Chapter

Prime Numbers Distribution Line

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Abstract

During the analysis of the fractal-primorial periodicity of the natural series of numbers, presented in the form of an alternation (sequence) of prime numbers (1 smallest prime factor > 1 of any integer), the regularity of prime numbers distribution was revealed. That is, the theorem is proved that for any integer = N on the segment of the natural series of numbers from 1 to N + $2\sqrt{N}$: (1) prime numbers are arranged in groups, by exactly three consecutive prime numbers of the form: (P_1-P_2-P_3). In this case, the distance from the first to the third prime number of any group is less than $2\sqrt{N}$ integers, that is, $P_3 – P_1 < 2\sqrt{N}$ integers. (2) These same prime numbers are redistributed in a line in groups, by exactly two consecutive prime numbers, on all segments of the natural series of numbers shorter than $2\sqrt{N}$ integers.

Keywords: residue groups, prime numbers, primorial, sieve of Eratosthenes, alternations, fractal

1. Introduction

1.1 Line-symmetrical primary-repeatable fractals of the positive integers

In the scientific works [7 pp. 142–147, 8 pp. 77–84, 9 pp. 109–116], positive integers are analyzed, hereinafter the P.I. is represented only as the alternance (array) of primes (according to the 1st least prime factor > 1 from every whole number). Type: 12.3.5.7.3.11.13.3.17.19.3.23.5.3.29 ... 3.*p*.*p*.3.*p* ... 3.*p*.*p*.3.*p* .... with for every recurrent prime = P_1, sieve of Eratosthenes formats the P.I., represented by alternance (array) of the first primes $\leq P_1$, in the form line-symmetrical repeating fractal-like structure, situated in the section of P.I. from 1 to $P_1\#$, with “eliminated” sections of P.I. and $\varphi(P_1\#)$ not eliminated odd numbers are line-symmetrical to the number = $P_1\#/2$ and are repeated without rearrangement of their position with the period = $P_1\#$, on the basis of rhythmical repeating of two even numbers. Every recurrent prime has its own line-symmetrical primary-repeatable fractal = P_1, then goes fractal = $P_1\#$ (see line 1 **Table 1**).

Every recurrent line-symmetrical fractal -$P_1\#$ is situated on the section of P.I. from 1 to $P_1\#$ and contains $\varphi(P_1\#)$ of the not eliminated odd numbers that are $\varphi(P_1\#)$ of the least residue, belonging to the indicated residue system (I.R.S) according to mod ($P_1\#$), type: C_n to the left from the number = $P_1\#/2$ and ($P_1\# – C_n$) to the right from the number = $P_1\#/2$, with C_n – is residue according to mod($P_1\#$). Hereinafter with the term mod($P_1\#$), we shall indicate the period of fractal $P_1\#$ repetition (I.R.S, sieve of Eratosthenes), equal to product of all first primes $\leq P_1$ (primorial = $P_1\#$) [1–6].

$C_1 = 1$	~~3,5,7..~~	$C_2 = P_2$	**~~ppp~~**	**..C_3 ..**	**~~ppp~~**	**.. Cn ..**	**~~ppp~~**	**$P_1\#$–Cn**	**~~ppp~~**	**$P_1\#–C_2$**	~~..7,5.3~~	$(P_1\#–1)$
1 + $P_1\#$	~~3,5,7..~~	$C_2 + P_1\#$	~~ppp~~	$C_3 + P_1\#$	~~ppp~~	Cn + $P_1\#$	~~ppp~~	$2P_1\#$–Cn	~~ppp~~	$2P_1\#–C_2$	**~~..7,5.3~~**	**$(2P_1\#–1)$**
1 + $2P_1\#$	~~3,5,7..~~	$C_2 + 2P_1\#$	~~ppp~~	$C_3 + 2P_1\#$	~~ppp~~	Cn + $2P_1\#$	~~ppp~~	$3P_1\#$–Cn	~~ppp~~	$3P_1\#–C_2$	**~~..7,5.3~~**	**$(3P_1\#–1)$**
...	~~3,5,7..~~	...	~~ppp~~	...	~~ppp~~	...	~~ppp~~	...	~~ppp~~	...	**~~..7,5.3~~**	...
...	~~3,5,7..~~	...	~~ppp~~	...	~~ppp~~	...	~~ppp~~	$P_2\#$–Cn	~~ppp~~	$P_2\#–C_2$	**~~..7,5.3~~**	**$(P_2\#–1)$**
1 + $P_2\#$	~~3,5,7..~~	$C_2 + P_2\#$	~~ppp~~	$C_3 + P_2\#$	~~ppp~~	Cn + $P_2\#$	~~ppp~~		~~ppp~~		**~~..7,5.3~~**	
And so on, repeating of fractal = $P_1\#$ with the period = $P_1\#$, with: ~~ppp~~ is alternance of $\leq P_1$												

Table 1.
P_2 *repeating of periodical fractal* = $\mathbf{P_1\#}$*, including I.R.S. according to the mod*$(P_1\#)$*.*

By term residue according to mod ($P_1\#$), we shall indicate every number, NOT eliminated by Sieve of Eratosthenes, not aliquot to the first primes $\le P_1$.

Alternance $\le P_1$ is the section of P.I. in the form of array of primes – NOT residues of mod($P_1\#$), (for the 1 least common factor > 1 from every NOT residue).

Eliminating (according to diagonals) 1 number multiple to P_2 in every column = Cn, we'll get in P_2 lines of **Table 1**: $\varphi(P_1\#)^*(P_2 \text{ lines}) — \varphi(P_1\#)_{\text{multiple}}$ $P_2 = \varphi(P_2\#)$ residue of mod($P_2\#$). Representing the section of P.I. from 1 to $P_2\#$ as one line, we'll get the fractal = $P_2\#$ with the period of repeating =$P_2\#$. And so on: every recurrent prime = P_n has got its periodical fractal = $P_n\#$ with n is the whole. The numerical illustration is indicated in the scientific works [7–10].

Fractal ($P_1\#$)-I.R.S. mod($P_1\#$) = (first line of **Table 1**).

Fractal ($P_2\#$)-I.R.S. mod($P_2\#$) = P_2 lines in **Table 1**. -$\varphi(P_1\#)$ numbers multiple to P_2.

Fractal ($P_3\#$) I.R.S. mod($P_3\#$) = P_3 lines in **Table 2**.-$\varphi(P_2\#)$ numbers multiple to P_3.

Fractal ($P_4\#$) I.R.S. mod($P_4\#$) = (P_4 repeating of fractal $P_3\#$)–$\varphi(P_3\#)$ numbers multiple to P_4.

Fractal ($P_5\#$) I.R.S. mod($P_5\#$) = (P_5 repeating of fractal $P_4\#$)–$\varphi(P_4\#)$ numbers multiple to P_5.

and so on according to cumulative primes.

1.2 Purpose and role of the overall length of the of alternance (array) of the all first primes ≤Pn

It is quite obvious that $\varphi(P_n\#)$ of the least residues of mod($P_n\#$) type = C and ($P_n\#$–C), of every recurrent fractal = $P_n\#$, gradate P.I. as $\varphi(P_n\#)$ "eliminated" sections of P.I. with different lengths of the type: *C*~~..3pp3.~~*C*~~.3pp3.~~*C* 3pp3 *C*., with ~~..3pp3..~~ "eliminated" sections of P.I. represented as array of "eliminated" NOT residues of mod($P_n\#$), or un the form of alternance (array) of the first primes $\le P_n$, (according to the 1st least prime factor >1 from every NOT residue of mod($P_n\#$)), hereinafter the alternance $\le P_n$. C – residue of mod($P_n\#$) (according to the 1st least $>P_n$ from every residue of mod($P_n\#$)), location from 1 to $P_n\#$ is line symmetrical relating to number = $P_n\#/2$. And further, repeated without rearrangement of their position with the period = $P_n\#$. Then, after we define the overall - maximal length of alternance that we can form using the fist primes p $\le P_{\mathbf{n}}$, (NOT residues of mod ($P_{\mathbf{n}}\#$)), type C_1 ... 3pp3pp3pp3 ... C_2 that is maximal amount of consequent odd numbers = maximal length of alternance -p $\le P_{\mathbf{n}}$ (one least NOT residue of mod ($P_{\mathbf{n}}\#$) >1 from the number), we can evaluate the distance between every two consequent residues of mod ($P_{\mathbf{n}}\#$) that is between two primes $<(P_{\mathbf{n+1}})^2$, according to formula: $(C_2 - C_1) - 2/2$ of the odd numbers ≤ maximal length of the alternance, (maximal amount of NOT residue of mod($P_{\mathbf{n}}\#$)).

