Chapter

Wavelet Filter Banks Using Allpass Filters

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Abstract

Allpass filter is a computationally efficient versatile signal processing building block. The interconnection of allpass filters has found numerous applications in digital filtering and wavelets. In this chapter, we discuss several classes of wavelet filter banks by using allpass filters. Firstly, we describe two classes of orthogonal wavelet filter banks composed of two real allpass filters or a complex allpass filter, and then consider design of orthogonal filter banks without or with symmetry, respectively. Next, we present two classes of filter banks by using allpass filters in lifting scheme. One class is causal stable biorthogonal wavelet filter bank and another class is orthogonal wavelet filter bank, all with approximately linear phase response. We also give several design examples to demonstrate the effectiveness of the proposed method.

Keywords: wavelet, filter bank, allpass filter, perfect reconstruction, symmetry, orthogonality

1. Introduction

The discrete wavelet transform (DWT), which is implemented by a two band perfect reconstruction (PR) filter bank, has been applied extensively to digital signal processing, image processing, medical and health care, economy and so on [1–4]. In many applications such as image processing, wavelets are required to be real since the signal is real-valued in general. We restrict ourselves to real-valued wavelet filter banks in this chapter.

In addition to orthogonality, one desirable property for wavelets is symmetry, which requires all filters in the filter bank to possess exactly linear phase, because the symmetric extension method is generally used to treat the boundaries of images [5, 6]. It is known in [1–4] that finite impulse response (FIR) filters (corresponding to the compactly supported wavelets) can easily realize exactly linear phase. However, it is widely appreciated that the only FIR solution that produces a real orthogonal symmetric wavelet basis is the Haar wavelet, which is not continuous and the corresponding filter is of order 1 only that is not enough for many practical applications. To obtain wavelet filter banks with higher degrees of freedom, infinite impulse response (IIR) filters have been used to construct wavelet filter banks with some of the desired properties [7–12]. Among the existing IIR wavelet filter banks, wavelet filter banks composed of allpass filters are attractive [7, 9, 10, 12], which can realize both of orthogonality and symmetry.

Allpass filter is a computationally efficient versatile signal processing building block and quite useful in many applications [13]. Allpass filter possesses unit magnitude at all frequencies (see Appendix) and is a basic scalar lossless building block. The interconnection of allpass filters has found numerous applications in practical filtering problems, such as low sensitivity filter structures, multirate filtering, filter banks and so on [7, 10, 12, 13]. The phase approximation of allpass filters has been also discussed in [13–15].

The lifting scheme proposed by W. Sweldens in [16, 17] is an efficient tool for constructing second generation wavelets, and has advantages such as faster implementation, fully in-place calculation, reversible integer-to-integer transforms, and so on. It has been proved in [18, 19] that every FIR wavelet filter bank can be decomposed into a finite number of lifting steps, thus this allows the construction of an integer version of the wavelet transform. Such integer wavelet transforms are invertible, and then are attractive in lossless coding applications. Due to these properties, the lifting implemantation has been adopted in the international standard JPEG2000 [5]. Conventionally, the lifting scheme is often used to construct a class of biorthogonal wavelet filter banks. It has been shown in [18] that orthogonal wavelet filter banks can also be realized by the lifting scheme. However, it is not always possible for IIR wavelet filter banks to be decomposed into a finite number of lifting steps.

In this chapter, we discuss several classes of wavelet filter banks by using allpass filters. Firstly, we describe two classes of orthogonal wavelet filter banks composed of two real allpass filters or a single complex allpass filter. We consider design of the proposed orthogonal wavelet filter banks without or with symmetry, respectively, and give the maximally flat solutions, where the orthogonal symmetric wavelet filter banks using real or complex allpass filter are corresponding to half sample symmetric (HSS) and whole sample symmetric (WSS) wavelets, respectively. Next, we present two classes of wavelet filter banks based on the lifting scheme with two lifting steps only. By using real allpass filters in the lifting steps, we can obtain one class of causal stable biorthogonal wavelet filter bank and another class of orthogonal wavelet filter bank, all with approximately linear phase response. In addition, we show some design examples to demonstrate the effectiveness of the proposed method.

2. Two band wavelet filter bank

It is well-known [1–4] that wavelet basis can be generated by two band filter bank shown in **Figure 1**. In **Figure 1**, $H_0(z)$ and $H_1(z)$ are analysis filters, and $G_0(z)$ and $G_1(z)$ are synthesis filters. The relationship of input $X(z)$ and output $Y(z)$ of the filter bank is given by

$$\begin{aligned} Y(z) = &\frac{1}{2}\{H_0(z)G_0(z) + H_1(z)G_1(z)\}X(z) \\ &+\frac{1}{2}\{H_0(-z)G_0(z) + H_1(-z)G_1(z)\}X(-z). \end{aligned} \tag{1}$$

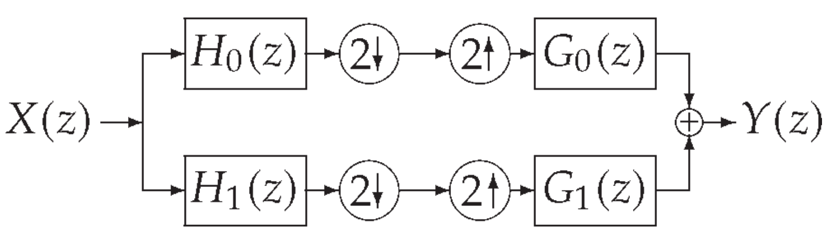

Figure 1.
Two band wavelet filter bank. $X(z)$ is input and $Y(z)$ is output. $H_0(z), H_1(z)$ are analysis filters and $G_0(z), G_1(z)$ are synthesis filters.

Therefore the PR condition is

$$\begin{cases} H_0(z)G_0(z) + H_1(z)G_1(z) = cz^{-I} \\ H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \end{cases}, \tag{2}$$

where c is constant and I is integer.

One desirable property for wavelets is orthogonality, which requires the filter bank is orthogonal, i.e., $|H_0(e^{j\omega})| = |G_0(e^{j\omega})| = |H_1(e^{j(\pi-\omega)})| = |G_1(e^{j(\pi-\omega)})|$. Another desirable property is symmetry, i.e., the wavelet basis is symmetric or antisymmetric. It requires all filters in the filter bank to possess exactly linear phase, whose impulse responses are symmetric or antisymmetric.

3. The proposed orthogonal wavelet filter banks using allpass filters

In this section, we describe several classes of orthogonal wavelet filter banks without or with symmetry. The proposed classes of orthogonal wavelet filter banks are composed of two real allpass filters or a complex allpass filter.

3.1 Orthogonal wavelet filter banks without symmetry

In some applications of signal processing, for example, speech and acoustic signal processing, wavelet filters are required to have minimal phase response rather than exactly linear phase. Therefore, wavelet basis is not necessarily symmetric or antisymmetric. In the following, we discuss two classes of orthogonal wavelet filter banks without symmetry [20].

3.1.1 Filter bank using real allpass filters

We firstly consider a pair of IIR filters $H_0(z)$ and $H_1(z)$ that are based on a parallel connection of two real allpass filters as shown in **Figure 2**, i.e.,

$$\begin{cases} H_0(z) = \dfrac{1}{2}\left\{A_{N_1}(z^2) + z^{-2K-1}A_{N_2}(z^2)\right\} \\ H_1(z) = \dfrac{1}{2}\left\{A_{N_1}(z^2) - z^{-2K-1}A_{N_2}(z^2)\right\} \end{cases}, \tag{3}$$

where K is integer, $A_{N_1}(z)$ and $A_{N_2}(z)$ are real allpass filters of order N_1 and N_2 respectively. Let the synthesis filters $G_0(z) = H_0(z^{-1})$ and $G_1(z) = H_1(z^{-1})$, then the PR condition in Eq.(2) is satisfied.

