients of polynomial are then related with the coefficients in (9) in order to determine a_1, a_2, a_3, b_6, b_7 as functions of the known b_1, b_2, b_3, b_4 and b_5 as follows,

$$
\begin{aligned}
& b_1 b_7^4 = a_1^2 b_7^4 \\
& 4b_1 b_6 b_7^3 + b_2 b_7^3 = 2a_1 b_7 (a_2 b_7 + b_{7\xi}) \\
& 6b_1 b_6^2 b_7^2 + 3b_2 b_6 b_7^2 + b_3 b_7^2 = 2a_1 b_7 (a_3 b_7 + b_{6\xi}) + (a_2 b_7 + b_{7\xi})^2 \\
& 4b_1 b_6^3 b_7 + 3b_2 b_6^2 b_7 + 2b_3 b_6 b_7 + b_4 b_7 = 2(a_2 b_7 + b_{7\xi})(a_3 b_7 + b_{6\xi}) \\
& b_1 b_6^4 + b_2 b_6^3 + b_3 b_6^2 + b_4 b_6 + b_5 = (a_3 b_7 + b_{6\xi})^2
\end{aligned} \tag{10}
$$

Hence, the first equation gives,

$$a_1 = f_0(b_1) \tag{11}$$

and the second equation is then,

$$a_2 b_7 + b_{7\xi} = \frac{4b_1 b_6 b_7^2 + b_2 b_7^2}{2f_0} \tag{12}$$

The next relation produces,

$$a_3 b_7 + b_{6\xi} = \frac{6b_1 b_6^2 b_7 + 3b_2 b_6 b_7 + b_3 b_7}{2f_0} - \frac{1}{8f_0^3}(4b_1 b_6 + b_2)^2 b_7^3 \tag{13}$$

Eqs. (12) and (13) are thus substituted into the fourth relation of (12) to form the third order polynomial equation in term of b_6 as follows,

$$
\begin{aligned}
& (32f_0^4 b_1 + 64b_1^3 b_7^4 - 192f_0^2 b_1^2 b_7^2) b_6^3 + (24f_0^4 b_2 + 32b_1^2 b_2 b_7^4 - 16b_1^2 b_2 b_7^4 - 96f_0^2 b_1 b_2 b_7^2 - 48f_0^2 b_1 b_2 b_7^2) b_6^2 \\
& + (16f_0^4 b_3 + 4b_1 b_2^2 b_7^4 + 8b_1 b_2^2 b_7^2 - 32f_0^2 b_1 b_3 b_7^2 - 24f_0^2 b_2^2 b_7^2) b_6 + 8f_0^4 b_4 + b_2^3 b_7^4 - 8f_0^2 b_2 b_3 b_7^2 = 0
\end{aligned} \tag{14}
$$

which the roots will determine the solution for b_6 as functions of b_1, b_2, b_3, b_4, b_7, or $b_6 = f_1(b_7)$ in simple unknown variable. The step now is to find the polynomial expression for b_7 from the last relation of (10) as,

$$
\begin{aligned}
& b_1 f_1^4(b_7) + b_2 f_1^3(b_7) + b_3 f_1^2(b_7) + b_4 f_1(b_7) + b_5 = \\
& \left[\frac{6b_1 f_1^2(b_7) b_7 + 3b_2 f_1(b_7) b_7 + b_3 b_7}{2f_0} - \frac{1}{8f_0^3}(4b_1 f_1(b_7) + b_2)^2 b_7^3 \right]^2
\end{aligned} \tag{15}
$$

Therefore, the last equation gives the expression for b_7 as polynomial equation of higher order, and the generated polynomial is,

$$a_n b_7^n + a_{n-1} b_7^{n-1} + a_{n-2} b_7^{n-2} + a_{n-3} b_7^{n-3} + \dots\dots\dots\dots + a_2 b_7^2 + a_1 b_7 + a_0 = 0 \tag{16}$$

In this case the higher order polynomial will be solved by reducing the order.

3. Reduction of higher order polynomial

Consider the sixth order polynomial equation as in the following,

$$b_7^6 + a_4 b_7^5 + a_5 b_7^4 + a_6 b_7^3 + a_7 b_7^2 + a_8 b_7 + a_9 = 0$$

First, multiply the above equation with the functionα and rearranged as,

$$B^6 + a_4\alpha B^5 + a_5\alpha^2 B^4 + a_6\alpha^3 B^3 + a_7\alpha^4 B^2 + a_8\alpha^5 B + a_9\alpha^6 + \varphi = \varphi \tag{17}$$

where $B = \alpha b_7$. The polynomial equation is cut as in the following,

$$\left(B^4 + b_1 B^3 + b_2 B^2 + b_3 B + b_4\right)B^2 + b_5 B^2 - \left(B^4 + b_1 B^3 + b_2 B^2 + b_3 B + b_4\right)b_6 = \varphi$$

Note that, the coefficients b_i in this section is different from the previous section. Expanding for the new coefficients,

$$B^6 + b_1 B^5 + (b_2 - b_5)B^4 + (b_3 - b_1 b_5)B^3 + (b_4 + b_5 - b_2 b_5)B^2 - b_3 b_5 B - b_4 b_6 = \varphi \tag{18}$$