In the scientific works [7 pp. 142–147, 8 pp. 77–84, 9 pp.109–116], the distribution of groups of 4 consequent residues in the form of "pairs of residues every two residue" is analyzed. But we have no information on distribution of groups of 4, 3, and 2 consequent residues of mod($P_n\#$) for every fractal $P_{\mathbf{n}}\#$.

In this scientific work, the $\varphi(P_n\#)$ of the least residue of mod($P_n\#$) of every recurrent fractal -$P_n\#$ is indexed as continuous sequence of groups: (a) No 4 has got 4 residues, or (b) No 3 has got 3 residues or (c) No 2 has got 2 consequent residues mod($P_n\#$). These groups No 4-3-2 are analyzed as subgroups with No 4-3-2 consequent residues of mod($P_{\mathbf{n}}\#$) that are surrounded by the maximal permissible amount of consequent NOT residues of mod($P_{\mathbf{n}}\#$).

We used the mathematical induction method to define the overall – maximal length of every kind of subgroups No 4, No 3, No 2 and overall maximal length of P.

$C_1 = 1$	~~3,5,7,,,~~	$C_2 = P_3$	**ppp**	**..C_3 ..**	**ppp**	**.. Cn ..**	**ppp**	**P_2#–Cn**	**ppp**	**P_2#–C_2**	~~..7,5.3~~	(P_2#–1)
1 + P_2#	**~~3,5,7,,,~~**	C_2 + P_2#	ppp	C_3 + P_2#	ppp	Cn + P_2#	ppp	2P_2#–Cn	ppp	2P_2#–C_2	**~~..7,5.3~~**	**(2P_2#–1)**
1 + 2P_2#	**~~3,5,7,,,~~**	C_2 + 2P_2#	ppp	C_3 + 2P_2#	ppp	Cn + 2P_2#	ppp	*3P_2#*–Cn	ppp	3P_2#–C_2	**~~..7,5.3~~**	**(3P_2#–1)**
...	**~~3,5,7,,,~~**	...	ppp	...	ppp	...	ppp	...	ppp	...	**~~..7,5.3~~**	...
...	**~~3,5,7,,,~~**	...	ppp	...	ppp	...	ppp	P_3#–Cn	ppp	P_3#–C_2	**~~..7,5.3~~**	**(P_3#–1)**
1 + P_3#	**~~3,5,7,,,~~**	C_2 + P_3#	ppp	C_3 + P_3#	ppp	Cn + P_3#	ppp	...	ppp	...	**~~..7,5.3~~**	...

And so on, repeating of fractal = P_2# with the period = P_2#, with: ppp is alternance of $\leq P_2$.
Representing the section of P.I. as line from 1 to P_3# we'll get the fractal P_3# and so on.

Table 2.
P_3 *repeating of periodical fractal* = **P_2#,** *including I.R.S. according to the mod*(P_2#).

I. sections in the form of maximal long alternances of all first primes $\leq P_n$, (that is maximal permissible amount of all NOT residues of mod($P_{\mathbf{n}}$#)), situated between two residues from C_A to C_B, between which, as subgroups are situated the groups of residues of mod (P_n#). Type:

a. -No 4: $C_{A..3pp3..}P_{1..}P_{2..}P_{3..}P_{4..3pp3..}C_B$.

b. No 3: $C_{A..3pp3..}P_{1..}P_{2..}P_{3..3pp3..}C_B$.

c. No 2: $C_{A..3\ pp3..}P_{1..}P_{2..3pp3..}C_B$.

As a result, we detected the loopback of these groups rearrangement from No 4 to No 3 up to No 2 according to the growing amount of the modulus, and the primes order distribution is defined.

2. Three groups of "eliminated" sections of every next fractal

It is quite obvious and requires no proof that indexing φ(P_1#) of the least residues of mod(P_1#) of every recurrent fractal-P_1#, or I.R.S mod(P_1#), is by groups, containing strictly 4 elements; three; two consequent residues of mod(P_1#), we have, that every recurrent fractal-P_1# would be represented as array of three groups of the residues of mod(P_1#), between are situated the alternances $\leq P_1$ (with different lengths) – the consequent NOT residues of mod(P_1#), **of types (a), (b), (c)** repeated without changes with period = P_1#.

a. φ(P_1#) groups No 4 containing strictly FOUR consequent residues of mod (P_1#)$_{A,B,C,D}$ **C,** between which the alternances of different amounts of different first primes $\leq P_1$, NOT residues of mod(P_1#), type: $_AC$..3pp3..$_BC$..3pp3..$_CC$ .. 3pp3..$_DC$. with: ~~..3pp3..~~ are alternances of the first primes $\leq P_1$ according to the 1st least common factor > 1 from every NOT residue of mod(P_1#). ($_{1\text{–}4}$C) – Four consequent residue of mod(P_1#), including the consequent primes of P.I. section from P_1 to $(P_2)^2$ of $_{A,B,C,D}\boldsymbol{P}$ type. Further, the fractal = P_1# represented as φ(P_1#) of No 4 groups (4 residues) of mod (P_1#). Three adjoined groups No 4 for every residue = C (**Table 3**).

   The length of group No 4, which means amount odd numbers, restricted by every group No 4 from $_AP$ to $_DP$ and from C_1 to C_4, for the mod(P_1#), is $(R_4–2)/2 \leq (P_2–1)$ of the odd numbers with: $R_4 = (_DP–_AP)$, $R_4 = (_4C–_1C)$, $R_4 \leq 2P_2$ (including 1 group of $R_4 = 2P_2$, detailed information is indicated in Section 5) (**Table 4**).

b. φ(P_1#) groups No 3 containing strictly THREE consequent residues of mod (P_1#). $_{A,B,C}$**C,** between which the alternances of different amounts of

1,3,5,7 …	$_AP$ ..3pp3..$_BP$..3pp3..$_CP$..3pp3..$_DP$. 3pp3…	P_2^2	$_1$C~~.3pp3.~~$_4$C.3pp3..	P_1#
3,5,7 …	————$_BP$..3pp3..$_CP$..3pp3..$_DP$..3pp3..$_AP$ …	P_2^2	—$_2$C~~..3pp3..~~$_5$C..$_2$C	P_1#
5,7 …	- ———$_CP$..3pp3..$_DP$.3pp3..$_AP$..3pp3..$_BP$	P_2^2	———$_3$C~~..3pp3..~~$_6$C..$_3$C	P_1#
And so on, repeating of fractal = P_1# with the period = P_1#, ppp - is alternance of ≤ P_1.				

Table 3.
Fractal = P_1#, represented as φ(P_1#) of No 4 groups (containing 4 residues) of mod(P_1#). Three adjoined groups No 4 for every residue = C.

Type-(a)	$\mathbf{C_1 = 1}$	7	11	$\mathbf{C_4 = 13}$	17	19	$\mathbf{C_7 = 23}$	29	31	C = 37	
Type-(b)		$\mathbf{C_2} = 7$	11	13	$\mathbf{C_5 = 17}$	19	23	$\mathbf{C_8 = 29}$	31	37	C = 41
Type-(c)			$\mathbf{C_3 = 11}$	13	17	$\mathbf{C_6 = 19}$	23	29	C = 31		
And so on, repeating of fractal =5# with the period = mod(5#)											

Table 4.
The numerical illustration of the fractal =5# in the form of $\varphi(5\#)$ = 8 groups No 4 (containing four residues). The three adjoined groups No 4 for every residue = C with $R_4 \le 2P_2 = 2*7$ (consult Section 5).

different first prime $\le P_1$, NOT residues of mod ($P_1\#$), type: $_AC$..3pp3..$_BC$..3pp3..$_CC$. with: ~~..3pp3..~~ are alternances of the first primes $\le P_1$ according to the 1st least common factor > 1 from every NOT residue of mod ($P_1\#$). ($_{1\text{-}3}$C) – three consequent residues of mod($P_1\#$), including the consequent primes of P.I. section from P_1 to $(P_2)^2$ of $_{A,B,C}$ ***P*** type**.** Further, the fractal = $P_1\#$ represented as $\varphi(P_1\#)$ of No 3 groups (3 residues) of mod($P_1\#$). Two adjoined groups No 3 for every residue = C (**Table 5**).

