Chapter

Mode-I and Mode-II Crack Tip Fields in Implicit Gradient Elasticity Based on Laplacians of Stress and Strain. Part II: Asymptotic Solutions

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Abstract

We develop asymptotic solutions for near-tip fields of Mode-I and Mode-II crack problems and for model responses reflected by implicit gradient elasticity. Especially, a model of gradient elasticity is considered, which is based on Laplacians of stress and strain and turns out to be derivable as a particular case of micromorphic (microstrain) elasticity. While the governing model equations of the crack problems are developed in Part I, the present paper addresses analytical solutions for near-tip fields by using asymptotic expansions of Williams' type. It is shown that for the assumptions made in Part I, the model does not eliminiate the well-known singularities of classical elasticity. This is in contrast to conclusions made elsewhere, which rely upon different assumptions. However, there are significant differences in comparison to classical elasticity, which are discussed in the paper. For instance, in the case of Mode-II loading conditions, the leading terms of the asymptotic solution for the components of the double stress exhibit the remarkable property that they include two stress intensity factors.

Keywords: implicit gradient elasticity, mode-I and mode-II crack problems, analytical solutions, asymptotic expansions of Williams' type, near-tip fields, order of singularity, stress intensity factors

1. Introduction

The 3-PG-Model, discussed in Part I, is a simple model of implicit gradient elasticity based on Laplacians of stress and strain and has been introduced by Gutkin and Aifantis [1]. It can be derived as a particular case of micromorphic (microstrain) elasticity (see, e. g., Forest and Sievert [2]), so that a free energy function and required boundary conditions are formulated rigorously. In the present paper, we are looking for near-tip asymptotic field solutions for Mode-I and Mode-II crack problems, in the context of plane strain states. The asymptotic solutions are obtained by using expansions of Williams' type (see Williams [3]).

For the assumptions made in Part I, it is found that both, conventional stress and conventional strain, are singular. This holds also for the nonconventional stress, the so-called double stress. All singular fields have an order of singularity $r^{-\frac{1}{2}}$. In particular, the leading terms of the asymptotic solutions of the conventional stress are exactly the same as in classical elasticity. Nevertheless, the results are quite interesting, since the two leading terms of the asymptotic solution of the macrostrain are different from the corresponding terms of classical elasticity, and since the form of the asymptotic solution of the double stress exhibits a remarkable feature. To be more specific, the leading term of the asymptotic solution of the double stress includes two stress intensity factors, which are independent of each other. This reflects, from a theoretical point of view, differences in the structure of the asymptotic solutions in comparison to classical elasticity as well as micropolar elasticity, where only one stress intensity factor is present in the solutions of Mode-II crack problems.

There are various works addressing singularities in the field variables. Among others, we mention for couple-stress elasticity the works of Muki and Sternberg [4], Sternberg and Muki [5], Bogy and Sternberg [6, 7], Xia and Hutchinson [8], Huang et al. [9–11] and Zhang et al. [12]. For micropolar elasticity the works of Paul and Sridharan [13], Chen et al. [14], Diegele et al. [15] and for gradient elasticity the works of Altan and Aifantis [16, 17], Ru and Aifantis [18], Unger and Aifantis [19–21], Chen et al. [22], Mousavi and Lazar [23], Shi et al. [24, 25], Vardoulakis et al. [26], Karlis et al. [27, 28], Georgiadis [29], Askes and Aifantis [30] and Gutkin and Aifantis [1] are to be mentioned. The latter is an interesting work and proves that use of the 3-PG-Model eliminates singularities from the"elastic stresses of defects" (see also Askes and Aifantis [30] as well as Aifantis [31]). This finding is in contrast to the conclusions of the present paper, but it should be emphasized that the form of the assumed boundary conditions in Gutkin and Aifantis [1] is different from the form assumed here.

The scope of the paper is organized as follows: Mode-I and Mode-II crack problems are considered in the setting of plane strain problems. For the 3-PG-Model, the reduced governing equations for plane strain states have been derived in Part I and are summarized in Section 2. Section 3 provides asymptotic solutions for the near-tip fields by starting from asymptotic expansions of the macrodisplacement and the microdeformation. An alternative and equivalent aproach, starting from asymptotic expansions of the stresses, is sketched in Section 4. The developed asymptotic solutions are summarized and discussed in Section 5. Finally, the paper closes with some conclusions in Section 6.

Throughout the paper, use is made of the notation introduced in Part I.

2. Summary of the governing equations for plane strain problems

Following equations of Part I will be employed to establish asymptotic solutions of the crack tip fields.

Free energy function (see section "The 3-PG-Model as particular case of micro-strain elasticity" in Part I)

$$\psi = \frac{1}{2}\,\varepsilon_{\alpha\beta}\,\mathbb{C}_{\alpha\beta\rho\zeta}\,\varepsilon_{\rho\zeta} + \frac{1}{2}\frac{c_2 - c_1}{c_1}\,\gamma_{\alpha\beta}\,\mathbb{C}_{\alpha\beta\rho\zeta}\,\gamma_{\rho\zeta} + \frac{1}{2}\,(c_2 - c_1)\,k_{\alpha\beta\gamma}\,\mathbb{C}_{\beta\gamma\rho\zeta}\,k_{\alpha\rho\zeta}. \tag{1}$$

Elasticity law for $\boldsymbol{\Sigma}$ (see section 3.1 "The 3-PG-Model as particular case of micro-strain elasticity" in Part I)

$$\Sigma_{\alpha\beta} = \frac{c_2}{c_1}\,\mathbb{C}_{\alpha\beta\gamma\rho}\,\varepsilon_{\gamma\rho} - \frac{c_2 - c_1}{c_1}\,\mathbb{C}_{\alpha\beta\gamma\rho}\,\Psi_{\gamma\rho} \tag{2}$$

or inversely

$$\varepsilon_{rr} = \frac{c_1}{2\mu c_2}\left[\Sigma_{rr} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_1}\Psi_{rr}, \tag{3}$$

$$\varepsilon_{\varphi\varphi} = \frac{c_1}{2\mu c_2}\left[\Sigma_{\varphi\varphi} - \nu\left(\Sigma_{rr} + \Sigma_{\varphi\varphi}\right)\right] + \frac{c_2 - c_1}{c_1}\Psi_{\varphi\varphi}, \tag{4}$$

$$\varepsilon_{r\varphi} = \frac{c_1}{2\mu c_2}\Sigma_{r\varphi} + \frac{c_2 - c_1}{c_1}\Psi_{r\varphi}. \tag{5}$$

Elasticity law for $\boldsymbol{\sigma}$ (see section 3.1 "The 3-PG-Model as particular case of micro-strain elasticity" in Part I)

$$\sigma_{\alpha\beta} = \frac{c_2 - c_1}{c_1}\mathbb{C}_{\alpha\beta\rho\zeta}\left(\varepsilon_{\rho\zeta} - \Psi_{\rho\zeta}\right). \tag{6}$$

Elasticity law for $\boldsymbol{\mu}$ (see section 4.5.1 "Elasticity law for double stress" in Part I)

$$\mu_{rrr} = (c_2 - c_1)\left[(\lambda + 2\mu)\partial_r\Psi_{rr} + \lambda\partial_r\Psi_{\varphi\varphi}\right], \tag{7}$$

$$\mu_{r\varphi\varphi} = (c_2 - c_1)\left[(\lambda + 2\mu)\partial_r\Psi_{\varphi\varphi} + \lambda\partial_r\Psi_{rr}\right], \tag{8}$$

$$\mu_{rzz} = (c_2 - c_1)\lambda\partial_r\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right) = \nu\left(\mu_{rrr} + \mu_{r\varphi\varphi}\right), \tag{9}$$

$$\mu_{rr\varphi} = (c_2 - c_1)2\mu\partial_r\Psi_{r\varphi}, \tag{10}$$

$$\mu_{\varphi rr} = \frac{c_2 - c_1}{r}\left[2\mu\left(\partial_\varphi\Psi_{rr} - 2\Psi_{r\varphi}\right) + \lambda\partial_\varphi\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right)\right], \tag{11}$$

$$\mu_{\varphi\varphi\varphi} = \frac{c_2 - c_1}{r}\left[2\mu\left(\partial_\varphi\Psi_{\varphi\varphi} + 2\Psi_{r\varphi}\right) + \lambda\partial_\varphi\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right)\right], \tag{12}$$

$$\mu_{\varphi zz} = \frac{c_2 - c_1}{r}\lambda\partial_\varphi\left(\Psi_{rr} + \Psi_{\varphi\varphi}\right) = \nu\left(\mu_{\varphi rr} + \mu_{\varphi\varphi\varphi}\right), \tag{13}$$

$$\mu_{\varphi r\varphi} = \frac{c_2 - c_1}{r}2\mu\left(\partial_\varphi\Psi_{r\varphi} + \Psi_{rr} - \Psi_{\varphi\varphi}\right), \tag{14}$$

$$\mu_{rrz} = \mu_{r\varphi z} = \mu_{\varphi rz} = \mu_{\varphi\varphi z} = 0, \tag{15}$$

$$\mu_{z\alpha\beta} = 0, \tag{16}$$

or inversely

$$(\nabla\Psi)_{rrr} = \frac{1}{(c_2 - c_1)E}\left[\left(1 - \nu^2\right)\mu_{rrr} - \nu(1 + \nu)\mu_{r\varphi\varphi}\right], \tag{17}$$

$$(\nabla\Psi)_{r\varphi\varphi} = \frac{1}{(c_2 - c_1)E}\left[\left(1 - \nu^2\right)\mu_{r\varphi\varphi} - \nu(1 + \nu)\mu_{rrr}\right], \tag{18}$$

$$(\nabla\Psi)_{rr\varphi} = \frac{1 + \nu}{(c_2 - c_1)E}\mu_{rr\varphi}, \tag{19}$$

$$(\nabla\Psi)_{\varphi rr} = \frac{1}{(c_2 - c_1)E}\left[\left(1 - \nu^2\right)\mu_{\varphi rr} - \nu(1 + \nu)\mu_{\varphi\varphi\varphi}\right], \tag{20}$$

$$(\nabla\Psi)_{\varphi\varphi\varphi} = \frac{1}{(c_2 - c_1)E}\left[\left(1 - \nu^2\right)\mu_{\varphi\varphi\varphi} - \nu(1 + \nu)\mu_{\varphi rr}\right], \tag{21}$$

$$(\nabla\Psi)_{\varphi r\varphi} = \frac{1 + \nu}{(c_2 - c_1)E}\mu_{\varphi r\varphi}. \tag{22}$$

Material parameters (see section 2 "Preliminaries—Notation" in Part I)

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \quad E = 2\mu(1+\nu). \tag{23}$$

Strain components (see section 4.1 "Kinematics" in Part I)

$$\varepsilon_{rr} = \partial_r u_r, \quad \varepsilon_{\varphi\varphi} = \frac{1}{r}\left(u_r + \partial_\varphi u_\varphi\right), \quad \varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{1}{r}\partial_\varphi u_r + \partial_r u_\varphi - \frac{1}{r}u_\varphi\right). \tag{24}$$

Microdeformation components (see section 4.1 "Kinematics" of Part I)

$$(\nabla\boldsymbol{\Psi})_{rrr} = \partial_r \boldsymbol{\Psi}_{rr}, \quad (\nabla\boldsymbol{\Psi})_{r\varphi\varphi} = \partial_r \boldsymbol{\Psi}_{\varphi\varphi}, \quad (\nabla\boldsymbol{\Psi})_{rr\varphi} = \partial_r \Psi_{r\varphi}, \tag{25}$$

$$(\nabla\boldsymbol{\Psi})_{\varphi rr} = \frac{1}{r}\left(\partial_\varphi \Psi_{rr} - 2\Psi_{r\varphi}\right), \tag{26}$$

$$(\nabla\boldsymbol{\Psi})_{\varphi\varphi\varphi} = \frac{1}{r}\left(\partial_\varphi \Psi_{\varphi\varphi} + 2\Psi_{r\varphi}\right), \tag{27}$$

$$(\nabla\boldsymbol{\Psi})_{\varphi r\varphi} = \frac{1}{r}\left(\partial_\varphi \Psi_{r\varphi} + \Psi_{rr} - \Psi_{\varphi\varphi}\right), \tag{28}$$

$$(\nabla\boldsymbol{\Psi})_{\alpha\beta z} = (\nabla\boldsymbol{\Psi})_{z\alpha\beta} = 0. \tag{29}$$

Classical equilibrium equations (see section 4.2 "Cauchy stress—Classical equilibrium equations" in Part I)

$$\partial_r \Sigma_{rr} + \frac{1}{r}\partial_\varphi \Sigma_{r\varphi} + \frac{1}{r}\left(\Sigma_{rr} - \Sigma_{\varphi\varphi}\right) = 0, \tag{30}$$

$$\partial_r \Sigma_{r\varphi} + \frac{1}{r}\partial_\varphi \Sigma_{\varphi\varphi} + \frac{2}{r}\Sigma_{r\varphi} = 0. \tag{31}$$

Nonclassical equilibrium equations (see section 4.5.2 "Nonclassical equilibrium conditions" in Part I)

$$\partial_r \mu_{rrr} + \frac{1}{r}\partial_\varphi \mu_{\varphi rr} + \frac{1}{r}\left(\mu_{rrr} - 2\mu_{\varphi r\varphi}\right) + \sigma_{rr} = 0, \tag{32}$$

$$\partial_r \mu_{r\varphi\varphi} + \frac{1}{r}\partial_\varphi \mu_{\varphi\varphi\varphi} + \frac{1}{r}\left(\mu_{r\varphi\varphi} + 2\mu_{\varphi r\varphi}\right) + \sigma_{\varphi\varphi} = 0, \tag{33}$$

$$\partial_r \mu_{rzz} + \frac{1}{r}\partial_\varphi \mu_{\varphi zz} + \frac{1}{r}\mu_{rzz} + \sigma_{zz} = 0, \tag{34}$$

$$\partial_r \mu_{rr\varphi} + \frac{1}{r}\partial_\varphi \mu_{\varphi r\varphi} + \frac{1}{r}\left(\mu_{rr\varphi} - \mu_{\varphi\varphi\varphi} + \mu_{\varphi rr}\right) + \sigma_{r\varphi} = 0. \tag{35}$$

Field equations for Ψ (see section 4.4 "Field equations for Ψ" in Part I)

$$\begin{aligned}\partial_{rr}\Psi_{rr} &+ \frac{1}{r^2}\partial_{\varphi\varphi}\Psi_{rr} + \frac{1}{r}\partial_r\Psi_{rr} - \frac{4}{r^2}\partial_\varphi\Psi_{r\varphi} - \left(\frac{2}{r^2} + \frac{1}{c_2}\right)\Psi_{rr} + \frac{2}{r^2}\Psi_{\varphi\varphi} \\ &+ \frac{1-\nu}{2\mu c_2}\Sigma_{rr} - \frac{\nu}{2\mu c_2}\Sigma_{\varphi\varphi} = 0,\end{aligned} \tag{36}$$

$$\begin{aligned}\partial_{rr}\Psi_{\varphi\varphi} &+ \frac{1}{r^2}\partial_{\varphi\varphi}\Psi_{\varphi\varphi} + \frac{1}{r}\partial_r\Psi_{\varphi\varphi} + \frac{4}{r^2}\partial_\varphi\Psi_{r\varphi} + \frac{2}{r^2}\Psi_{rr} - \left(\frac{2}{r^2} + \frac{1}{c_2}\right)\Psi_{\varphi\varphi} \\ &+ \frac{1-\nu}{2\mu c_2}\Sigma_{\varphi\varphi} - \frac{\nu}{2\mu c_2}\Sigma_{rr} = 0,\end{aligned} \tag{37}$$

$$\partial_{rr}\Psi_{r\varphi}+\frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{r\varphi}+\frac{1}{r}\,\partial_r\Psi_{r\varphi}+\frac{2}{r^2}\,\partial_\varphi\Psi_{rr}-\frac{2}{r^2}\,\partial_\varphi\Psi_{\varphi\varphi}-\left(\frac{4}{r^2}+\frac{1}{c_2}\right)\Psi_{r\varphi}$$
$$+\frac{1}{2\mu c_2}\,\Sigma_{r\varphi}=0. \tag{38}$$

