Chapter

Solving Second-Order Differential Equations by Decomposition

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Abstract

The subject of this article are linear and quasilinear differential equations of second order that may be decomposed into a first-order component with guaranteed solution procedure for obtaining closed-form solutions. These are homogeneous or inhomogeneous linear components, special Riccati components, Bernoulli, Clairaut or d'Alembert components. Procedures are described how they may be determined and how solutions of the originally given second order equation may be obtained from them. This makes it possible to solve new classes of differential equations and opens up a new area of research. Applying decomposition to linear *inhomogeneous* equations a simple procedure for determining a special solution follows. It is not based on the method of *variation of constants* of Lagrange, and consequently does not require the knowledge of a fundamental system. Algorithms based on these results are implemented in the computer algebra system ALLTYPES which is available on the website www.alltypes.de.

Keywords: ordinary differential equations, decomposition, exact solutions, computer algebra

1. Introduction

The history of differential equations begins shortly after the establishment of the analysis by Newton and Leibniz in the 17th century. A brief overview of its first hundred years can be found in Appendix A of Ince's book [1]. These early investigations were mainly limited to first-order equations, associated with the names Riccati, Bernoulli and Euler. Starting in the early 18th century special linear equations of higher order were also investigated.

A more systematic search for solution methods was initiated by the results of Galois for solving algebraic equations in the early 19th century. Inspired by these results, Picard and Pessiot in Paris founded a solution theory for *linear* differential equations, known as Picard-Vessiot theory or differential Galois theory. A good introduction into their work and its extensions by Loewy may be found in the books [2, 3]. Completely independent of these activities Sophus Lie in Leipzig founded the so-called symmetry analysis for solving *nonlinear* differential Equations [4, 5]. Its main weaknesses are that most differential equations have no symmetries and therefore it cannot be applied. Furthermore, there are many differential equations with fairly simple closed form solutions that have no symmetries. That was essentially the status in the early twentieth century, which did not fundamentally change until its end.

In this situation, a new solution method based on *decompositions* was proposed [6]. Essentially a decomposition means to find a *component* of lower order such that the original equation may be represented as a differential polynomial in terms of this component. Its existence is based on the following observation. Let $F(x, y, y', y'') = 0$ be a second-order differential equation for a function y depending on a variable x, and $\omega(x, y, C_1, C_2) = 0$ its general solution depending on two undetermined constants. It describes a two-parameter family of curves in the $x - y$-plane. If C_1 and C_2 are constrained by a relation $\varphi(C_1, C_2) = 0$ the resulting expression for ω contains effectively a single parameter C. It describes a family of curves that may obey a first-order differential equation called a component. Its solutions are also solutions of the originally given second-order equation.

Every second-order equation has an infinite number of first-order components corresponding to the choice of $\varphi(C_1, C_2)$. Any such component has the form

$$F(x, y, y', y'') = f(x, y, z, z', C)(z \equiv g(x, y, y', C)). \tag{1}$$

Its meaning may be described as follows. If $z \equiv g(x, y, y', C)$ is substituted into $f(x, y, z, z', C)$ the second-order equation on the left-hand side is obtained. The constant C does not necessarily occur in f and g, the same is true for y and its occurence in f.

Solving a second-order equation by decomposition involves two steps. First a decomposition of a certain type has to be found. Then the first order equation has to be solved in order to get the solutions of the original second-order equation. Of particular interest are those components the solution of which can always be determined. These are linear homogeneous and inhomogeneous components, special Riccati components, Bernoulli, Clairaut or d'Alembert components.

In this article equations of second order for an unknown function y depending on x with leading term y'' or $y'y''$ are considered. They are assumed to be linear in y'', polynomial in the derivatives y', and rational in y and x. Equations of this kind are fairly common in applications, therefore many special examples of them are given in the collections by Kamke [7], Murphy [8], Polyanin [9], Sachdev [10] and Zwillinger [11]. Many interesting applications of such differential equations can be found in the textbooks by MacCluer et al. [12] and Swift and Wirkus [13].

In the following Section 2 equations with leading term y'' are considered, and possible linear or Bernoulli components are determined. For linear inhomogeneous equations it is shown how decomposition leads to a new procedure for determining a special solution without first having to know a fundamental system. Equations with leading term $y'y''$ and possible components of Clairaut or d'Alembert type are the subject of Section 3. Most of the examples do not have Lie symmetries, so decomposition is the only way to solve them. The last Section 4 discusses various possible generalizations of the decomposition method, on the one hand more general equations to be solved, on the other hand more general first-order components.