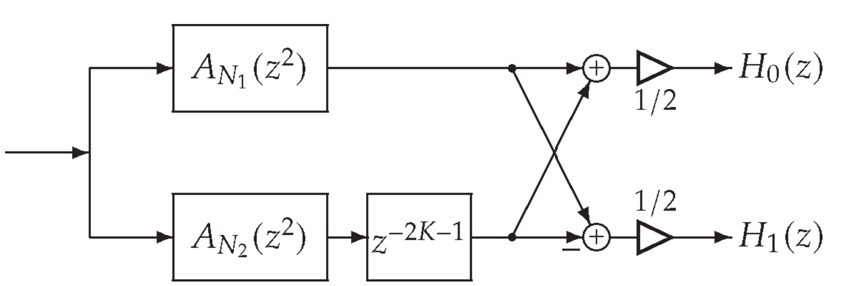

Figure 2.
Filter bank using real allpass filters. $A_{N_1}(z), A_{N_2}(z)$ are real allpass filters of order N_1 and N_2. $H_0(z), H_1(z)$ are lowpass and highpass filters.

From Eq.(3), we have

$$H_0(z) = \frac{1}{2} A_{N_2}\left(z^2\right)\left\{A_N\left(z^2\right) + z^{-2K-1}\right\}, \tag{4}$$

where $A_N(z)$ is a real allpass filter of order $N = N_1 + N_2$, and defined as

$$A_N(z) = \frac{A_{N_1}(z)}{A_{N_2}(z)} = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}}, \tag{5}$$

where a_n is real coefficient, and $a_0 = 1$.

Let $\theta(\omega)$ be the phase response of $A_N(z)$, the magnitude responses of $H_0(z)$ and $H_1(z)$ are given by

$$\begin{cases} \left|H_0\left(e^{j\omega}\right)\right| = \left|\cos\left(\dfrac{\theta(2\omega)}{2} + \left(K + \dfrac{1}{2}\right)\omega\right)\right| \\ \left|H_1\left(e^{j\omega}\right)\right| = \left|\sin\left(\dfrac{\theta(2\omega)}{2} + \left(K + \dfrac{1}{2}\right)\omega\right)\right| \end{cases}. \tag{6}$$

It is clear that the magnitude responses satisfy $\left|H_0\left(e^{j\omega}\right)\right| = \left|H_1\left(e^{j(\pi-\omega)}\right)\right|$ and the following power-complementary relation;

$$\left|H_0\left(e^{j\omega}\right)\right|^2 + \left|H_1\left(e^{j\omega}\right)\right|^2 = 1, \tag{7}$$

which means that the filter bank is orthogonal.

For $H_0(z)$ and $H_1(z)$ to be a pair of lowpass and highpass filters, the desired phase response of $A_N(z)$ is given by

$$\theta_d(\omega) = -\left(K + \frac{1}{2}\right)\omega = -\tau\omega, \tag{8}$$

where $\tau = K + \frac{1}{2}$. From the regularity of wavelets, it is known that an additional flatness condition is required to impose on $H_0(z)$, i.e.,

$$\left.\frac{\partial^k \left|H_0\left(e^{j\omega}\right)\right|}{\partial\omega^k}\right|_{\omega=\pi} = 0 \quad (k = 0, 1, \cdots, L-1), \tag{9}$$

where L is integer. Hence, the resulting wavelet function will have L consecutive vanishing moments. This flatness condition can be obtained if $H_0(z)$ contains L zeros located at $z = -1$.

For the maximally flat filters, the closed-form formula is given by

$$a_n = \binom{N}{n} \prod_{i=1}^{n} \frac{N - \tau - i + 1}{\tau + i}. \tag{10}$$

Once a set of filter coefficients a_n are obtained, we compute poles of $A_N(z)$ and then assign the poles inside the unit circle to $A_{N_1}(z)$ as its poles and the poles outside the unit circle to $A_{N_2}(z)$ as its zeros. Therefore, we can obtain causal stable analysis filters $H_0(z)$ and $H_1(z)$, then the synthesis filters $G_0(z)$ and $G_1(z)$ are anti-causal stable.

In many applications of signal processing, frequency selectivity is also thought of as a useful property from the viewpoint of signal band-splitting. However, regularity and frequency selectivity somewhat contradict each other. For this reason, design of $H_0(z)$ that has the best possible frequency selectivity for the given flatness condition has been also discussed in [20].

Example 1: We consider design of filter banks using two real allpass filters with $N_1 = N_2 = 2$ and $K = 0$. By setting $L = 9, 5, 1$, we have designed $H_0(z)$ by using the design method proposed in [20]. The magnitude responses are shown in **Figure 3**, and the scaling and wavelet functions are shown in **Figure 4**, respectively. When $L = 9$, it is seen that $H_0(z)$ is the maximally flat filter, and it is the elliptic filter if $L = 1$. It is clear in **Figure 3** that the magnitude error increases with an increasing L, and in **Figure 4** that the scaling and wavelet functions decline more rapidly.

3.1.2 Filter bank using complex allpass filter

We consider a pair of $H_0(z)$ and $H_1(z)$ using a single complex allpass filter as shown in **Figure 5**, i.e.,

$$\begin{cases} H_0(z) = \dfrac{1}{2}\left\{A_N(z) + \hat{A}_N(z)\right\} \\ H_1(z) = \dfrac{z^{-1}}{2j}\left\{A_N(z) - \hat{A}_N(z)\right\} \end{cases}, \tag{11}$$

where $A_N(z)$ and $\hat{A}_N(z)$ are complex allpass filters of order N, and their coefficients are mutually complex conjugate. Let $G_0(z) = H_0(z^{-1})$ and $G_1(z) = H_1(z^{-1})$ similarly. From the orthogonality, $A_N(z)$ and $\hat{A}_N(z)$ must satisfy [7]

$$A_N(z) = \pm j\hat{A}_N(-z), \tag{12}$$

which means that if α is a pole of $A_N(z)$, then $-\alpha^*$ is a pole of $A_N(z)$ also. Consequently, $A_N(z)$ has a pair of poles $(\alpha, -\alpha^*)$ or a single pole $j\beta$, where β is real, α is complex and α^* denotes the complex conjugate of α.

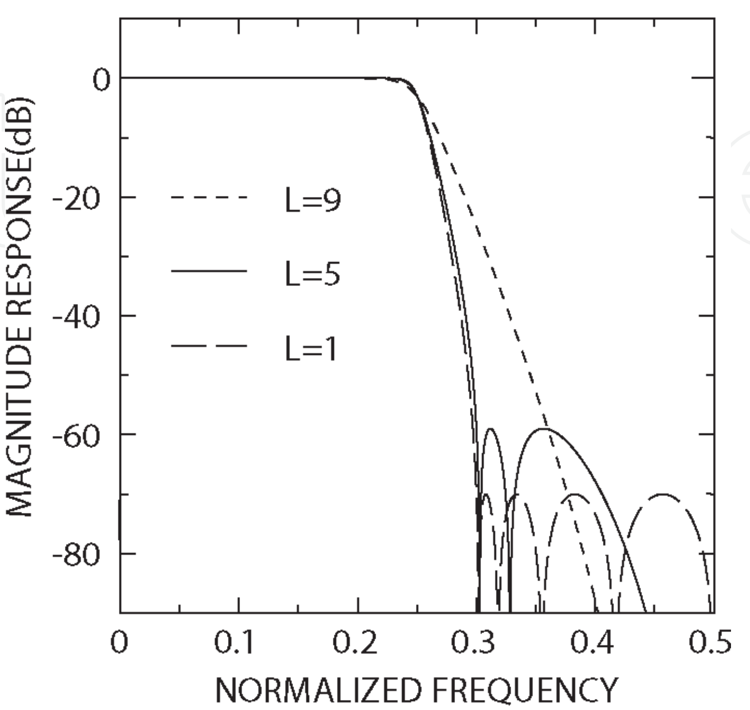

Figure 3.
Magnitude responses of $H_0(z)$ in example 1.