Hence, the relation for coefficients is,

$$\begin{aligned}
&b_1 = a_1\alpha \\
&b_2 - b_6 = a_2\alpha^2 \\
&b_3 - b_1 b_6 = a_3\alpha^3 \\
&b_4 + b_5 - b_2 b_6 = a_4\alpha^4 \qquad\qquad \text{or} \\
&-b_3 b_6 = a_5\alpha^5 \\
&-b_4 b_6 = a_6\alpha^6 + \varphi \\
&b_6 = \frac{\varphi}{b_5}
\end{aligned}$$

$$\begin{aligned}
&b_1 = a_1\alpha \\
&b_2 - b_6 = a_2\alpha^2 \\
&b_3 - a_1\alpha\left(b_2 - a_2\alpha^2\right) = a_3\alpha^3 \\
&b_4 + b_5 - \left(b_2 - a_2\alpha^2\right)^2 = a_4\alpha^4 + a_2\alpha^2\left(b_2 - a_2\alpha^2\right) \\
&-\left[a_3\alpha^3 + a_1\alpha\left(b_2 - a_2\alpha^2\right)\right]\left(b_2 - a_2\alpha^2\right) = a_5\alpha^5 \\
&-\left[a_4\alpha^4 + a_2\alpha^2\left(b_2 - a_2\alpha^2\right) + \left(b_2 - a_2\alpha^2\right)^2 - b_5\right]\left(b_2 - a_2\alpha^2\right) = a_6\alpha^6 + \varphi \\
&\left(b_2 - a_2\alpha^2\right) = -\frac{\left[a_4\alpha^4 + a_2\alpha^2(b_2 - a_2\alpha^2) + (b_2 - a_2\alpha^2)^2 - b_5\right](b_2 - a_2\alpha^2) + a_6\alpha^6}{b_5}
\end{aligned}.$$

The fifth coefficient relation is rearranged as,

$$\left(b_2 - a_2\alpha^2\right)^2 + \frac{a_3}{a_1}\alpha^2\left(b_2 - a_2\alpha^2\right) + \frac{a_5}{a_1}\alpha^4 = 0 \tag{19}$$

and the roots are,

$$\left(b_2 - a_2\alpha^2\right) = \frac{1}{2}\alpha^2\left[-\frac{a_3}{a_1} \pm \left(\frac{a_3^2}{a_1^2} - 4\frac{a_5}{a_1}\right)^{\frac{1}{2}}\right] = f_0\alpha^2 \tag{20}$$

Also, the last relation is rewritten as,

$$\left(b_2 - a_2\alpha^2\right)^3 + a_2\alpha^2\left(b_2 - a_2\alpha^2\right)^2 + a_4\alpha^4\left(b_2 - a_2\alpha^2\right) + a_6\alpha^6 = 0 \tag{21}$$

Note that performing (19) into (21) will remove b_5 and α. Thus, it is necessary to take other relation, i.e. $a_6\alpha^6 = \alpha^{12} + b_5$, which will produce the cubic equation as follows,

$$\alpha^{12} + \left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)\alpha^6 + b_5 = 0 \tag{22}$$

which has the roots as,

$$\alpha^6 = -\frac{1}{2}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}} \tag{23}$$

Substituting back into $a_6\alpha^6 = \alpha^{12} + b_5$ to get,

$$\begin{aligned}&\left\{-\frac{1}{2}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}}\right\}^2 \\ &+b_5 = -\frac{1}{2}a_6\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}a_6\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}}\end{aligned} \quad \text{or}$$

$$\begin{aligned}&\frac{1}{4}\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right] - \frac{1}{2}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}} \\ &+\frac{1}{4}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 + b_5 = -\frac{1}{2}a_6\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}a_6\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}}\end{aligned}$$

or

$$b_5 = \frac{1}{4}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - \frac{1}{4}\left[\frac{\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 + a_6\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)}{\left(f_0^3 + a_2 f_0^2 + a_4 f_0 + a_6\right)}\right]^2 \tag{24}$$

Therefore,α is also determined by (24) and so all the coefficients, $b_1, b_2, b_3, b_4, b_6, \varphi$. The polynomial equation of sixth order is then re-expressed as,

$$\left(B^4 + b_1 B^3 + b_2 B^2 + b_3 B + b_4\right)\left(B^2 - b_6\right) = -b_5\left(B^2 - \frac{\varphi}{b_5}\right) \tag{25}$$

as reduced into the quartic equation the roots can be obtained by radical solution.

The procedure described by (17–25) can be applied and iterated into (16) until the polynomial equation of b_7 is reduced into quartic equation. Hence, all the coefficients for Riccati equation the first order series, i.e. a_1, a_2, a_3, b_6, b_7 are determined and produce the solution as,

$$\begin{aligned}&\phi = b_6 + b_7 A \text{ or} \\ &\phi = b_6 + \frac{\sqrt{2}}{2} b_7 e^{\int_\xi a_2 d\xi}\left(\int_\xi a_3 d\xi + C_2\right)\left[\int_\xi a_1 b_7 e^{2\int_\xi a_2 d\xi}\left(\int_\xi a_3 d\xi + C_2\right) d\xi + C_3\right]^{-\frac{1}{2}}\end{aligned} \tag{26}$$

Thus, following the method explained by (2–6) and (17–25), we have arrived at the solution of Jacobi and Weierstrass equations with variable coefficients.

4. Solution examples

4.1 The elliptic-like equation

As an application, consider the elliptic-like equation with forcing function,

$$\phi_{\xi\xi} + b_1(\xi)\phi^3 + b_2(\xi)\phi = b_3(\xi) \tag{27}$$

The balance principle suggests that the solution should be in the form,

$$\phi = b_4 + b_5 A \tag{28}$$

Substituting into (27) will reproduce the following expression,

$$b_5 A_{\xi\xi} + 2b_{5\xi}A_\xi + b_1 b_5^3 A^3 + 3b_1 b_4 b_5^2 A^2 + \left(b_{5\xi\xi} + 3b_1 b_4^2 b_5 + b_2 b_5\right)A + b_{4\xi\xi} + b_1 b_4^3 + b_2 b_4 = b_3 \tag{29}$$

The next step is to differentiate the Riccati equation once,

$$A_{\xi\xi} = 2a_1^2 A^3 + (3a_1 a_2 + a_{1\xi})A^2 + \left(2a_1 a_3 + a_2^2 + a_{2\xi}\right)A + a_2 a_3 + a_{3\xi}$$