With the unknown to us length of the group No 3 from $_AP$ to $_CP$ and from C_1 to C_3, for the mod ($P_1\#$), is $(R_3–2)/2$ of the odd numbers with: $R_3 = (_CP–_AP)$., $R_3 = (_3C–_1C)$., R_3 =? (it is quite obvious that for mod ($P_1\#$) $R_3 < < R_4$) (**Table 6**).

c. $\varphi(P_1\#)$ groups No 2 containing strictly TWO consequent residues of mod $(P_1\#)_{A,B}$**C,** between which the alternances of different amounts of different first prime $\le P_1$, NOT residues of mod($P_1\#$), type: $_AC$..3pp3..$_BC$. with: ~~..3pp3..~~ are alternances of the first primes $\le P_1$ according to the 1st least common factor > 1 from every NOT residue of mod($P_1\#$). ($_{1–2}$C) – two consequent residue of mod($P_1\#$), including the consequent primes of P.I. section from P_1 to $(P_2)^2$ of $_{A,B}$***P*** type**.** Further, the fractal = $P_1\#$ represented as $\varphi(P_1\#)$ of No 2 groups (2 residues) of mod($P_1\#$) (**Tables 7–9**).

1.,3.,5.,7..	**$_AP$..3pp3..$_BP$ ..3pp3..$_CP$**	P_2^2	**$_1$C~~..3pp3..~~$_3$C ... 3pp3.. $_1$C ... $_3$C**	$P_1\#$
3.,5.,7 ...	——————..$_BP$ ..3pp3..$_CP$..3pp3..$_AP$..3pp3..	P_2^2	——————$_2$C~~..3pp3..~~$_4$C.. $_2$C .. $_4$C	$P_1\#$
And so on, repeating of fractal = $P_1\#$ with the period = $P_1\#$, with: ppp is alternance of $\le P_1$.				

Table 5.
Fractal = $P_1\#$, represented as $\varphi(P_1\#)$ of No 3 groups (containing 3 residues) of mod($P_1\#$). Two adjoined groups No 3 for every residue = C.

Type-(a)	$\mathbf{C_1 = 1}$	7	$C_3 = 11$	13	$\mathbf{C_5 = 17}$	19	$\mathbf{C_7 = 23}$	29	C =31	
Type-(b)		$C_2 = 7$	11	$\mathbf{C_4 = 13}$	17	$\mathbf{C_6 = 19}$	23	$\mathbf{C_8 = 29}$	31	C = 37
And so on, repeating of fractal =5# with the period =5#.										

Table 6.
*The numerical illustration of the fractal =5# in the form of $\varphi(5\#)$ = 8 groups No 3 (containing two residues). The two adjoined groups No 3 for every residue = C with R_3 =? (=2*5 consult Section 6).*

1.,3.,5.,7..	**$_AP$..3pp3..$_BP$..3pp3..$_AP$.3pp3..**	P_2^2	**$_1$C~~..3pp3..~~$_2$C~~..3pp3..~~$_1$C~~..3pp3..~~$_2$C..$_1$C..$_2$C...**	$P_1\#$
And so on, repeating of fractal = $P_1\#$ with the period = $P_1\#$, ppp - is alternance of $\le P_1$.				

Table 7.
Fractal = $P_1\#$, represented as $\varphi(P_1\#)$ of No 2 groups of mod($P_1\#$).

1.,3.,5.,7..	**$_{A}P$..~~3pp3~~..$_{B}P$..~~3pp3~~..$_{A}P$.~~3pp3~~..**	**P_3^2**	**$_1$C~~..3pp3..~~$_2$C~~..3pp3..~~$_1$C~~..3pp3..~~$_2$C..$_1$C..$_2$C...**	**P_2#**
And so on, repeating of fractal = P_2# with the period = P_2#, ~~ppp~~ - is alternance of $\leq P_2$.				

Table 8.
Fractal = P_2#, represented as $\varphi(P_2\#)$ of No 2 groups of mod(P_2#).

1.,3.,5.,7..	**$_{A}P$..~~3pp3~~..$_{B}P$..~~3pp3~~..$_{A}P$.~~3pp3~~..**	**P_4^2**	**$_1$C~~..3pp3..~~$_2$C~~..3pp3..~~$_1$C~~..3pp3..~~$_2$C..$_1$C..$_2$C...**	**P_3#**
And so on, repeating of fractal = P_3# with the period = P_3#, ~~ppp~~ - is alternance of $\leq P_3$.				

Table 9.
Fractal = P_3#, represented as $\varphi(P_3\#)$ of No 2 groups of mod(P_3#).

With the unknown to us, length of the group No 2 from $_{A}P$ to $_{B}P$ and from C_1 to C_2, for the mod(P_1#), is $(R_2–2)/2$ of the odd numbers with: $R_2 = ({}_{B}P–{}_{A}P)$., $R_2 = ({}_{2}C–{}_{1}C)$, R_2 =? (it is quite obvious that for mod(P_1#) $R_2 << R_3$).

Herewith for each group No 4-3-2 according to mod(P_1#), there are two residues of mod(P_1#): C_A – to the left and C_B – to the right, that is every group No 4-3-2 is the subgroup on the P.I. sections of the length unknown to us from C_A to C_B: (a)C_A-(C_1-C_2-C_3-C_4)-C_B. (b) C_A-(C_1-C_2-C_3)-C_B. (c) C_A-(C_1-C_2)-C_B.

3. Correlations of length limits of the subgroups No 4, No 3, No 2

In the scientific works [7 pp. 142–147, 8 pp. 77–84, 9 p. 109–116, 10 p. 1805], including Section 5 of this work, the overall – maximal length of the subgroup No 4 (containing 4 residual for every recurrent fractal -P_n#), of type is defined:

max $R_4 = (C_4 – C_1) = 2P_{n+1}$ of whole numbers.

Herewith, it is quite obvious and it is beyond argument that relations of limits, unknown to us of groups No 4-3-2 length according to the increasing modulus are indicated in **Table 10**.

Prime value P_n	**Fractal - P_n#**	**Period of fractal repetition =mod (P_n#).**	**Max length of $\varphi(P_n\#)$ of the subgroups No 4 max$R_4 = (C_4–C_1)$**	**>>**	**Max length $\varphi(P_n\#)$ of the subgroups No 3 max$R_3 = (C_3–C_1)$**	**>>**	**Max length $\varphi(P_n\#)$ of the subgroups No 2 max$R_2 = (C_2–C_1)$**
P_1	P_1#	mod(P_1#).	max R_4 = $(C_4–C_1) = 2P_2$	>>	max R_3 = $(C_3–C_1)$ =?	>>	max R_2 = $(C_2–C_1)$ =?
P_2	P_2#	mod(P_2#).	max R_4 = $(C_4–C_1) = 2P_3$	>>	max R_3 = $(C_3–C_1)$ =?	>>	max R_2 = $(C_2–C_1)$ =?
P_4	P_4#	mod(P_4#).	max R_4 = $(C_4–C_1) = 2P_5$	>>	max R_3 = $(C_3–C_1)$ =?	>>	max R_2 = $(C_2–C_1)$ =?
P_5	P_5#	mod(P_5#).	max R_4 = $(C_4–C_1) = 2P_6$	>>	max R_3 = $(C_3–C_1)$ =?	>>	max R_2 = $(C_2–C_1)$ =?
...	...	...	...	>>	...	>>	...
P_n	P_n#	mod(P_n#).	max R_4 = $(C_4–C_1) = 2P_{n+1}$	>>	max R_3 = $(C_3–C_1)$ =?	>>	max R_2= $(C_2–C_1)$ =?
And so on, repeating of fractal = P_n# with the period = P_n#. $C_{1–4}$ - residue of mod (P_n#).							