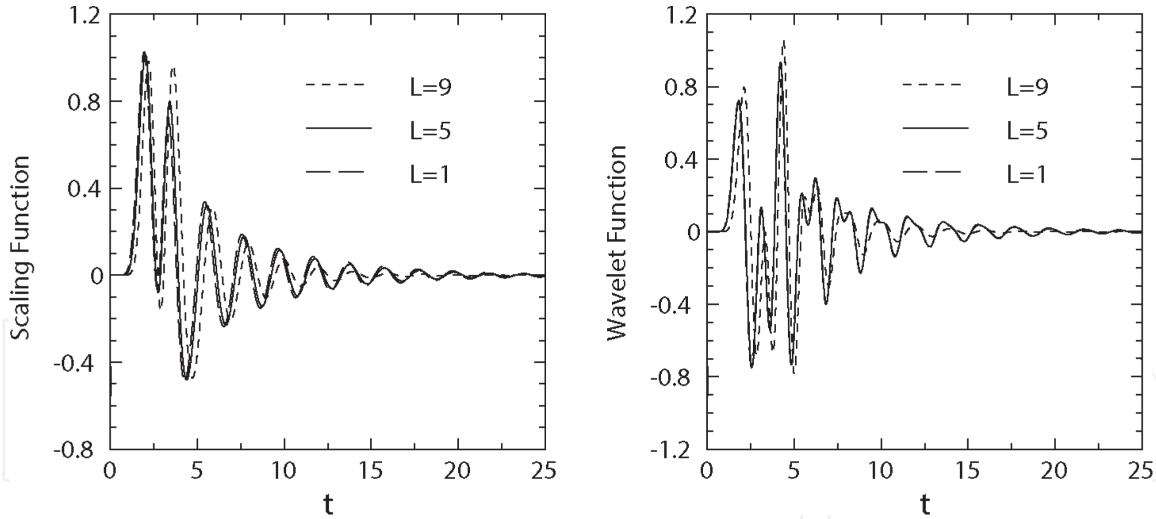

Figure 4.
Scaling and wavelet functions in example 1.

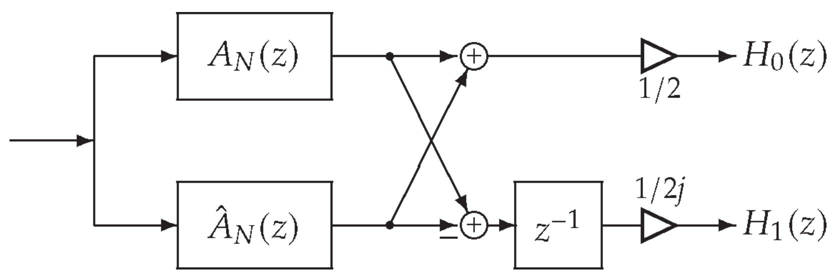

Figure 5.
Filter bank using complex allpass filter. $A_N(z), \hat{A}_N(z)$ are complex allpass filters of order N. $H_0(z), H_1(z)$ are lowpass and highpass filters.

From Eq.(11),

$$H_0(z) = \frac{1}{2}\hat{A}_N(z)\{A_{2N}(z) + 1\}, \tag{13}$$

where $A_{2N}(z)$ is a complex allpass filter of order $2N$, and defined by

$$A_{2N}(z) = \frac{A_N(z)}{\hat{A}_N(z)} = \frac{e^{j\eta}}{e^{-j\eta}} \frac{\sum_{n=0}^{N_1} a_{2n}\{z^{2n} + z^{-2n}\} + j\sum_{n=0}^{N_2} a_{2n+1}\{z^{2n+1} + z^{-2n-1}\}}{\sum_{n=0}^{N_1} a_{2n}\{z^{2n} + z^{-2n}\} - j\sum_{n=0}^{N_2} a_{2n+1}\{z^{2n+1} + z^{-2n-1}\}}, \tag{14}$$

where $\eta = \pm\pi/4$ or $\pm 3\pi/4$, a_n is real and $a_0 = 1/2$, $N_1 = N/2$ and $N_2 = N/2 - 1$ if N is even, and $N_1 = N_2 = (N-1)/2$ if N is odd.

Therefore, the phase response $\theta(\omega)$ of $A_{2N}(z)$ is given by

$$\theta(\omega) = 2\eta + 2\tan^{-1} \frac{\sum_{n=0}^{N_2} a_{2n+1}\cos((2n+1)\omega)}{\sum_{n=0}^{N_1} a_{2n}\cos(2n\omega)}, \tag{15}$$

and the magnitude responses of $H_0(z)$ and $H_1(z)$ are

$$\begin{cases} |H_0(e^{j\omega})| = |\cos\dfrac{\theta(\omega)}{2}| \\ |H_1(e^{j\omega})| = |\sin\dfrac{\theta(\omega)}{2}| \end{cases}, \tag{16}$$

which satisfies the power-complementary relation in Eq.(7).

The closed-form formula of the maximally flat filters can be given by

$$a_n = C_n \prod_{i=1}^{n} \frac{i - N - 1}{i + N}, \tag{17}$$

where $C_{2n} = 1$ and $C_{2n+1} = \tan\eta$. Therefore, we compute poles of $A_{2N}(z)$ and assign the poles inside the unit circle to $A_N(z)$ as its poles to obtain causal stable analysis filters $H_0(z)$ and $H_1(z)$. Thus, the synthesis filters $G_0(z)$ and $G_1(z)$ are anti-causal stable. Design of $H_0(z)$ having the best possible frequency selectivity for the given degrees of flatness has been also discussed in [20].

Example 2: We consider design of filter banks using a complex allpass filter with $N = 4$ and $L = 8, 4, 0$. We have designed $H_0(z)$ by using the design method proposed in [20]. The magnitude responses are shown in **Figure 6**, and the scaling and wavelet functions are shown in **Figure 7**, respectively. It is seen in **Figure 6** that $H_0(z)$ is the maximally flat filter if $L = 8$, and the elliptic filter if $L = 0$ that does not have any zero located at $z = -1$ and is different from that in *Example 1*.

3.2 Orthogonal symmetric wavelet filter banks

In many applications of image processing, digital filters are required to have exactly linear phase. Therefore, the impulse responses of wavelet filters need to be symmetric or antisymmetric, and the generated wavelet bases are symmetric or antisymmetric also. In the following, we discuss two classes of orthogonal symmetric wavelet filter banks composed of allpass filters: HSS [21] and WSS [22] wavelet filter banks.

3.2.1 Filter bank using real allpass filters

To obtain exactly linear phase, we constitute a pair of $H_0(z)$ and $H_1(z)$ in **Figure 2** by using an allpass filter $A_N(z)$ as

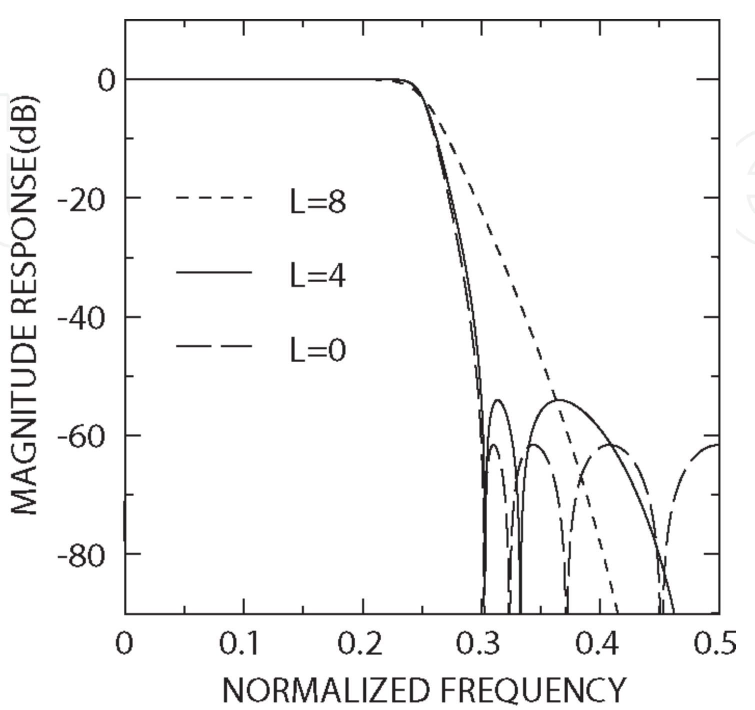

Figure 6.
Magnitude responses of $H_0(z)$ in example 2.