Substituting into Eq. (29) and it will produce the polynomial equation as in the following,

$$\begin{aligned}&\left(2a_1^2 b_5 + b_1 b_5^3\right)A^3 + \left(3a_1 a_2 b_5 + a_{1\xi} b_5 + 2a_1 b_{5\xi} + 3b_1 b_4 b_5^2\right)A^2\\ &+\left(2a_1 a_3 b_5 + a_2^2 b_5 + a_{2\xi} b_5 + 2a_2 b_{5\xi} + b_{5\xi\xi} + 3b_1 b_4^2 b_5 + b_2 b_5\right)A\\ &+a_2 a_3 b_5 + a_{3\xi} b_5 + 2a_3 b_{5\xi} + b_{4\xi\xi} + b_1 b_4^3 + b_2 b_4 = b_3\end{aligned}$$

or the next step is to relate the coefficients as,

$$\begin{aligned}&2a_1^2 b_5 + b_1 b_5^3 = 0\\ &3a_1 a_2 b_5 + a_{1\xi} b_5 + 2a_1 b_{5\xi} + 3b_1 b_4 b_5^2 = 0\\ &2a_1 a_3 b_5 + a_2^2 b_5 + a_{2\xi} b_5 + 2a_2 b_{5\xi} + b_{5\xi\xi} + 3b_1 b_4^2 b_5 + b_2 b_5 = 0\\ &a_2 a_3 b_5 + a_{3\xi} b_5 + 2a_3 b_{5\xi} + b_{4\xi\xi} + b_1 b_4^3 + b_2 b_4 = b_3\end{aligned}$$

In this case, the first equation gives,

$$a_1 = f(b_1) b_5 \tag{30}$$

For the second equation,

$$3f a_2 b_5 + f_\xi b_5 + 3f b_{5\xi} + 3b_1 b_4 b_5 = 0$$

Thus, provide the expression for b_5 as,

$$b_5 = C_4 f^{-\frac{1}{3}} e^{-\int_\xi \left(a_2 + \frac{b_1 b_4}{f}\right) d\xi} \tag{31}$$

The third and fourth equations produce,

$$a_3 b_5 = b_5^{-1} e^{-\int_\xi a_2 d\xi} \left[\int_\xi b_5 e^{\int_\xi a_2 d\xi} \left(b_3 - b_{4\xi\xi} - b_1 b_4^3 - b_2 b_4 \right) d\xi + C_4 \right] \tag{32}$$

Substituting (32) into (31),

$$-a_2^2 - a_{2\xi} - 2a_2 \frac{b_{5\xi}}{b_5} - \frac{b_{5\xi\xi}}{b_5} - 3b_1 b_4^2 - b_2 = 2f b_5^{-1} e^{-\int_\xi a_2 d\xi} \left[\int_\xi b_5 e^{\int_\xi a_2 d\xi} \left(b_3 - b_{4\xi\xi} - b_1 b_4^3 - b_2 b_4 \right) d\xi + C_4 \right]$$

Replace b_5 with (10),

$$\begin{aligned} &\frac{2}{3}\frac{f_\xi}{f} a_2 + \frac{1}{3}\frac{f_{\xi\xi}}{f} - \frac{4}{9}\left(\frac{f_\xi}{f}\right)^2 - \left(\frac{b_1 b_4}{f}\right)^2 + \left(\frac{b_1 b_4}{f}\right)_\xi - 3b_1 b_4^2 - b_2 = \\ &2f e^{\int_\xi \frac{b_1 b_4}{f} d\xi} \left[\int_\xi e^{-\int_\xi \frac{b_1 b_4}{f} d\xi} \left(b_3 - b_{4\xi\xi} - b_1 b_4^3 - b_2 b_4 \right) d\xi + C_4 \right] \end{aligned} \tag{33}$$

which then solves a_2 regardless of b_4. In this case, we take b_4 as the chosen fundamental variable and the resulted coefficients, b_5, a_1, a_2, a_3 depend on b_4 and with the known coefficients b_1, b_2, b_3. Therefore, the solution of (27) is generated as,

$$\begin{aligned} \phi(\xi) = b_5(b_4) \left\{ \frac{\sqrt{2}}{2} e^{\int_\xi a_2(b_4) d\xi} \left(\int_\xi a_3(b_4) d\xi + C_2 \right) \left[\int_\xi a_1(b_4) e^{2\int_\xi a_2(b_4) d\xi} \left(\int_\xi a_3(b_4) d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}} \right\} \\ + b_4 \end{aligned} \tag{34}$$

4.2 Korteweg de Vries equation

The next example is for the Korteweg de Vries type equation,

$$\phi_{\xi\xi\xi} + b_1(\xi)\phi\phi_\xi + b_2(\xi)\phi_\xi + b_3(\xi)\phi + b_4(\xi) = 0 \tag{35}$$

The balancing principle with application of Riccati equation will determined the ansatz,

$$\phi = b_5 + b_6 A + b_7 A^2 \tag{36}$$

Performing into (35) will produce,

$$\begin{aligned} &2b_7 A A_{\xi\xi\xi} + b_6 A_{\xi\xi\xi} + 6b_7 A_\xi A_{\xi\xi} + 6b_{7\xi} A_\xi^2 + 6b_{7\xi} A A_{\xi\xi} + 2b_1 b_7^2 A^3 A_\xi + 3b_1 b_6 b_7 A^2 A_\xi \\ &+ \left(6b_{7\xi\xi} + b_1 b_6^2 + 2b_1 b_5 b_7 + 2b_2 b_7\right) A A_\xi + 3b_{6\xi} A_{\xi\xi} + \left(3b_{6\xi\xi} + b_1 b_5 b_6 + b_2 b_6\right) A_\xi + b_1 b_7 b_{7\xi} A^4 \\ &+ \left(b_1 b_{6\xi} b_7 + b_1 b_6 b_{7\xi}\right) A^3 + \left(b_{7\xi\xi\xi} + b_1 b_5 b_{7\xi} + b_1 b_6 b_{6\xi} + b_1 b_{5\xi} b_7 + b_2 b_{7\xi} + b_3 b_7\right) A^2 \\ &+ \left(b_{6\xi\xi\xi} + b_1 b_5 b_{6\xi} + b_1 b_{5\xi} b_6 + b_2 b_{6\xi} + b_3 b_6\right) A + b_{5\xi\xi\xi} + b_1 b_5 b_{5\xi} + b_2 b_{5\xi} + b_3 b_5 + b_4 = 0 \end{aligned} \tag{37}$$