Classical compatibility condition (see section 4.3 "Classical compatibility condition" in Part I)

$$\chi_1(\Psi_{\alpha\beta})+\chi_2(\Sigma_{\alpha\beta})=0, \tag{39}$$

$$\chi_1(\Psi_{\alpha\beta}) := \frac{c_2-c_1}{c_2}\left[\partial_{rr}\Psi_{\varphi\varphi}-\frac{2}{r}\,\partial_{r\varphi}\Psi_{r\varphi}+\frac{1}{r^2}\,\partial_{\varphi\varphi}\Psi_{rr}\right.$$
$$\left.-\frac{1}{r}\,\partial_r\Psi_{rr}+\frac{2}{r}\,\partial_r\Psi_{\varphi\varphi}-\frac{2}{r^2}\,\partial_\varphi\Psi_{r\varphi}\right], \tag{40}$$

$$\chi_2(\Sigma_{\alpha\beta}) := \frac{(1-\nu)\,c_1}{2\mu c_2}\left[\partial_{rr}(\Sigma_{rr}+\Sigma_{\varphi\varphi})+\frac{1}{r^2}\,\partial_{\varphi\varphi}(\Sigma_{rr}+\Sigma_{\varphi\varphi})\right.$$
$$\left.+\frac{1}{r}\,\partial_r(\Sigma_{rr}+\Sigma_{\varphi\varphi})\right]. \tag{41}$$

Nonclassical compatibility conditions (see section 4.6 "Nonclassical compatibility conditions" in Part I)

$$\partial_\varphi\mu_{rr\varphi}-\mu_{\varphi r\varphi}-r\,\partial_r\mu_{\varphi r\varphi}+\mu_{rrr}-\mu_{r\varphi\varphi}=0, \tag{42}$$

$$\partial_\varphi\mu_{r\varphi\varphi}+\partial_\varphi\mu_{rrr}-\mu_{\varphi\varphi\varphi}-\mu_{\varphi rr}-r\,\partial_r\mu_{\varphi\varphi\varphi}-r\,\partial_r\mu_{\varphi rr}=0, \tag{43}$$

$$\partial_\varphi\mu_{r\varphi\varphi}-\partial_\varphi\mu_{rrr}-\mu_{\varphi\varphi\varphi}+\mu_{\varphi rr}-r\,\partial_r\mu_{\varphi\varphi\varphi}+r\,\partial_r\mu_{\varphi rr}+4\mu_{rr\varphi}=0. \tag{44}$$

Classical boundary conditions (see section 4.7 "Boundary conditions" in Part I)

$$[\Sigma_{r\varphi}]_{\varphi=\pm\pi}=0, \tag{45}$$

$$[\Sigma_{\varphi\varphi}]_{\varphi=\pm\pi}=0. \tag{46}$$

Nonclassical boundary conditions (see section 4.7 "Boundary conditions" in Part I)

$$[\mu_{\varphi rr}]_{\varphi=\pm\pi}=[\mu_{\varphi\varphi\varphi}]_{\varphi=\pm\pi}=[\mu_{\varphi r\varphi}]_{\varphi=\pm\pi}=0, \tag{47}$$

or equivalently

$$[\partial_\varphi\Psi_{rr}-2\Psi_{r\varphi}]_{\varphi=\pm\pi}=0, \tag{48}$$

$$[\partial_\varphi\Psi_{\varphi\varphi}+2\Psi_{r\varphi}]_{\varphi=\pm\pi}=0, \tag{49}$$

$$[\partial_\varphi\Psi_{r\varphi}+\Psi_{rr}-\Psi_{\varphi\varphi}]_{\varphi=\pm\pi}=0. \tag{50}$$

Symmetry conditions—Mode-I (see section 4.8 "Symmetry conditions" in Part I)

$$\Sigma_{rr}(r,\varphi)=\Sigma_{rr}(r,-\varphi),\quad \Sigma_{\varphi\varphi}(r,\varphi)=\Sigma_{\varphi\varphi}(r,-\varphi), \tag{51}$$

$$\Sigma_{r\varphi}(r,\varphi)=-\Sigma_{r\varphi}(r,-\varphi), \tag{52}$$

$$\Psi_{rr}(r,\varphi)=\Psi_{rr}(r,-\varphi),\quad \Psi_{\varphi\varphi}(r,\varphi)=\Psi_{\varphi\varphi}(r,-\varphi), \tag{53}$$

$$\Psi_{r\varphi}(r,\varphi)=-\Psi_{r\varphi}(r,-\varphi), \tag{54}$$

$$\mu_{rrr}(r,\varphi)=\mu_{rrr}(r,-\varphi),\quad \mu_{\varphi rr}(r,\varphi)=-\mu_{\varphi rr}(r,-\varphi), \tag{55}$$

$$\mu_{r\varphi\varphi}(r,\varphi)=\mu_{r\varphi\varphi}(r,-\varphi),\quad \mu_{\varphi\varphi\varphi}(r,\varphi)=-\mu_{\varphi\varphi\varphi}(r,-\varphi), \tag{56}$$

$$\mu_{rzz}(r,\varphi)=\mu_{rzz}(r,-\varphi),\quad \mu_{\varphi zz}(r,\varphi)=-\mu_{\varphi zz}(r,-\varphi), \tag{57}$$

$$\mu_{rr\varphi}(r,\varphi)=\mu_{rr\varphi}(r,-\varphi),\quad \mu_{\varphi r\varphi}(r,\varphi)=-\mu_{\varphi r\varphi}(r,-\varphi). \tag{58}$$

Symmetry conditions—Mode-II (see section 4.8 “Symmetry conditions” in Part I)

$$\Sigma_{rr}(r,\varphi)=-\Sigma_{rr}(r,-\varphi),\quad \Sigma_{\varphi\varphi}(r,\varphi)=-\Sigma_{\varphi\varphi}(r,-\varphi), \tag{59}$$

$$\Sigma_{r\varphi}(r,\varphi)=\Sigma_{r\varphi}(r,-\varphi), \tag{60}$$

$$\Psi_{rr}(r,\varphi)=-\Psi_{rr}(r,-\varphi),\quad \Psi_{\varphi\varphi}(r,\varphi)=-\Psi_{\varphi\varphi}(r,-\varphi), \tag{61}$$

$$\Psi_{r\varphi}(r,\varphi)=\Psi_{r\varphi}(r,-\varphi), \tag{62}$$

$$\mu_{rrr}(r,\varphi)=-\mu_{rrr}(r,-\varphi),\quad \mu_{\varphi rr}(r,\varphi)=\mu_{\varphi rr}(r,-\varphi), \tag{63}$$

$$\mu_{r\varphi\varphi}(r,\varphi)=-\mu_{r\varphi\varphi}(r,-\varphi),\quad \mu_{\varphi\varphi\varphi}(r,\varphi)=\mu_{\varphi\varphi\varphi}(r,-\varphi), \tag{64}$$

$$\mu_{rzz}(r,\varphi)=-\mu_{rzz}(r,-\varphi),\quad \mu_{\varphi zz}(r,\varphi)=\mu_{\varphi zz}(r,-\varphi), \tag{65}$$

$$\mu_{rr\varphi}(r,\varphi)=-\mu_{rr\varphi}(r,-\varphi),\quad \mu_{\varphi r\varphi}(r,\varphi)=\mu_{\varphi r\varphi}(r,-\varphi). \tag{66}$$

3. Near-tip asymptotic solutions for Mode-I and Mode-II loading conditions

We shall solve the given problems by employing asymptotic expansions of Williams’ type (see Williams [3]).

3.1 Expansions of Williams’ type

As the components of the macrodisplacement and the microdeformation reflect the independent kinematical degrees of freedom, we assume for u_α and $\Psi_{\alpha\beta}$ asymptotic expansions of the same form. We fix the crack tip at the origin O of the coordinate system (see Figure 1 in Part I) and set

$$u_\alpha=r^p\,u_\alpha^{(0)}+r^{p+\frac{1}{2}}u_\alpha^{(1)}+\ \dots\ =\sum_{k=0}^{\infty}r^{p+\frac{k}{2}}u_\alpha^{(k)}, \tag{67}$$

$$\Psi_{\alpha\beta}=\overline{\Psi}_{\alpha\beta}+r^p\,\Psi_{\alpha\beta}^{(0)}+r^{p+\frac{1}{2}}\Psi_{\alpha\beta}^{(1)}+\ \dots\ =\overline{\Psi}_{\alpha\beta}+\sum_{k=0}^{\infty}r^{p+\frac{k}{2}}\Psi_{\alpha\beta}^{(k)}, \tag{68}$$

with

$$u_\alpha^{(k)}=u_\alpha^{(k)}(\varphi),\quad \Psi_{\alpha\beta}^{(k)}=\Psi_{\alpha\beta}^{(k)}(\varphi),\quad \overline{\Psi}_{\alpha\beta}=\overline{\Psi}_{\alpha\beta}(\varphi), \tag{69}$$

and p being a real number. Since the crack tip is fixed at O, no constant term is present in the expansion of $\boldsymbol{u}$ in Eq. (67). However, we allow a constant term $\overline{\boldsymbol{\Psi}}=$ const., with physical components $\overline{\Psi}_{\alpha\beta}$ in conjunction with cylindrical coordinates, to be present in the expansion of $\boldsymbol{\Psi}$. While the Cartesian components $\overline{\Psi}_{ij}$ are constant, the physical components $\overline{\Psi}_{\alpha\beta}$ are functions of φ. There are the following well known transformation rules between $\overline{\Psi}_{\alpha\beta}$ and $\overline{\Psi}_{ij}$ (see any textbook)

$$\overline{\Psi}_{rr} = \frac{1}{2}\left(\overline{\Psi}_{11} + \overline{\Psi}_{22}\right) + \frac{1}{2}\left(\overline{\Psi}_{11} - \overline{\Psi}_{22}\right)\cos 2\varphi + \overline{\Psi}_{12}\sin 2\varphi, \tag{70}$$

$$\overline{\Psi}_{\varphi\varphi} = \frac{1}{2}\left(\overline{\Psi}_{11} + \overline{\Psi}_{22}\right) - \frac{1}{2}\left(\overline{\Psi}_{11} - \overline{\Psi}_{22}\right)\cos 2\varphi - \overline{\Psi}_{12}\sin 2\varphi, \tag{71}$$

$$\overline{\Psi}_{r\varphi} = -\frac{1}{2}\left(\overline{\Psi}_{11} - \overline{\Psi}_{22}\right)\sin 2\varphi + \overline{\Psi}_{12}\cos 2\varphi. \tag{72}$$

For later reference, we note the relations

$$\partial_\varphi \overline{\Psi}_{rr} - 2\overline{\Psi}_{r\varphi} = 0, \quad \partial_\varphi \overline{\Psi}_{\varphi\varphi} + 2\overline{\Psi}_{r\varphi} = 0, \tag{73}$$

$$\partial_\varphi \overline{\Psi}_{r\varphi} + \overline{\Psi}_{rr} - \overline{\Psi}_{\varphi\varphi} = 0, \tag{74}$$

which imply that the physical components $\overline{\Psi}_{\alpha\beta}$ trivially obey the nonclassical boundary conditions (32)–(35). Anticipating the results in Section 5, we decompose $\overline{\boldsymbol{\Psi}}$ into parts $\overline{\boldsymbol{\Psi}}^I$ and $\overline{\boldsymbol{\Psi}}^{II}$, reflecting symmetries according to Mode-I and Mode-II:

$$\overline{\Psi}_{\alpha\beta} = \overline{\Psi}^I_{\alpha\beta} + \overline{\Psi}^{II}_{\alpha\beta}, \tag{75}$$

with

$$\overline{\Psi}^I_{rr} := \overline{L}_{I,1} + \overline{L}_{I,2}\cos 2\varphi, \quad \overline{\Psi}^{II}_{rr} := \overline{L}_{II}\sin 2\varphi, \tag{76}$$

$$\overline{\Psi}^I_{\varphi\varphi} := \overline{L}_{I,1} - \overline{L}_{I,2}\cos 2\varphi, \quad \overline{\Psi}^{II}_{\varphi\varphi} := -\overline{L}_{II}\sin 2\varphi, \tag{77}$$

$$\overline{\Psi}^I_{r\varphi} := -\overline{L}_{I,2}\sin 2\varphi, \qquad \overline{\Psi}^{II}_{r\varphi} := \overline{L}_{II}\cos 2\varphi \tag{78}$$

and

$$\overline{L}_{I,1} := \frac{1}{2}\left(\overline{\Psi}_{11} + \overline{\Psi}_{22}\right), \quad \overline{L}_{I,2} := \frac{1}{2}\left(\overline{\Psi}_{11} - \overline{\Psi}_{22}\right), \quad \overline{L}_{II} := \overline{\Psi}_{12}. \tag{79}$$

The main idea in Williams' approach is to expand each field variable $f(r,\varphi)$ in a sum of products as in Eqs. (67) and (68). We say that f is of the order r^m, and write $f \sim r^m$, whenever r^m is the power function of r in the leading term of the expansion of f. It can be deduced, from Eq. (67), that $\varepsilon_{\alpha\beta} \sim r^{p-1}$. From this, in turn, together with Eq. (68) and the elasticity laws (3)–(5), we can deduce, that $\Sigma_{\alpha\beta} \sim r^{p-1}$. Thus,

$$\Sigma_{\alpha\beta} = r^{p-1}\Sigma^{(0)}_{\alpha\beta} + r^{p-\frac{1}{2}}\Sigma^{(1)}_{\alpha\beta} + \ldots = \sum_{k=0}^{\infty} r^{p-1+\frac{k}{2}}\Sigma^{(k)}_{\alpha\beta}, \tag{80}$$

with

$$\Sigma^{(k)}_{\alpha\beta} = \Sigma^{(k)}_{\alpha\beta}(\varphi). \tag{81}$$

Expansion (67) suggests that the necessary and sufficient condition for u_α to vanish at the crack tip is

$$p > 0. \tag{82}$$

This restriction is in agreement with energetic considerations. To verify, we remark that $\nabla\overline{\boldsymbol{\Psi}} = \mathbf{0}$, as $\overline{\boldsymbol{\Psi}}$ is constant. Therefore, from Eq. (68) together with Eqs. (25)–(29), we may infer that $(\nabla\boldsymbol{\Psi})_{\alpha\beta\gamma} \sim r^{p-1}$. For the free energy per unit macrovolume ψ it follows that $\psi \sim r^{2p-2}$ [cf. Eq. (1)]. Now, consider a small circular area $r \le R$, enclosing the crack tip. The total free energy (per unit length in z–direction) of this area is

$$\int_0^{2\pi}\int_0^R \psi\, r\, \mathrm{d}r\, \mathrm{d}\varphi. \tag{83}$$

Since $\psi r \sim r^{2p-1}$, restriction (82) is the necessary and sufficient condition for the energy in Eq. (83) to be bounded.