2. Equations with leading term y''

Equations that are linear in the highest derivative y'', but may contain powers of y' with coefficients that are rational in y and x are considered in this section. Moreover it is assumed that they are primitive, i.e. the leading coefficient is unity. Their general form is

$$y'' + \sum_{k=0}^{K} c_k(x, y) y'^k = 0 \,\text{with}\, c_k(x, y) \in \mathbb{Q}(x, y), \quad K \in \mathbb{N}. \tag{2}$$

Equations of this form appear in numerous applications, as can be seen in the collections of solved examples quoted above. The following proposition has been proved in [6], it is the basis for generating quasilinear first-order components; as usual $y' \equiv \frac{dy}{dx}$ and $D \equiv \frac{d}{dx}$.

Proposition 1 Let a second-order quasilinear Eq. (2) be given. A first-order component $z \equiv y' + r(x,y)$ exists if $r(x,y)$ satisfies

$$r_x - rr_y - \sum_{k=0}^{K} (-1)^k c_k(x,y) r^k = 0. \tag{3}$$

Then the original second-order equation can be decomposed as

$$\left(z' - r_y z + \sum_{k=1}^{K} c_k(x,y) \Big((z-r)^k - (-1)^k r^k \Big) \right) (z \equiv y' + r) = 0. \tag{4}$$

The proof may be found in Section 2 of [6]. As a first application linear first-order components of the form $z \equiv y' + a(x)y + b(x)$ are searched for, i.e. with the above notation $r(x,y) = a(x)y + b(x)$; its coefficients a and b are solutions of the so-called *determining system*, they may be in any field extension of $\mathbb{Q}(x)$. The following proposition describes how they may be obtained.

Proposition 2. Let a second-order quasilinear Eq. (2) be given. In order that it has a linear first-order component $z \equiv y' + a(x)y + b(x)$ the coefficients $a(x)$ and $b(x)$ have to satisfy

$$(a' - a^2)y + b' - ab - \sum_{k=0}^{K} (-1)^k c_k(x,y)(ay+b)^k = 0. \tag{5}$$

Then (2) may be written as follows

$$\left(z' - az + \sum_{k=1}^{K} c_k(x,y) \Big((z - ay - b)^k - (-1)^k (ay+b)^k \Big) \right) (z \equiv y' + ay + b) = 0. \tag{6}$$

The coefficients a and b are solutions of a first-order algebro-differential system. Its general form is

$$a' - a^2 + p(a,b,x) = 0, \;\; b' - ab + q(a,b,x) = 0, \;\; r_i(a,b,x) = 0 \tag{7}$$

for $i = 1, 2 \ldots$; $p(a,b,x)$, $q(a,b,x)$ and $r_i(a,b,x)$ are polynomials in a and b, and rational in x; their maximal degree in a and b is K. The $r_i(a,b,x)$ generate an ideal $I_{ab} \in \mathbb{Q}(x)[a,b]$.

Proof. Substituting $r = a(x)y + b(x)$ into (3) yields (5). At this point y is considered as an undetermined function. Therefore the left-hand side of (5) is represented as a partial fraction in y. Equating its coefficients to zero yields sufficient conditions in order that (5) vanishes and z is a component of (2). The first order ode's for a and b in the determining system (7) originate from the coefficients of first and zeroth degree in y of (5). The polynomials in a and b, i.e. $p(a,b,x)$, $q(a,b,x)$ and $r_i(a,b,x)$ in (7) originate from the powers of $ay + b$ and the rational coefficients $c_k(x,y)$ in (5), i.e. exclusively from the nonlinearities of (2). Substitution of $y' = z - ay - b$ into (4) yields (6). As a result, the sums at the left-hand side of (6) are a polynomial in z the coefficients of which may depend explicitly on y.

It is important to represent the left side of (5) as a partial fraction in y, only in this way the structure of the system (7) is assured.

2.1 Linear equations

If $K = 1$, $c_1(x,y) = c_1(x)$ and $c_0(x,y) = c_0(x)y + c_r(x)$ the above proposition contains the decomposition of linear equations as a special case as shown next.

Corollary 1 Let $K = 1$, $c_1(x,y) = c_1(x)$, $c_0(x,y) = c_0(x)y + c_r(x)$ and the linear inhomogeneous second-order equation

$$y'' + c_1(x)y' + c_0(x)y + c_r(x) = 0 \tag{8}$$

be given. A first-order component $z \equiv y' + a(x)y + b(x)$ exists if a and b are solutions of the determining system

$$a' - a^2 + c_1(x)a - c_0(x) = 0 \;\text{ and }\; b' + (c_1(x) - a)b - c_r(x) = 0. \tag{9}$$

If it is satisfied Eq. (8) may be written as

$$(z' + (c_1 - a)z)(z \equiv y' + ay + b) = 0. \tag{10}$$

Proof. The system (9) follows from (5) for the given special values of K and the coefficients c_k. Then reduction of (8) w.r.t. z yields (10). □

It is remarkable that in the case of linear equations the algebraic conditions $r_i(a,b,x)$ are missing, i.e. they are the most significant contributions originating from possible nonlinearities in (2).