Performing the Riccati equation into (37) will produce the following polynomial,

$$
\begin{aligned}
&\left(12b_7a_1^3+2b_1b_7^2a_1+12a_1^3b_7\right)A^5+\begin{pmatrix}18a_1a_{1\xi}b_7+54a_1^2a_2b_7+6a_1^3b_6+b_1b_7b_{7\xi}+2a_2b_1b_7^2\\ +18a_1^2b_{7\xi}+3a_1b_1b_6b_7\end{pmatrix}A^4\\
&+\begin{pmatrix}40a_1^2a_3b_7+32a_1a_2^2b_7+12a_{1\xi}a_2b_7+18a_1a_{2\xi}b_7+2a_{1\xi\xi}b_7+6a_1a_{1\xi}b_6+12a_1^2a_2b_6+b_1b_{6\xi}b_7\\ +b_1b_6b_{7\xi}+2a_3b_1b_7^2+6a_2^3b_7+30a_1a_2b_{7\xi}+6a_{1\xi}b_{7\xi}+6a_1^2b_{6\xi}+3a_2b_1b_6b_7+6a_1b_{7\xi\xi}\\ +a_1b_1b_6^2+2a_1b_1b_5b_7+2a_1b_2b_7\end{pmatrix}A^3\\
&+\begin{pmatrix}52a_1a_2a_3b_7+14a_{1\xi}a_3b_7+10a_1a_{3\xi}b_7+12a_2a_{2\xi}b_7+2a_{2\xi\xi}b_7+8a_2^3b_7+8a_1^2a_3b_6+7a_1a_2^2b_6\\ +3a_{1\xi}a_2b_6+6a_1a_{2\xi}b_6+a_{1\xi\xi}b_6+24a_1a_3b_{7\xi}+12a_2^2b_{7\xi}+6a_{2\xi}b_{7\xi}+9a_1a_2b_{6\xi}+3a_{1\xi}b_{6\xi}+b_{7\xi\xi\xi}\\ +b_1b_5b_{7\xi}+b_1b_6b_{6\xi}+b_1b_{5\xi}b_7+b_2b_{7\xi}+b_3b_7+3a_3b_1b_6b_7+6a_2b_{7\xi\xi}+a_2b_1b_6^2+2a_2b_1b_5b_7\\ +2a_2b_2b_7+3a_1b_{6\xi\xi}+a_1b_1b_5b_6+a_1b_2b_6\end{pmatrix}A^2\\
&+\begin{pmatrix}16a_1a_3^2b_7+14a_2^2a_3b_7+10a_{2\xi}a_3b_7+8a_2a_{3\xi}b_7+2a_{3\xi\xi}b_7+8a_1a_2a_3b_6+4a_{1\xi}a_3b_6+2a_1a_{3\xi}b_6\\ +3a_2a_{2\xi}b_6+a_{2\xi\xi}b_6+a_2^3b_6+18a_2a_3b_{7\xi}+6a_{3\xi}b_{7\xi}+6a_1a_3b_{6\xi}+3a_2^2b_{6\xi}+3a_{2\xi}b_{6\xi}+b_{6\xi\xi\xi}\\ +b_1b_5b_{6\xi}+b_1b_{5\xi}b_6+b_2b_{6\xi}+b_3b_6+6a_3b_{7\xi\xi}+a_3b_1b_6^2+2a_3b_1b_5b_7+2a_3b_2b_7+3a_2b_{6\xi\xi}\\ +a_2b_1b_5b_6+a_2b_2b_6\end{pmatrix}A\\
&+\begin{pmatrix}2a_1a_3^2b_6+a_2^2a_3b_6+2a_{2\xi}a_3b_6+a_2a_{3\xi}b_6+a_{3\xi\xi}b_6+6a_2a_3^2b_7+6a_3a_{3\xi}b_7+3a_2a_3b_{6\xi}\\ +3a_{3\xi}b_{6\xi}+b_{5\xi\xi\xi}+b_1b_5b_{5\xi}+b_2b_{5\xi}+6a_3^2b_{7\xi}+3a_3b_{6\xi\xi}+a_3b_1b_5b_6+a_3b_2b_6+b_3b_5+b_4\end{pmatrix}=0
\end{aligned}
\tag{38}
$$

From this step on, there is a little hope to solve all the coefficients as they are equal to zero. As it has also to be reduced, it is important to note that the problem of reduction here is different from the case of Jacobi equation since all the coefficients are in principle solvable in algebraic form. In this case, it is not practical to reduce the fifth order polynomial as the even highest power, i.e., as a tenth order polynomial equation. The calculation will become too tedious as the detail expression is needed in the reduced polynomial equation. The next sub section will illustrate the reduction of an odd highest power polynomial equation.

4.3 Reduction of fifth order polynomial

Consider Eq. (38) as follows,

$$d_1A^5+d_2A^4+d_3A^3+d_4A^2+d_5A+d_6=0$$

Multiply by the function, β and rearrange,

$$d_1B^5+d_2\beta B^4+d_3\beta^2B^3+d_4\beta^3B^2+d_5\beta^4B+d_6\beta^5+\varphi=\varphi \tag{39}$$

where, $B=\beta A$. Rearranged Eq. (39) as given by,

$$\left(b_1B^3+b_2B^2+b_3B+b_4\right)B^2+b_5B^2-\left(b_1B^3+b_2B^2+b_3B+b_4\right)b_6=\varphi \tag{40}$$