3.2 Consequences of the classical equilibrium equations

Substitute the expansion (80) into Eqs. (30) and (31) and collect coefficients of like powers of r, to obtain

$$\begin{aligned}
&r^{p-2}\left\{p\,\Sigma_{rr}^{(0)} + \partial_\varphi \Sigma_{r\varphi}^{(0)} - \Sigma_{\varphi\varphi}^{(0)}\right\}\\
&+r^{p-\frac{3}{2}}\left\{\left(p+\frac{1}{2}\right)\Sigma_{rr}^{(1)} + \partial_\varphi \Sigma_{r\varphi}^{(1)} - \Sigma_{\varphi\varphi}^{(1)}\right\}\\
&+r^{p-1}\left\{(p+1)\,\Sigma_{rr}^{(2)} + \partial_\varphi\, \Sigma_{r\varphi}^{(2)} - \Sigma_{\varphi\varphi}^{(0)}\right\}\\
&+\ldots = 0.
\end{aligned} \tag{84}$$

Similarly, we find from Eq. (31) that

$$\begin{aligned}
&r^{p-2}\left\{(p+1)\,\Sigma_{r\varphi}^{(0)} + \partial_\varphi \Sigma_{\varphi\varphi}^{(0)}\right\}\\
&+r^{p-\frac{3}{2}}\left\{\left(p+\frac{3}{2}\right)\Sigma_{r\varphi}^{(1)} + \partial_\varphi \Sigma_{\varphi\varphi}^{(1)}\right\}\\
&+r^{p-1}\left\{(p+2)\,\Sigma_{r\varphi}^{(2)} + \partial_\varphi \Sigma_{\varphi\varphi}^{(2)}\right\}\\
&+\ldots = 0.
\end{aligned} \tag{85}$$

3.3 Consequences of the classical compatibility condition

A look at $\chi_1(\cdot)$ in Eq. (40) reveals that χ_1 is a linear differential operator, i. e.,

$$\chi_1\left(\Psi_{\alpha\beta} - \overline{\Psi}_{\alpha\beta}\right) = \chi_1\left(\Psi_{\alpha\beta}\right) - \chi_1\left(\overline{\Psi}_{\alpha\beta}\right). \tag{86}$$

Since $\overline{\Psi}_{\alpha\beta}$ is independent of r, we infer from Eq. (40) that

$$\chi_1\left(\overline{\Psi}_{\alpha\beta}\right) = \frac{c_2 - c_1}{c_2}\frac{1}{r^2}\,\partial_\varphi\left(\partial_\varphi \overline{\Psi}_{rr} - 2\overline{\Psi}_{r\varphi}\right), \tag{87}$$

and by virtue of Eq. (73),

$$\chi_1\left(\overline{\Psi}_{\alpha\beta}\right) = 0. \tag{88}$$

Therefore, from Eq. (86),

$$\chi_1(\Psi_{\alpha\beta}) = \chi_1(\Psi_{\alpha\beta} - \overline{\Psi}_{\alpha\beta}), \tag{89}$$

and by appealing to expansion (68), we infer from Eq. (40) that

$$\begin{aligned}\chi_1(\Psi_{\alpha\beta}) = \frac{c_2 - c_1}{c_2} \sum_{k=0}^{\infty} r^{p-2+\frac{k}{2}} \Bigg\{ &\left(p + \frac{k}{2}\right)\left(p + \frac{k}{2} - 1\right) \Psi^{(k)}_{\varphi\varphi} + \partial_{\varphi\varphi} \Psi^{(k)}_{rr} \\ &-2\left(p + \frac{k}{2}\right) \partial_\varphi \Psi^{(k)}_{r\varphi} - \left(p + \frac{k}{2}\right) \Psi^{(k)}_{rr} \\ &+2\left(p + \frac{k}{2}\right) \Psi^{(k)}_{\varphi\varphi} - 2\partial_\varphi \Psi^{(k)}_{r\varphi} \Bigg\}.\end{aligned} \tag{90}$$

Similarly, by appealing to expansion (80), we infer from Eq. (41) that

$$\begin{aligned}\chi_2(\Sigma_{\alpha\beta}) = \frac{(1-\nu)c_1}{2\mu c_2} \sum_{k=0}^{\infty} r^{p-2+\frac{k}{2}} \Bigg\{ &\left(p - 1 + \frac{k}{2}\right)\left(p - 2 + \frac{k}{2}\right)\left(\Sigma^{(k)}_{rr} + \Sigma^{(k)}_{\varphi\varphi}\right) \\ &+\partial_{\varphi\varphi}\left(\Sigma^{(k)}_{rr} + \Sigma^{(k)}_{\varphi\varphi}\right) \\ &+\left(p - 1 + \frac{k}{2}\right)\left(\Sigma^{(k)}_{rr} + \Sigma^{(k)}_{\varphi\varphi}\right) \Bigg\}.\end{aligned} \tag{91}$$

Inserting Eqs. (90) and (91) into Eq. (39) and collecting coefficients of like powers of r gives, after some lengthy but straightforward manipulations,

$$\begin{aligned}&r^{p-3}\,\frac{(1-\nu)c_1}{2\mu c_2}\left\{(p-1)^2\left(\Sigma^{(0)}_{rr} + \Sigma^{(0)}_{\varphi\varphi}\right) + \partial_{\varphi\varphi}\left(\Sigma^{(0)}_{rr} + \Sigma^{(0)}_{\varphi\varphi}\right)\right\} \\ &+r^{p-\frac{5}{2}}\,\frac{(1-\nu)c_1}{2\mu c_2}\left\{\left(p - \frac{1}{2}\right)^2\left(\Sigma^{(1)}_{rr} + \Sigma^{(1)}_{\varphi\varphi}\right) + \partial_{\varphi\varphi}\left(\Sigma^{(1)}_{rr} + \Sigma^{(1)}_{\varphi\varphi}\right)\right\} \\ &+r^{p-2}\left\{\frac{(1-\nu)c_1}{2\mu c_2}\left[p^2\left(\Sigma^{(2)}_{rr} + \Sigma^{(2)}_{\varphi\varphi}\right) + \partial_{\varphi\varphi}\left(\Sigma^{(2)}_{rr} + \Sigma^{(2)}_{\varphi\varphi}\right)\right]\right. \\ &\qquad\left.+\frac{c_2 - c_1}{c_2}\left[p(p+1)\Psi^{(0)}_{\varphi\varphi} + \partial_{\varphi\varphi}\Psi^{(0)}_{rr} - 2(p+1)\Psi^{(0)}_{r\varphi}\right]\right\} \\ &+r^{p-\frac{3}{2}}\left\{\frac{(1-\nu)c_1}{2\mu c_2}\left[\left(p + \frac{1}{2}\right)^2\left(\Sigma^{(3)}_{rr} + \Sigma^{(3)}_{\varphi\varphi}\right) + \partial_{\varphi\varphi}\left(\Sigma^{(3)}_{rr} + \Sigma^{(3)}_{\varphi\varphi}\right)\right]\right. \\ &\qquad\left.+\frac{c_2 - c_1}{c_2}\left[\left(p + \frac{1}{2}\right)\left(p + \frac{3}{2}\right)\Psi^{(1)}_{\varphi\varphi} + \partial_{\varphi\varphi}\Psi^{(1)}_{rr} - 2\left(p + \frac{3}{2}\right)\Psi^{(1)}_{r\varphi}\right]\right\} \\ &+\ldots = 0.\end{aligned} \tag{92}$$

3.4 Consequences of the classical boundary conditions

By invoking the asymptotic expansion (80) in the classical boundary conditions (45) and (46), we conclude that

$$r^{p-1}\left[\Sigma^{(0)}_{r\varphi}\right]_{\varphi=\pm\pi} + r^{p-\frac{1}{2}}\left[\Sigma^{(1)}_{r\varphi}\right]_{\varphi=\pm\pi} + \ldots = 0, \tag{93}$$

$$r^{p-1}\left[\Sigma^{(0)}_{\varphi\varphi}\right]_{\varphi=\pm\pi} + r^{p-\frac{1}{2}}\left[\Sigma^{(1)}_{\varphi\varphi}\right]_{\varphi=\pm\pi} + \ldots = 0. \tag{94}$$

3.5 Cauchy stress

Before going any further, it is convenient to evaluate the results so far. The necessary and sufficient conditions for the equilibrium Eqs. (84) and (85), the compatibility condition (92) and the boundary conditions (93) and (94) to hold for arbitrary r in the vicinity of the crack tip are vanishing coefficients of all powers of r. For $\Sigma_{\alpha\beta}^{(k)}, k = 0, 1, 2$, this leads to the following systems of differential equations and associated boundary conditions.

Terms $\Sigma_{\alpha\beta}^{(0)}$

$$\partial_\varphi \Sigma_{r\varphi}^{(0)} + p\,\Sigma_{rr}^{(0)} - \Sigma_{\varphi\varphi}^{(0)} = 0, \tag{95}$$

$$\partial_\varphi \Sigma_{\varphi\varphi}^{(0)} + (p+1)\,\Sigma_{r\varphi}^{(0)} = 0, \tag{96}$$

$$\partial_{\varphi\varphi}\left(\Sigma_{rr}^{(0)} + \Sigma_{\varphi\varphi}^{(0)}\right) + (p-1)^2\left(\Sigma_{rr}^{(0)} + \Sigma_{\varphi\varphi}^{(0)}\right) = 0, \tag{97}$$

with boundary conditions

$$\left[\Sigma_{r\varphi}^{(0)}\right]_{\varphi=\pm\pi} = 0, \quad \left[\Sigma_{\varphi\varphi}^{(0)}\right]_{\varphi=\pm\pi} = 0. \tag{98}$$

Terms $\Sigma_{\alpha\beta}^{(1)}$

$$\partial_\varphi \Sigma_{r\varphi}^{(1)} + \left(p + \frac{1}{2}\right)\Sigma_{rr}^{(1)} - \Sigma_{\varphi\varphi}^{(1)} = 0, \tag{99}$$

$$\partial_\varphi \Sigma_{\varphi\varphi}^{(1)} + \left(p + \frac{3}{2}\right)\Sigma_{r\varphi}^{(1)} = 0, \tag{100}$$

$$\partial_{\varphi\varphi}\left(\Sigma_{rr}^{(1)} + \Sigma_{\varphi\varphi}^{(1)}\right) + \left(p - \frac{1}{2}\right)^2\left(\Sigma_{rr}^{(1)} + \Sigma_{\varphi\varphi}^{(1)}\right) = 0, \tag{101}$$

with boundary conditions

$$\left[\Sigma_{r\varphi}^{(1)}\right]_{\varphi=\pm\pi} = 0, \quad \left[\Sigma_{\varphi\varphi}^{(1)}\right]_{\varphi=\pm\pi} = 0. \tag{102}$$

Terms $\Sigma_{\alpha\beta}^{(2)}$

$$\partial_\varphi \Sigma_{r\varphi}^{(2)} + (p+1)\,\Sigma_{rr}^{(2)} - \Sigma_{\varphi\varphi}^{(2)} = 0, \tag{103}$$

$$\partial_\varphi \Sigma_{\varphi\varphi}^{(2)} + (p+2)\,\Sigma_{r\varphi}^{(2)} = 0, \tag{104}$$

$$\begin{aligned}&\frac{1-\nu}{2\mu c_2}\left\{\partial_{\varphi\varphi}\left(\Sigma_{rr}^{(2)} + \Sigma_{\varphi\varphi}^{(2)}\right) + p^2\left(\Sigma_{rr}^{(2)} + \Sigma_{\varphi\varphi}^{(2)}\right)\right\}\\ &+\frac{c_2 - c_1}{c_2}\left\{\partial_{\varphi\varphi}\Psi_{\varphi\varphi}^{(0)} - 2(p+1)\,\partial_\varphi\Psi_{r\varphi}^{(0)} + p\,(p+1)\,\Psi_{\varphi\varphi}^{(0)} - p\,\Psi_{rr}^{(0)}\right\} = 0,\end{aligned} \tag{105}$$

with boundary conditions

$$\left[\Sigma_{r\varphi}^{(2)}\right]_{\varphi=\pm\pi} = 0, \quad \left[\Sigma_{\varphi\varphi}^{(2)}\right]_{\varphi=\pm\pi} = 0. \tag{106}$$

It can be recognized that coupling between components of $\boldsymbol{\Sigma}$ and components of $\boldsymbol{\Psi}$ arises for the first time in the equations for $\Sigma_{\alpha\beta}^{(2)}$. Therefore, we shall focus attention only on the terms $\Sigma_{\alpha\beta}^{(0)}$ and $\Sigma_{\alpha\beta}^{(1)}$.

The solution of the systems of differential equations for $\Sigma_{\alpha\beta}^{(0)}$ and $\Sigma_{\alpha\beta}^{(1)}$, subjected to the restriction (82), can be established by well known methods (see, e. g., A) and turns out to be identical to the solution of the corresponding problems in classical elasticity. That means that the stress components $\Sigma_{\alpha\beta}$ are singular, with order of singularity $r^{-\frac{1}{2}}$, or equivalently,

$$p = \frac{1}{2}. \tag{107}$$

The coefficients of the singular terms, $\Sigma_{\alpha\beta}^{(0)}$, are given by

$$\Sigma_{\alpha\beta}^{(0)} = \frac{\tilde{K}_I}{\sqrt{2\pi}} f_{\alpha\beta}^{I}(\varphi) + \frac{\tilde{K}_{II}}{\sqrt{2\pi}} f_{\alpha\beta}^{II}(\varphi), \tag{108}$$

where the constants $\tilde{K}_I$ and $\tilde{K}_{II}$ are the stress intensity factors. Here and in the following, the indices I and II stand for Mode-I and Mode-II, respectively. Moreover, we use the notations $\tilde{K}_I$ and $\tilde{K}_{II}$, in order to distinguish the stress intensity factors of the microstrain continuum from the stress intensity factors K_I and K_{II} of classical continua.