For linear *homogeneous* ode's, i.e. for $c_r = 0$ and $b = 0$, Loewy decompositions have been shown to be an effective method for determining a fundamental system [3]. It is based on a factorization of the linear differential operator corresponding to the given equation over its base field, i.e. restricting the coefficients of the factors to the field of the coefficients of the given second-order equation. This restriction does not apply in the above corollary, the coefficients may be in any field extension.

For linear *inhomogeneous* equations in addition to a fundamental system a special solution has to be found. The above corollary avoids the usual method of variation of constants that somehow appears like an *ad hoc* method. The method described in the above corollary requires only a special solution of a Riccati equation and subsequently solving a linear first-order equation in order to obtain the general solution of the second-order Eq. (8). The following example applies this procedure.

Example 1 The equation

$$y'' - y' - \frac{1}{x}y = (x+1)e^x \tag{11}$$

is Equation 2.109 in Kamke's collection [7]. Here $c_1 = -1$, $c_0 = -\frac{1}{x}$ and $c_r = -(x+1)e^x$. The Riccati equation $a' - a^2 - a + \frac{1}{x} = 0$ has the special solution $a = -1 - \frac{1}{x}$. From $b' + \frac{1}{x}b = (x+1)e^x$ follows $b = \frac{C}{x} + \frac{1}{x}(x^2 - x + 1)e^x$ and leads to the component

$$z = y' - \left(1 + \frac{1}{x}\right)y + \frac{1}{x}\left(C + \left(x^2 - x + 1\right)e^x\right).$$

Integration yields the general solution

$$y = C_1 x e^x + C_2 x e^x \int \exp(-x) \frac{dx}{x^2} + \left(x^2 - x\log(x) - 1\right) e^x.$$

This is also the general solution of Eq. (11).

It may occur that a fundamental system of a second-order equation is rather complicated. Usually this is the case when the Riccati equation for a in (9) does not have a special rational solution and the usual algorithms for solving it do not apply, but one of the special cases of Section 4.9 (a), ... (e) in [7]. Then it may be advantageous to assume that all integration constants in (9) are zero and only a special solution is determined as shown next.

Example 2 Consider the equation

$$y'' - \frac{1}{2x} y' + xy + 1 = 0. \tag{12}$$

Here $c_1 = -\frac{1}{2x}$, $c_0 = x$ and $c_r = 1$. The Riccati equation $a' - a^2 - \frac{1}{2x} a - x = 0$ has the special solution $a = \sqrt{x} \tan\left(-\frac{2}{3} x\sqrt{x}\right)$, it yields

$$b' - \left(\frac{1}{2x} + \sqrt{x}\tan\left(-\frac{2}{3}\sqrt{x}\right)\right) b - 1 = 0.$$

Its special solution leads to the component

$$z \equiv y' + \sqrt{x}\tan\left(-\frac{2}{3}x\sqrt{x}\right) y + \frac{\sqrt{x}}{\cos\left(-\frac{2}{3}x\sqrt{x}\right)} \int \cos\left(-\frac{2}{3}x\sqrt{x}\right) \frac{dx}{\sqrt{x}}.$$

One more integration yields a special solution of (12).

$$y_0 = -\cos\left(-\frac{2}{3}x\sqrt{x}\right) \int \frac{\sqrt{x}}{\cos\left(-\frac{2}{3}x\sqrt{x}\right)^2} \int \cos\left(-\frac{2}{3}x\sqrt{x}\right) \frac{dx}{\sqrt{x}} dx. \qquad \square$$

The application of Corollary 1 is particularly convenient if the coefficients c_1 and c_0 are constant and the solutions of the algebraic equation $a^2 - c_1 a + c_0 = 0$ are also solutions of the Riccati equation for a. The following example is of this type.