Expanding the all the coefficients as,

$$b_1B^5+b_2B^4+(b_3-b_1b_6)B^3+(b_4+b_5-b_2b_6)B^2-b_3b_6B-b_4b_6=\varphi$$

Relate the coefficients as in the following,

$$
\begin{aligned}
&b_1 = d_1 \\
&b_2 = d_2\beta \\
&b_3 - b_1 b_6 = d_3\beta^2 \\
&b_4 + b_5 - d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = d_4\beta^3 \\
&-\frac{1}{d_1^2}\left(b_3 - d_3\beta^2\right)^2 = d_5\beta^4 + d_3\beta^2\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) \\
&-\left[d_4\beta^3 + d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) - b_5\right]\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = d_6\beta^5 + \varphi \\
&\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = -\left\{\frac{\left[d_4\beta^3 + d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) - b_5\right]\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) + d_5\beta^5}{b_5}\right\}
\end{aligned}
\tag{41}
$$

The fifth equation of (41) gives the roots as,

$$
\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = \frac{1}{2}\beta^2\left[d_3 \pm \left(d_3^2 - 4d_5\right)^{\frac{1}{2}}\right] = \beta^2 f_0 \tag{42}
$$

Moving to the last equation, the functions b_5 and β disappear from the operation. In this case we will consider the test function, $b_5 + \beta^{10} = d_6\beta^5$, and will perform as,

$$
\beta^{10} + d_4\beta^3\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) + d_2\beta\frac{1}{d_1^2}\left(b_3 - d_3\beta^2\right)^2 + b_5 = 0 \tag{43}
$$

Substituting for b_3, the expression for β is,

$$
\begin{aligned}
&\beta^{10} + \left(d_4 f_0 + d_2 f_0^2\right)d_4\beta^5 + b_5 = 0 \\
&\beta^5 = -\frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}}
\end{aligned}
\tag{44}
$$

Substitute back to, $b_5 + \beta^{10} = d_6\beta^5$ as follows

$$
\begin{aligned}
&\left\{-\frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}}\right\}^2 + b_5 \\
&= -\frac{1}{2}d_6\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}d_6\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}} \text{ or}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right] - \frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right)\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}} \\
&+\frac{1}{4}\left(d_4 f_0 + d_2 f_0^2\right)^2 + b_5 = -\frac{1}{2}d_6\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}d_6\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}} \\
&\text{or}
\end{aligned}
$$

$$
b_5 = \frac{1}{4}\left(d_4 f_0 + d_2 f_0^2\right)^2 - \frac{1}{4}\left[\frac{\left(d_4 f_0 + d_2 f_0^2\right)^2 + d_6\left(d_4 f_0 + d_2 f_0^2\right)}{\left(d_4 f_0 + d_2 f_0^2 + d_6\right)}\right]^2 \tag{45}
$$

which then solves b_5, β, φ and thus generates all the coefficients of b_i. The polynomial is then rewritten as,

$$\left(b_1B^3+b_2B^2+b_3B+b_4\right)\left(B^2-b_6\right)=-b_5\left(B^2-\frac{\varphi}{b_5}\right)$$

which is reduced as,

$$d_1A^3+d_2A^2+\left\{\frac{1}{2}d_1\left[d_3\pm\left(d_3^2-4d_5\right)^{\frac{1}{2}}\right]+d_3\right\}A+d_4+\frac{1}{2}d_2\left[d_3\pm\left(d_3^2-4d_5\right)^{\frac{1}{2}}\right]=0 \tag{46}$$

Eq. (46) dictates that the relations, $d_1=d_2=d_3=d_4=0$ will satisfy for the solution. Hence, the coefficients are then,

$$\begin{aligned}
&12b_7a_1^3+2b_1b_7^2a_1+12a_1^3b_7=0\\
&18a_1a_{1\xi}b_7+54a_1^2a_2b_7+6a_1^3b_6+b_1b_7b_{7\xi}+2a_2b_1b_7^2+18a_1^2b_{7\xi}+3a_1b_1b_6b_7=0\\
&40a_1^2a_3b_7+32a_1a_2^2b_7+12a_{1\xi}a_2b_7+18a_1a_{2\xi}b_7+2a_{1\xi\xi}b_7+6a_1a_{1\xi}b_6+12a_1^2a_2b_6\\
&+b_1b_{6\xi}b_7+b_1b_6b_{7\xi}+2a_3b_1b_7^2+6a_2^3b_7+30a_1a_2b_{7\xi}+6a_{1\xi}b_{7\xi}+6a_1^2b_{6\xi}+3a_2b_1b_6b_7\\
&+6a_1b_{7\xi\xi}+a_1b_1b_6^2+2a_1b_1b_5b_7+2a_1b_2b_7=0\\
&52a_1a_2a_3b_7+14a_{1\xi}a_3b_7+10a_1a_{3\xi}b_7+12a_2a_{2\xi}b_7+2a_{2\xi\xi}b_7+8a_2^3b_7+8a_1^2a_3b_6\\
&+7a_1a_2^2b_6+3a_{1\xi}a_2b_6+6a_1a_{2\xi}b_6+a_{1\xi\xi}b_6+24a_1a_3b_{7\xi}+12a_2^2b_{7\xi}+6a_{2\xi}b_{7\xi}+9a_1a_2b_{6\xi}\\
&+3a_{1\xi}b_{6\xi}+b_{7\xi\xi\xi}+b_1b_5b_{7\xi}+b_1b_6b_{6\xi}+b_1b_{5\xi}b_7+b_2b_{7\xi}+b_3b_7+3a_3b_1b_6b_7+6a_2b_{7\xi\xi}\\
&+a_2b_1b_6^2+2a_2b_1b_5b_7+2a_2b_2b_7+3a_1b_{6\xi\xi}+a_1b_1b_5b_6+a_1b_2b_6=0
\end{aligned} \tag{47}$$