The so-called angular functions $f_{\alpha\beta}^{I}$ and $f_{\alpha\beta}^{II}$ are defined through

$$\begin{pmatrix} f_{rr}^{I} \\ f_{\varphi\varphi}^{I} \\ f_{r\varphi}^{I} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5\cos\dfrac{\varphi}{2} - \cos\dfrac{3\varphi}{2} \\ 3\cos\dfrac{\varphi}{2} + \cos\dfrac{3\varphi}{2} \\ \sin\dfrac{\varphi}{2} + \sin\dfrac{3\varphi}{2} \end{pmatrix}, \quad \begin{pmatrix} f_{rr}^{II} \\ f_{\varphi\varphi}^{II} \\ f_{r\varphi}^{II} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -5\sin\dfrac{\varphi}{2} + 3\sin\dfrac{3\varphi}{2} \\ -3\sin\dfrac{\varphi}{2} - 3\sin\dfrac{3\varphi}{2} \\ \cos\dfrac{\varphi}{2} + 3\cos\dfrac{3\varphi}{2} \end{pmatrix}, \tag{109}$$

and are normalized by the conditions

$$\left[f_{\varphi\varphi}^{I}\right]_{\varphi=0} = 1, \quad \left[f_{r\varphi}^{II}\right]_{\varphi=0} = 1. \tag{110}$$

The constant terms $\Sigma_{\alpha\beta}^{(1)}$ are given by

$$\begin{pmatrix} \Sigma_{rr}^{(1)} \\ \Sigma_{\varphi\varphi}^{(1)} \\ \Sigma_{r\varphi}^{(1)} \end{pmatrix} = \tilde{k}_I \begin{pmatrix} \cos^2\varphi \\ \sin^2\varphi \\ -\dfrac{1}{2}\sin 2\varphi \end{pmatrix} \tag{111}$$

with $\tilde{k}_I$ being constant. Constant terms for Mode-II are not present. The first two terms of the asymptotic expansion of $\Sigma_{\alpha\beta}$ are summarized in Section 5.

3.6 Strain

Although the first two terms in the expansion of $\Sigma_{\alpha\beta}$ are identical to the ones of classical elasticity, the corresponding terms of $\varepsilon_{\alpha\beta}$ differ from those of classical elasticity. This follows from the fact that the elasticity laws (3)–(5) are not classical.

Evidently, the components $\varepsilon_{\alpha\beta}$ obey the asymptotic expansion

$$\varepsilon_{\alpha\beta} = r^{-\frac{1}{2}}\varepsilon_{\alpha\beta}^{(0)} + \varepsilon_{\alpha\beta}^{(1)} + \ \dots \, . \tag{112}$$

We use this and the asymptotic expansions of Section 3.1, with $p = \frac{1}{2}$, in the elasticity laws (3)–(5), and collect coefficients of like powers of r. Thus, we derive the following solutions for $\varepsilon_{\alpha\beta}^{(0)}$ and $\varepsilon_{\alpha\beta}^{(1)}$.

Terms $\varepsilon_{\alpha\beta}^{(0)}$

$$\varepsilon_{rr}^{(0)} = \frac{c_1}{2\mu c_2}\left[\Sigma_{rr}^{(0)} - \nu\left(\Sigma_{rr}^{(0)} + \Sigma_{\varphi\varphi}^{(0)}\right)\right], \tag{113}$$

$$\varepsilon_{\varphi\varphi}^{(0)} = \frac{c_1}{2\mu c_2}\left[\Sigma_{\varphi\varphi}^{(0)} - \nu\left(\Sigma_{rr}^{(0)} + \Sigma_{\varphi\varphi}^{(0)}\right)\right], \tag{114}$$

$$\varepsilon_{r\varphi}^{(0)} = \frac{c_1}{2\mu c_2}\,\Sigma_{r\varphi}^{(0)}. \tag{115}$$

By taking into account the solutions for $\Sigma_{\alpha\beta}^{(0)}$ of the last section, we find that

$$\begin{aligned}\varepsilon_{rr}^{(0)} &= \frac{c_1\tilde{K}_I}{8\mu c_2\sqrt{2\pi}}\left[(5-8\nu)\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2}\right] \\ &\quad + \frac{c_1\tilde{K}_{II}}{8\mu c_2\sqrt{2\pi}}\left[-(5-8\nu)\sin\frac{\varphi}{2} + 3\sin\frac{3\varphi}{2}\right],\end{aligned} \tag{116}$$

$$\begin{aligned}\varepsilon_{\varphi\varphi}^{(0)} &= \frac{c_1\tilde{K}_I}{8\mu c_2\sqrt{2\pi}}\left[(3-8\nu)\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2}\right] \\ &\quad + \frac{c_1\tilde{K}_{II}}{8\mu c_2\sqrt{2\pi}}\left[-(3-8\nu)\sin\frac{\varphi}{2} - 3\sin\frac{3\varphi}{2}\right],\end{aligned} \tag{117}$$

$$\varepsilon_{r\varphi}^{(0)} = \frac{c_1\tilde{K}_I}{8\mu c_2\sqrt{2\pi}}\left[\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right] + \frac{c_1\tilde{K}_{II}}{8\mu c_2\sqrt{2\pi}}\left[\cos\frac{\varphi}{2} + 3\cos\frac{3\varphi}{2}\right]. \tag{118}$$

Terms $\varepsilon_{\alpha\beta}^{(1)}$

$$\varepsilon_{rr}^{(1)} = \frac{c_1}{2\mu c_2}\left[\Sigma_{rr}^{(1)} - \nu\left(\Sigma_{rr}^{(1)} + \Sigma_{\varphi\varphi}^{(1)}\right)\right] + \frac{c_2 - c_1}{c_2}\,\overline{\Psi}_{rr}, \tag{119}$$

$$\varepsilon_{\varphi\varphi}^{(1)} = \frac{c_1}{2\mu c_2}\left[\Sigma_{\varphi\varphi}^{(1)} - \nu\left(\Sigma_{rr}^{(1)} + \Sigma_{\varphi\varphi}^{(1)}\right)\right] + \frac{c_2 - c_1}{c_2}\,\overline{\Psi}_{\varphi\varphi}, \tag{120}$$

$$\varepsilon_{r\varphi}^{(1)} = \frac{c_1}{2\mu c_2}\,\Sigma_{r\varphi}^{(1)} + \frac{c_2 - c_1}{c_2}\,\overline{\Psi}_{r\varphi}. \tag{121}$$

Now, we take into account the solutions for $\Sigma_{\alpha\beta}^{(1)}$, established in the last section, as well as the representations for $\overline{\Psi}_{\alpha\beta}$, given by Eqs. (75)–(79), to obtain

$$\varepsilon_{rr}^{(1)} = \tilde{k}_{I,1}^{\varepsilon} + \tilde{k}_{I,2}^{\varepsilon}\cos 2\varphi + \tilde{k}_{II}^{\varepsilon}\sin 2\varphi, \tag{122}$$

$$\varepsilon_{\varphi\varphi}^{(1)} = \tilde{k}_{I,1}^{\varepsilon} - \tilde{k}_{I,2}^{\varepsilon}\cos 2\varphi - \tilde{k}_{II}^{\varepsilon}\sin 2\varphi, \tag{123}$$

$$\varepsilon_{r\varphi}^{(1)} = -\tilde{k}_{I,2}^{\varepsilon}\sin 2\varphi + \tilde{k}_{II}^{\varepsilon}\cos 2\varphi. \tag{124}$$

The constants $\tilde{k}^{\varepsilon}_{I,1}$, $\tilde{k}^{\varepsilon}_{I,2}$ and $\tilde{k}^{\varepsilon}_{II}$ are defined as follows:

$$\tilde{k}^{\varepsilon}_{I,1} := \frac{c_1 \tilde{k}_I (1-2\nu)}{4\mu c_2} + \frac{(c_2 - c_1)\overline{L}_{I,1}}{c_2}, \tag{125}$$

$$\tilde{k}^{\varepsilon}_{I,2} := \frac{c_1 \tilde{k}_I}{4\mu c_2} + \frac{(c_2 - c_1)\overline{L}_{I,2}}{c_2}, \tag{126}$$

$$\tilde{k}^{\vec{\varepsilon}}_{II} := \frac{(c_2 - c_1)\overline{L}_{II}}{c_2}. \tag{127}$$

The first two terms of the asymptotic expansion of $\varepsilon_{\alpha\beta}$ are summarized in Section 5.

3.7 Macrodisplacements

The macrodisplacement components u_r and u_φ will be determined by integrating Eqs. (24). For plane strain elasticity, it is well known that the constants of integration represent rigid body motions. Omitting such motions, we conclude for the radial component u_r that

$$u_r = \int \varepsilon_{rr} \mathrm{d}r = \int \left(r^{-\frac{1}{2}} \varepsilon^{(0)}_{rr} + \varepsilon^{(1)}_{rr} + \ldots \right) \mathrm{d}r, \tag{128}$$

or

$$r^{\frac{1}{2}} u^{(0)}_r + r u^{(1)}_r + \ldots = 2 r^{\frac{1}{2}} \varepsilon^{(0)}_{rr} + r \varepsilon^{(1)}_{rr} + \ldots. \tag{129}$$

For the circumferential component u_φ, we conclude that

$$u_\varphi = \int \left(r \varepsilon_{\varphi\varphi} - u_r \right) \mathrm{d}\varphi, \tag{130}$$

or

$$r^{\frac{1}{2}} u^{(0)}_\varphi + r u^{(1)}_\varphi + \ldots = r^{\frac{1}{2}} \int \left(\varepsilon^{(0)}_{\varphi\varphi} - u^{(0)}_r \right) \mathrm{d}\varphi + r \int \left(\varepsilon^{(1)}_{\varphi\varphi} - u^{(1)}_r \right) \mathrm{d}\varphi + \ldots. \tag{131}$$

By employing steps similar to those in the last section, we get the following solutions for $u^{(0)}_\alpha$ and $u^{(1)}_\alpha$.

Terms $u^{(0)}_\alpha$

$$u^{(0)}_r = 2 \varepsilon^{(0)}_{rr}, \tag{132}$$

$$u^{(0)}_\varphi = \int \left(\varepsilon^{(0)}_{\varphi\varphi} - u^{(0)}_r \right) \mathrm{d}\varphi = \int \left(\varepsilon^{(0)}_{\varphi\varphi} - 2\varepsilon^{(0)}_{rr} \right) \mathrm{d}\varphi. \tag{133}$$

Invoking Eqs. (116) and (117), we get, after some straightforward manipulations,

$$u^{(0)}_r = \frac{c_1 \tilde{K}_I}{4\mu c_2 \sqrt{2\pi}} \left[(5 - 8\nu) \cos \frac{\varphi}{2} - \cos \frac{3\varphi}{2} \right]$$

$$+\frac{c_1\tilde{K}_{II}}{4\mu c_2\sqrt{2\pi}}\left[-(5-8\nu)\sin\frac{\varphi}{2}+3\sin\frac{3\varphi}{2}\right], \tag{134}$$

$$u_\varphi^{(0)}=\frac{c_1\tilde{K}_I}{4\mu c_2\sqrt{2\pi}}\left[-(7-8\nu)\sin\frac{\varphi}{2}+\sin\frac{3\varphi}{2}\right]$$
$$+\frac{c_1\tilde{K}_{II}}{4\mu c_2\sqrt{2\pi}}\left[-(7-8\nu)\cos\frac{\varphi}{2}+3\cos\frac{3\varphi}{2}\right]. \tag{135}$$

Terms $u_\alpha^{(1)}$

$$u_r^{(1)}=\varepsilon_{rr}^{(1)}, \tag{136}$$

$$u_\varphi^{(1)}=\int\left(\varepsilon_{\varphi\varphi}^{(1)}-u_r^{(1)}\right)\mathrm{d}\varphi=\int\left(\varepsilon_{\varphi\varphi}^{(1)}-\varepsilon_{rr}^{(1)}\right)\mathrm{d}\varphi, \tag{137}$$

from which, by virtue of Eqs. (119) and (120),

$$u_r^{(1)}=\tilde{k}_{I,1}^{\vec{\varepsilon}}+\tilde{k}_{I,2}^{\vec{\varepsilon}}\cos2\varphi+\tilde{k}_{II}^{\vec{\varepsilon}}\sin2\varphi, \tag{138}$$

$$u_\varphi^{(1)}=-\tilde{k}_{I,2}^{\vec{\varepsilon}}\sin2\varphi+\tilde{k}_{II}^{\vec{\varepsilon}}\cos2\varphi. \tag{139}$$

The first two terms of the asymptotic expansion of u_α are also summarized in Section 5.

3.8 Microdeformation

We shall derive differential equations for $\Psi_{\alpha\beta}^{(0)}$ and $\Psi_{\alpha\beta}^{(1)}$ by inserting the asymptotic expansions of $\Psi_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ (see Eqs. (68) and (80), with $p=\frac{1}{2}$) into Eqs. (36)–(38). Note that, by virtue of Eqs. (73) and (74) and since $\overline{\Psi}_{\alpha\beta}$ is independent of r, the identity

$$\partial_{rr}\overline{\Psi}_{rr}+\frac{1}{r^2}\partial_{\varphi\varphi}\overline{\Psi}_{rr}+\frac{1}{r}\partial_r\overline{\Psi}_{rr}-\frac{4}{r^2}\partial_\varphi\overline{\Psi}_{r\varphi}-\frac{2}{r^2}\overline{\Psi}_{rr}+\frac{2}{r^2}\overline{\Psi}_{\varphi\varphi}=0 \tag{140}$$

applies. Keeping this in mind and collecting terms of like powers of r, after some lengthy but otherwise straightforward manipulations, Eq. (36) yields

$$\begin{aligned}&r^{-\frac{3}{2}}\left\{\partial_{\varphi\varphi}\Psi_{rr}^{(0)}-4\partial_\varphi\Psi_{r\varphi}^{(0)}-\frac{7}{4}\Psi_{rr}^{(0)}+2\Psi_{\varphi\varphi}^{(0)}\right\}\\&+r^{-1}\left\{\partial_{\varphi\varphi}\Psi_{rr}^{(1)}-4\partial_\varphi\Psi_{r\varphi}^{(1)}-\Psi_{rr}^{(1)}+2\Psi_{\varphi\varphi}^{(1)}\right\}+\ldots=0.\end{aligned} \tag{141}$$

Similarly, from Eqs. (37) and (38), we get

$$\begin{aligned}&r^{-\frac{3}{2}}\left\{\partial_{\varphi\varphi}\Psi_{\varphi\varphi}^{(0)}+4\partial_\varphi\Psi_{r\varphi}^{(0)}-\frac{7}{4}\Psi_{\varphi\varphi}^{(0)}+2\Psi_{rr}^{(0)}\right\}\\&+r^{-1}\left\{\partial_{\varphi\varphi}\Psi_{\varphi\varphi}^{(1)}+4\partial_\varphi\Psi_{r\varphi}^{(1)}-\Psi_{\varphi\varphi}^{(1)}+2\Psi_{rr}^{(1)}\right\}+\ldots=0,\end{aligned} \tag{142}$$

$$\begin{aligned}&r^{-\frac{3}{2}}\left\{\partial_{\varphi\varphi}\Psi_{r\varphi}^{(0)}+2\partial_\varphi\left[\Psi_{rr}^{(0)}-\Psi_{\varphi\varphi}^{(0)}\right]-\frac{15}{4}\Psi_{r\varphi}^{(0)}\right\}\\&+r^{-1}\left\{\partial_{\varphi\varphi}\Psi_{r\varphi}^{(1)}+2\partial_\varphi\left[\Psi_{rr}^{(0)}-\Psi_{\varphi\varphi}^{(0)}\right]-\frac{15}{4}\Psi_{r\varphi}^{(0)}\right\}+\ldots=0.\end{aligned} \tag{143}$$

It is worth remarking that if only terms up to order r^{-1} are retained in Eqs. (141)–(143), then the terms $\Psi^{(0)}_{\alpha\beta}$ and $\Psi^{(1)}_{\alpha\beta}$ are uncoupled from the terms $\overline{\Psi}_{\alpha\beta}$ and $\Sigma^{(k)}_{\alpha\beta}$.