Example 3 The equation $y'' + 4y' + 4y = \cosh(x)$ has coefficients $c_1 = c_0 = 4$ and $c_r = -\cosh(x)$. The solution of $(a-2)^2 = 0$ is $a = 2$. It leads to $b' + 2b = \cosh(x)$ and the component

$$z = y' + 2y + C\exp(-2x) + \frac{1}{3}\sinh(x) - \frac{2}{3}\sinh(x).$$

Its general solution

$$y = C_1 \exp(-2x) + C_2 x \exp(-2x) + \frac{5}{9}\cosh(x) - \frac{4}{9}\sinh(x)$$

is also the general solution of the given second-order equation. $\square$

2.2 Quasilinear equations

The most interesting applications of Proposition 2 relate to nonlinear equations, of course. They differ from the linear case mainly by the occurence of the ideal I_{ab}

in (7), which defines algebraic conditions $r_i(a,b,x) = 0$ for the coefficients of a possible component. Furthermore, the first-order ode's for a and b are modified due to the nonlinearity by additional terms. The structure of the determining system (7) suggests the following solution procedure.

At first the algebraic system $r_i(a,b,x) = 0$ is established and a Gröbner basis for the ideal I_{ab} is generated. Usually it may be determined rather efficiently.

If it is inconsistent a linear component does not exist in any field extension. This applies to a generic nonlinear equation of the form (2).

If the ideal I_{ab} is finite-dimensional each solution that satisfies the two first-order ode's yields a component that may be integrated and leads to a one-parameter family of solutions of the given second-order equation.

Finally, the algebraic equations may generate a relation between a and b; substitution into the first-order differential equations may lead to one of the above cases, or to a solution depending on a parameter. In the latter case a one-parameter family of linear components exists, integrating the corresponding equation yields the general solution of the given second-order equation containing two undetermined constants.

Subsequently this proceeding will be illustrated by several examples. They show that all of the alternatives mentioned actually exist.

Example 4 Consider the equation

$$y'' + xy'^2 + (x-1)yy' + \frac{x}{x+1}y' - y^2 - \frac{1}{x+1}y = 0.$$

Its coefficients $c_2 = x$, $c_1 = (x-1)y + \frac{x}{x+1}$ and $c_0 = -y^2 - \frac{y}{x+1}$ result in the system

$$a' - a^2 - 2abx + \frac{ax}{x+1} + b(x-1) = 0, \;\; b' - ab - b^2x + \frac{bx}{x+1} + \frac{1}{x+1} = 0,$$
$$a^2 - \frac{x-1}{x}a - \frac{1}{x} = 0.$$

The single algebraic equation has the solutions $a = -\frac{1}{x}$ and $a = 1$ and the decompositions

$$\left(z' + xz^2 + \frac{x^3y + 2x^2y + x^2 + xy + x + 1}{x^2 + x}z\right)\left(z \equiv y' - \frac{1}{x}y\right) = 0,$$
$$\left(z' + xz^2 - \frac{x^2y + 2xy + y + 1}{x+1}z\right)(z \equiv y' + y) = 0$$

follow. Integration of the two components yields the two one-parameter families $y = C\exp(-x)$ and $y = Cx$ of solution curves. It is not obvious how the general solution of the second-order equation involving two constants is supported by them. □

The most interesting, of course, are equations that allow a one-parameter family of linear components and whose integration gives the general solution. The next example is of this type.

Example 5 Consider the equation

$$yy'' - y'^2 + \frac{2}{3}y' - \frac{1}{x}y^2 = 0 \qquad (13)$$

with $K = 2$ and the coefficients $c_2 = -\frac{1}{y}$, $c_1 = \frac{2}{3y}$ and $c_0 = -\frac{y}{x}$; they generate the determining system

$$a' + \frac{1}{x} = 0, \;\; b' + ab + \frac{2}{3}a = 0, \;\; b^2 + \frac{2}{3}b = 0$$

with solution $a = -\log(x) + C$, $b = -\frac{2}{3}$ from which the decomposition

$$\left(z' - \frac{1}{y}z^2 - \frac{1}{y}\left(\log(x)y - Cy + \frac{2}{3}\right)z\right)\left(z \equiv y' - (\log(x) - C)y - \frac{2}{3}\right) = 0$$

is obtained. Integration of the first-order component leads to the general solution

$$y = \frac{2}{3}x^x \exp(C_1 x)\left(\int \exp(-C_1 x)\frac{dx}{x^x} + C_2\right)$$

of Eq. (13), it does not have a Lie symmetry.

Here the question arises how exceptional are the equations that have a one-parameter family of linear components of the first order and thus have a general solution in closed form. The following example is a generalization of the previous one. A family of second order equations is constructed whose general solution can be given explicitly.