The first equation gives,

$$b_7=f(b_1)a_1^2 \tag{48}$$

The second equation is rewritten as,

$$18a_{1\xi}fa_1^3+54fa_1^3a_2+6a_1^3b_6+b_1ff_{\xi}a_1^4+2b_1f^2a_1^3a_{1\xi}+2a_2b_1f^2a_1^4+18f_{\xi}a_1^4+18fa_1^3a_{1\xi}+3b_1b_6fa_1^2=0 \text{ or}$$

$$18a_{1\xi}f+54fa_2+6b_6+b_1ff_{\xi}a_1+2b_1f^2a_{1\xi}+2a_2b_1f^2a_1+18f_{\xi}a_1+18fa_{1\xi}+3b_1b_6f=0 \text{ or}$$

$$\left(36f+2b_1f^2\right)a_{1\xi}+54fa_2+6b_6+\left(b_1ff_{\xi}+2a_2b_1f^2+18f_{\xi}\right)a_1+3b_1b_6f=0$$

The solution for b_6 is then,

$$\left(36f+2b_1f^2\right)a_{1\xi}=-\left(b_1ff_{\xi}+2a_2b_1f^2+18f_{\xi}\right)a_1-54fa_2-\left(6+3fb_1\right)b_6$$

$$b_6=-\frac{1}{\left(6+3fb_1\right)}\left[\left(b_1ff_{\xi}+2a_2b_1f^2+18f_{\xi}\right)a_1+\left(36f+2b_1f^2\right)a_{1\xi}+54fa_2\right]=h_1(a_1,a_2) \tag{49}$$

The third equation will produce,

$$40a_1^2a_3f + 32a_1^2a_2^2f + 12a_{1\xi}a_2fa_1 + 18a_{2\xi}fa_1^2 + 2a_{1\xi\xi}fa_1 + 6a_{1\xi}b_6 + 12a_1a_2b_6 + b_1b_{6\xi}fa_1 + b_1b_6f_{\xi}a_1$$
$$+2fa_{1\xi}b_1b_6 + 2a_3b_1f^2a_1^3 + 6a_2^3fa_1 + 30f_{\xi}a_1^2a_2 + 30ff_{\xi}a_1^2a_2 + 60f^2a_{1\xi}a_1a_2 + 6a_{1\xi}f_{\xi}a_1 + 12fa_{1\xi}^2$$
$$+6a_1b_{6\xi} + 3a_2b_1b_6fa_1 + 6f_{\xi\xi}a_1^2 + 24f_{\xi}a_{1\xi}a_1 + 12fa_{1\xi\xi}a_1 + 12fa_{1\xi}^2 + b_1b_6^2 + 2b_1b_5fa_1^2 + 2b_2fa_1^2 = 0$$

Take the expression for b_5 as,

$$b_5 = -\frac{1}{2b_1fa_1^2}\left[h_2(a_1, a_2) + \left(40fa_1^2 + 2b_1f^2a_1^3\right)a_3\right]$$

with,

$$h_2(a_1, a_2) = 32a_1^2a_2^2f + 12a_{1\xi}a_2fa_1 + 18a_{2\xi}fa_1^2 + 2a_{1\xi\xi}fa_1 + 6a_{1\xi}h_1 + 12a_1a_2h_1$$
$$+b_1h_{1\xi}fa_1 + b_1h_1f_{\xi}a_1 + 24f_{\xi}a_{1\xi}a_1 + 6a_2^3fa_1 + 30f_{\xi}a_1^2a_2 + 30ff_{\xi}a_1^2a_2 + 60f^2a_{1\xi}a_1a_2$$
$$+6a_{1\xi}f_{\xi}a_1 + 12fa_{1\xi}^2 + 6a_1h_{1\xi} + 3a_2b_1h_1fa_1 + 6f_{\xi\xi}a_1^2 + 2fa_{1\xi}b_1h_1 + 12fa_{1\xi\xi}a_1 + 12fa_{1\xi}^2$$
$$+b_1h_1^2 + 2b_2fa_1^2 \tag{50}$$

The fourth relation of (47) will generate,

$$h_3(a_1, a_2) = \left(24f_{\xi}a_1^3 + 48fa_1^2a_{1\xi} + 3b_1b_6fa_1^2\right)a_3 + 6fa_{1\xi}a_{1\xi\xi}12a_2a_{2\xi}fa_1^2 + 2a_{2\xi\xi}fa_1^2$$
$$+8a_2^3fa_1^2 + 7a_1a_2^2h_1 + 3a_{1\xi}a_2h_1 + 6a_1a_{2\xi}h_1 + a_{1\xi\xi}h_1 + 12f_{\xi}a_1^2a_2^2 + 24fa_1a_{1\xi}a_2^2 + 6f_{\xi}a_1^2a_{2\xi}$$
$$+12fa_1a_{1\xi}a_{2\xi} + 9a_1a_2h_{1\xi} + 3a_{1\xi}h_{1\xi} + f_{\xi\xi\xi}a_1^2 + 2f_{\xi\xi}a_1a_{1\xi} + 4f_{\xi\xi}a_1a_{1\xi} + 6f_{\xi}a_{1\xi}^2 + 6f_{\xi}a_1a_{1\xi\xi}$$
$$+2fa_1a_{1\xi\xi\xi} + b_1h_1h_{1\xi} + f_{\xi}a_1^2b_2 + 2fa_1a_{1\xi}b_2 + b_3fa_1^2 + 2a_2b_2fa_1^2 + 3a_1h_{1\xi\xi} + a_1b_2h_1$$
$$+6f_{\xi\xi}a_1^2a_2 + 24f_{\xi}a_1a_{1\xi}a_2 + 12fa_{1\xi}^2a_2 + 12fa_1a_{1\xi\xi}a_2 + a_2b_1h_1^2 + \frac{h_2}{2b_1fa_1^2} + b_1fa_1^2\left(\frac{h_2}{2b_1fa_1^2}\right)_{\xi} \tag{51}$$

which then produce the solution of a_3.