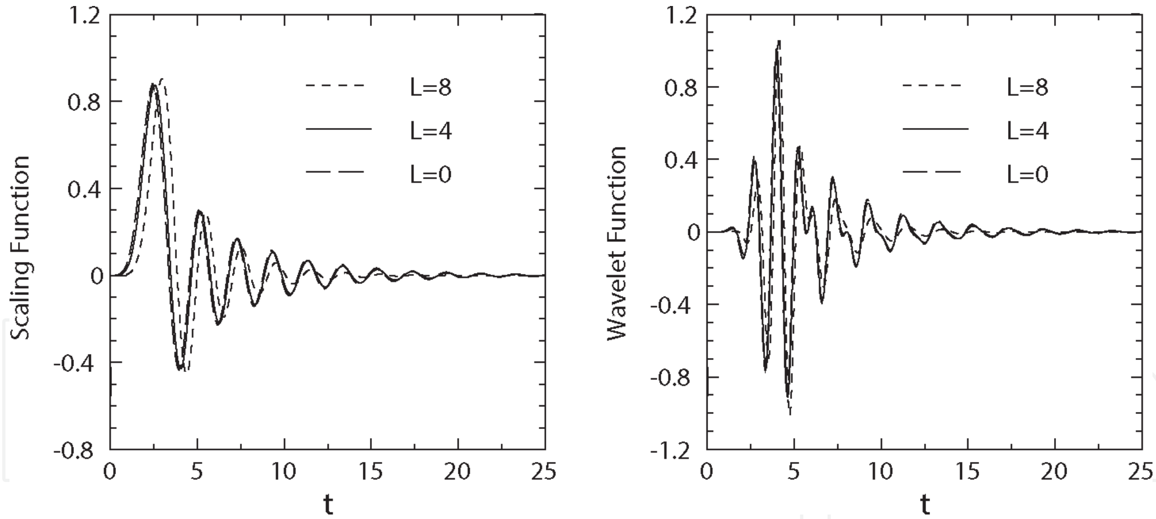

Figure 7.
Scaling and wavelet functions in example 2.

$$\begin{cases} H_0(z) = \frac{1}{2}\left\{A_N\left(z^2\right) + z^{-2K-1}A_N\left(z^{-2}\right)\right\} \\ H_1(z) = \frac{1}{2}\left\{A_N\left(z^2\right) - z^{-2K-1}A_N\left(z^{-2}\right)\right\} \end{cases}. \tag{18}$$

Let $\theta(\omega)$ be the phase response of $A_N(z)$, then the frequency responses of $H_0(z)$ and $H_1(z)$ are given by

$$\begin{cases} H_0\left(e^{j\omega}\right) = e^{-j\left(K+\frac{1}{2}\right)\omega}\cos\left\{\theta(2\omega) + \left(K+\frac{1}{2}\right)\omega\right\} \\ H_1\left(e^{j\omega}\right) = je^{-j\left(K+\frac{1}{2}\right)\omega}\sin\left\{\theta(2\omega) + \left(K+\frac{1}{2}\right)\omega\right\} \end{cases}. \tag{19}$$

It is clear in Eq.(19) that $H_0(z)$ and $H_1(z)$ have exact linear phase response and satisfy the power-complementary relation in Eq.(7). The filter has a group delay of $K + \frac{1}{2}$, and its impulse response is HSS. Therefore, the design problem of the wavelet filter banks becomes the phase approximation of allpass filter $A_N(z)$. For $H_0(z)$ and $H_1(z)$ to be a pair of lowpass and highpass filters, the desired phase response of $A_N(z)$ is

$$\theta_d(\omega) = -\left(\frac{K}{2} + \frac{1}{4}\right)\omega = -\tau\omega, \tag{20}$$

where $\tau = \frac{K}{2} + \frac{1}{4}$. The filter coefficients a_n of the maximally flat filters can be computed by Eq.(10). Design of wavelet filters having the best possible frequency selectivity for the given degrees of flatness has been also discussed in [21]. It has been pointed out in [21] that we must choose $K = \cdots, -7, -6, -3, -2, 1, 2, 5, 6, \cdots$ if N is odd and $K = \cdots, -5, -4, -1, 0, 3, 4, 7, \cdots$ if N is even, in order to obtain a pair of reasonable lowpass and highpass filters to avoid the undesired zero and bump.

Example 3: We consider design of the maximally flat wavelet filter banks with $N = 4$ and $L = 9$. We have designed $A_N(z)$ with $K = 0$ and $K = 1$. The magnitude responses of $H_0(z)$ are shown in **Figure 8**. It is seen in **Figure 8** that $H_0(z)$ with

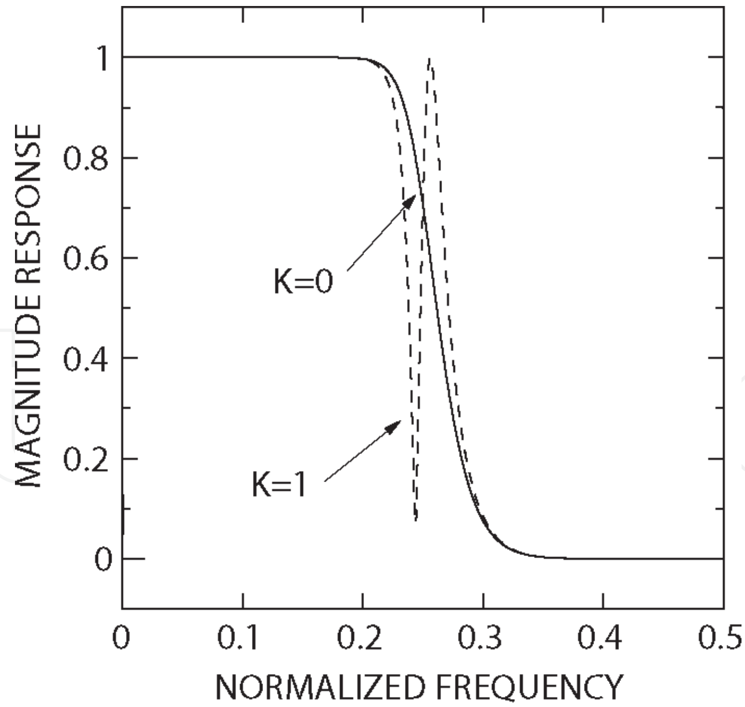

Figure 8.
Magnitude responses of $H_0(z)$ in example 3. $H_0(z)$ has an undesired zero and bump when $K = 1$.

$K = 1$ has the undesired zero and bump nearby $\omega = \pi/2$. The generated scaling and wavelet functions are shown in **Figure 9** respectively. It is seen in **Figure 9** that the scaling functions are symmetric, while the wavelet functions are antisymmetric. Although $H_0(z)$ with $K = 0$ and $K = 1$ have the same degrees of flatness, it is seen that the scaling and wavelet functions of $K = 1$ decline more slowly than that of $K = 0$, because of the undesired zero and bump. Therefore, we should not choose $K = 1$ in this case.