Example 6 The equation

$$yy'' - y'^2 + p(x)y' + q(x)y^2 = 0 \tag{14}$$

with undetermined coefficients $p(x)$ and $q(x)$ generalizes the preceding example. Here $c_2 = \frac{1}{y}$, $c_1 = \frac{p(x)}{y}$ and $c_0 = q(x)y$. A first-order linear component $z \equiv y' + ay + b$ exists if a and b are solutions of the system

$$a' - q(x) = 0, \;\; b' + ab + p(x)a = 0, \;\; b(b + p(x)) = 0.$$

The result may be described as follows. If $p(x) = k$ is a constant, and $q(x)$ is an undetermined function then $a = \int q(x)dx + C$, $b = -k$ and the decomposition

$$y'' - \frac{1}{y}y'^2 + \frac{k}{y}y' + q(x)y$$
$$= \left(z' - \frac{1}{y}z^2 + \left(\int q(x)dx + C\right)z - \frac{k}{y}z\right)\left(z \equiv y' + \left(\int q(x)dx + C\right)y - k\right) = 0$$

exists. Defining $Q(x, C_1) \equiv \int q(x)dx + C_1$ integration of the first-order component yields

$$y = \exp\left(-\int Q(x, C_1)dx\right)\left(k\int \exp\left(\int Q(x, C_1)dx\right) + C_2\right).$$

This is the general solution of Eq. (14). □

It turns out that a behavior similar to that in the previous example often applies, i.e. first-order linear components often exist not only for isolated equations, but for entire families, which are parameterized by indefinite functions. This explains the existence of families of solvable equations as those given in the collections mentioned above.

Bernoulli equations are another class of first-order ode's with guaranteed closed form general solutions. In addition to a term linear in y they contain a nonlinearity y^n where n is an integer; $n = 1$ or $n = 0$ correspond to linear homogeneous or linear inhomogeneous equations, respectively. Similar as for linear components, a special Bernoulli component guarantees a one-parameter set of solution curves of a given second-order equation, and a one-parameter family of such components guarantees the general solution of the latter. The main result of this section is the following proposition.

Proposition 3 Let a second-order quasilinear Eq. (2) be given. In order that it has a first-order Bernoulli component $z \equiv y' + a(x)y^n + b(x)y$, $n \in \mathbb{N}$, the coefficients a and b have to satisfy

$$\left(a' - (n+1)ab\right)y^n + \left(b' - b^2\right)y - na^2y^{2n-1} - \sum_{k=0}^{K}(-1)^k c_k(x,y)(ay^n + by)^k = 0. \tag{15}$$

Then (2) may be written as follows

$$\left(z' - \left(nay^{n-1} + b\right)z + \sum_{k=1}^{K}\left((z - ay^n - by)^k + (-1)^{k+1}(ay^n + by)^k\right)\right) \tag{16}$$
$$(z \equiv y' + ay^n + by) = 0.$$

The coefficients a and b may be obtained from a first-order algebro-differential system; its general form is

$$a' - (n+1)ab + p(a,b,x) = 0, \quad b' - b^2 + q(a,b,x) = 0, \quad r_i(a,b,x) = 0; \tag{17}$$

$p(a,b,x)$, $q(a,b,x)$ and $r_i(a,b,x)$ are polynomials in a and b, and rational in x; the maximal degree in a and b is K, they generate an ideal I_{ab} in the ring $\mathbb{Q}(x)[a,b]$.

Proof. Substituting $r = a(x)y^n + b(x)y$ into (3) yields condition (15). Representing its left-hand side as partial fraction in the variable y, the coefficients of the various terms yield sufficient conditions for its vanishing. They form the algebro-differential system (17). The first order ode's for a and b originate from the coefficients of nth and first degree in y, respectively; $p(a,b,x)$, $q(a,b,x)$ and $r_i(a,b,x)$ originate from the coefficients $c_k(x,y)$ and the powers of $ay^n + by$.

Substitution of $y' = z - ay^n - by$ into (4) yields (16). As a result, the sums at the left-hand side of (16) are a polynomial in z and z' the coefficients of which may depend explicitly on y. □

The structure of the system (17) is similar as for linear components considered above, and consequently also the proceeding for its solution. The following examples applies the above proposition.