Note that a_1 and a_2 are the chosen fundamental variables and according to (48–51) and with the known coefficients b_1, b_2, b_3, b_4, they will define, $b_5, b_6, b_7, a_3, \beta$. Therefore, the solution of Korteweg de Vries equation is generated as,

$$\phi(\xi) = b_5(a_1, a_2) + b_6(a_1, a_2)\left\{\frac{\sqrt{2}}{2}\frac{e^{\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1, a_2)d\xi + C_2\right)}{\left[\int_{\xi}a_1e^{2\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1, a_2)d\xi + C_2\right)d\xi + C_3\right]^{\frac{1}{2}}}\right\}$$
$$+b_7(a_1, a_2)\left\{\frac{1}{2}\frac{e^{2\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1, a_2)d\xi + C_2\right)^2}{\left[\int_{\xi}a_1e^{2\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1, a_2)d\xi + C_2\right)d\xi + C_3\right]}\right\} \tag{52}$$

Since only a few of the considered equation has a special polynomial to be solved by equating all the variable coefficients to zero, it is important to note that the reduction of polynomial order would be an important step. Solving all coefficients to zero often be an obstacle because the difficulty would be the same or even more than the original nonlinear ODEs. In this case, the reduction of polynomial manipulates and reduces the need for solving all coefficients.

However, it is possible not to search for the expression of variable coefficients, i.e., b_5, b_6, b_7, a_1, a_2 and a_3. First the roots of Eq. (37) are determined first as ϕ, and then Eq. (1) is decomposed as,

$$D_\xi = a_1 B D^2 + \left(a_2 - \frac{B_\xi}{B}\right) D + \frac{a_3}{B} \tag{53}$$

with $A = BD$. The solution of (53) is then,

$$D = \frac{\sqrt{2}}{2B} e^{\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) d\xi + C_3\right]^{-\frac{1}{2}} \text{ or}$$

$$A = BD = \frac{\sqrt{2}}{2} e^{\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) d\xi + C_3\right]^{-\frac{1}{2}} \tag{54}$$

The definition for B is determined by substituting the polynomial solution, ϕ into (54) as in the following,

$$A = \phi = \frac{\sqrt{2}}{2} e^{\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) d\xi + C_3\right]^{-\frac{1}{2}}$$

Rearranging the above equation as,

$$\phi^2 \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) d\xi + C_3\right] = \frac{1}{2} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right)^2$$

Differentiating once,

$$\begin{aligned} \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) = & \left(\frac{a_2}{\phi^2} e^{2\int_\xi a_2 d\xi} - \frac{\phi_\xi}{\phi^3} e^{2\int_\xi a_2 d\xi}\right) \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right)^2 \\ & + \frac{a_3}{B} \frac{1}{\phi^2} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) \text{ or} \end{aligned}$$

$$\frac{a_1}{B} = \left(\frac{a_2}{\phi^2} - \frac{\phi_\xi}{\phi^3}\right) \left(\int_\xi \frac{a_3}{B} d\xi + C_2\right) + \frac{a_3}{B} \frac{1}{\phi^2} \tag{55}$$

Eq. (55) is a first order ODE in B and can be easily solved, which then prove that $A = \phi$ without establishing the explicit expression for variable coefficients.

5. Generalized method

In this section, the method of solution to the Riccati equation is extended for the class of the first order polynomial differential equation as,

$$A_\xi = a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \ldots\ldots\ldots + a_3 A^3 + a_2 A^2 + a_1 A + a_0 \tag{56}$$

The above equation can be always re-expressed as,

$$A_\xi = \left(b_n A^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \text{.........} + b_3 A + b_2\right)A^2 + b_1 A \\ + \left(b_n A^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \text{.........} + b_3 A + b_2\right)b_0$$

or

$$A_\xi = b_n A^n + b_{n-1}A^{n-1} + (b_{n-2} + b_n b_0)A^{n-2} + \text{......} + (b_3 + b_{n-1}b_0)A^3 \\ + (b_2 + b_{n-2}b_0)A^2 + (b_1 + b_3 b_0)A + b_2 b_0 \tag{57}$$

which the coefficients will be reformulated as,

$$\begin{aligned} & b_n = a_n, b_{n-1} = a_{n-1}, b_2 b_0 = a_0 \\ & b_{n-2} + b_n b_0 = a_{n-2} \\ & b_3 + b_{n-1} b_0 = a_3 \\ & b_2 + b_{n-2} b_0 = a_2 \\ & b_1 + b_3 b_0 = a_1 \end{aligned} \quad \text{or} \quad \begin{aligned} & b_n = a_n, b_{n-1} = a_{n-1}, b_2 b_0 = a_0 \\ & a_2 b_2^2 - b_2^3 + a_n a_0 = a_{n-2} a_0 b_2 \\ & b_3 + a_{n-1}\frac{a_0}{b_2} = a_3 \\ & b_2^2 + b_{n-2} a_0 = a_2 b_2 \\ & b_1 + b_3 \frac{a_0}{b_2} = a_1 \end{aligned} \tag{58}$$

In this case, we will always obtain the new coefficients b_i. Proceeding into the other equations andmultiply the equation by the function α, to get,

$$\alpha A_\xi = \left(b_n A^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \text{.........} + b_3 A + b_2\right)\alpha A^2 + b_1 \alpha A \\ + \left(b_n A^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \text{.........} + b_3 A + b_2\right)\alpha b_0 \quad \text{or}$$

$$B_\xi = \left(b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} + \text{.........} + b_3 \alpha^{-1} B + b_2\right)\alpha^{-1} B^2 + \left(b_1 + \frac{\alpha_t}{\alpha}\right) B \\ + \left(b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} + \text{.........} + b_3 \alpha^{-1} B + b_2\right)\alpha b_0 \tag{59}$$

where $B = \alpha A$. Then, all the new coefficients in b_i will be determined. The step is now to solve the Riccati equation. Let $B = \beta_2 \beta_3$, the equation can be rearranged as,

$$\beta_3 \beta_{2\xi} + \beta_2 \beta_{3\xi} = \left(b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} + \text{.........} + b_3 \alpha^{-1} B + b_2\right)\alpha^{-1}\beta_2^2 \beta_3^2 \\ + \left(b_1 + \frac{\alpha_\xi}{\alpha}\right)\beta_2\beta_3 + \left(b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} + \text{.........} + b_3 \alpha^{-1} B + b_2\right)\alpha b_0 \quad \text{or}$$