3.2.2 Filter bank using complex allpass filter

We consider again $H_0(z)$ and $H_1(z)$ using a complex allpass filter in **Figure 5**,

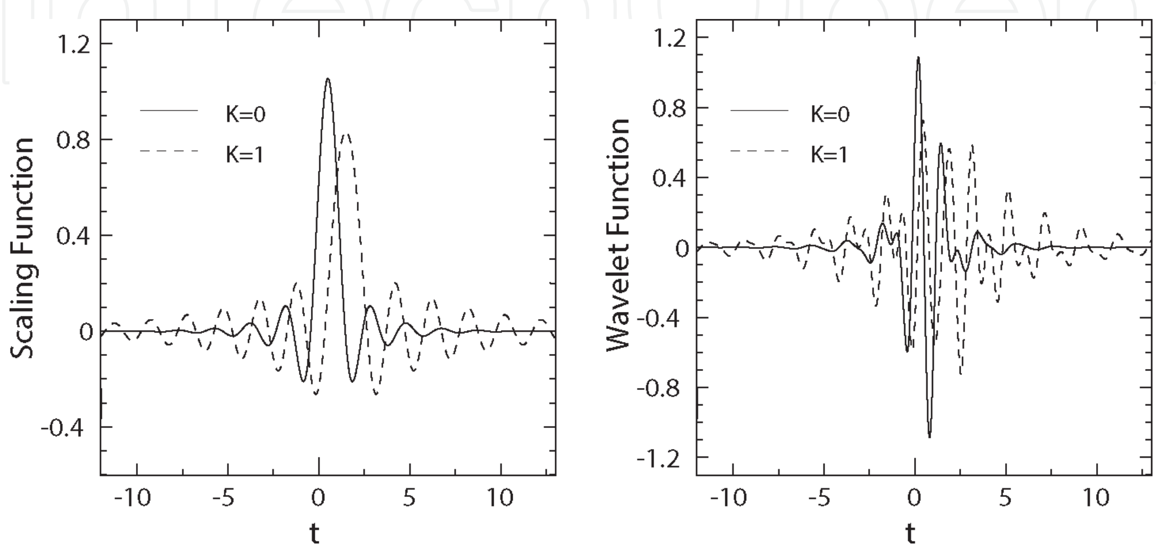

Figure 9.
Scaling and wavelet functions in example 3. The symmetric point is dependent on the group delay of $K + \frac{1}{2}$.

$$\begin{cases} H_0(z) = \dfrac{1}{2}\left\{A_N(z) + \hat{A}_N(z)\right\} \\ H_1(z) = \dfrac{z^{-1}}{2j}\left\{A_N(z) - \hat{A}_N(z)\right\} \end{cases}. \tag{21}$$

To obtain exactly linear phase, $A_N(z)$ and $\hat{A}_N(z)$ must satisfy

$$\hat{A}_N(z) = \frac{1}{A_N(z)}, \tag{22}$$

which means that if α is a pole of $A_N(z)$, then $1/\alpha$ is a pole of $A_N(z)$ too. In addition to orthogonality, $A_N(z)$ has a quadruplet of poles $(\alpha, -\alpha^*, 1/\alpha, -1/\alpha^*)$ or a pair of poles $(j\beta, 1/j\beta)$. Therefore, we have

$$A_N(z) = e^{j\eta}z^{-N}\frac{a_0 + ja_1z + a_2z^2 + \cdots + a_2z^{N-2} + ja_1z^{N-1} + a_0z^N}{a_0 - ja_1z^{-1} + a_2z^{-2} + \cdots + a_2z^{-N+2} - ja_1z^{-N+1} + a_0z^{-N}}, \tag{23}$$

where N is even, a_n is real and $a_0 = 1$. The phase response $\theta(\omega)$ of $A_N(z)$ is given by

$$\theta(\omega) = \eta + 2\varphi(\omega), \tag{24}$$

where if $M = N/2$ is even,

$$\varphi(\omega) = \tan^{-1}\frac{\sum_{n=0}^{M/2-1} a_{2n+1}\cos(M-2n-1)\omega}{\frac{a_M}{2} + \sum\limits_{n=0}^{M/2-1} a_{2n}\cos(M-2n)\omega}, \tag{25}$$

and if $M = N/2$ is odd,

$$\varphi(\omega) = \tan^{-1}\frac{\frac{a_M}{2} + \sum_{n=0}^{(M-3)/2} a_{2n+1}\cos(M-2n-1)\omega}{\sum_{n=0}^{(M-1)/2} a_{2n}\cos(M-2n)\omega}. \tag{26}$$

Thus, we have

$$\begin{cases} H_0(e^{j\omega}) = \cos\theta(\omega) \\ H_1(e^{j\omega}) = e^{-j\omega}\sin\theta(\omega) \end{cases}. \tag{27}$$

It is clear that $H_0(z)$ and $H_1(z)$ have exactly linear phase responses and satisfy the power-complementary relation in Eq.(7). Its impulse response is WSS. Therefore, the design problem of wavelet filter banks becomes the phase approximation of $A_N(z)$ in Eq.(23).

For the maximally flat filters, the closed-form formula is given in [22] by

$$a_n = C_n\binom{N}{n}, \tag{28}$$

where $C_{2n} = 1$ and $C_{2n+1} = -\tan\frac{\eta}{2}$. Design of wavelet filters having the best possible frequency selectivity for the given degrees of flatness has been also discussed in [22]. It has been pointed out in [22] that we must choose $\eta = \pm\pi/4$ if M is even and $\eta = \pm 3\pi/4$ if M is odd.

Example 4: We consider design of the filter banks with $N = 6$ and $\eta = -3\pi/4$. We have designed $A_N(z)$ with $L = 0, 2, 4, 6$ by using the design method proposed in [22]. The magnitude responses of $H_0(z)$ are shown in **Figure 10**, and the scaling and

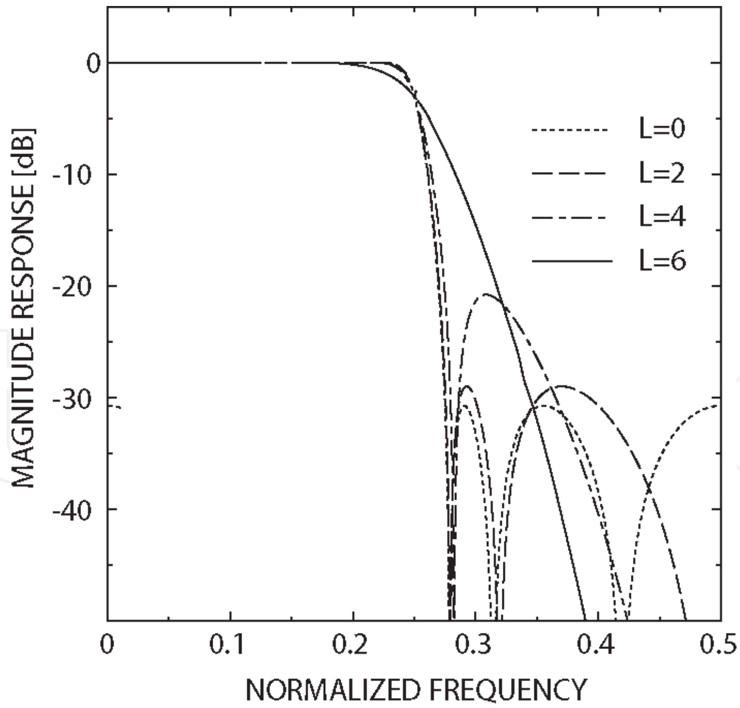

Figure 10.
Magnitude responses of $H_0(z)$ in example 4.