In an analogous manner, by substituting the asymptotic expansion (68) into the nonclassical boundary conditions (48)–(50), we show that

$$r^{\frac{1}{2}}\left[\partial_{\varphi}\Psi^{(0)}_{rr}-2\Psi^{(0)}_{r\varphi}\right]_{\varphi=\pm\pi}+r\left[\partial_{\varphi}\Psi^{(1)}_{rr}-2\Psi^{(1)}_{r\varphi}\right]_{\varphi=\pm\pi}+\ldots=0, \tag{144}$$

$$r^{\frac{1}{2}}\left[\partial_{\varphi}\Psi^{(0)}_{\varphi\varphi}+2\Psi^{(0)}_{r\varphi}\right]_{\varphi=\pm\pi}+r\left[\partial_{\varphi}\Psi^{(1)}_{\varphi\varphi}+2\Psi^{(1)}_{r\varphi}\right]_{\varphi=\pm\pi}+\ldots=0, \tag{145}$$

$$r^{\frac{1}{2}}\left[\partial_{\varphi}\Psi^{(0)}_{r\varphi}+\Psi^{(0)}_{rr}-\Psi^{(0)}_{\varphi\varphi}\right]_{\varphi=\pm\pi}+r\left[\partial_{\varphi}\Psi^{(1)}_{r\varphi}+\Psi^{(1)}_{rr}-\Psi^{(1)}_{\varphi\varphi}\right]_{\varphi=\pm\pi}+\ldots=0. \tag{146}$$

3.8.1 Differential equations for $\Psi^{(0)}_{\alpha\beta}$

Equating to zero the coefficients of power $r^{-\frac{3}{2}}$ in Eqs. (141)–(143) leads to the system of ordinary differential equations

$$\partial_{\varphi\varphi}\Psi^{(0)}_{rr}-4\partial_{\varphi}\Psi^{(0)}_{r\varphi}-\frac{7}{4}\Psi^{(0)}_{rr}+2\Psi^{(0)}_{\varphi\varphi}=0, \tag{147}$$

$$\partial_{\varphi\varphi}\Psi^{(0)}_{\varphi\varphi}+4\partial_{\varphi}\Psi^{(0)}_{r\varphi}-\frac{7}{4}\Psi^{(0)}_{\varphi\varphi}+2\Psi^{(0)}_{rr}=0, \tag{148}$$

$$\partial_{\varphi\varphi}\Psi^{(0)}_{r\varphi}+2\partial_{\varphi}\left[\Psi^{(0)}_{rr}-\Psi^{(0)}_{\varphi\varphi}\right]-\frac{15}{4}\Psi^{(0)}_{r\varphi}=0. \tag{149}$$

Similarly, equating to zero the coefficients of power $r^{\frac{1}{2}}$ in the boundary conditions (144)–(146) leads to

$$\left[\partial_{\varphi}\Psi^{(0)}_{rr}-2\Psi^{(0)}_{r\varphi}\right]_{\varphi=\pm\pi}=0, \tag{150}$$

$$\left[\partial_{\varphi}\Psi^{(0)}_{\varphi\varphi}+2\Psi^{(0)}_{r\varphi}\right]_{\varphi=\pm\pi}=0, \tag{151}$$

$$\left[\partial_{\varphi}\Psi^{(0)}_{r\varphi}+\Psi^{(0)}_{rr}-\Psi^{(0)}_{\varphi\varphi}\right]_{\varphi=\pm\pi}=0. \tag{152}$$

Proceeding to solve the system (147)–(149), we note that Eqs. (147) and (148) imply the ordinary differential equation

$$\partial_{\varphi\varphi}\left[\Psi^{(0)}_{rr}+\Psi^{(0)}_{\varphi\varphi}\right]+\frac{1}{4}\left[\Psi^{(0)}_{rr}+\Psi^{(0)}_{\varphi\varphi}\right]=0 \tag{153}$$

for the sum $\Psi^{(0)}_{rr}+\Psi^{(0)}_{\varphi\varphi}$, which has the solution

$$\Psi^{(0)}_{rr}+\Psi^{(0)}_{\varphi\varphi}=A^{(0)}\cos\frac{\varphi}{2}+B^{(0)}\sin\frac{\varphi}{2}. \tag{154}$$

For determining the constants of integration $A^{(0)}$ and $B^{(0)}$, we utilize the boundary conditions. From Eqs. (150) and (151), we derive the equation

$$\left[\partial_{\varphi}\left(\Psi^{(0)}_{rr}+\Psi^{(0)}_{\varphi\varphi}\right)\right]_{\varphi=\pm\pi}=0. \tag{155}$$

By substituting the solution (154), we see that

$$A^{(0)} = 0. \tag{156}$$

To go further, we notice that Eqs. (147) and (148) imply

$$\partial_\varphi \Psi^{(0)}_{r\varphi} = \frac{1}{8}\left[\partial_{\varphi\varphi}\left(\Psi^{(0)}_{rr} - \Psi^{(0)}_{\varphi\varphi}\right) - \frac{15}{4}\left(\Psi^{(0)}_{rr} - \Psi^{(0)}_{\varphi\varphi}\right)\right]. \tag{157}$$

Next, we differentiate Eq. (149) with respect to φ and use Eq. (157). Rearrangement of terms leads to the ordinary differential equation

$$\begin{aligned}&\frac{1}{2}\,\partial_{\varphi\varphi\varphi\varphi}\left(\Psi^{(0)}_{rr} + \Psi^{(0)}_{\varphi\varphi}\right) + \frac{17}{4}\,\partial_{\varphi\varphi}\left(\Psi^{(0)}_{rr} + \Psi^{(0)}_{\varphi\varphi}\right) + \frac{225}{32}\left(\Psi^{(0)}_{rr} + \Psi^{(0)}_{\varphi\varphi}\right)\\ &-\partial_{\varphi\varphi\varphi\varphi}\Psi^{(0)}_{\varphi\varphi} - \frac{17}{2}\,\partial_{\varphi\varphi}\Psi^{(0)}_{\varphi\varphi} - \frac{225}{16}\,\Psi^{(0)}_{\varphi\varphi} = 0.\end{aligned} \tag{158}$$

By substituting the solutions (154) and (156), we gain an ordinary differential equation for $\Psi^{(0)}_{\varphi\varphi}$,

$$\partial_{\varphi\varphi\varphi\varphi}\Psi^{(0)}_{\varphi\varphi} + \frac{17}{2}\,\partial_{\varphi\varphi}\Psi^{(0)}_{\varphi\varphi} + \frac{225}{16}\,\Psi^{(0)}_{\varphi\varphi} = 6B^{(0)}\sin\frac{\varphi}{2}, \tag{159}$$

which obeyes the solution

$$\begin{aligned}\Psi^{(0)}_{\varphi\varphi} = {}&\frac{1}{2}B^{(0)}\sin\frac{\varphi}{2} + E^{(0)}\sin\frac{3\varphi}{2} + F^{(0)}\sin\frac{5\varphi}{2} + C^{(0)}\cos\frac{3\varphi}{2}\\ &+D^{(0)}\cos\frac{5\varphi}{2},\end{aligned} \tag{160}$$

with $C^{(0)}, D^{(0)}, E^{(0)}$ and $F^{(0)}$ being new constants of integration. Further, from Eqs. (154), (156) and (160),

$$\begin{aligned}\Psi^{(0)}_{rr} = {}&\frac{1}{2}B^{(0)}\sin\frac{\varphi}{2} - E^{(0)}\sin\frac{3\varphi}{2} - F^{(0)}\sin\frac{5\varphi}{2} - C^{(0)}\cos\frac{3\varphi}{2}\\ &-D^{(0)}\cos\frac{5\varphi}{2}.\end{aligned} \tag{161}$$

Finally, using the solutions (161) and (160) in Eq. (157), we obtain the solution $\Psi^{(0)}_{r\varphi}$ in the form

$$\Psi^{(0)}_{r\varphi} = C^{(0)}\sin\frac{3\varphi}{2} + D^{(0)}\sin\frac{5\varphi}{2} - E^{(0)}\cos\frac{3\varphi}{2} - F^{(0)}\cos\frac{5\varphi}{2} + G^{(0)}, \tag{162}$$

where $G^{(0)}$ is a further constant of integration. For the constants of integration in the solutions (160)–(162) we can verify, by evaluating the boundary conditions (150)–(152) that

$$G^{(0)} = 0, \quad -D^{(0)} = C^{(0)}, \quad -F^{(0)} = E^{(0)}. \tag{163}$$

In accordance with the symmetry conditions (53) and (54) for Mode-I as well as (61) and (62) for Mode-II, we set

$$C^{(0)} \equiv C_I^{(0)}, \quad B^{(0)} \equiv B_{II}^{(0)}, \quad E^{(0)} \equiv E_{II}^{(0)}. \tag{164}$$

Then, the solutions (160)–(162) become

$$\Psi_{rr}^{(0)} = -C_I^{(0)}\left(\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2}\right) + \frac{1}{2}B_{II}^{(0)}\sin\frac{\varphi}{2} - E_{II}^{(0)}\left(\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2}\right), \tag{165}$$

$$\Psi_{\varphi\varphi}^{(0)} = C_I^{(0)}\left(\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2}\right) + \frac{1}{2}B_{II}^{(0)}\sin\frac{\varphi}{2} + E_{II}^{(0)}\left(\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2}\right), \tag{166}$$

$$\Psi_{r\varphi}^{(0)} = C_I^{(0)}\left(\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2}\right) - E_{II}^{(0)}\left(\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2}\right). \tag{167}$$

It is of interest to comment the following issue. Obviously not all constants of integration may be determined, because boundary conditions are prescribed only on the crack faces. Nevertheless, it is remarkable that the solutions of Mode-I include only one unknown constant, whereas the solutions of Mode-II depend on two unknown constants. We shall come back to this specific feature in the next section as well as in Section 5, while discussing the asymptotic solutions of the double stresses.

3.8.2 Differential equations for $\Psi_{\alpha\beta}^{(1)}$

Equating to zero the coefficients of power r^{-1} in Eqs. (141)–(143) and the coefficients of power r in the boundary conditions (144)–(146) furnish the system of ordinary differential equations

$$\partial_{\varphi\varphi}\Psi_{rr}^{(1)} - 4\partial_\varphi\Psi_{r\varphi}^{(1)} - \Psi_{rr}^{(1)} + 2\Psi_{\varphi\varphi}^{(1)} = 0, \tag{168}$$

$$\partial_{\varphi\varphi}\Psi_{\varphi\varphi}^{(1)} + 4\partial_\varphi\Psi_{r\varphi}^{(1)} - \Psi_{\varphi\varphi}^{(1)} + 2\Psi_{rr}^{(1)} = 0, \tag{169}$$

$$\partial_{\varphi\varphi}\Psi_{r\varphi}^{(1)} + 2\partial_\varphi\left(\Psi_{rr}^{(1)} - \Psi_{\varphi\varphi}^{(1)}\right) - 3\Psi_{r\varphi}^{(1)} = 0, \tag{170}$$

and corresponding boundary conditions

$$\left[\partial_\varphi\Psi_{rr}^{(1)} - 2\Psi_{r\varphi}^{(1)}\right]_{\varphi=\pm\pi} = 0, \tag{171}$$

$$\left[\partial_\varphi\Psi_{\varphi\varphi}^{(1)} + 2\Psi_{r\varphi}^{(1)}\right]_{\varphi=\pm\pi} = 0, \tag{172}$$

$$\left[\partial_\varphi\Psi_{r\varphi}^{(1)} + \Psi_{rr}^{(1)} - \Psi_{\varphi\varphi}^{(1)}\right]_{\varphi=\pm\pi} = 0. \tag{173}$$

Since the steps for solving the above system of differential equations are quite similar to those in the last section, we omit the details and present only the final solutions

$$\Psi_{rr}^{(1)} = \frac{1}{2}A^{(1)}\cos\varphi - D_I^{(1)}(\cos\varphi + \cos 3\varphi) - E_{II}^{(1)}\sin\varphi - F_{II}^{(1)}\sin 3\varphi, \tag{174}$$

$$\Psi_{\varphi\varphi}^{(1)} = \frac{1}{2}A^{(1)}\cos\varphi + D_I^{(1)}(\cos\varphi + \cos 3\varphi) + E_{II}^{(1)}\sin\varphi + F_{II}^{(1)}\sin 3\varphi, \tag{175}$$

$$\Psi_{r\varphi}^{(1)} = D_I^{(1)}\left(\sin\varphi + \sin 3\varphi\right) - E_{II}^{(1)}\left(\frac{1}{2} + \cos\varphi\right) - F_{II}^{(1)}\left(\frac{1}{2} - \cos 3\varphi\right). \tag{176}$$

With regard to the symmetry conditions (53), (54), (61) and (62), the constants $A_I^{(1)}, D_I^{(1)}, E_{II}^{(1)}$ and $F_{II}^{(1)}$ are attributed to loading conditions of Mode-I and Mode-II, respectively. The solutions $\Psi_{\alpha\beta}^{(0)}$ and $\Psi_{\alpha\beta}^{(1)}$ are summarized and discussed in Section 5.