Table 10.
The relation of length limits of the subgroups according to the increasing modulus.

4. Distribution of prime number

Correlation of length limits of the subgroups in **Table 10** and distribution of groups of the indexed residues No 4-3-2, in every respective fractal -$P_n\#$ according to the increasing modulus is defined by theorem 1.

Theorem 1. The loopback of prime number distribution.

Every prescribed prime number squared = $(P_2)^2$ defines the distribution of all previous prime numbers < $(P_2)^2$, as all first prime numbers are less than every prescribed prime number squared = $(P_2)^2$, are situated in the P.I. as part of fractal $P_1\#$, where they are distributed by subgroups of **(a)**, **(b)**, **(c)** types.

a. $\varphi(P_1\#)$ of subgroups No 4 having pure FOUR consequent prime number (${}_{A,B,C,D}C$) of $P_1 < ({}_AC\text{-}{}_BC\text{-}{}_CC\text{-}{}_DC) < P_2^2; P_3^2$ type. At the P.I. section from P_1 to P_2^2 (including from $(P_2–2)^2$ to P_2^2), and further, from P_2^2 to $P_1\#$ pure FOUR consequent residues of mod($P_1\#$) on all P.I. sections with length not exceeding $2P_3$ of whole numbers with length of every subgroup No 4 at every section is:

$$R_4 = ({}_DC–{}_AC) \le 2P_2.$$

In that case, these primes of fractal = $\boldsymbol{P_1\#}$, by loopback, are distributed by groups:

b. $\varphi(P_1\#)$ of subgroups No 3 having pure THREE consequent prime number (${}_{A,B,C}C$) of $P_1 < ({}_AC\text{-}{}_BC\text{-}{}_CC) < P_2^2$ type. At the P.I. section from P_1 to P_2^2 and further, from P_2^2 to $P_1\#$ pure THREE consequent residues of mod($P_1\#$) on all P.I. sections with length not exceeding $2P_2$ of whole numbers with length of every subgroup No 3 at every section is: $R_3 = ({}_CC–{}_AC) \le \mathbf{2}P_1$.

c. $\varphi(P_1\#)$ of subgroups No 2 having pure TWO consequent prime number (${}_{A,B}C$) of P_0, $P_1 < ({}_AC\text{-}{}_BC) < P_2^2$ type. At the P.I. section from P_1 to P_2^2 and further, from P_2^2 to $P_1\#$ pure TWO consequent residues of mod($P_1\#$) on all P. I. sections with length not exceeding $2P_1$ of whole numbers with length of every subgroup No 2 at every section is: $R_2 = ({}_BC–{}_AC) \le \mathbf{2}P_0$.

With: ${}_{A,B,C,D}C$ are consequent residues of mod($P_1\#$) including the primes < $(P_2)^2$. $R_{4\text{-}3\text{-}2}$ is the remainder of the first and the last number of every group No 4-3-2 (of the fractal $P_1\#$). Further, the length of the subgroup No 4-3-2 as the amount of odd numbers, restricted by every group from ${}_AC$ to ${}_{B\text{-}C\text{-}D}C$, from ${}_AP$ to ${}_{B\text{-}C\text{-}D}P$ are $(R_{4,3,2}–2)/2$ odd numbers.

The order of groups (a), (b), (c) rearrangement according to the increasing modulus for visual clarity is indicated in **Table 11**.

5. Proof of theorem

5.1 Proof of section a of the Theorem 1

It is feasible that in P.I. using the first prime number $\le P_n$ (NOT residues of mod ($P_n\#$)), by the only single way, we can form the maximal long P.I. section as the maximal long alternance – array for the 1 least common factor > 1 from every NOT residue of mod($P_{\mathbf{n}}\#$). That is that maximal amount of NOT residues of mod($P_n\#$), maximal long alternance $\le P_{\mathbf{n}}$.

1	2		3		4		5	
Fractal from 1 to $P_n\#$ Composition and its repeating =mod($P_n\#$)	Length of P.I. section which defines values of primes number of this fractal including on the P.I section: $(P_{n+1})^2-(P_{n+1}-2)^2 = 4P_{n+1}$ of whole numbers		φ($P_n\#$) groups No 4 for 4 residues of mod ($P_n\#$) (containing 4 simple) $_AC$-$_BC$-$_CC$-$_DC$ $< P_{n+1}^2$		φ($P_n\#$) groups No 3 for 3 residues of mod ($P_n\#$) (containing 3 simple) $_AC$-$_BC$-$_CC$ $< P_{n+1}^2$		φ($P_n\#$) groups No 2 for 2 residues of mod ($P_n\#$) (containing 2 simple) ($_AC$-$_BC$) $< P_{n+1}^2$	
			Subgroup No 4 length. $R_4 = {}_DC{-}_AC$	Length of every section for the group No 4	Subgroup No 3 length. $R_3 = {}_CC{-}_AC$	Length of every section for the group No 3	Subgroup No 2 length. $R_2 = {}_BC{-}_AC$	Length of every section for the group No 2
1 ÷ $P_n\#$ mod($P_n\#$)	from $(P_n)^2$ to $(P_{n+1})^2$	$\geq 4P_{n+1}$ number	$R_4 \leq 2P_{n+1} < (P_n\# \pm P_{n+1})$	$\leq 2P_{n+2} < (P_n\# \pm P_{n+2})$	$R_3 \leq 2P_n$ (**Table 12**)	$\leq 2P_{n+1}$ number	$R_2 \leq 2P_{n-1}$ (**Table 13**)	$\leq 2P_n$ number
...	...	...	...	...	...	...	...	...
1 ÷ $P_1\#$ mod($P_1\#$)	from $(P_1)^2$ to $(P_2)^2$	$\geq 4P_2$ number	$R_4 \leq 2P_2$	$\leq 2P_3$ number	$R_3 \leq 2P_1$	$\leq 2P_2$ number	$R_2 \leq 2P_0$	$\leq 2P_1$ number
1 ÷ $P_2\#$ mod($P_2\#$)	from $(P_2)^2$ to $(P_3)^2$	$\geq 4P_3$ number	$R_4 \leq 2P_3$	$\leq 2P_4$ number	$R_3 \leq 2P_2$	$\leq 2P_3$ number	$R_2 \leq 2P_1$	$\leq 2P_2$ number
1 ÷ $P_3\#$ mod($P_3\#$)	from $(P_3)^2$ to $(P_4)^2$	$\geq 4P_4$ number	$R_4 \leq 2P_4$	$\leq 2P_5$ number	$R_3 \leq 2P_3$	$\leq 2P_4$ number	$R_2 \leq 2P_2$	$\leq 2P_3$ number
1 ÷ $P_4\#$ mod($P_4\#$)	from $(P_4)^2$ to $(P_5)^2$	$\geq 4P_5$ number	$R_4 \leq 2P_5$	$\leq 2P_6$ number	$R_3 \leq 2P_4$	$\leq 2P_5$ number	$R_2 \leq 2P_3$	$\leq 2P_4$ number
1 ÷ $P_5\#$ mod($P_5\#$)	from $(P_5)^2$ to $(P_6)^2$	$\geq 4P_6$ number	$R_4 \leq 2P_6$	$\leq 2P_7$ number	$R_3 \leq 2P_5$	$\leq 2P_6$ number	$R_2 \leq 2P_4$	$\leq 2P_5$ number
... and so on according to the increasing meanings of the modulus ...								

Table 11.
Loopback of primes number subgroups distribution according to the increasing meanings of the modulus.