Example 7 The equation

$$yy'' - y'^2 + 2y^3y' + xy^2 = 0 \tag{18}$$

with $K = 2$ has coefficients $c_2 = -\frac{1}{y}$, $c_1 = 2y^2$ and $c_0 = xy$. For generic n the condition

$$\left(a' - (n-1)ab\right)y^n + \left(b' - x\right)y - (n-1)a^2y^{2n-1} + 2ay^{n+2} + 2by^3 = 0 \tag{19}$$

follows. The two equations $b' - x = 0$ and $b = 0$ originating from the coefficients of y and y^3, respectively, are inconsistent. In order for a Bernoulli component to exist, this inconsistency must be compensated by other coefficients for a suitable choice of n. To this end either $n = 1$ or $n = 3$ is required. The former leads to the inconsistency $x = 0$, whereas the latter yields

$$a' - 2ab + 2b = 0, \quad b' - x = 0, \quad a^2 - a = 0.$$

This system has the solution $a = 1$, $b = -\frac{1}{2}x^2 + C$ from which the decomposition

$$\left(z' - \frac{1}{y}z^2 + \left(y^2 + \frac{1}{2}x^2 + C\right)z\right)\left(z \equiv y' + y^3 + \left(\frac{1}{2}x^2 + C\right)y\right) = 0$$

follows. Integrating the right component yields the general solution

$$y = \frac{1}{\sqrt{(2)}\exp\left(C_1x + \frac{1}{6}x^3\right)\left(\int \exp\left(-2C_1x - \frac{1}{3}x^3\right)dx + \frac{1}{2}C_2\right)^{1/2}}$$

of Eq. (18). It does not have a Lie symmetry. □

The next example deals with a problem in hydrodynamics. The boundary layer at a circular cylinder immersed in the uniform flow of liquid is considered [14], see also Eq. 6.210 of [7].

Example 8 The equation

$$y^3y'' + yy'' - 3y^2y'^2 + y'^2 = 0 \tag{20}$$

has the only nonvanishing coefficient $c_2 = -\frac{3y^2-1}{y(y^2+1)}$. Substitution into (15) yields

$$\begin{aligned}((a' - (n+1)ab)y^n + (b' - b^2)y - na^2y^{2n-1}\\ -\left(\frac{1}{y} - \frac{4y}{y^2+1}\right)\left(a^2y^{2n} + 2aby^{n+1} + b^2y^2\right) = 0.\end{aligned} \tag{21}$$

It turns out that for $n = 3$ this condition specializes to

$$\left(\left(a' - 4a^2 + 2ab\right)y^3 + \left(b' + 4a^2 - 8ab + 2b^2\right)y - 4(a-b)^2\frac{y}{y^2+1} = 0.$$

After some simplifications the resulting system for a and b is $a' - 2a^2 = 0$ and $b = a$; Its solution $a = b = -\frac{\frac{1}{2}}{x+C}$ leads to the Bernoulli equation $y' - -\frac{\frac{1}{2}}{x+C}y^3 - \frac{\frac{1}{2}}{x+C} = 0$ with general solution

$$y = \frac{\sqrt{1 - C_1x - C_2}}{\sqrt{C_1x + C_2}}.$$

This is also the general solution of Eq. (20) □

In general it is *a priori* not known whether there exists a Bernoulli component of any order. If a component for small values of n cannot be found it is desirable to determine bounds for its possible existence. The next example shows that this is possible in special cases.

Example 9 Consider the equation

$$y'' + \frac{x}{y-1}y'^2 + y' + xy = 0 \tag{22}$$

with $K = 2$ and non-vanishing coefficients $c_2 = \frac{x}{y-1}$, $c_1 = 1$ and $c_0 = xy$. Substitution into (15) yields

$$(a' - (n+1)ab)y^n + \left(b' - b^2\right)y - na^2y^{2n-1} - xy + ay^n + by - \frac{x}{y-1}(ay^n + by)^2.$$

Expanding the last term into unique partial fractions by using the general formula

$$\frac{y^n}{y-k} = \sum_{\nu=0}^{n-1} k^{n-\nu-1}y^\nu + \frac{k^n}{y-k} \tag{23}$$

leads to

$$\begin{aligned}(a' - (n+1)ab)y^n + \left(b' - b^2\right)y - na^2y^{2n-1} - xy + ay^n + by \\ -a^2x\sum_{\nu=0}^{2n-1} y^\nu - 2abx\sum_{\nu=0}^{n} y^\nu - \frac{(a+b)^2x}{y-1} = 0.\end{aligned}$$

The coefficients of the various terms yield a system for the unknowns a, b and n. There is always the subsystem $a' + a^2 + a + x = 0$, $a + b = 0$ independent of n, it originates from the coefficient of y and the term independent of y. Furthermore, the leading term of the first sum in the above equation requires $a = 0$ for for any $n \geq 2$. These equations combined are inconsistent, i.e. the above Eq. (22) does not allow a Bernoulli component for any nonnegative natural number n. A similar reasoning exists for negative values of n. □

At the moment an algorithm for determining bounds for n is not known, it is not even clear whether the existence of bounds is decidable in general.

