Chapter

Positive Periodic Solutions for First-Order Difference Equations with Impulses

Mesliza Mohamed, Gafurjan Ibragimov and Seripah Awang Kechil

Abstract

This paper investigates the first-order impulsive difference equations with periodic boundary conditions

$$\Delta x(n) = f(n, x(n)), \quad n \in J, \;\; n \neq n_k,$$
$$\Delta x(n_k) = I_k(x(n_k)), \quad k = 1, 2, \ldots, m,$$
$$x(0) = x(T),$$

where $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$. By using a cone theoretic fixed point theorem, new existence theorems on positive periodic solutions are established. Our main results enrich and complement those available in the literature.

Keywords: Impulse, difference equation, boundary value problem, Green's function, fixed point theorem

1. Introduction

Let $\mathbb{R}$ denote the real numbers and $\mathbb{R}^+$ the positive real numbers. Let $J = [0, T] = \{0, 1, \cdots, T\}$. In this paper we investigate the existence of positive periodic solutions for nonlinear impulsive difference equations

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \in J, \;\; n \neq n_k, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad k = 1, 2, \ldots, m, \\ x(0) &= x(T), \end{aligned} \tag{1}$$

where Δ denotes the forward difference operators, i.e., $\Delta x_n = x_{n+1} - x_n$, $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$.

Boundary value problems for difference equations and impulsive differential equation have been widely received attentions from many authors (see [1–12]) and reference therein. However, as far as we know, the theory of difference equation for boundary value problems (BVPs) with impulses is rather less, there are still lots of work and research that should be done. In [3], He and Zhang obtained the criteria on the existence of minimal and maximal solutions of (1) by using the method of upper and lower solutions and monotone iterative technique. The similar techniques were applied in [11] for the problem

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \in J, \quad n \neq n_k, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad k = 1, 2, \cdots, p, \\ Mx(0) &- Nx(T) = C, \end{aligned} \tag{2}$$

where $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$. Motivated by the work of [3, 13] we establish the existence of positive periodic solutions for (1) by using the fixed point-theorem in cones following the ideas of [13]. The results herein improve some of the results in [3, 11].

Throughout the paper, we make the following assumptions:

1. $0 < M < 1$, $0 < L_k < M$, $f(n, x(n)) + Mx(n) \geq 0$.

2. $(M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k) \geq 0$.

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, by applying the fixed point theorem in cones, we obtain some new existence theorems for the impulsive difference equations with periodic boundary conditions.

2. Preliminaries

Let

$$E = \{x : J \to \mathbb{R} : x(0) = x(T)\}$$

with the norm $\|x\| = \max_{n \in J} |x(n)|$. Then E is a Banach space.

Consider the following linear impulsive difference equations with periodic boundary condition

$$\begin{aligned} \Delta x(n) + Mx(n) &= \sigma(n), \quad n \in J, n \neq n_k, \\ \Delta x(n_k) = -L_k x(n_k) &+ I_k(x(n_k)) + L_k x(n_k), \quad k = 1, 2, \ldots, m, \\ x(0 &= x(T), \end{aligned} \tag{3}$$

where M is a constant, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\sigma \in C(J, \mathbb{R}^+)$.

The following lemma transforms the analysis of PBVP (3) to the analysis of summation equation. We denote $G(n,j)$, the Green's function of the problem (3).

Lemma 1. *Let (A1) and (A2) hold, $\sigma \in C(J, \mathbb{R}^+)$. Then $x \in E$ is a solution of PBVP (3) if and only if x is a solution of the following impulsive summation equation*

$$\begin{aligned} x(n) = &\sum_{j=0, j \neq n_k}^{T-1} G(n,j)\sigma(j) \\ &+ \sum_{0 < n_k \leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + +L_k x(n_k)), \end{aligned} \tag{4}$$

where

$$G(n,j) = \frac{1}{1 - (1-M)^T} \begin{cases} \dfrac{(1-M)^n}{(1-M)^{j+1}}, & 0 \leq j \leq n-1, \\[2ex] \dfrac{(1-M)^{T+n}}{(1-M)^{j+1}}, & n \leq j \leq T-1. \end{cases} \tag{5}$$