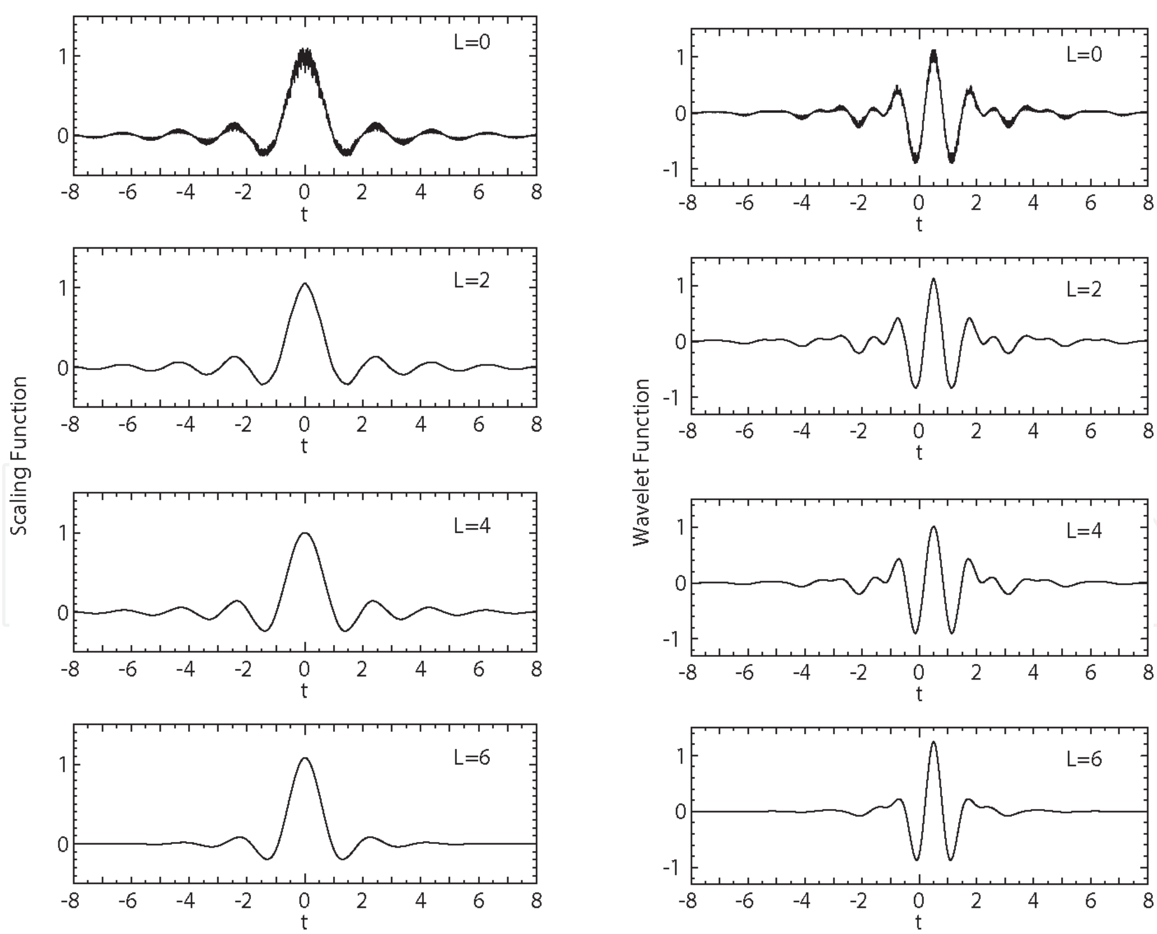

Figure 11.
Scaling and wavelet functions in example 4.

wavelet functions are shown in **Figure 11**, respectively. It is seen in **Figure 10** that $H_0(z)$ with $L = 6$ corresponds to the maximally flat filter, and $H_0(z)$ with $L = 0$ is the minimax filter that has no zero located at $z = -1$. In **Figure 11**, the scaling and wavelet functions are not continuous because the regularity condition is not satisfied when $L = 0$, and become more smooth with an increasing L. Both the scaling and wavelet functions are symmetric.

4. Lifting-based wavelet filter banks using allpass filters

The lifting scheme proposed in [16] and [17] is an efficient tool for constructing second generation wavelets, and has advantages such as faster implementation, fully in-place calculation, reversible integer-to-integer transforms, and so on. It has been proved in [18] and [19] that every FIR wavelet filter bank can be decomposed into a finite number of lifting steps, thus this allows the construction of an integer version of the wavelet transform. Such integer wavelet transforms are invertible, and then are attractive in lossless coding applications. Conventionally, the lifting scheme is often used to construct a class of biorthogonal wavelet filter banks. It has been shown in [18] that the orthogonal wavelet filter banks can also be realized by the lifting scheme. However, it is not always possible for IIR wavelet filter banks to be decomposed into a finite number of lifting steps. For example, it is difficult to realize the IIR orthogonal wavelet filter banks discussed in Section 3 by using a finite number of lifting steps.

Now, we restrict ourselves to the lifting scheme with two lifting steps [10] as shown in **Figure 12**. Let $H_0(z)$ and $H_1(z)$ be a pair of lowpass and highpass filters,

$$\begin{cases} H_0(z) = \dfrac{1}{2}\left\{z^{-2K_1-1} + P(z^2)\right\} \\ H_1(z) = z^{-2K_2} - Q(z^2)H_0(z) \end{cases}, \tag{29}$$

then $G_0(z) = H_1(-z)$ and $G_1(z) = -H_0(-z)$. It is clear in **Figure 12** that the PR condition is structurally satisfied. Therefore, the design of $H_0(z)$ and $H_1(z)$ becomes how to determine $P(z)$ and $Q(z)$. In the following, we describe two classes of near symmetric wavelet filter banks by using real allpass filters in the lifting scheme: causal stable biorthogonal wavelet filter bank [23] and orthogonal wavelet filter bank [24].

4.1 Causal stable wavelet filter banks

We use two real allpass filters in lifting steps, i.e., $P(z) = A_{N_1}(z)$ and $Q(z) = A_{N_2}(z)$, and thus,

$$\begin{cases} H_0(z) = \dfrac{1}{2}\left\{z^{-2K_1-1} + A_{N_1}(z^2)\right\} \\ H_1(z) = z^{-2K_2} - A_{N_2}(z^2)H_0(z) \end{cases}. \tag{30}$$

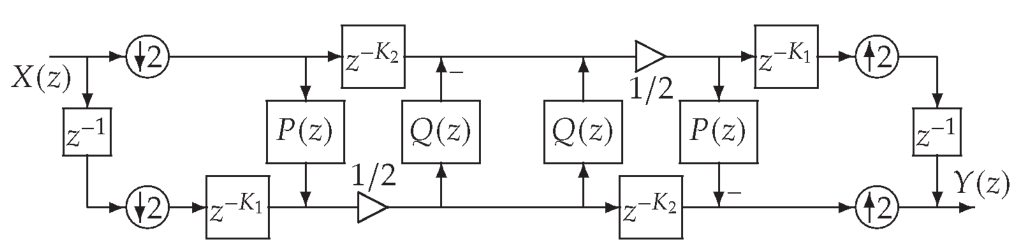

Figure 12.
Lifting scheme with two lifting steps [10]. $X(z)$ is input and $Y(z)$ is output. $P(z), Q(z)$ are filters.

Let $\theta_1(\omega)$ be the phase response of $A_{N_1}(z)$, the frequency response of $H_0(z)$ is given by

$$H_0\left(e^{j\omega}\right) = e^{j\left(\frac{\theta_1(2\omega)}{2} - \left(K_1 + \frac{1}{2}\right)\omega\right)} \cos\left(\frac{\theta_1(2\omega)}{2} + \left(K_1 + \frac{1}{2}\right)\omega\right). \tag{31}$$

For $H_0(z)$ to be lowpass filter, the desired phase response of $A_{N_1}(z)$ is

$$\theta_{1d}(\omega) = -\left(K_1 + \frac{1}{2}\right)\omega = -\tau_1\omega, \tag{32}$$

where $\tau_1 = K_1 + \frac{1}{2}$. According to Appendix, the order of $A_{N_1}(z)$ is required to be $N_1 = K_1$ or $N_1 = K_1 + 1$ to obtain causal stable allpass filter.

Ideally, $H_0\left(e^{j\omega}\right) = 0$ in the stopband of $H_0(z)$, then $H_1\left(e^{j\omega}\right) = e^{-j2K_2\omega}$, having linear phase response from Eq.(30). In the passband of $H_0(z)$, $H_0\left(e^{j\omega}\right) = e^{-j(2K_1+1)\omega}$ ideally, thus,

$$\begin{aligned} H_1\left(e^{j\omega}\right) &= e^{-j2K_2\omega} - e^{j\theta_2(2\omega)}e^{-j(2K_1+1)\omega} \\ &= -2je^{j\left(\frac{\theta_2(2\omega)}{2} - \left(K_1 + K_2 + \frac{1}{2}\right)\omega\right)} \sin\left(\frac{\theta_2(2\omega)}{2} - \left(K_1 - K_2 + \frac{1}{2}\right)\omega\right), \end{aligned} \tag{33}$$

where $\theta_2(\omega)$ is the phase response of $A_{N_2}(z)$. Therefore, in the stopband of $H_1(z)$, the desired phase response of $A_{N_2}(z)$ is

$$\theta_{2d}(\omega) = \left(K_1 - K_2 + \frac{1}{2}\right)\omega = -\tau_2\omega, \tag{34}$$

where $\tau_2 = K_2 - K_1 - \frac{1}{2}$. Similarly, the order of $A_{N_2}(z)$ is required to be $N_2 = K_2 - K_1$ or $N_2 = K_2 - K_1 - 1$ to obtain causal stable allpass filter. Therefore, once N_1 and N_2 are given, we can obtain causal stable wavelet filter banks by appropriately choosing K_1 and K_2. The maximally flat filters can be designed by using Eq.(10). $H_0(z)$ and $H_1(z)$ have approximately linear phase response.