3. Equations with leading term $y'y''$

Another important class of differential equations are those with leading term $y'y''$, they are considered in this section. Their general form is

$$y'y'' + c(x,y)y'' + \sum_{k=0}^{K} c_k(x,y)y'^k = 0 \text{ with } c(x,y), c_k(x,y) \in \mathbb{Q}(x)[y], \quad K \in \mathbb{N}. \tag{24}$$

Components of Clairaut or d'Alembert type $z \equiv y - xf(y') - g(y')$ may lead to partial or even general solutions in closed form, mostly in a parameter representation. The main result of this section is given in the following proposition.

Proposition 4 Let a second-order differential Eq. (24) be given. A first-order component $z \equiv y - xf(y') - g(y')$ exists if $f(y')$ and $g(y')$ satisfy

$$\begin{aligned}(p - f(p))(c(x, xf(p) + g(p)) + p) \\ +\left(xf'(p) + g'(p)\right)\sum_{k=0}^{K} c_k(x, xf(p) + g(p))p^k = 0\end{aligned} \tag{25}$$

where $p \equiv y'$ has been defined. Representing the left hand side of (25) as a partial fraction w.r.t. x and equating the coefficients of the various terms to zero, a system of first-order quasilinear ode's for $f(p)$ and $g(p)$ is obtained; its degree in $f(p)$ and $g(p)$ is not higher than the degree in y of the coefficients $c(x,y)$ and $c_k(x,y)$.

Proof. Reduction of (24) w.r.t. $z \equiv y - xf(y') - g(y')$ leads to Eq. (25). Their properties follow directly from the assumptions about the coefficients $c(x,y)$ and $c_k(x,y)$ in (24), and representing the left hand side of (25) as a partial fraction in x.

The determining system for the two functions $f(p)$ and $g(p)$ may be obtained explicitly from (25) if the coefficients $c(x,y)$ and $c_k(x,y)$ are known. Without restrictions on the coefficients $c_k(x,y)$ the derivatives $f'(p)$ and $g'(p)$ may occur linearly in any equation obtained after separation w.r.t. x, and an algebraic system in $p, f(p), g(p)$, $f'(p)$ and $g'(p)$ follows. It turns out that an algebraic Gröbner basis algorithm including factorization is a suitable tool for solving them in many cases. If a solution has been obtained the corresponding component may be applied for generating the decomposition of the given equation explicitly. The following example uses this proceeding.

Example 10 Consider the equation

$$y'y'' + \frac{1}{2}yy'' + \frac{x-1}{2x}y'^2 - \frac{1}{2x}yy' + \frac{1}{2x} = 0. \tag{26}$$

Here $J = 1$ and $K = 2$, its nonvanishing coefficients are $c(x,y) = -\frac{1}{2}y$, $c_2 = \frac{x-1}{2x}$, $c_1 = -\frac{y}{2x}$ and $c_0 = \frac{1}{2x}$. A linear or Bernoulli component does not exist. Proposition 4 leads to the system

$$g'gp + g'p^2 - g' = 0, \quad f'fp - f'p^2 + f^2 - fp = 0,$$

$$f'gp + f'p^2 - f' + fg'p + fg + 2fp - g'p^2 - gp - 2p^2 = 0.$$

Transforming the left-hand sides into algebraic Gröbner bases in the term order $f'(p) > g'(p) > f(p) > g(p) > p$, the following two systems and their solutions are obtained.

$$f - p = 0, gp + p^2 - 1 = 0, \ \to f = p, g = \frac{1}{p} - p,$$

$$f'p + f = 0, gp + p^2 - 1 = 0, \ \to f = C\frac{1}{p}, g = \frac{1}{p} - p.$$

They lead to the decompositions

$$\left(zz' + \frac{xy'^2 + y'^2 + 1}{y'}z' + \frac{xy'^2 - y'^2 - 1}{xy'}z\right)\left(z \equiv y - xy' + y' - \frac{1}{y'}\right) = 0,$$

$$\left(zz' + \frac{y'^2 + Cx + 1}{y'}z' - \frac{xy'^2 + y'^2 + 1}{xy'}z\right)\left(z \equiv y - \frac{C}{y'}x + y' - \frac{1}{y'}\right) = 0,$$

respectively. The former decomposition generates a Clairaut component. It yields the solution $y = Cx + \frac{1}{C} - C$ of (26), C is an undetermined constant. Its parameter solution $x = \frac{1}{p^2} + 1, y = \frac{2}{p}$ does not solve it, it annihilates a lower-order factor of the expression in the left-hand bracket of this decomposition and has to be discarded.