Then for every recurrent prime number $P_n = P_1$ at the P.I., formed as recurrent line symmetrical, primary-repeatable periodical fractal = $P_1\#$ or I.R.S. according to mod($P_1\#$), (see the first line of **Table 1**), the maximal long P.I. section, formed as the alternance of the all first primes number $\le P_1$, (NOT residues of mod($P_n\#$)), shall be situated within the P.I. section from C_A to C_B, with as the subgroup is the only maximal long maximal subgroup No 4 (C_1-C_2-C_3-C_4) with the 4 consequent residue of mod($P_1\#$): Type: $C_A = (P_1\#–P_3)..C_1 = (P_1\#–P_2)$.

... $C_2 = (P_1\#–1), C_3 = (P_1\# + 1)$ ... $C_4 = (P_1\# + P_2)$ ... $C_B = (P_1\# + P_3)$. The length of such maximally long subgroup No 4 of mod($P_1\#$), is: max
$R_4 = (C_4–C_1) = (P_1\# + P_2)–(P_1\#–P_2) = 2P_2$ of the whole numbers.

The limit of length of the P.I. section within which from C_A to C_B would be situated maximal as well as all the other $\varphi(P_1\#)$ subgroups No 4 (with 4 residues) of mod($P_1\#$), is: $(C_B–C_A) = (P_1\# + P_3)–(P_1\#–P_3) = 2P_3$ of the whole numbers.

It is genuinely:

At the line-symmetrical, primary-repeatable fractal-$P_1\#$ or I.R.S. according to mod($P_1\#$), $\varphi(P_1\#)$ of the least residue (indexed in the form of $\varphi(P_1\#)$ groups No 4 of mod($P_1\#$), with the alternances $\le P_1$ with different lengths), are situated line-symmetrically relating to the center of symmetry of the fractal-$P_1\#$, of the number = ($P_1\#/2$). That is they are situated reflecting in pairs and are formed by two different ways: (the left and the right sieve of Eratosthenes), to the left and to the right from the symmetry center of the fractal = $P_1\#$ of number = $P_1\#/2$. To the right – for the increasing values numbers of the P.I. from $P_1\#/2$ to $P_1\#$ to the left for the decreasing values of the P.I. $P_1\#/2$ up to 1.

To every left group No 4 of mod($P_1\#$), with the remainder $R_4 = (C_4–C_1)$, is matched by line-symmetrical right group No 4 mod($P_1\#$), with reflecting location of the same first primes number in the same amount and of the same length of the alternance $\le P_1$: $R_4 = (P_1\#–C_1)–(P_1\#–C_4) = (C_4–C_1)$ consult [7 pp. 142–147, 8 pp. 77–84, 9 pp. 109–116].

Besides two not reflecting that is formed solely subgroup of group No 4: with constant reminder for every P_n of type: $R_4 = (P_1\#/2 + 4)–(P_1\#/2–4) = 8$.

And the section of P.I. fractal $P_1\#$ (I.R.S. of mod($P_1\#$) from C_A to C_B), represented by alternance $\le \mathbf{P_1}$ with using of all NOT residues of mod($P_1\#$), (according to the 1 the least > 1), with the subgroup is situated the only maximally long - maximal group No 4 with 4 residues of mod($P_1\#$) (C_1-C_2-C_3-C_4). Type:
$C_A = (P_1\#-P_3)$ $_{...\,3pp3\,...}$ $C_1 = (P_1\#-P_2)$ $_{...\,3pp3\,...}$ $C_2 = (P_1\#-1)$, $C_3 = (P_1\# + 1)$ $_{...\,3pp3\,...}$
$C_4 = (P_1\# + P_2)$ ... $_{...\,3pp3\,...}$ $C_B = (P_1\# + P_3)$.

Thus in the fractal-$P_1\#$-I.R.S. of the mod($P_1\#$), there is only one maximally long subgroup No 4, situated within the maximally long alternance $\le P_1$, using all NOT residues of mod($P_1\#$), at the P.I. section ($P_1\# \pm P_2$) with length maximal $R_4 = 2P_2$ restricting $(R_4–2)/2 = (2P_2–2)/2 = (P_2–1)$ of the odd numbers, situated within the P. I. section, formed solely from ($P_1\#–P_3$) to ($P_1\# + P_3$) with length of (P_3–1) of odd numbers.

It is quite obvious that all the other, line-symmetrical subgroups No 4 of mod ($P_1\#$), situated within the alternances $\le P_1$ with different lengths or NOT residues, of mod($P_1\#$), cannot have the maximal length as they are formed by two different ways, that is they would be shorter than $R_4 < 2P_2$, and situated within the P.I. sections from C_A to C_B with length not exceeding the maximal long P.I. section $(C_B–C_A) \le 2P_3$ of the whole numbers, not exceeding (P_3–1) of the odd numbers.

And so on, for all posterior prime numbers = P_n, at the increasing fractals -$P_n\#$ with n - as the whole number and proves the reality of the values of column No 3 of **Table 11** and item (a) of Theorem 1.

It is feasible that there is such a prime number $P_n = P_{(1)}$, for which the P.I. is the line-symmetrical fractal $P_{(1)}\#$, situated at the P.I. section from 1 to $P_{(1)}\#$ with subgroup No 4 (containing 4 residues) of mod($P_{(1)}\#$) with length = $R_4 > 2^*P_{(2)}$, situated within the alternance of all first primes number $\leq P_{(1)}$ within the P.I. section with length $> 2^*P_{(3)}$, $>(P_{(3)}–1)$ of the odd numbers. Then every subgroup No 4 would be line-symmetrical to the left and to the right from the center of the fractal $P_{(1)}\#$ symmetry of the number $P_{(1)}\#/2$. That is, in the result, we'll get in fractal $P_{(1)}\#$ using all primes number $\leq P_{(1)}$, − we can by more than by one way from the maximally long alternance of all the prime numbers $\leq P_{(1)}$, that is by the sieve of Eratosthenes, focused to the left and to the right (to the left and to the right from the number = $P_{(1)}\#/2$), that is contrary to the taken axiom.

6. The maximal length of P.I. section with maximal long subgroups No 3 (with 3 residues) for the mod($P_2\#$)

At the fractal -$P_1\#$, there are $\varphi(P_1\#)$ subgroups No 4 of mod($P_1\#$), with length $R_4 \leq 2P_2$ of the whole numbers including one maximal long subgroup No 4 of mod ($P_1\#$) with length max $R_4 = 2P_2$ of the whole numbers. At the transition from mod ($P_1\#$) to mod($P_2\#$), the fractal $P_1\#$ and $\varphi(P_1\#)$ of groups No 4 repeat P_2 times. Then at the P.I. section from 1 to $P_2\#$ (at P_2 lines of **Table 1**), we'll get $\varphi(P_1\#)$ columns of groups No 4 of mod($P_1\#$), with length $R_4 \leq 2P_2$, (P_2 lines at the column No 4).

It is quite obvious that in Section 8.1, it is proved that if by number P_2, "eliminate," that is moved to mod($P_2\#$) 1 time every elimination C_2 and C_3, in the column of every $\varphi(P_1\#)$ group No 4 of mod($P_1\#$), than at the P.I. section from 1 to $P_2\#$, (that is at the fractal $P_2\#$), we'll get =$2\varphi(P_1\#)$ groups No 3 of mod($P_2\#$) of the same length, that is $R_4 \leq 2P_2$ of mod($P_1\#$) would become = $R_3 \leq 2P_2$ of mod($P_2\#$) with changing of the alternances composition from $\leq P_1$ to $\leq P_2$.