Example 5: We consider design of the maximally flat wavelet filter banks with $N_1 = N_2 = 6$. We have designed $A_{N_1}(z)$ with $K_1 = 5$, and the magnitude response of $H_0(z)$ is shown in **Figure 13**. We then designed $A_{N_2}(z)$ with $K_2 = 11$ and $K_2 = 12$, and the magnitude responses of $H_1(z)$ are shown also in **Figure 13**. It is seen in **Figure 13** that $H_1(z)$ with $K_2 = 11$ has a large overshoot nearby $\omega = \pi/2$. To avoid this overshoot, we should choose $K_2 = N_2 + K_1 + 1$ if $K_1 = N_1 - 1$ and $K_2 = N_2 + K_1$ if $K_1 = N_1$. The scaling and wavelet functions generated by analysis and synthesis filters with $K_1 = 5$ and $K_2 = 12$ are shown in **Figure 14** respectively.

4.2 Orthogonal wavelet filter banks

The above-mentioned causal stable wavelet filter banks are biorthogonal (not orthogonal). Here we discuss a class of orthogonal wavelet filter banks using the lefting scheme. We use $P(z) = A_N(z)$ and $Q(z) = A_N(z^{-1})$, then,

$$\begin{cases} H_0(z) = \dfrac{1}{2}\left\{z^{-2K_1-1} + A_N\left(z^2\right)\right\} \\ H_1(z) = z^{-2K_2} - A_N(z^{-2})H_0(z) \end{cases}. \tag{35}$$

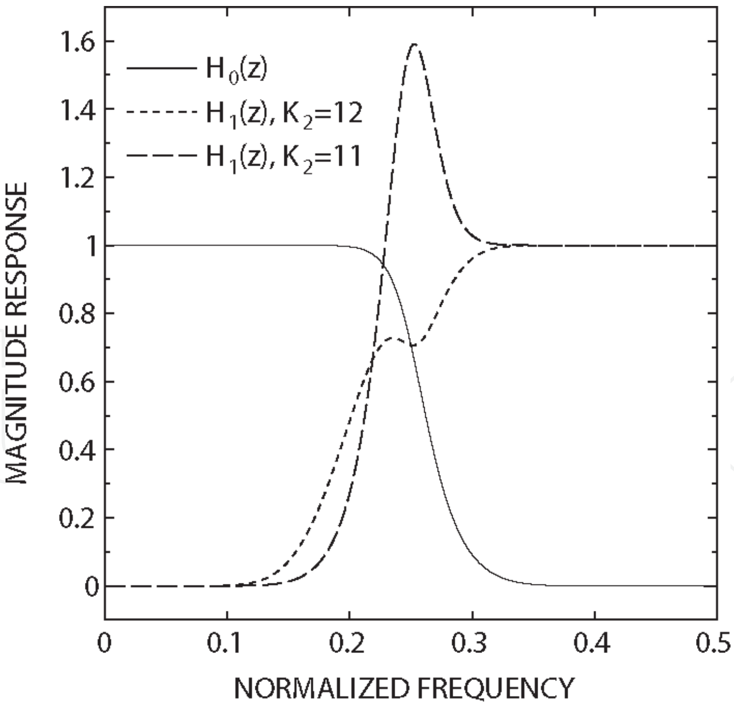

Figure 13.
Magnitude responses of $H_0(z)$ and $H_1(z)$ in example 5.

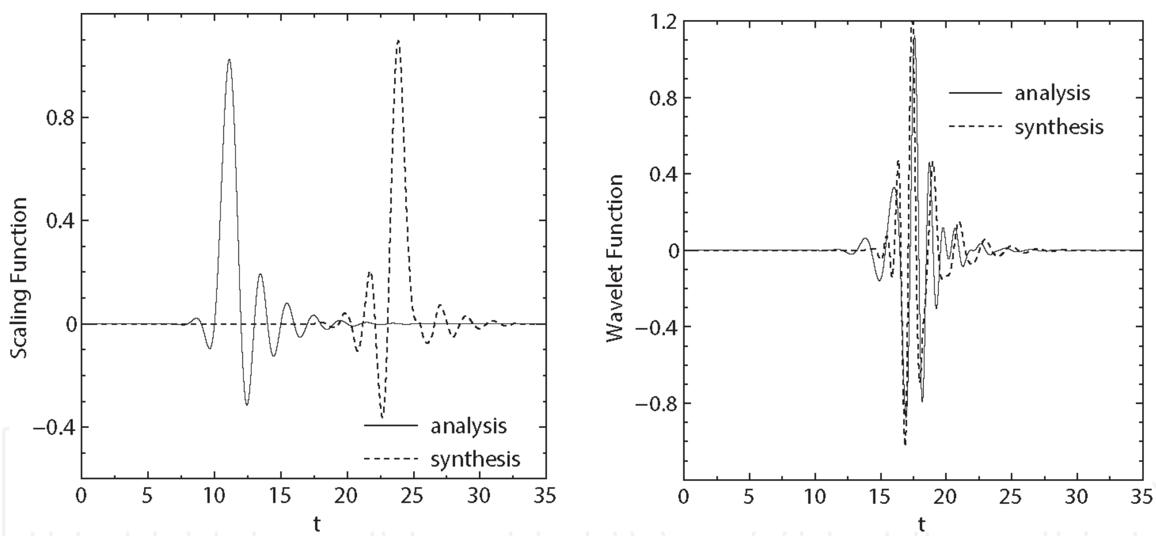

Figure 14.
Scaling and wavelet functions generated by analysis and synthesis filters in example 5. It is because the wavelet filter bank is biorthogonal, but not orthogonal.

Let $\theta(\omega)$ be the phase response of $A_N(z)$, the frequency response of $H_0(z)$ is the same as in Eq.(31), thus the desired phase response of $A_N(z)$ is

$$\theta_d(\omega) = -\left(K_1 + \frac{1}{2}\right)\omega = -\tau\omega, \tag{36}$$

where $\tau = K_1 + \frac{1}{2}$. To be orthogonal, we set $K_2 = 0$ and have

$$H_1(z) = 1 - A_N\left(z^{-2}\right)\frac{1}{2}\left\{z^{-2K_1-1} + A_N\left(z^2\right)\right\} = \frac{1}{2}\left\{1 - z^{-2K_1-1}A_N\left(z^{-2}\right)\right\}, \tag{37}$$

whose frequency response is

$$H_1(e^{j\omega}) = je^{-j\left(\frac{\theta(2\omega)}{2}+\left(K_1+\frac{1}{2}\right)\omega\right)} \sin\left(\frac{\theta(2\omega)}{2} + \left(K_1 + \frac{1}{2}\right)\omega\right). \tag{38}$$

It is clear that the magnitude responses satisfy $|H_0(e^{j\omega})| = |H_1(e^{j(\pi-\omega)})|$ and the power-complementary relation in Eq.(7). Therefore, this class of wavelet filter banks is orthogonal and both $H_0(z)$ and $H_1(z)$ have approximately linear phase response. Design of this class of orthogonal wavelet filter banks has been discussed and applied to lossy to lossless image coding in [24].