3.9 Double stress

The considerations of Section 3.1, together with $p = \frac{1}{2}$ (see Eq. (107)), and the elasticity laws for $\boldsymbol{\mu}$ [see Eqs. (7)–(16)] suggest the asymptotic expansion

$$\mu_{\alpha\beta\gamma} = r^{-\frac{1}{2}}\mu_{\alpha\beta\gamma}^{(0)} + \mu_{\alpha\beta\gamma}^{(1)} + \ldots = \sum_{k=0}^{\infty} r^{-\frac{1}{2}+\frac{k}{2}}\mu_{\alpha\beta\gamma}^{(k)}, \tag{177}$$

with

$$\mu_{\alpha\beta\gamma}^{(k)} = \mu_{\alpha\beta\gamma}^{(k)}(\varphi). \tag{178}$$

The goal is to determine $\mu_{\alpha\beta\gamma}^{(0)}$ and $\mu_{\alpha\beta\gamma}^{(1)}$ by substituting the asymptotic expansion for $\Psi_{\alpha\beta}$ into the elasticity laws (7)–(16). It is readily verified that in view of the conditions (73) and (74), the terms $\overline{\Psi}_{\alpha\beta}$ of the expansion (68) will disappear in the subsequent equations. Thus, we conclude from Eqs. (7)–(16), by equating the coefficients of power $r^{-\frac{1}{2}}$ that

$$\mu_{rrr}^{(0)} = (c_2 - c_1)\left(\frac{\lambda + 2\mu}{2}\Psi_{rr}^{(0)} + \frac{\lambda}{2}\Psi_{\varphi\varphi}^{(0)}\right), \tag{179}$$

$$\mu_{r\varphi\varphi}^{(0)} = (c_2 - c_1)\left(\frac{\lambda + 2\mu}{2}\Psi_{\varphi\varphi}^{(0)} + \frac{\lambda}{2}\Psi_{rr}^{(0)}\right), \tag{180}$$

$$\mu_{rzz}^{(0)} = (c_2 - c_1)\frac{\lambda}{2}\left(\Psi_{rr}^{(0)} + \Psi_{\varphi\varphi}^{(0)}\right), \tag{181}$$

$$\mu_{rr\varphi}^{(0)} = (c_2 - c_1)\mu\Psi_{r\varphi}^{(0)}, \tag{182}$$

$$\mu_{\varphi rr}^{(0)} = (c_2 - c_1)\left[2\mu\left(\partial_\varphi\Psi_{rr}^{(0)} - 2\Psi_{r\varphi}^{(0)}\right) + \lambda\partial_\varphi\left(\Psi_{rr}^{(0)} + \Psi_{\varphi\varphi}^{(0)}\right)\right], \tag{183}$$

$$\mu_{\varphi\varphi\varphi}^{(0)} = (c_2 - c_1)\left[2\mu\left(\partial_\varphi\Psi_{\varphi\varphi}^{(0)} + 2\Psi_{r\varphi}^{(0)}\right) + \lambda\partial_\varphi\left(\Psi_{rr}^{(0)} + \Psi_{\varphi\varphi}^{(0)}\right)\right], \tag{184}$$

$$\mu_{\varphi zz}^{(0)} = (c_2 - c_1)\lambda\partial_\varphi\left(\Psi_{rr}^{(0)} + \Psi_{\varphi\varphi}^{(0)}\right), \tag{185}$$

$$\mu_{\varphi r\varphi}^{(0)} = (c_2 - c_1)2\mu\left(\partial_\varphi\Psi_{r\varphi}^{(0)} + \Psi_{rr}^{(0)} - \Psi_{\varphi\varphi}^{(0)}\right), \tag{186}$$

and by equating the coefficients of power r^0 that

$$\mu_{rrr}^{(1)} = (c_2 - c_1)\left((\lambda + 2\mu)\Psi_{rr}^{(1)} + \lambda\Psi_{\varphi\varphi}^{(1)}\right), \tag{187}$$

$$\mu_{r\varphi\varphi}^{(1)} = (c_2 - c_1)\left((\lambda + 2\mu)\Psi_{\varphi\varphi}^{(1)} + \lambda\Psi_{rr}^{(1)}\right), \tag{188}$$

$$\mu_{rzz}^{(1)} = (c_2 - c_1)\lambda\left(\Psi_{rr}^{(1)} + \Psi_{\varphi\varphi}^{(1)}\right), \tag{189}$$

$$\mu^{(1)}_{r\varphi\varphi} = (c_2 - c_1)\, 2\mu \Psi^{(1)}_{r\varphi}, \tag{190}$$

$$\mu^{(1)}_{\varphi rr} = (c_2 - c_1)\left[2\mu\left(\partial_\varphi \Psi^{(1)}_{rr} - 2\Psi^{(1)}_{r\varphi}\right) + \lambda \partial_\varphi\left(\Psi^{(1)}_{rr} + \Psi^{(1)}_{\varphi\varphi}\right)\right], \tag{191}$$

$$\mu^{(1)}_{\varphi\varphi\varphi} = (c_2 - c_1)\left[2\mu\left(\partial_\varphi \Psi^{(1)}_{\varphi\varphi} + 2\Psi^{(1)}_{r\varphi}\right) + \lambda \partial_\varphi\left(\Psi^{(1)}_{rr} + \Psi^{(1)}_{\varphi\varphi}\right)\right], \tag{192}$$

$$\mu^{(1)}_{\varphi zz} = (c_2 - c_1)\, \lambda \partial_\varphi\left(\Psi^{(1)}_{rr} + \Psi^{(1)}_{\varphi\varphi}\right), \tag{193}$$

$$\mu^{(1)}_{r\varphi\varphi} = (c_2 - c_1)\, 2\mu\left(\partial_\varphi \Psi^{(1)}_{r\varphi} + \Psi^{(1)}_{rr} - \Psi^{(1)}_{\varphi\varphi}\right). \tag{194}$$

If we introduce the solutions (160)–(162) into Eqs. (179)–(186) and rearrange terms, then, for $\mu^{(0)}_{\alpha\beta\gamma}$, we obtain the representations

$$\begin{aligned}\mu^{(0)}_{rrr} = (c_2 - c_1)\Bigg[&-\mu C^{(0)}_I\left(\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2}\right) + \frac{\lambda+\mu}{2} B^{(0)}_{II} \sin\frac{\varphi}{2}\\ &-\mu E^{(0)}_{II}\left(\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2}\right)\Bigg],\end{aligned} \tag{195}$$

$$\begin{aligned}\mu^{(0)}_{r\varphi\varphi} = (c_2 - c_1)\Bigg[&\mu C^{(0)}_I\left(\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2}\right) + \frac{\lambda+\mu}{2} B^{(0)}_{II} \sin\frac{\varphi}{2}\\ &+\mu E^{(0)}_{II}\left(\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2}\right)\Bigg],\end{aligned} \tag{196}$$

$$\mu^{(0)}_{rzz} = (c_2 - c_1)\frac{\lambda}{2} B^{(0)}_{II} \sin\frac{\varphi}{2}, \tag{197}$$

$$\begin{aligned}\mu^{(0)}_{rr\varphi} = (c_2 - c_1)\Bigg[&\mu C^{(0)}_I\left(\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2}\right)\\ &-\mu E^{(0)}_{II}\left(\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2}\right)\Bigg],\end{aligned} \tag{198}$$

$$\begin{aligned}\mu^{(0)}_{\varphi rr} = (c_2 - c_1)\Bigg[&-\mu C^{(0)}_I\left(\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2}\right) + \frac{\lambda+\mu}{2} B^{(0)}_{II} \cos\frac{\varphi}{2}\\ &+\mu E^{(0)}_{II}\left(\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2}\right)\Bigg],\end{aligned} \tag{199}$$

$$\begin{aligned}\mu^{(0)}_{\varphi\varphi\varphi} = (c_2 - c_1)\Bigg[&\mu C^{(0)}_I\left(\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2}\right) + \frac{\lambda+\mu}{2} B^{(0)}_{II} \cos\frac{\varphi}{2}\\ &-\mu E^{(0)}_{II}\left(\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2}\right)\Bigg],\end{aligned} \tag{200}$$

$$\mu^{(0)}_{\varphi zz} = (c_2 - c_1)\frac{\lambda}{2} B^{(0)}_{II} \cos\frac{\varphi}{2}, \tag{201}$$

$$\mu^{(0)}_{\varphi r\varphi} = (c_2 - c_1)\left[-\mu C^{(0)}_I\left(\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2}\right) - \mu E^{(0)}_{II}\left(\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2}\right)\right]. \tag{202}$$

The fact that the solutions $\mu^{(0)}_{\alpha\beta\gamma}$ depend on two unknown constants in case of Mode-II is a characteristic property. As we shall see in Section 5, this feature leads to the existence of two stress intensity factors for the double stresses in case of Mode-II.

Using steps similar to those above we obtain for $\mu^{(1)}_{\alpha\beta\gamma}$ the representations

$$\mu^{(1)}_{rrr} = (c_2 - c_1)\Big[(\lambda+\mu)A^{(1)}_I \cos\varphi - 2\mu D^{(1)}_I(\cos\varphi + \cos 3\varphi) \\ -2\mu E^{(1)}_{II} \sin\varphi - 2\mu F^{(1)}_{II} \sin 3\varphi\Big], \tag{203}$$

$$\mu^{(1)}_{r\varphi\varphi} = (c_2 - c_1)\Big[(\lambda+\mu)A^{(1)}_I \cos\varphi + 2\mu D^{(1)}_I(\cos\varphi + \cos 3\varphi) \\ +2\mu E^{(1)}_{II} \sin\varphi + 2\mu F^{(1)}_{II} \sin 3\varphi\Big], \tag{204}$$

$$\mu^{(1)}_{\varphi zz} = (c_2 - c_1)A^{(1)}_I \cos\varphi, \tag{205}$$

$$\mu^{(1)}_{rr\varphi} = (c_2 - c_1)\Big[2\mu D^{(1)}_I(\sin\varphi + \sin 3\varphi) \\ - 2\mu E^{(1)}_{II}\left(\frac{1}{2} + \cos\varphi\right) - 2\mu F^{(1)}_{II}\left(\frac{1}{2} - \cos 3\varphi\right)\Big], \tag{206}$$

$$\mu^{(1)}_{\varphi rr} = (c_2 - c_1)\Big[-(\lambda+\mu)A^{(1)}_I \sin\varphi - 2\mu D^{(1)}_I(\sin\varphi - \sin 3\varphi) \\ +2\mu E^{(1)}_{II}(1 + \cos\varphi) - 2\mu F^{(1)}_{II}(1 + \cos 3\varphi)\Big], \tag{207}$$

$$\mu^{(1)}_{\varphi\varphi\varphi} = (c_2 - c_1)\Big[-(\lambda+\mu)A^{(1)}_I \sin\varphi + 2\mu D^{(1)}_I(\sin\varphi - \sin 3\varphi) \\ -2\mu E^{(1)}_{II}(1 + \cos\varphi) + 2\mu F^{(1)}_{II}(1 + \cos 3\varphi)\Big], \tag{208}$$

$$\mu^{(1)}_{\varphi zz} = (c_2 - c_1)\lambda A^{(1)}_I \sin\varphi, \tag{209}$$

$$\mu^{(1)}_{\varphi r\varphi} = (c_2 - c_1)\Big[-2\mu D^{(1)}_I(\cos\varphi - \cos 3\varphi) \\ -2\mu E^{(1)}_{II} \sin\varphi + 2\mu F^{(1)}_{II} \sin 3\varphi\Big]. \tag{210}$$

Before going to discuss the obtained solutions, it is perhaps of interest to rederive the analytical solutions by an alternative approach, starting from asymptotic expansions of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ rather than the asymtptotic expansions of $\boldsymbol{u}$ and $\boldsymbol{\Psi}$ used in this section.

4. Alternative approach for the determination of the near-tip fields

In Section 3 we determined the near-tip fields by starting from asymptotic expansions of the same form for the kinematical variables $\boldsymbol{u}$ and $\boldsymbol{\Psi}$ [see Eqs. (67) and (68)]. Alternatively, it is instructive to start from asymptotic expansions of the same type for the stresses $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$, i. e.,

$$\Sigma_{\alpha\beta} = r^{p-1}\Sigma^{(0)}_{\alpha\beta} + r^{p-\frac{1}{2}}\Sigma^{(1)}_{\alpha\beta} + \dots, \tag{211}$$

$$\mu_{\alpha\beta\gamma} = r^{p-1}\mu^{(0)}_{\alpha\beta\gamma} + r^{p-\frac{1}{2}}\mu^{(1)}_{\alpha\beta\gamma} + \dots, \tag{212}$$

where $\Sigma^{(k)}_{\alpha\beta} = \Sigma^{(k)}_{\alpha\beta}(\varphi)$ and $\mu^{(k)}_{\alpha\beta\gamma} = \mu^{(k)}_{\alpha\beta\gamma}(\varphi)$, $k = 0, 1, 2, \dots$. Then, from the elasticity laws (17)–(22), we recognize that $(\nabla\boldsymbol{\Psi})_{\alpha\beta\gamma} \sim r^{p-1}$ and hence the components $\Psi_{\alpha\beta}$ are of form (67). It follows that all outcomes of sections 3.2–3.6 apply as well and, in particular, that $p = \frac{1}{2}$. Then, it remains to show, how to determine the terms $\mu^{(0)}_{\alpha\beta\gamma}$ and $\mu^{(1)}_{\alpha\beta\gamma}$. The corresponding terms of $\mathbf{\Psi}$ will then be established by integrating the elasticity laws (17)–(22). For the purposes of the present section, however, it suffices to demonstrate only how to determine the terms $\mu^{(0)}_{\alpha\beta\gamma}$. To this end, we shall involve the nonclassical equilibrium Eqs. (32)–(35), in conjunction with the elasticity law (6) for $\boldsymbol{\sigma}$, as well as the nonclassical compatibility conditions (42)–(44). It is necessary to involve the latter for we are directly seeking for solutions of $\mu_{\alpha\beta\gamma}$.

4.1 Nonclassical equilibrium equations

Since $\varepsilon_{\alpha\beta} \sim r^{-\frac{1}{2}}$ and $\Psi_{\alpha\beta} \sim r^0$, we recognize from the elasticity law (6) that $\sigma_{\alpha\beta} \sim r^{-\frac{1}{2}}$. On the other hand, by virtue of the expansion (212), $\partial_r \mu_{\alpha\beta\gamma} \sim r^{-\frac{3}{2}}$ and $\frac{1}{r}\mu_{\alpha\beta\gamma} \sim r^{-\frac{3}{2}}$. Therefore, up to terms of order r^{-1} there will be no contributions of $\boldsymbol{\sigma}$ present in the nonclassical equilibrium Eqs. (32)–(35) and we conclude that

$$r^{-\frac{3}{2}}\left\{-\frac{1}{2}\mu^{(0)}_{rrr} + \partial_\varphi \mu^{(0)}_{\varphi rr} + \mu^{(0)}_{rrr} - 2\mu^{(0)}_{\varphi r\varphi}\right\} + \ldots = 0, \tag{213}$$

$$r^{-\frac{3}{2}}\left\{-\frac{1}{2}\mu^{(0)}_{r\varphi\varphi} + \partial_\varphi \mu^{(0)}_{\varphi\varphi\varphi} + \mu^{(0)}_{r\varphi\varphi} + 2\mu^{(0)}_{\varphi r\varphi}\right\} + \ldots = 0, \tag{214}$$

$$r^{-\frac{3}{2}}\left\{-\frac{1}{2}\mu^{(0)}_{rzz} + \partial_\varphi \mu^{(0)}_{\varphi zz} + \mu^{(0)}_{rzz}\right\} + \ldots = 0, \tag{215}$$

$$r^{-\frac{3}{2}}\left\{-\frac{1}{2}\mu^{(0)}_{rr\varphi} + \partial_\varphi \mu^{(0)}_{\varphi r\varphi} + \mu^{(0)}_{rr\varphi} - \mu^{(0)}_{\varphi\varphi\varphi} + \mu^{(0)}_{\varphi rr}\right\} + \ldots = 0. \tag{216}$$

Equating to zero the coefficients of power $r^{-\frac{3}{2}}$ leads to

$$2\partial_\varphi \mu^{(0)}_{\varphi rr} + \mu^{(0)}_{rrr} - 4\mu^{(0)}_{\varphi r\varphi} = 0, \tag{217}$$

$$2\partial_\varphi \mu^{(0)}_{\varphi\varphi\varphi} + \mu^{(0)}_{r\varphi\varphi} + 4\mu^{(0)}_{\varphi r\varphi} = 0, \tag{218}$$

$$2\partial_\varphi \mu^{(0)}_{\varphi r\varphi} + \mu^{(0)}_{rr\varphi} - 2\mu^{(0)}_{\varphi\varphi\varphi} + 2\mu^{(0)}_{\varphi rr} = 0, \tag{219}$$

and

$$2\partial_\varphi \mu^{(0)}_{\varphi zz} + \mu^{(0)}_{rzz} = 0. \tag{220}$$

The last equation will not be considered further, for it can be established from Eqs. (217) and (218). To see this, we recall Eqs. (9) and (13) to recast Eq. (220) equivalently in the form

$$2\partial_\varphi \mu^{(0)}_{\varphi rr} + 2\partial_\varphi \mu^{(0)}_{\varphi\varphi\varphi} + \mu^{(0)}_{rrr} + \mu^{(0)}_{r\varphi\varphi} = 0. \tag{221}$$

But this equations can also be obtained by adding up Eqs. (217) and (218).