$$\beta_3 \beta_{2\xi} - \left(b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} + \text{.........} + b_3 \alpha^{-1} B + b_2\right)\alpha^{-1}\beta_2^2 \beta_3^2 \\ - \left(b_1 + \frac{\alpha_\xi}{\alpha}\right)\beta_2\beta_3 = -\beta_2 \beta_{3\xi} + \begin{pmatrix} b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} \\ + \text{.........} + b_3 \alpha^{-1} B + b_2 \end{pmatrix}\alpha b_0 = \gamma \beta_2 \beta_3 \tag{60}$$

and is separated as,

$$\beta_{2\xi} - \left(b_n \alpha^{2-n} B^{n-2} + b_{n-1}\alpha^{3-n} B^{n-3} + b_{n-2}\alpha^{4-n} B^{n-4} + \text{.........} + b_3 \alpha^{-1} B + b_2\right)\alpha^{-2}\beta_2^2 \beta_3 \\ - \left(b_1 + \frac{\alpha_\xi}{\alpha} + \gamma\right)\beta_2 = 0$$

and

$$\beta_{3\xi} + \gamma\beta_3 - \frac{1}{\beta_2}\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right)\alpha b_0 = 0 \tag{61}$$

The solutions for β_2 and β_3 are,

$$\begin{array}{l}\beta_2 = -e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}+\gamma\right)d\xi}x \\ \left[\int_\xi e^{\int_t\left(b_1+\frac{\alpha_\xi}{\alpha}+\gamma\right)d\xi}\begin{pmatrix} b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} \\ + \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\end{pmatrix}\alpha^{-1}\beta_3 d\xi + C_1\right]^{-1} \text{ and} \\ \beta_3 = e^{-\int_\xi \gamma d\xi}\left[\int_\xi e^{\int_\xi \gamma d\xi}\begin{pmatrix} b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} \\ + \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\end{pmatrix}\alpha\frac{b_0}{\beta_2}d\xi + C_2\right]\end{array} \tag{62}$$

The relation for $B = \beta_2\beta_3$ is thus,

$$\begin{array}{l}B = \beta_2\beta_3 = -e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\left[\begin{array}{l}\int_\xi e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}+\gamma\right)d\xi}\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right)\alpha^{-1}\beta_3 d\xi \\ +C_1\end{array}\right]^{-1} x \\ \left[\int_\xi e^{\int_\xi \gamma d\xi}\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right)\alpha b_0 d\xi + C_2\right]\end{array}$$

Without loss of generality, suppose that $\beta_2 = \varphi e^{\int_\xi \gamma d\xi}$ and the above relation is performed as,

$$\begin{array}{l}\beta_2\beta_3 = -e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\left[\int_\xi e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\begin{pmatrix} b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} \\ +b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B \\ +b_2\end{pmatrix}\alpha^{-1}\varphi\beta_2\beta_3 d\xi + C_1\right]^{-1} \text{ or} \\ \left[\int_\xi\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right)\alpha\varphi^{-1}b_0 d\xi + C_2\right]\end{array}$$

$$\begin{array}{l}\beta_2\beta_3\left[\int_\xi e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\begin{pmatrix} b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} \\ + \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\end{pmatrix}\alpha^{-1}\varphi\beta_2\beta_3 d\xi + C_1\right] = \\ -e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\left[\int_t\begin{pmatrix} b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} \\ + \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\end{pmatrix}\alpha\varphi^{-1}b_0 d\xi + C_2\right]\end{array}$$

Rearrange the above equation as,

$$\begin{array}{l}e^{\int_\xi b_1 d\xi}\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right)\varphi\beta_2\beta_3 \\ \left[\int_\xi e^{\int_\xi b_1 d\xi}\left(b_n y^{n-2} + b_{n-1}y^{n-3} + b_{n-2}y^{n-4} + \ \ldots\ldots\ldots + b_3 y + b_2\right)\varphi\beta_2\beta_3 d\xi + C_1\right] = \\ -e^{2\int_\xi b_1 d\xi}\alpha\varphi\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right) \\ \left[\int_\xi\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots + b_3\alpha^{-1}B + b_2\right)\alpha\varphi^{-1}b_0 d\xi + C_2\right]\end{array}$$

Let $e^{2\int_\xi b_1 d\xi}\alpha\varphi = \alpha\varphi^{-1}b_0$ and integrate the above equation to get,

$$\left[\int_\xi e^{\int_\xi b_1 d\xi}\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B + b_2\right)\varphi\beta_2\beta_3 dt + C_1\right]^2 =$$
$$-\left[\int_\xi \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B + b_2\right)\alpha\varphi^{-1}b_0 d\xi + C_2\right]^2$$

or

$$e^{\int_\xi b_1 d\xi}\left(b_n\alpha^{2-n}B^{n-1} + b_{n-1}\alpha^{3-n}B^{n-2} + b_{n-2}\alpha^{4-n}B^{n-3} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B^2 + b_2 B\right)\varphi^2 =$$
$$-\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B + b_2\right)\alpha b_0$$

The solution for B is then reduced into the solution of the polynomial equation. Thus, let $A = \alpha^{-1}B = \phi$, whereϕ is the expression from the solution of the resulting polynomial equation which is similar to (38). The expression for α can be determined by the inverse method as in (53–55) for the first order polynomial differential Eq. (56).

6. Conclusion

In this chapter, we propose the method of the simplest or the auxiliary equation to solve the nonlinear differential equation with variable coefficients. The method is based on the solution of the generalized Riccati equation as the simplest equation. It is found that the other known simplest equations, i.e., Jacobi and Weierstrass equation, are also solved by the Riccati equation. The applications with the variable coefficients elliptic-like and Korteweg de Vries equations show that the problem of solving nonlinear differential equations with variable coefficients are simplified, especially by the reduction of the resulting polynomial equation in solving the Korteweg de Vries equation. The generalization of the method is also derived in detail.

Conflict of interest

Authors declare that there is no conflict of interest.

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