Example 6: We consider design of the maximally flat orthogonal wavelet filter banks with $N = 2, 4, 6$. We have designed $A_N(z)$ with $K_1 = N - 1$. The magnitude

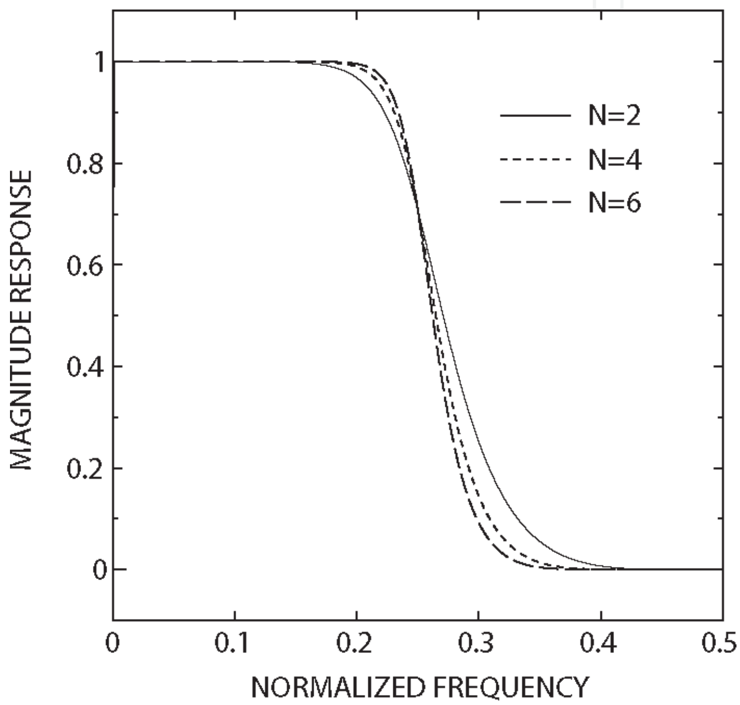

Figure 15.
Magnitude responses of $H_0(z)$ in example 6.

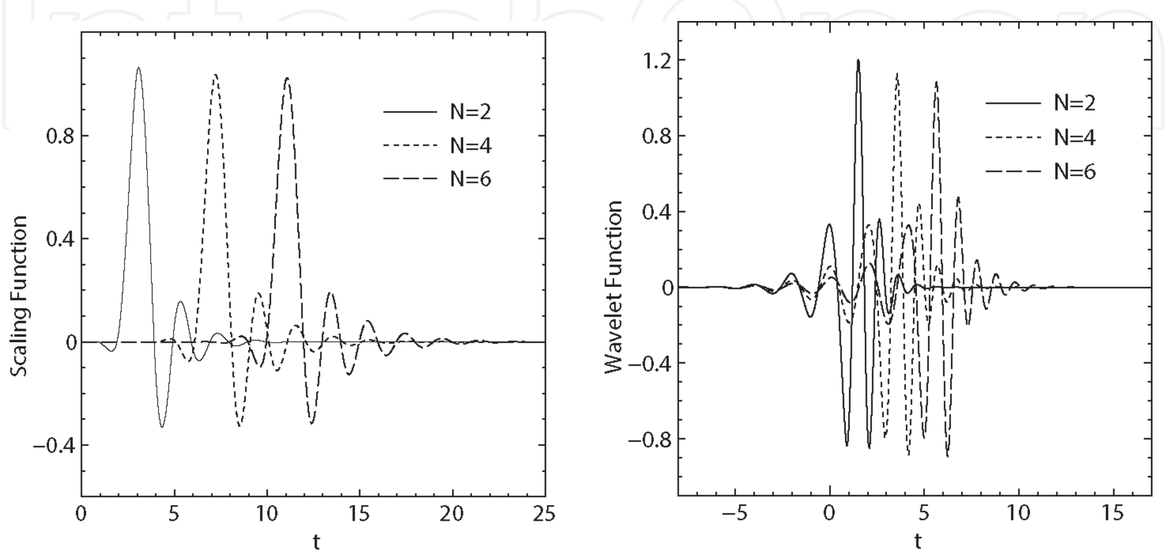

Figure 16.
Scaling and wavelet functions in example 6.

responses of $H_0(z)$ are shown in **Figure 15**. The generated scaling and wavelet functions are shown in **Figure 16**, respectively. It is seen in **Figure 16** that the wavelet functions are near symmetric.

5. Conclusions

In this chapter, we have proposed several new classes of wavelet filter banks with some properties of orthogonality, symmetry and causal stablity by using allpass filters, which are potential options for readers to choose wavelet basis in practical applications. As shown in **Table 1**, first class of wavelet filter banks in Section 3.1 is orthogonal, but asymmetric, its analysis filters is causal stable. Second class of wavelet filter banks in Section 3.2 is orthogonal and symmetric, but not causal. Third and fourth classes of wavelet filter banks are based on the lifting scheme. Third class in Section 4.1 is biorthogonal, causal stable and near symmetric, while fourth class in Section 4.2 is orthogonal and near symmetric, but not causal. There is no solution to all of orthogonality, symmetry and causal stablity. The wavlet filter banks using allpass filters have been extended to Hilbert transform pair of wavelets [25], 2D wavelet filter banks [26], and applied to lossy to lossless image coding [27–30] and scalable video compression [31]. It is possible also to extend them to higher dimension and irregural signal processing and to apply them to wavelet denoising, image fusion and so on.

Filter Bank Class	**Sec.3.1**	**Sec.3.2**	**Sec.4.1**	**Sec.4.2**	**D-8/8**	**D-9/7**
Filter Type	IIR	IIR	IIR	IIR	FIR	FIR
Orthogonality	○	○	×	○	○	×
Symmetry	×	○	Δ	Δ	×	○
Causal stablity	Δ	×	○	×	○	○

Table 1.
Comparison of the proposed classes of wavelet filter banks with the conventional wavelets D-8/8, D-9/7 in [1].

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Abbreviations

DWT	discrete wavelet transform
PR	perfect reconstruction
FIR	finite impulse response
IIR	infinite impulse response
HSS	half sample symmetric
WSS	whole sample symmetric

Appendix

Digital allpass filter is a computationally efficient signal processing building block and quite useful in many signal processing applications. One of the most widely used applications is phase or delay equalizer. Allpass filter possesses unit magnitude at all frequencies and is a basic scalar lossless building block. The interconnection of allpass filters has found numerous applications in practical filtering problems, such as low sensitivity filter structures, multirate filtering, filter banks and so on [13].

The transfer function of an Nth-order allpass filter is defined as

$$A_N(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n^* z^{-n}}, \tag{39}$$

where $a_n = a_{nr} + ja_{ni}$ is a complex coefficient in general, and a_n^* denotes the complex conjugate of a_n. When $a_{ni} = 0$, a_n is a real coefficient and $A_N(z)$ is a real allpass filter. Thus the real allpass filter is a special case of complex allpass filter. All poles and zeros of $A_N(z)$ occur in mirror-image pairs with respect to the unit circle, and then the frequency response $A_N(e^{j\omega})$ exhibits unit magnitude at all frequencies, i.e., $|A_N(e^{j\omega})| \equiv 1$ for all ω. The phase response of $A_N(z)$ is given by

$$\theta(\omega) = -N\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N} \{a_{nr} \sin n\omega + a_{ni} \cos n\omega\}}{\sum_{n=0}^{N} \{a_{nr} \cos n\omega - a_{ni} \sin n\omega\}}. \tag{40}$$

If all poles locate inside the unit circle, then $A_N(z)$ is causal stable. The phase response decreases monotonically with an increasing frequency and $\theta(\pi) = \theta(-\pi) - 2N\pi$. If $A_N(z)$ is real allpass filter, $\theta(0) = 0$ and $\theta(\pi) = -N\pi$. When one pole locates at the origin, it is seen that $A_N(z) = z^{-1}A_{N-1}(z)$ due to $a_N = 0$. Then z^{-N} is a special case of $A_N(z)$ if all poles locate at the origin. When k poles locate outside the unit circle, we can divide $A_N(z)$ into two causal stable allpass filters $A_{N-k}(z)$ and $A_k(z)$, i.e.,

$$A_N(z) = \frac{A_{N-k}(z)}{A_k(z)}. \tag{41}$$

The phase response $\theta(\omega)$ of $A_N(z)$ is the phase difference between $A_{N-k}(z)$ and $A_k(z)$, and $\theta(\pi) = \theta(-\pi) - 2(N-2k)\pi$. The design problem of allpass filters to approximate the specified phase response in the Chebyshev sense has been discussed in [14] and [15].

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