4.2 Nonclassical compatibility conditions

We insert the asymptotic expansion (212) into the nonclassical compatibility conditions (42)–(44) and collect terms of like powers of r, to get

$$r^{-\frac{1}{2}}\left\{\partial_\varphi\mu^{(0)}_{rr\varphi}-\mu^{(0)}_{\varphi r\varphi}+\frac{1}{2}\mu^{(0)}_{\varphi r\varphi}+\mu^{(0)}_{rrr}-\mu^{(0)}_{r\varphi\varphi}\right\}+\ldots=0, \tag{222}$$

$$r^{-\frac{1}{2}}\left\{\partial_\varphi\mu^{(0)}_{r\varphi\varphi}+\partial_\varphi\mu^{(0)}_{rrr}-\mu^{(0)}_{\varphi\varphi\varphi}-\mu^{(0)}_{\varphi rr}+\frac{1}{2}\mu^{(0)}_{\varphi\varphi\varphi}+\frac{1}{2}\mu^{(0)}_{\varphi rr}\right\}+\ldots=0, \tag{223}$$

$$r^{-\frac{1}{2}}\left\{\partial_\varphi\mu^{(0)}_{r\varphi\varphi}-\partial_\varphi\mu^{(0)}_{rrr}-\mu^{(0)}_{\varphi\varphi\varphi}+\mu^{(0)}_{\varphi rr}+\frac{1}{2}\mu^{(0)}_{\varphi\varphi\varphi}-\frac{1}{2}\mu^{(0)}_{\varphi rr}+4\mu^{(0)}_{rr\varphi}\right\}+\ldots=0. \tag{224}$$

Again, equating to zero the coefficients of power $r^{-\frac{1}{2}}$ leads to

$$\partial_\varphi\mu^{(0)}_{rr\varphi}-\frac{1}{2}\mu^{(0)}_{\varphi r\varphi}+\mu^{(0)}_{rrr}-\mu^{(0)}_{r\varphi\varphi}=0, \tag{225}$$

$$\partial_\varphi\mu^{(0)}_{r\varphi\varphi}+\partial_\varphi\mu^{(0)}_{rrr}-\frac{1}{2}\mu^{(0)}_{\varphi\varphi\varphi}-\frac{1}{2}\mu^{(0)}_{\varphi rr}=0, \tag{226}$$

$$\partial_\varphi\mu^{(0)}_{r\varphi\varphi}-\partial_\varphi\mu^{(0)}_{rrr}-\frac{1}{2}\mu^{(0)}_{\varphi\varphi\varphi}+\frac{1}{2}\mu^{(0)}_{\varphi rr}+4\mu^{(0)}_{rr\varphi}=0. \tag{227}$$

4.3 Determination of $\mu^{(0)}_{\alpha\beta\gamma}$

Eqs. (217)–(219) and (225)–(227) are 6 differential equations for the 6 unknowns $\mu^{(0)}_{rrr}, \mu^{(0)}_{r\varphi\varphi}, \mu^{(0)}_{rr\varphi}, \mu^{(0)}_{\varphi rr}, \mu^{(0)}_{\varphi\varphi\varphi}$ and $\mu^{(0)}_{\varphi r\varphi}$. The required boundary conditions can be verified to be [cf. Eq. (47)].

$$\left[\mu^{(0)}_{\varphi rr}\right]_{\varphi=\pm\pi}=\left[\mu^{(0)}_{\varphi\varphi\varphi}\right]_{\varphi=\pm\pi}=\left[\mu^{(0)}_{\varphi r\varphi}\right]_{\varphi=\pm\pi}=0. \tag{228}$$

It can be shown (cf. A) that the solutions are given by

$$\mu^{(0)}_{rrr}=\frac{\overline{B}}{2}\sin\frac{\varphi}{2}+\overline{C}\left(\sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\right)-\overline{A}\left(\cos\frac{3\varphi}{2}-\cos\frac{5\varphi}{2}\right), \tag{229}$$

$$\mu^{(0)}_{r\varphi\varphi}=\frac{\overline{B}}{2}\sin\frac{\varphi}{2}-\overline{C}\left(\sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\right)+\overline{A}\left(\cos\frac{3\varphi}{2}-\cos\frac{5\varphi}{2}\right), \tag{230}$$

$$\mu^{(0)}_{rr\varphi}=\overline{C}\left(\cos\frac{3\varphi}{2}-\cos\frac{5\varphi}{2}\right)+\overline{A}\left(\sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\right), \tag{231}$$

$$\mu^{(0)}_{\varphi rr}=\frac{\overline{B}}{2}\cos\frac{\varphi}{2}-\overline{C}\left(\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\right)-\overline{A}\left(\sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\right), \tag{232}$$

$$\mu^{(0)}_{\varphi\varphi\varphi}=\frac{\overline{B}}{2}\cos\frac{\varphi}{2}+\overline{C}\left(\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\right)+\overline{A}\left(\sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\right), \tag{233}$$

$$\mu^{(0)}_{\varphi r\varphi}=\overline{C}\left(\sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\right)-\overline{A}\left(\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\right). \tag{234}$$

If we define

$$\overline{A}:=(c_2-c_1)\mu C^{(0)}_I,\quad \overline{B}:=(c_2-c_1)(\lambda+\mu)B^{(0)}_{II}, \tag{235}$$

$$\overline{C}:=-(c_2-c_1)\mu E^{(0)}_{II} \tag{236}$$

then these are nothing more but the solutions for $\mu^{(0)}_{\alpha\beta\gamma}$ of Section 3.9.

5. Discussion of the asymptotic solutions

As suggested in Section 3.5, it is common to represent the leading terms of the asymptotic expansion of stresses by introducing stress intensity factors and angular functions. For the Cauchy stress, this is indicated in Eq. (108). Eqs. (108)–(110) also reveal that

$$\frac{\tilde{K}_I}{\sqrt{2\pi}} = \left[\Sigma_{\varphi\varphi}^{(0)}\right]_{\varphi=0}, \quad \frac{\tilde{K}_{II}}{\sqrt{2\pi}} = \left[\Sigma_{r\varphi}^{(0)}\right]_{\varphi=0}. \tag{237}$$

To accomplish a representation for $\mu_{\alpha\beta\gamma}^{(0)}$ similar to the one for $\Sigma_{\alpha\beta}^{(0)}$ in Eq. (108), we remark that there is only one unknown constant for Mode-I, namely $C_I^{(0)}$, but there are two unknown constants for Mode-II, $B_{II}^{(0)}$ and $E_{II}^{(0)}$ [cf. Eqs. (195)–(202)]. Therefore, in analogy to Eq. (108), we set

$$\mu_{\alpha\beta\gamma}^{(0)} = \frac{\tilde{L}_I}{\sqrt{2\pi}} g_{\alpha\beta\gamma}^{I}(\varphi) + \frac{\tilde{L}_{II,1}}{\sqrt{2\pi}} g_{\alpha\beta\gamma}^{II,1}(\varphi) + \frac{\tilde{L}_{II,2}}{\sqrt{2\pi}} g_{\alpha\beta\gamma}^{II,2}(\varphi), \tag{238}$$

and define for Mode–I (cf. Eq. (202))

$$\frac{\tilde{L}_I}{\sqrt{2\pi}} := \left[\mu_{\varphi r\varphi}^{(0)}\right]_{\varphi=0} = -(c_2 - c_1)\, 2\mu C_I^{(0)}, \tag{239}$$

rendering $\left[g_{\varphi r\varphi}^{I}\right]_{\varphi=0}$ to be normalized,

$$\left[g_{\varphi r\varphi}^{I}\right]_{\varphi=0} = 1. \tag{240}$$

To define $\tilde{L}_{II,1}$ and $\tilde{L}_{II,2}$ unambiguously, we note that $B_{II}^{(0)}$ can be determined by adding Eqs. (199) and (200) while taking $\varphi = 0$. Similarly, $E_{II}^{(0)}$ can be determined by substracting Eqs. (199) and (200) from each other while taking $\varphi = 0$. We intend to normalize the angular functions $g_{\alpha\beta\gamma}^{II,1}$ and $g_{\alpha\beta\gamma}^{II,2}$ by

$$\left[g_{\varphi rr}^{II,1}\right]_{\varphi=0} = \left[g_{\varphi rr}^{II,2}\right]_{\varphi=0} = 1, \tag{241}$$

and therefore define the stress intensity factors $\tilde{L}_{II,1}$ and $\tilde{L}_{II,2}$ by (cf. Eqs. (199) and (200))

$$\frac{\tilde{L}_{II,1}}{\sqrt{2\pi}} := \frac{1}{2}\left[\mu_{\varphi rr}^{(0)} + \mu_{\varphi\varphi\varphi}^{(0)}\right]_{\varphi=0} = \frac{1}{2}(c_2 - c_1)(\lambda + \mu) B_{II}^{(0)}, \tag{242}$$

$$\frac{\tilde{L}_{II,2}}{\sqrt{2\pi}} := \frac{1}{2}\left[\mu_{\varphi rr}^{(0)} - \mu_{\varphi\varphi\varphi}^{(0)}\right]_{\varphi=0} = (c_2 - c_1)\, 2\mu E_{II}^{(0)}. \tag{243}$$

The angular functions will be determined by comparison of Eqs. (238)–(243) with Eqs. (195), (196), (198)–(200) ,and (202). Explicitely, we find that

$$\begin{pmatrix} g^I_{rrr} \\ g^I_{r\varphi\varphi} \\ g^I_{rr\varphi} \\ g^I_{\varphi rr} \\ g^I_{\varphi\varphi\varphi} \\ g^I_{\varphi r\varphi} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2} \\ -\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2} \\ -\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2} \\ \sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2} \\ -\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2} \\ \cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2} \end{pmatrix}, \tag{244}$$

$$\begin{pmatrix} g^{II,1}_{rrr} \\ g^{II,1}_{r\varphi\varphi} \\ g^{II,1}_{rr\varphi} \\ g^{II,1}_{\varphi rr} \\ g^{II,1}_{\varphi\varphi\varphi} \\ g^{II,1}_{\varphi r\varphi} \end{pmatrix} = \begin{pmatrix} \sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} \\ 0 \\ \cos\frac{\varphi}{2} \\ \cos\frac{\varphi}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} g^{II,2}_{rrr} \\ g^{II,2}_{r\varphi\varphi} \\ g^{II,2}_{rr\varphi} \\ g^{II,2}_{\varphi rr} \\ g^{II,2}_{\varphi\varphi\varphi} \\ g^{II,2}_{\varphi r\varphi} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2} \\ \sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2} \\ -\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2} \\ \cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2} \\ -\cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2} \\ -\sin\frac{3\varphi}{2} - \sin\frac{5\varphi}{2} \end{pmatrix}. \tag{245}$$

Some comments addressing Mode-I and Mode-II crack problems are in order at this stage. In classical elasticity, there are two intensity factors in the expansion of the Cauchy stress, one for each mode. In micropolar elasticity (see, e. g., Diegele et al. [15]), there are also two stress intensity factors in the expansion of the Cauchy stress and in addition two nonclassical intensity factors in the expansion of the couple stress, one for each mode. In the present case of microstrain elasticity, there are also two stress intensity factors in the expansion of the Cauchy stress, one for each mode. However, in the expansions of the double stress there is one intensity factor for Mode-I, but there are two intensity factors for Mode-II. Actually, there are no further conditions to relate $\tilde{L}_{II,1}$ and $\tilde{L}_{II,1}$ and the numerical simulations in Part III confirm this fact.

It is also convenient to replace the constants $A_I^{(1)}, D_I^{(1)}, E_{II}^{(1)}$ and $F_{II}^{(1)}$ by the definitions

$$\tilde{l}_{I,1} := (c_2 - c_1)(\lambda + \mu) A_I^{(1)}, \tag{246}$$

$$\tilde{l}_{I,2} := -(c_2 - c_1) 2\mu D_I^{(1)}, \tag{247}$$

$$\tilde{l}_{II,1} := -(c_2 - c_1) 2\mu E_{II}^{(1)}, \tag{248}$$

$$\tilde{l}_{II,2} := -(c_2 - c_1) 2\mu F_{II}^{(1)}. \tag{249}$$

Evidently, the new constants for Mode-I and Mode-II in the expansions of $\mu^{(0)}_{\alpha\beta\gamma}$ and $\mu^{(1)}_{\alpha\beta\gamma}$ can be employed to rewrite $\Psi^{(0)}_{\alpha\beta}$ and $\Psi^{(1)}_{\alpha\beta}$. In particular, we can conclude from Eqs. (160)–(162) and (239)–(243) that

$$\Psi^{(0)}_{\alpha\beta} = \frac{\tilde{L}_I}{(c_2-c_1)\,2\mu\sqrt{2\pi}}h^I_{\alpha\beta} + \frac{\tilde{L}_{II,1}}{(c_2-c_1)\,(\lambda+\mu)\sqrt{2\pi}}h^{II,1}_{\alpha\beta} + \frac{\tilde{L}_{II,2}}{(c_2-c_1)\,2\mu\sqrt{2\pi}}h^{II,2}_{\alpha\beta}, \tag{250}$$

with

$$\begin{pmatrix} h^I_{rr} \\ h^I_{\varphi\varphi} \\ h^I_{r\varphi} \end{pmatrix} = \begin{pmatrix} \cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2} \\ -\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2} \\ -\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2} \end{pmatrix}. \tag{251}$$

Table 1 summarizes the first two terms of the asymptotic solutions of the near-tip fields. All stresses are singular with order of singularity $r^{-\frac{1}{2}}$. Especially, the terms