Proof For convenience, we give the proof for the corresponding linear case (3). Consider first the homogeneous equation

$$\Delta u(n) + Mu(n) = 0, \qquad n \in J, \;\; n \neq n_k,$$

which is easily solved by iteration. We have

$$u(n) = (1-M)^n,$$

with $u(0) = 1$. Now the first Eq. (3) can be solved by substituting $x(n) = u(n)y(n)$ into Eq. (3), where y is to be determined:

$$u(n+1)y(n+1) - u(n)y(n)(1-M) = \sigma(n)$$

or

$$\Delta y(n) = \frac{\sigma(n)}{Eu(n)}, n \neq n_k.$$

So from (3), we see that $y(n)$ satisfies

$$\begin{gathered} \Delta y(n) = \frac{\sigma(n)}{(1-M)^{n+1}}, n \neq n_k, \\ \Delta y(n_k) = \frac{M-L_k}{1-M} y(n_k) + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}}, \;\; k = 1, 2, \ldots, m, \\ y(0) = y(T)(1-M)^T. \end{gathered} \tag{6}$$

From (6), we have

$$\begin{aligned} y(n) = y(0) &+ \sum_{j=0, j\neq n_k}^{n-1} \frac{\sigma(j)}{(1-M)^{j+1}} \\ &+ \sum_{0<n_k\leq n-1} \left(\frac{M-L_k}{1-M} y(n_k) + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}} \right). \end{aligned} \tag{7}$$

Letting $n = T$ in (7), we have

$$\begin{aligned} y(T) = y(0) &+ \sum_{j=0, j\neq n_k}^{T-1} \frac{\sigma(j)}{(1-M)^{j+1}} \\ &+ \sum_{0<n_k\leq T-1} \left(\frac{M-L_k}{1-M} y(n_k) + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}} \right). \end{aligned} \tag{8}$$

Applying (8) and the boundary condition $y(0) = y(T)(1-M)^T$, we get

$$\begin{aligned} y(0) = \frac{(1-M)^T}{1-(1-M)^T} &\left(\sum_{j=0, j\neq n_k}^{T-1} \frac{\sigma(j)}{(1-M)^{j+1}} + \sum_{0<n_k\leq T-1} \left(\frac{M-L_k}{1-M} y(n_k) \right.\right. \\ &\left.\left. + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}} \right) \right). \end{aligned} \tag{9}$$

Substituting (9) into (7) and using $y(n) = \frac{x(n)}{(1-M)^n}, n \in J$, we get the unique solution of (3)

$$x(n) = \sum_{j=0, j \neq n_k}^{T-1} G(n,j)\sigma(j) + \sum_{0 < n_k \leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))),$$

where $G(n,j)$ is given in (5). The proof is complete. □.
Consider the PBVP (1). To define a cone, we observe that

$$\frac{(1-M)^n}{1-(1-M)^T} \leq |G(n, n_j)| \leq \frac{1}{1-(1-M)^T}.$$

Define

$$A := \frac{(1-M)^n}{1-(1-M)^T},$$
$$B := \frac{1}{1-(1-M)^T}.$$

We denote the cone in E by

$$K = \{x \in E \ , \ n \in [0, T] \text{ and } x(n) \geq \sigma \|x\|\},$$

where $\sigma = \frac{A}{B} \in (0, 1)$. Define a mapping $T : E \to E$ by

$$Tx(n) = \sum_{j=0, j \neq n_k}^{T-1} G(n,j)\sigma(j) + \sum_{0 < n_k \leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))). \tag{10}$$

Since f is continuous, T is also a continuous map. Before starting the main results, we shall give some important lemmas. The next Lemma is essential in obtaining our results.

Lemma 2. *The mapping T maps K into K, i.e $TK \subset K$.*

Proof For any $x \in K$, it is easy to see that $Tx \in E$. From (10) we have

$$\begin{aligned} Tx(n) &= \sum_{j=0, j \neq n_k}^{T-1} G(n,j)\sigma(j) + \sum_{0 < n_k \leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\ &\leq \sum_{j=0, j \neq n_k}^{T-1} B\sigma(j) + \sum_{0 < n_k \leq T-1} B((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))). \end{aligned}$$

Noting that

$$G(n,j)\sigma(j) \geq 0, \quad G(n,n_k)((M-L_k)x(n_k)+I_k(x(n_k))+L_k x(n_k)) \geq 0.$$

We can also obtain

$$\begin{aligned}
Tx(n) = & \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\
& + \sum_{0<n_k\leq T-1} G(n,n_k)((M-L_k)x(n_k)+I_k(x(n_k))+L_k x(n_k))) \\
& \geq \sum_{j=0, j\neq n_k}^{T-1} A\sigma(j) + \sum_{0<n_k\leq T-1} A((M-L_k)x(n_k)+I_k(x(n_k))+L_k x(n_k))) \\
& \geq \frac{A}{B}|Tx(n)| \\
& \geq \sigma\|Tx\|.
\end{aligned}$$

Hence $TK\subset K$. The proof is complete. □.

The following lemma is crucial to prove our main results.