Integration of the d'Alembert component $z \equiv y'^2 + yy' - Cx - 1$ leads to the general solution of (26) in a parameter representation

$$x = \frac{1}{C_1\sqrt{C_1 + p^2}}\left(\sqrt{C_1 + p^2} - C_1 p \log\frac{\sqrt{C_1 + p^2} + p}{\sqrt{C_1}} + (C_1C_2 + 1)p\right),$$

$$y = -\frac{1}{\sqrt{C_1 + p^2}}\left(p\sqrt{C_1 + p^2} - C_1 \log\frac{\sqrt{C_1 + p^2} + p}{\sqrt{C_1}} + C_1C_2 + 1\right).$$

Eq. (26) does not have a Lie symmetry. □

This example shows that solutions of a component must be tested to see if they meet the second order equation, otherwise they have to be discarded; this phenomenon seems to be quite common.

4. Conclusions

The structure of the determining systems for linear or Bernoulli components of a nonlinear Eq. (2) given in Propositions 2 or 3, respectively, show clearly its relation to the corresponding system for the decomposition of a linear equation. For a generic equation of the second order this appears to be the best possible result. The same applies to the verious solution steps given on page 6. The corresponding result for determining Clairaut and d'Alembert components given in Proposition 4 is less specific. However, it should be possible, to obtain more detailed results if special classes of second-order equations are considered. In general, this area is only at an early stage and a better understanding of the underlying mechanisms generating the solutions and also its limitations would be highly desirable.

There are numerous possible generalizations fairly obvious. On the one hand, this concerns the equations to be solved. More general function fields for its coefficients like e.g. algebraic or elementary functions may be allowed. Equations of order three or four would be interesting in many applications. The greatest challenge however is certainly to develop similar procedures for partial differential equations as it has been indicated in Section 5 of [6].

On the other hand, the component type offers space for extensions too. In principle all equations of first order, as described for example in Kamke's book [7], Part A, Section 4, are possible components. Components that guarantee at least a partial solution are of course particularly useful, the most important of them have been discussed in this article.

In order to apply decompositions to concrete problems the implementation of the procedures described in this article are available on the website www.alltypes.de [15].

Beyond that there are a number of general problems related to decompositions. For instance the question how rare are equations that allow a particular decomposition, Example 6 provides a partial answer. If two or more one-parameter families of solution curves are known as in Example 4, does it faciliate generating the general solution? The exact relation between Lie's symmetry analysis and solution by decompositions is another subject of interest.

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References

[1] E. L. Ince, Ordinary Differential Equations, Longmans, Green and Co., 1926 [Reprint by Dover Publications Inc., 1960].

[2] A. Magid, Lectures on differential Galois theory, AMS university lecture series, vol 7, AMS Press, Providence, 1994.

[3] F. Schwarz, Loewy Decomposition of Linear Differential Equations, Springer-Verlag, Wien, 2012 and Bulletin of Mathematical Sciences **3**, page 19–71 (2012); DOI 10.1007/s13373-012-0026-7.

[4] S. Lie, Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, Leipzig, 1891; reprinted by Chelsea Publishing Company, New York, 1967.

[5] F. Schwarz, Algorithmic Lie Theory for Solving Ordinary Differential Equations, Chapman & Hall/CRC, 2007.

[6] F. Schwarz, Decomposition of ordinary differential equations, Bulletin of MathematicalSciences **7**, 565–613 (2017); doi.org/10.1007/s13373-017-0110-0.

[7] E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen I. Gewöhnliche Differentialgleichungen, Akademische Verlagsgesellschaft, Leipzig, 1967.

[8] G. M. Murphy, Ordinary Differential Equations and Their Solutions, Van Nostrand Company, 1960.

[9] A. D. Polyanin, V. F. Zaitsev, Handbook of Ordinary Differential Equations, Exact Solutions, Methods and Problems, CRC Press, 2018.

[10] P. L. Sachdev, A Compendium on Nonlinear Ordinary Differential Equations, John Wiley & Sons, 1997.

[11] D. Zwillinger, Handbook of Differential Equations, Academic Press, 1998.

[12] B. D. MacCluer, P. S. Bourdon, T. L. Kriete, Differential Equations, Techniques, Theory, and Applications, American Mathematical Society, 2019.

[13] R. J. Swift, S. A. Wirkus, A Course in Ordinary Differential Equations, Chapman & Hall/CRC, 2007.

[14] K. Hiemenz, Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder, Dinglers Polytechnisches Journal 326(1911).

[15] F. Schwarz, ALLTYPES in the Web, ACM Communications in Computer Algebra **42**, No. 3, 185–187(2008).