---

$$\Sigma_{\alpha\beta} = \frac{\tilde{K}_I}{\sqrt{2\pi r}}f^I_{\alpha\beta} + \frac{\tilde{K}_{II}}{\sqrt{2\pi r}}f^{II}_{\alpha\beta} + \Sigma^{(1)}_{\alpha\beta} + \dots, \tag{252}$$

$$\begin{pmatrix} \Sigma_{rr} \\ \Sigma_{\varphi\varphi} \\ \Sigma_{r\varphi} \end{pmatrix} = \frac{\tilde{K}_I}{4\sqrt{2\pi r}}\begin{pmatrix} 5\cos\frac{\varphi}{2} - \cos\left(\frac{3}{2}\varphi\right) \\ 3\cos\frac{\varphi}{2} + \cos\left(\frac{3}{2}\varphi\right) \\ \sin\frac{\varphi}{2} + \sin\left(\frac{3}{2}\varphi\right) \end{pmatrix} + \frac{\tilde{K}_{II}}{4\sqrt{2\pi r}}\begin{pmatrix} -5\sin\frac{\varphi}{2} + 3\sin\left(\frac{3}{2}\varphi\right) \\ -3\sin\frac{\varphi}{2} - 3\sin\left(\frac{3}{2}\varphi\right) \\ \cos\frac{\varphi}{2} + 3\cos\left(\frac{3}{2}\varphi\right) \end{pmatrix} + \tilde{k}_I\begin{pmatrix} \cos^2\varphi \\ \sin^2\varphi \\ -\frac{1}{2}\sin(2\varphi) \end{pmatrix} + \dots. \tag{253}$$

$$\Psi_{\alpha\beta} = \overline{\Psi}_{\alpha\beta} + \sqrt{\frac{r}{2\pi}}\frac{\tilde{L}_I}{(c_2-c_1)\,2\mu}h^I_{\alpha\beta} + \sqrt{\frac{r}{2\pi}}\frac{\tilde{L}_{II,1}}{(c_2-c_1)\,(\lambda+\mu)}h^{II,1}_{\alpha\beta} + \sqrt{\frac{r}{2\pi}}\frac{\tilde{L}_{II,2}}{(c_2-c_1)\,2\mu}h^{II,2}_{\alpha\beta} + r\Psi^{(1)}_{\alpha\beta} + \dots, \tag{254}$$

$$\begin{pmatrix} \Psi_{rr} - \overline{\Psi}_{rr} \\ \Psi_{\varphi\varphi} - \overline{\Psi}_{\varphi\varphi} \\ \Psi_{r\varphi} - \overline{\Psi}_{r\varphi} \end{pmatrix} = \sqrt{\frac{r}{2\pi}}\frac{\tilde{L}_I}{(c_2-c_1)\,2\mu}\begin{pmatrix} \cos\frac{3\varphi}{2} - \cos\frac{5\varphi}{2} \\ -\cos\frac{3\varphi}{2} + \cos\frac{5\varphi}{2} \\ -\sin\frac{3\varphi}{2} + \sin\frac{5\varphi}{2} \end{pmatrix}$$

$$+\sqrt{\frac{r}{2\pi}}\frac{\tilde{L}_{II,1}}{(c_2-c_1)(\lambda+\mu)}\begin{pmatrix}\sin\frac{\varphi}{2}\\ \sin\frac{\varphi}{2}\\ 0\end{pmatrix}$$

$$+\sqrt{\frac{r}{2\pi}}\frac{\tilde{L}_{II,2}}{(c_2-c_1)2\mu}\begin{pmatrix}-\sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\\ \sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\\ -\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\end{pmatrix}$$

$$+r\frac{\tilde{l}_{I,1}}{2(c_2-c_1)(\lambda+\mu)}\begin{pmatrix}\cos\varphi\\ \cos\varphi\\ 0\end{pmatrix} \tag{255}$$

$$+r\frac{\tilde{l}_{I,2}}{(c_2-c_1)2\mu}\begin{pmatrix}\cos\varphi+\cos3\varphi\\ -\cos\varphi-\cos3\varphi\\ -\sin\varphi-\sin3\varphi\end{pmatrix}$$

$$+r\frac{\tilde{l}_{II,1}}{(c_2-c_1)2\mu}\begin{pmatrix}\sin\varphi\\ -\sin\varphi\\ \frac{1}{2}+\cos\varphi\end{pmatrix}$$

$$+r\frac{\tilde{l}_{II,2}}{(c_2-c_1)2\mu}\begin{pmatrix}\sin3\varphi\\ -\sin3\varphi\\ \frac{1}{2}-\cos3\varphi\end{pmatrix}+\dots.$$

$$\mu_{\alpha\beta\gamma}=\frac{\tilde{L}_I}{\sqrt{2\pi r}}g^{I}_{\alpha\beta\gamma}+\frac{\tilde{L}_{II,1}}{\sqrt{2\pi r}}g^{II,1}_{\alpha\beta\gamma}+\frac{\tilde{L}_{II,2}}{\sqrt{2\pi r}}g^{II,2}_{\alpha\beta\gamma}+\mu^{(1)}_{\alpha\beta\gamma}+\dots, \tag{256}$$

$$\begin{pmatrix}\mu_{rrr}\\ \mu_{r\varphi\varphi}\\ \mu_{rr\varphi}\\ \mu_{\varphi rr}\\ \mu_{\varphi\varphi\varphi}\\ \mu_{\varphi r\varphi}\end{pmatrix}=\frac{\tilde{L}_I}{\sqrt{2\pi r}}\frac{1}{2}\begin{pmatrix}\cos\frac{3\varphi}{2}-\cos\frac{5\varphi}{2}\\ -\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\\ -\sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\\ \sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\\ -\sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\\ \cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\end{pmatrix}$$

$$
+\frac{\tilde{L}_{II,1}}{\sqrt{2\pi r}}\begin{pmatrix}\sin\frac{\varphi}{2}\\ \sin\frac{\varphi}{2}\\ 0\\ \cos\frac{\varphi}{2}\\ \cos\frac{\varphi}{2}\\ 0\end{pmatrix}+\frac{\tilde{L}_{II,2}}{\sqrt{2\pi r}}\frac{1}{2}\begin{pmatrix}-\sin\frac{3\varphi}{2}+\sin\frac{5\varphi}{2}\\ \sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\\ -\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\\ \cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}\\ -\cos\frac{3\varphi}{2}-\cos\frac{5\varphi}{2}\\ -\sin\frac{3\varphi}{2}-\sin\frac{5\varphi}{2}\end{pmatrix}
$$

$$
+\tilde{l}_{I,1}\begin{pmatrix}\cos\varphi\\ \cos\varphi\\ 0\\ -\sin\varphi\\ -\sin\varphi\\ 0\end{pmatrix}+\tilde{l}_{I,2}\begin{pmatrix}\cos\varphi+\cos 3\varphi\\ -\cos\varphi-\cos 3\varphi\\ -\sin\varphi-\sin 3\varphi\\ \sin\varphi-\sin 3\varphi\\ -\sin\varphi+\sin 3\varphi\\ \cos\varphi-\cos 3\varphi\end{pmatrix}
$$

$$
+\tilde{l}_{II,1}\begin{pmatrix}\sin\varphi\\ -\sin\varphi\\ \frac{1}{2}+\cos\varphi\\ -1-\cos\varphi\\ 1+\cos\varphi\\ \sin\varphi\end{pmatrix}+\tilde{l}_{II,2}\begin{pmatrix}\sin 3\varphi\\ -\sin 3\varphi\\ \frac{1}{2}-\cos 3\varphi\\ 1+\cos 3\varphi\\ -1-\cos 3\varphi\\ -\sin 3\varphi\end{pmatrix}+\dots, \tag{257}
$$

$$
\begin{pmatrix}\varepsilon_{rr}\\ \varepsilon_{\varphi\varphi}\\ \varepsilon_{r\varphi}\end{pmatrix}=\frac{\tilde{K}_I}{\sqrt{2\pi r}}\frac{c_1}{8\mu c_2}\begin{pmatrix}(5-8\nu)\cos\frac{\varphi}{2}-\cos\frac{3\varphi}{2}\\ (3-8\nu)\cos\frac{\varphi}{2}+\cos\frac{3\varphi}{2}\\ \sin\frac{\varphi}{2}+\sin\frac{3\varphi}{2}\end{pmatrix}
$$

$$
+\frac{\tilde{K}_{II}}{\sqrt{2\pi r}}\frac{c_1}{8\mu c_2}\begin{pmatrix}-(5-8\nu)\sin\frac{\varphi}{2}+3\sin\frac{3\varphi}{2}\\ -(3-8\nu)\sin\frac{\varphi}{2}-3\sin\frac{3\varphi}{2}\\ \cos\frac{\varphi}{2}+3\cos\frac{3\varphi}{2}\end{pmatrix}
$$

$$
+\tilde{k}^{\vec{\varepsilon}}_{I,1}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}+\tilde{k}^{\vec{\varepsilon}}_{I,2}\begin{pmatrix}\cos 2\varphi\\ -\cos 2\varphi\\ -\sin 2\varphi\end{pmatrix}+\tilde{k}^{\vec{\varepsilon}}_{II}\begin{pmatrix}\sin 2\varphi\\ -\sin 2\varphi\\ \cos 2\varphi\end{pmatrix} \tag{258}
$$

$$
+\dots,
$$

$$\begin{pmatrix} u_r \\ u_\varphi \end{pmatrix} = \sqrt{\frac{r}{2\pi}} \frac{c_1 \tilde{K}_I}{4\mu c_2} \begin{pmatrix} (5-8\nu)\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2} \\ -(7-8\nu)\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2} \end{pmatrix}$$

$$+\sqrt{\frac{r}{2\pi}} \frac{c_1 \tilde{K}_{II}}{4c_2\mu} \begin{pmatrix} -(5-8\nu)\sin\frac{\varphi}{2} + 3\sin\frac{3\varphi}{2} \\ -(7-8\nu)\cos\frac{\varphi}{2} + 3\cos\frac{3\varphi}{2} \end{pmatrix}$$

$$+r\left[\tilde{k}^{\vec{\varepsilon}}_{I,1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{k}^{\vec{\varepsilon}}_{I,2}\begin{pmatrix} \cos 2\varphi \\ -\sin 2\varphi \end{pmatrix} + \tilde{k}^{\vec{\varepsilon}}_{II}\begin{pmatrix} \sin 2\varphi \\ \cos 2\varphi \end{pmatrix}\right] \tag{259}$$
$$+\dots.$$

Table 1.
Analytical solutions of the fields.

$\Sigma^{(0)}_{\alpha\beta}$ and $\Sigma^{(1)}_{\alpha\beta}$ are identical to those of classical elasticity. However, the terms $\varepsilon^{(0)}_{\alpha\beta}$ and $\varepsilon^{(1)}_{\alpha\beta}$ are different from the corresponding terms of classical elasticity. In particular, $\varepsilon^{(1)}_{\alpha\beta}$ includes terms arising from $\overline{\Psi}_{\alpha\beta}$. There are also qualititative differences to micropolar elasticity. For instance, terms of couple stresses corresponding to $\mu^{(0)}_{\alpha\beta\gamma}$ in Mode-II and to $\mu^{(1)}_{\alpha\beta\gamma}$ in Mode-I do not exist.

6. Concluding remarks

Closed form analytical solutions, predicted by the 3-PG-Model for Mode-I and Mode-II crack problems, have been developed in the present paper. The solutions are based on asymptotic expansions of Williams' type of the near-tip fields. The main conclusions, which can be drawn on the basis of the preceding developements, can be briefly stated as follows.

1. The first two terms in the asymptotic expansion of the components of the Cauchy stress are identical to the ones of classical elasticity. In particular, the Cauchy stress is singular with order of singularity $r^{-\frac{1}{2}}$.

2. This is in contrast to statements in other works, which rely upon boundary conditions different from the ones adopted here.

3. There are, however, significant differences in comparison to classical elasticity, in what concerns the components of macrostrain and macrodisplacement.

4. There are also significant qualitative differences in comparison to micropolar elasticity concerning the nonclassical stresses.

5. For instance, the leading terms of the double stress of Mode-II problems include two different stress intensity factors. This is a remarkable feature of the 3-PG-Model.

Acknowledgements

We would like to thank the TU Darmstadt for support of publishing this work in open access.

Authors' contributions

Each one of the authors Broese, Frischmann, and Tsakmakis contributed the same amount of work for the three papers "Mode-I and Mode-II crack tip fields in implicit gradient elasticity based on Laplacians of stress and strain." Part I–III.

Dr. C. Broese: Theory and numerical simulations.

Dr. J. Frischmann: Analytical solution and numerical simulations.

Prof. Dr. Tsakmakis: Theory and analytical solution.

Therefore, it is a joint work by all three authors.

Funding

The first and second authors acknowledge and thank the Deutsche Forschungsgemeinschaft (DFG) for partial support of this work under Grant TS 29/13–1.

Consent for publication

Not applicable.

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics approval and consent to participate

Not applicable.

Availability of data and material

Not applicable.

List of abbreviations

eq. = equation

Appendix

In order to make the present work self-contained, we sketch briefly how to ascertain the solutions (107)–(111) from Eqs. (95)–(102). We start with the system of differential Eqs. (95)–(97), which can be proved to posses the solutions

$$\begin{aligned}\Sigma_{rr}^{(0)} &= -C\cos([p+1]\varphi) - D\sin([p+1]\varphi)\\ &\quad + \frac{3-p}{4}A\cos([p-1]\varphi) + \frac{3-p}{4}B\sin([p-1]\varphi)\ ,\end{aligned} \tag{A1}$$

$$\begin{aligned}\Sigma_{\varphi\varphi}^{(0)} &= C\cos([p+1]\varphi) + D\sin([p+1]\varphi)\\ &\quad + \frac{p+1}{4}A\cos([p-1]\varphi) + \frac{p+1}{4}B\sin([p-1]\varphi)\ ,\end{aligned} \tag{A2}$$

$$\begin{aligned}\Sigma_{r\varphi}^{(0)} &= C\sin([p+1]\varphi) - D\cos([p+1]\varphi)\\ &\quad + \frac{p-1}{4}A\sin([p-1]\varphi) + \frac{p-1}{4}B\cos([p-1]\varphi)\ .\end{aligned} \tag{A3}$$

Here, A, B, C and D are constants of integration. In order to determine these constants, we incorporate the solutions in the boundary conditions (98). After some manipulations, we gain the following two homogeneous systems for the constants A, B, C, and D:

$$\begin{pmatrix} 2\cos([p+1]\pi) & \dfrac{p-1}{2}\cos([p-1]\pi) \\ 2\sin([p+1]\pi) & \dfrac{p+1}{2}\sin([p-1]\pi) \end{pmatrix}\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A4}$$

$$\begin{pmatrix} 2\sin([p+1]\pi) & \dfrac{p-1}{2}\sin([p-1]\pi) \\ 2\cos([p+1]\pi) & \dfrac{p+1}{2}\cos([p-1]\pi) \end{pmatrix}\begin{pmatrix} C \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A5}$$

The conditions for the existence of nontrivial solutions are vanishing determinants of the coefficient matrices of Eqs. (A4) and (A5). It turns out that both conditions lead to the same equation

$$2\cos(p\pi)\sin(p\pi) = \sin(2p\pi) = 0, \tag{A6}$$

which has the solutions

$$p = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \dots\ . \tag{A7}$$

The smallest value of p compatible with the restriction (82) is $p = \frac{1}{2}$, as stated in Eq. (107). For this case, the systems (A4) and (A5) imply

$$D = -\frac{3}{8}B, \quad C = \frac{1}{8}A, \tag{A8}$$

and the solutions (A1)–(A3) become

$$\Sigma_{rr}^{(0)} = \frac{1}{8}A\left(5\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2}\right) + \frac{1}{8}B\left(-5\sin\frac{\varphi}{2} + 3\sin\frac{3\varphi}{2}\right), \tag{A9}$$

$$\Sigma_{\varphi\varphi}^{(0)} = \frac{1}{8}A\left(3\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2}\right) + \frac{1}{8}B\left(-3\sin\frac{\varphi}{2} - 3\sin\frac{3\varphi}{2}\right), \tag{A10}$$

$$\Sigma_{r\varphi}^{(0)} = \frac{1}{8}A\left(\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right) + \frac{1}{8}B\left(\cos\frac{\varphi}{2} + 3\cos\frac{3\varphi}{2}\right). \tag{A11}$$

These solutions, in turn, are equivalent to those of Eqs. (108)–(109). Moreover, it can be shown that for $p = \frac{1}{2}$, the solutions of Eqs. (99)–(102) might be expressed in the form (111).

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