Lemma 3. *(Guo−Krasnoselskii's fixed point theorem [14])* Let E be a Banach space and let $K\subset E$ be a cone E. Assume Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1$, $\bar{\Omega}_1\subset\Omega_2$, and let

$$T : K\cap(\bar{\Omega}_2\backslash\Omega_1) \to K$$

be a completely continuous operator such that either.

i. $\|Tx\| \leq \|x\|$ for $x\in K\cap\partial\Omega_1$ and $\|Tx\| \geq \|x\|$ for $x\in K\cap\partial\Omega_2$; or

ii. $\|Tx\| \geq \|x\|$ for $x\in K\cap\partial\Omega_1$ and $\|Tx\| \leq \|x\|$ for $x\in K\cap\partial\Omega_2$.

Then T has a fixed point in $K\cap(\bar{\Omega}_2\backslash\Omega_1)$.

We are now in a position to apply the preceding results to obtain the existence of positive periodic solutions to (1).

3. Main results

In this section, we state and prove our main findings.

Theorem 1. *Suppose that the following conditions hold*

$$lim_{x\to 0^+}\frac{f(n,x)}{x} = -M, \qquad lim_{x\to 0^+}\sum_{k=1}^{m}\frac{I_k(x)}{x} = 0, \tag{11}$$

$$lim_{x\to +\infty}\frac{f(n,x)}{x} = \infty, \qquad lim_{x\to +\infty}\sum_{k=1}^{m}\frac{I_k(x)}{x} = \infty. \tag{12}$$

Then the problem (1) has at least one positive periodic solution.

Proof $0<r<R<\infty$, setting

$$\Omega_1 = \{x\in E : \|x\|<r\}, \quad \Omega_2 = \{x\in E : \|x\|<R\}.$$

We have $0 \in \Omega_1$, $\Omega_1 \subseteq \Omega_2$. It follows from (11) that there exists $r > 0$ so that for any $0 < x \le r$,

$$f(n, x(n)) \le c_1 x - Mx, \quad \sum_{k=1}^{m} I_k \le c_2 x,$$

where c_1, c_2 are positive constants satisfying

$$\sigma B(Tc_1 + (T-m)(M + c_2)) < 1.$$

Therefore for $x \in K$, with $\|x\| = r$,

$$f(n, x(n)) + Mx \le c_1 x, \quad \sum_{k=1}^{m} I_k \le c_2 x.$$

Moreover $0 < \sigma\|x\| \le x(n) \le \|x\| = r$. Thus

$$\begin{aligned}
Tx(n) = & \sum_{j=0, j\ne n_k}^{T-1} G(n,j)\sigma(j) \\
& + \sum_{0<n_k \le T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \le \sum_{n=0, n\ne n_k}^{T-1} B(f(n, x(n)) + Mx(n)) \\
& + \sum_{0<n_k \le T-1} B((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \le TBc_1\sigma\|x\| + (T-m)B\sigma(M - L_k + c_2 + L_k)\|x\| \\
& \le \sigma B(Tc_1 + (T-m)(M + c_2))\|x\| \\
& \le \|x\|,
\end{aligned}$$

which implies $\|Tx\| \le \|x\|$ for $\forall x \in K \cap \partial\Omega_1$.

On the other hand (12) yields the existence of $\hat{R} > 0$ such that for any $x \ge \hat{R}$

$$f(n, x(n)) \ge \eta_1 x, \quad \sum_{k=1}^{m} I_k \ge \eta_2 x$$

where $\eta_1, \eta_2 > 0$ are constants large enough such that

$$A\sigma(T(\eta_1 + M) + (T-m)(M + \eta_2)) > 1.$$

Fixing $R \ge \max\left\{r, \frac{\hat{R}}{\sigma}\right\}$. and letting $x \in K$ with $\|x\| = R$, we get $x(n) \ge \sigma\|x\| = \sigma R > \hat{R}$ and $f(n, x(n)) + Mx \ge \eta_1 x + Mx \ge \sigma(\eta_1 + M)\|x\|$.

Thus

$$\begin{aligned}
Tx(n) = & \sum_{j=0, j\ne n_k}^{T-1} G(n,j)\sigma(j) \\
& + \sum_{0<n_k \le T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \ge \sum_{n=0, n\ne n_k}^{T-1} A(f(n, x(n)) + Mx(n))
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0<n_k\leq T-1} A((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k)))\\
&\geq TA\sigma(\eta_1+M)\|x\| + A(T-m)\sigma(M-L_k+\eta_2+L_k)\|x\|\\
&\geq A\sigma(T(\eta_1+M)+(T-m)(M+\eta_2))\|x\|\\
&\geq \|x\|.
\end{aligned}
$$

In particular $\|Tx\| \geq \|x\|$, $\forall x \in K \cap \partial\Omega_2$.