As all one by one eliminated residues C_2 or C_3, at rearrangement of the groups from No 4 to No3 for the mod($P_2\#$), cannot change the length of none of the subgroups, that is all R_3 would permanently be $\leq 2P_2$, included in $2\varphi(P_1\#)$ groups No 3 of mod ($P_2\#$) there is only one P_2 times repeated, maximally long subgroup No 4 of mod($P_1\#$) with the alternance $\leq P_1$, with length max $R_4 = 2P_2$, that would be

(a) Fractal = $P_2\#$, $\varphi(P_2\#)$ group No 3 of mod($P_2\#$), alternance $\leq P_2$, maxR_3 = **2**P_2, with n = (multiple $P_2 \pm 1)/P_1\#$ = the whole. At the P.I. sections (n$P_1\# \pm P_2$) and $(P_2–n)P_1\# \pm P_2$. Within the limits of P.I. section (n$P_1\# \pm P_3$) with: n and (P_2–n) is line number in **Table 1**								
$\underline{\ldots .C_{A \ldots .}}$	.	$\underline{\ldots \ldots C_1 \ldots}$	11,	$\underline{(кр.P_2) \text{ и } C_2}$	3,	$\underline{\ldots \ldots C_3 \ldots}$	..	$\underline{\ldots .C_B \ldots \ldots}$
$=nP_1\#–P_3$	.	$=nP_1\#–P_2$	3,	$=nP_1\#. \pm .1$	5,	$=nP_1\# + P_2$		$=nP_1\# + P_3$
$\underline{{}_AC = P_2\#–C_B}$		$\underline{{}_1C = P_2\#–C_3}$	7,	$\underline{{}_2C \text{ и } (кр.P_2)}$	7,	$\underline{{}_3C = P_2\#–C_1}$		$\underline{{}_BC = P_2\#–C_A}$
$(P_2–n)P_1\#–P_3$		$(P_2–n)P_1\#–P_2$	5,	$(P_2\text{-}n)P_1\# \pm 1$	3,	$(P_2–n)P_1\# + P_2$		$(P_2–n)P_1\# + P_3$
			3		11.			
(b) Fractal = $P_3\#$, $\varphi(P_3\#)$ group No 3 of mod($P_3\#$), alternance $\leq P_3$, maxR_3 = **2**P_3, with n = (multiple $P_3 \pm 1)/P_2\#$ = the whole. At the P.I. sections (n$P_2\# \pm P_3$) and $(P_3–n)P_2\# \pm P_3$. Within the limits of P.I. section (n$P_2\# \pm P_4$) with: n and (P_3–n) is line number on **Table 2**								
$\underline{\ldots C_{A \ldots .}}$	.	$\underline{\ldots \ldots C_1 \ldots}$	11,	$\underline{(кр.P_3) \text{ и } C_2}$	3,	$\underline{\ldots \ldots C_3 \ldots}$	..	$\underline{\ldots .C_B \ldots \ldots}$
$=nP_2\#–P_4$	.	$=nP_2\#–P_3$	3,	$=nP_2\#. \pm .1$	5,	$=nP_2\# + P_3$		$=nP_2\# + P_4$
$\underline{{}_AC = P_3\#–C_B}$		$\underline{{}_1C = P_2\#–C_3}$	7,	$\underline{{}_2C \text{ и } (кр.P_3)}$	7,	$\underline{{}_3C = P_2\#–C_1}$		$\underline{{}_BC = P_2\#–C_A}$
$(P_3–n)P_2\#–P_4$		$(P_3–n)P_2\#–P_3$	5,	$(P_3–n)P_2\# \pm 1$	3,	$(P_3–n)P_2\# + P_3$		$(P_3–n)P_2\# + P_4$
			3		11,			
(c) And so on for every mod($P_n\#$), maxR_3 = $2P_n$, n = $(кр.P_n \pm 1)/P_{n\text{-}1}\#$ = the whole, $P_{n\text{-}(1)}$-primes								

Table 12.
*Type and formula for indexing of two line-symmetrical, maximally long subgroups No 3 (having 3 residues) at the increasing fractal according to the increasing modulus (**Tables 11** and **14**).*

(b) Fractal = $P_3\#$, $\varphi(P_3\#)$ group No 2 of mod$(P_3\#)$, alternance $\leq P_3$, maxR_2 = $\mathbf{2}P_2$, with n = (multiple $P_2{*}P_3 \pm 1)/P_1\#$ = the whole. At the P.I. sections $(nP_1\# \pm P_2)$ and $(P_2{*}P_3-n)P_1\# \pm P_2$. Within the limits of P.I. section $(nP_1\# \pm P_3)$., with: n and $(P_2{*}P_3-n)$ is line number on **Table 1**								
… …C_A … . $=nP_1\#-P_3$	. .	… …C_1 … $=nP_1\#-P_2$	.. ..	(кр.P_2) (кр.P_3) $=nP_1\#. \pm .1$	3, 5,	… … C_2 … $=nP_1\# + P_2$	. .	… .C_B … … $=nP_1\# + P_3$
$_AC = P_3\#-C_B$ $(P_2P_3-n)P_1\#-P_3$		$_1C = P_3\#-C_3$ $(P_2P_3-n)P_1\#-P_2$	7, 5, 3	(кр.P_3) (кр.P_2) $(P_2P_3-n)P_1\# \pm 1$	7.	$_2C = P_3\#-C_1$ $(P_2P_3-n)P_1\# + P_2$		$_BC = P_3\#-C_A$ $(P_2P_3-n)P_1\# + P_3$
(c) Fractal = $P_4\#$, $\varphi(P_4\#)$ group No 2 of mod$(P_4\#)$, alternance $\leq P_4$, maxR_2 = $\mathbf{2}P_3$, with n = (multiple $P_4{*}P_3 \pm 1)/P_2\#$ = the whole. At the P.I. sections $(nP_2\# \pm P_3)$ and $(P_4{*}P_3-n)P_2\# \pm P_3$. Within the limits of P.I. section $(nP_2\# \pm P_4)$., with: n-and $(P_4{*}P_3-n)$ is line number on **Table 2**								
… …C_A … . $=nP_2\#-P_4$	. .	… …C_1 … $=nP_2\#-P_3$	.. ..	(кр.P_4) (кр.P_3) $=nP_2\#. \pm .1$	3, 5,	… … C_2 … $=nP_2\# + P_3$	. .	… .C_B … … $=nP_2\# + P_4$
$_AC = P_4\#-C_B$ $(P_4P_3-n)P_2\#-P_4$		$_1C = P_4\#-C_3$ $(P_4P_3-n)P_2\#-P_3$	7, 5, 3	(кр.P_3) (кр.P_4) $(P_4P_3-n)P_2\# \pm 1$	7.	$_2C = P_4\#-C_1$ $(P_4P_3-n)P_2\# + P_3$		$_BC = P_4\#-C_A$ $(P_4P_3-n)P_2\# + P_4$
(d) And so on for every mod$(P_{n+1}\#)$, maxR_2 = $2P_n$, n = $(P_{n+1}{*}P_n \pm 1)/P_{n-1}\#$ = the whole, P_n-primes								

Table 13.
*Type and formula for indexing of two line-symmetrical, maximally long subgroups No 2 (having 2 residues) at the increasing fractal according to the increasing modulus (**Tables 11** and **14**).*

restructured into two maximally long line-symmetrical groups No 3 of mod$(P_2\#)$ by "eliminating" the residues C_2 and C_3 by number multiple to P_2, (1 time at P_2 lines). As all the other $2\varphi(P_1\#)-2$ subgroups, changed from No 4 to No 3 for mod$(P_2\#)$ are shorter than (P_2-1) of the odd numbers that is: $R_3 < 2P_2$. In Sections 8.1 and 8.3, there are no other ways of making or changing the subgroups No 3 of mod$(P_2\#)$ with length $R_3 > 2P_2$.

Order, type, and formula of indexing of two subgroups No 3 according to the increasing modulus are represented in **Table 12** a, b, c.

The length of these two line-symmetrical subgroups No 3 of mod$(P_2\#)$, that is the length of alternance $\leq P_2$ from C_1 to C_3, is maximal $R_3 = (C_3-C_1) = (nP_1\#. \pm P_2)-(nP_1\#.-P_2) = 2P_2$; (P_2-1) of the odd numbers. Two of these subgroups No 3 are situated within P.I. section $(nP_1\#. \pm P_3)$ with length $(C_B-C_A) = 2P_3$; (P_3-1) of the odd numbers from C_A to C_B. Numerical values of these two maximal subgroups No 3 are defined according to the formula (multiple P_2 and C_2 = multiple $P_2 \pm 2$) is $(nP_1\#. \pm .1)$ and $(P_2-n)P_1\# \pm 1$, with n and (P_2-n) define the number of the line for the group No 3 of mod$(P_2\#)$ in column P_2 and the repeated maximal of the group No 4 of mod$(P_1\#)$ with the period = $P_1\#$ (consult **Table 1**). That is n = (multiple $P_2. \pm 1)/P_1\#$ = the whole $< P_2/2$.