Consequently by Lemma 3(i), T has a fixed point in

$$K \cap (\bar{\Omega}_2 \backslash \{\Omega_1\}),$$

which is a positive periodic solution of (1). The proof is complete. □.

Theorem 2. *Suppose that the following conditions hold*

$$lim_{\,x\to 0^+} \frac{f(n,x)}{x} = \infty, \qquad lim_{\,x\to 0^+} \sum_{k=1}^{m} \frac{I_k(x)}{x} = \infty, \tag{13}$$

$$lim_{\,x\to +\infty} \frac{f(n,x)}{x} = -M, \qquad lim_{\,x\to +\infty} \sum_{k=1}^{m} \frac{I_k(x)}{x} = 0. \tag{14}$$

Then the problem (1) has at least one positive periodic solution.

Proof We follow the same strategy and notations as in the proof of Theorem 1. Firstly, we show that for $r > 0$ sufficiently large

$$\|Tx\| \geq \ \|x\|, \ \ \forall x \in K \cap \Omega_1.$$

From (13) it follows that there exists $0 < x < \hat{r}$, where β_1, β_2 are constants large enough such that $\sigma A(T(\beta_1+M)+(T-m)(M+\beta_2)) > 1$. Therefore, for $0 < x < \hat{r}$, if $x \in K$ and $\|x\| = r$, then from (12),

$$f(n,x(n)) + Mx \geq \beta_1 x + Mx \geq \sigma(\beta_1+M)\|x\|,$$

$$\sum_{k=1}^{m} I_k \geq \beta_2 x \geq \sigma\beta_2\|x\|.$$

Furthermore, we obtain

$$
\begin{aligned}
Tx(n) = &\sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j)\\
&+ \sum_{0<n_k\leq T-1} G(n,n_k)((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k)))\\
&\geq \sum_{n=0, n\neq n_k}^{T-1} A(f(n,x(n))+Mx(n))\\
&+ \sum_{0<n_k\leq T-1} A((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k)))\\
&\geq TA\sigma(\beta_1+M)\|x\| + (T-m)A\sigma(M-L_k+\beta_2+L_k)\|x\|\\
&\geq \sigma A(T(\beta_1+M)+(T-m)(M+\beta_2))\|x\|\\
&\geq \|x\|,
\end{aligned}
$$

which implies $\|Tx\| \geq \|x\|$, for each $x \in K \cap \partial\Omega_1$.

Next we show that for $R > 0$ sufficiently large, $\|Tx\| \leq \|x\|$, $\forall x \in K \cap \partial\Omega_2$. On the other hand (14) yields the existence of $\hat{R} > 0$ such that for any $x \geq \hat{R}$

$$f(n, x(n)) \leq \eta_1 x - Mx, \quad \sum_{k=1}^{m} I_k \leq \eta_2 x,$$

where $\eta_1, \eta_2 > 0$ are constants such that

$$B\sigma(T\eta_1 + (T - m)(M + \eta_2)) < 1.$$

Fixing $R \geq \max\left\{r, \frac{\hat{R}}{\sigma}\right\}$. and letting $x \in K$ with $\|x\| = R$, we get $x(n) \geq \sigma\|x\| = \sigma R > \hat{R}$ and $f(n, x(n)) + Mx \leq \eta_1 x \leq \eta_1 \sigma\|x\|$ and

$$\begin{aligned}
Tx(n) = & \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\
& + \sum_{0<n_k\leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \leq \sum_{n=0, n\neq n_k}^{T-1} B(f(n, x(n)) + Mx(n)) \\
& + \sum_{0<n_k\leq T-1} B((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \leq TB\sigma\eta_1\|x\| + B(T - m)\sigma((M - L_k)\|x\| + \eta_2\|x\| + L_k\|x\|) \\
& \leq B\sigma(T\eta_1 + (T - m)(M + \eta_2))\|x\| \\
& \leq \|x\|.
\end{aligned}$$

which implies $\|Tx\| \leq \|x\|$, $\forall x \in K \cap \partial\Omega_2$.

Finally, it follows from Lemma 3(ii) that T has a fixed point in

$$K \cap (\bar{\Omega}_2 \backslash \Omega_1),$$

which is a positive periodic solution of (1). The proof is complete. □

4. Conclusion

By applying the fixed point theorem in cones, we establish new existence theorems on positive periodic solutions for impulsive difference equations with periodic boundary conditions. Our main findings enrich and complement those available in the literature.

Additional classifications

AMS subject classification: 39A10, 34B37

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