Whereas, it is quite obvious that in and proved in Section 8.2, the other subgroups No 3 of mod$(P_2\#)$, with different lengths, changed from groups No 4 would be within P.I. section, with limit length = $2P_3$ of the whole numbers.

And so on, for every of all posterior primes = P_n, at the increasing fractals $P_n\#$, with n is the whole, represented in **Tables 11** and **14** (the proof is indicated in Section 10). (Numerical illustrations are in **Table 15**).

7. The maximal length of the P.I. section, where two maximally long subgroups No 2 (with 2 residues) for the mod$(P_3\#)$

Representing as one line, the first P_2 lines in **Table 1** we'll get the fractal $P_2\#$ according to the mod$(P_2\#)$ – I.R.S. at mod$(P_2\#)$, that is situated at the P.I. section

1	2	3	4
Fractal composition - $P_n\#$	$\varphi(P_n\#)$ groups No 4 of mod $(P_n\#)$. Including 1 group = maxR_4 **Proof in Section 5**	Loopback (P_n-3) $\varphi(P_{n-1}\#)$ groups No 3 + loopback $2\varphi(P_{n-1}\#)$ group No 4 $[\mathrm{mod}P_{n-1}\#]=$ $=\varphi(P_n\#)$ groups No 3 of mod$(P_n\#)$ including 2 groups No 3 = max R_3 of mod $(P_n\#)$ **Proved in Sections 6 and 8**	Loopback (P_n-2) $\varphi(P_{n-1}\#)$ Groups No 2 + loopback $\varphi(P_{n-1}\#)$ groups No 3 $[\mathrm{mod}P_{n-1}\#]=$ $=\varphi(P_n\#)$ groups No 2 of mod$(P_n\#)$ including 2 groups No 2 = max R_2 mod $(P_n\#)$ **Proved in Sections 7 and 9**
By repeating P_1 times the line of fractal -$P_0\#$ and group No 4-3-2, we'll get $\varphi(P_0\#)$ columns of groups No 4-3-2 of mod$(P_0\#)$ within the alternances $\leq P_0$ (P_1 line in the column). By "eliminating" 1 number multiple -P_1 (in line of every column No 4-3-2), that is by transiting this group for mod$(P_1\#)$ with changing of its length from $R_{(4,3)}$ to $R_{3,2}$ and alternances composition from $\leq P_0$ to $\leq P_1$. At the P.I. section from 1 to $P_1\#$ we'll get the fractal -$P_1\#$ of mod$(P_1\#)$. $\varphi(P_1\#)$ groups No 4-3-2			
Amount of groups of Length and location	$\varphi(P_1\#)$ groups No 4. R_4. $\leq 2P_2$ at every sectione $<2P_3$ is group No 4	$\varphi(P_0\#)^*$ $^*(P_1-3)$ $R_{(3)} \leq 2P_0$ + $2\varphi(P_0\#)$ $R_{(4)} \leq 2P_1$ = $\varphi(P_1\#)$ group No 3 $R_3 \leq 2P_1$	$\varphi(P_0\#)^*$ $^*(P_1-2)$ $R_{(2)} \leq 2P_{..}$ + $\varphi(P_0\#)$ $R_{(3)} \leq 2P_0$ = $\varphi(P_1\#)$ group No 2 $R_2 \leq 2P_0$
		At every section $<2P_2$ is group No 3	At every section $<2P_1$ is group No 2
Formula = n Type max R Groups and location	$(P_1\# \pm P_2)$ max $R_4 = 2P_2$ At the section with length = $(P_1\# \pm P_3)$	n = (multiple $P_1 \pm 1$) / $P_0\#$ = the whole $(nP_0\#. \pm .P_1)$ and $(P_1-n)P_0\#. \pm .P_1$ 2 groups No 3- max $R_3 = 2P_1$. On the segment of length = $(nP_0\# \pm P_2)$	n = (multiple $P_0{}^*P_1 \pm 1)/P_{(.)}\#$ = the whole $(nP_{(.)}\# \pm P_0)$ and $(P_0P_1-n)P_{(.)}\# \pm P_0$ 2 groups No2- max $R_2 = 2P_0$. On the segment of length = $(nP_{(..)}\# \pm P_1)$
By repeating P_2 times the line of fractal –$P_1\#$ and group No 4-3-2, we'll get $\varphi(P_1\#)$ columns of groups No 4-3-2 of mod$(P_1\#)$ within the alternances $\leq P_1$ (P_2 line in the column). By "eliminating" 1 number multiple -P_2 (in line of every column No 4-3-2), that is by transiting this group for mod$(P_2\#)$ with changing of its length from $R_{(4,3)}$ to $R_{3,2}$ and alternances composition from $\leq P_1$ to $\leq P_2$. At the P.I. section from 1 to $P_2\#$ we'll get the fractal –$P_2\#$ of mod$(P_2\#)$. $\varphi(P_2\#)$ groups No 4-3-2			
Amount of groups of length and location	$\varphi(P_2\#)$ groups No 4. R_4. $\leq 2P_3$ at every sectione $<2P_4$ is group No 4	$\varphi(P_1\#)^*$ $^*(P_2-3)$ $R_{(3)} \leq 2P_1$ + $2\varphi(P_1\#)$ $R_{(4)} \leq 2P_2$ = $\varphi(P_2\#)$ group No 3 $R_3 \leq 2P_2$	$\varphi(P_1\#)^*$ $^*(P_2-2)$ $R_{(2)} \leq 2P_0$ + $\varphi(P_1\#)$ $R_{(3)} \leq 2P_1$ = $\varphi(P_2\#)$ group No 2 $R_2 \leq 2P_1$
		At every section $<2P_3$ is group No 3	At every section $<2P_2$ is group No 2
Formula = n Type max R Groups and location	$(P_2\# \pm P_3)$ max $R_4 = 2P_3$ At the section with length = $(P_2\# \pm P_4)$	n = (multiple $P_2 \pm 1$) / $P_1\#$ = the whole $(nP_1\#. \pm .P_2)$ and $(P_2-n)P_1\#. \pm .P_2$ 2 groups No 3- max $R_3 = 2P_2$. On the segment of length = $(nP_1\# \pm P_3)$	n = (multiple $P_2{}^*P_1 \pm 1)/P_0\#$ = the whole $(nP_0\# \pm P_1)$ and $(P_2P_1-n)P_0\# \pm P_1$ 2 groups No2- max $R_2 = 2P_1$. On the segment of length = $(nP_0\# \pm P_2)$
By repeating P_3 times the line of fractal –$P_2\#$ and group No 4-3-2, we'll get $\varphi(P_2\#)$ columns of groups No 4-3-2 of mod$(P_2\#)$ within the alternances $\leq P_2$ (P_3 line in the column). By "eliminating" 1 number multiple–P_3 (in line of every column No 4–3-2), that is by transiting this group for mod$(P_3\#)$ with changing of its length from $R_{(4,3)}$ to $R_{3,2}$ and alternances composition from $\leq P_2$ to $\leq P_3$. At the P.I. section from 1 to $P_3\#$ we'll get the fractal –$P_3\#$ of mod$(P_3\#)$. $\varphi(P_3\#)$ groups No 4-3-2:			
Amount of groups of length and location	$\varphi(P_3\#)$ groups No 4. R_4. $\leq 2P_4$ at every section $<2P_5$ is group No 4	$\varphi(P_2\#)^*$ $^*(P_3-3)$ $R_{(3)} \leq 2P_2$ + $2\varphi(P_2\#)$ $R_{(4)} \leq 2P_3$ = $\varphi(P_3\#)$ group No 3 $R_3 \leq 2P_3$	$\varphi(P_2\#)^*$ $^*(P_3-2)$ $R_{(2)} \leq 2P_1$ + $\varphi(P_2\#)$ $R_{(3)} \leq 2P_2$ = $\varphi(P_3\#)$ group No 2 $R_2 \leq 2P_2$
		At every section $<2P_4$ is group No 3	At every section $<2P_3$ is group No 2
Formula = n Type max R Groups and location	$(P_3\# \pm P_4)$ max $R_4 = 2P_4$ At the section with length = $(P_3\# \pm P_5)$	n = (multiple $P_3 \pm 1$) / $P_2\#$ = the whole $(nP_2\#. \pm .P_3)$ and $(P_3-n)P_2\#. \pm .P_3$ 2 groups No 3- max $R_3 = 2P_3$. On the segment of length = $(nP_2\# \pm P_4)$	n = (multiple $P_2{}^*P_3 \pm 1)/P_1\#$ = the whole $(nP_1\# \pm P_2)$ and $(P_2P_3-n)P_1\# \pm P_2$ 2 groups No 2- max $R_2 = 2P_2$. On the segment of length = $(nP_1\# \pm P_3)$
...	...	...	...
And so on for the increasing meanings of modulus = mod $P_n\#$. with: $P_n\#$ - primorial. $P_{(\dots)} < P_0 < P_1 < P_2 \dots < P_n$ are the consequent primes. $C_{1\text{-}2\text{-}3\text{-}4}$ are primes and residues of mod$(P_n\#)$ $R_{4\text{-}3\text{-}2} = (C_{4-3-2} - C_1)$ that is length of the group = $(R_{4\text{-}3\text{-}2}-2)/2$ of odd numbers.			

Table 14.
The loopback of prime's groups and residues rearrangement according to the increasing modulus. With max R = const from mod($P_1\#$*) to mod(*$P_3\#$*).*

from 1 to $P_2\#$ and represented in 1 line of **Table 2** where $\varphi(P_2\#)$ of groups No 3 of $\mathrm{mod}(P_2\#)$ are situated with length $R_3 \le 2P_2$ of the whole numbers. In this number, the two maximally long groups No 3 of $\mathrm{mod}(P_2\#)$ with length max $R_3 = 2P_2$ of the whole numbers are represented. Then on the P.I. section from 1 to $P_3\#$ (at P_3 lines of **Table 2**), we'll get $\varphi(P_2\#)$ columns of group No 3 of $\mathrm{mod}(P_2\#)$, with length $R_3 \le 2P_2$, (P_3 lines in columns of groups No 3).

It is quite obvious, that in Section 9.1, it is proved that if by number P_3, "eliminate," that is to change for the model $\mathrm{mod}(P_3\#)$ residue C_2 in the column of every $\varphi(P_2\#)$ group No 3 of $\mathrm{mod}(P_2\#)$, then at the P.I. section from 1 to $P_3\#$, that is at the fractal $P_3\#$, we'll get $\varphi(P_2\#)$ groups No 2 of $\mathrm{mod}(P_3\#)$ of the same length, that is $R_3 \le 2P_2$ $\mathrm{mod}(P_2\#)$, would become $R_2 \le 2P_2$ of $\mathrm{mod}(P_3\#)$ with changing the structure of alternance from $\le P_2$ to $\le P_3$.

As any eliminated residue C_2, during the rearrangement of groups from No 3 to No 2 for the $\mathrm{mod}(P_3\#)$ cannot change the length on no subgroup that is all R_2 would permanently $\le 2P_2$, and included in $\varphi(P_2\#)$ groups No 3 of $\mathrm{mod}(P_2\#)$ there are two uncial, repeated P_3 times maximally long subgroups No 3 of $\mathrm{mod}(P_2\#)$ with the alternance $\le P_2$ with length maximal $R_3 = 2P_2$, that would be rearranged into two maximally long line-symmetrical groups No 2 of $\mathrm{mod}(P_3\#)$ by "eliminating" the residues C_2 with the number multiple P_3, (1 time in P_3 lines). As all the other $\varphi(P_2\#)$–2 subgroups, rearranged from No 3 to No 2 for the $\mathrm{mod}(P_3\#)$, are shorter than (P_2–1) of the off numbers, that is: $R_2 < 2P_2$, and in Sections 9.1 and 9.3, it is proved that there are no other ways of comparing or rearranging of the subgroups No 2 of $\mathrm{mod}(P_2\#)$ with length $R_3 > 2P_2$. The order, type, and formula of indexing of two subgroups No 2 according to the increasing modulus are represented in **Table 13** b, c, d.

The length of these two line-symmetrical subgroups No 2 of $\mathrm{mod}(P_3\#)$, the length of alternance $\le P_3$, from C_1 to C_2, that is max $R_2 = (C_2–C_1) = (nP_1\# + P_2)–(nP_1\#–P_2) = 2P_2$; ($P_2$–1) of odd numbers. Two of these subgroups No 2 are situated within the P.I. section ($nP_1\#. \pm P_3$) with length $(C_B–C_A) = 2P_3$; (P_3–1) of odd numbers from C_A to C_B. The numerical values of these maximal groups No 2 is defined according to the formula: (multiple P_2 and multiple P_3) is ($nP_1\#. \pm .1$) and $(P_2P_3–n)P_1\#. \pm .1.$, with n and $(P_2P_3–n)$ define the line number for the group No 2 of $\mathrm{mod}(P_3\#)$ in the column $-P_2{}^*P_3$ of the duplication of the max group No 4 of mod ($P_1\#$) with the period = $P_1\#$ (**Table 1**). That is n = (multiple $P_2{}^*P_3. \pm 1)/P_1\#$ = the whole $< P_2{}^*P_3/2$.

Herewith, it is quite obvious, and proved in Section 9.3, that all the other subgroups No 2 of $\mathrm{mod}(P_3\#)$, with different lengths, rearranges from groups No 3 would be within the P.I., with length not exceeding the limit $=2P_3$ of the wholes.

And so on, for every of all posterior primes = P_n, at the increasing fractals $P_n\#$, with n is the whole, represented in **Tables 11** and **14** (the proof is indicated in Section 10) (numerical illustrations are in **Table 16**).

8. The loopback of rearrangement for $\varphi(P_n\#)$ groups No 3 (with 3 residues) from $\mathrm{mod}(P_{n-1}\#)$ for $\mathrm{mod}(P_n\#)$

The loopback order of rearrangement of $\varphi(P_n\#)$ groups No 3, according to the increasing modulus, that are represented in column No 3 of **Table 14** at Section 8 are examined by steps, for every recurrent increasing fractal $-P_n\#$:

During the transition from $\mathrm{mod}(P_1\#)$ to $\mathrm{mod}(P_2\#)$ of the fractal $-P_1\#$ (1 line of **Table 1**) and every from $\varphi(P_1\#)$ groups No 4-3-2 of $\mathrm{mod}(P_1\#)$ are repeated P_2 times. That at the P.I. section from 1 to $P_2\#$, we'll get $\varphi(P_1\#)$ columns of No 4-3-2 groups (P_2 limes at the column). Number P_2 according to diagonals of P_2 lines

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for the fractal −7# = 210, mod(7#), maxR$_3$ = 2*7, at the P.I. section (C_B–C_A) = 2*11, n = 3								
C_A = 79 **${}_AC$ = 109**	**...**	**C_1 = 83.** **${}_1C$ = 113**	**5,3**	**C_2 = 89 и (7*13)** **(7*17) и ${}_2C$ = 121**	**3,5**	**C_3 = 97** **${}_3C$ = 127**	**..**	**C_B = 101** **${}_BC$ = 131**
for the fractal −*11#* = 2310, mod(*11#*), maxR$_3$ = 2*11, at the P.I. section (C_B–C_A) = 2*13, n = 1								
.C_A = 197. =n7#–*13* .${}_AC$ = 2087. =11#–C_B	...	..C_1 = 199. =n7#–*11* ${}_1C$ = 2089. =11#–C_3	3,7, 5,3	(11*19),C_2 = 211 =n7#. ± .*1* ${}_2C$ = 2099,(11*191) =(11–n)7# ± 1	3,5, 7,3,	..C_3 = 221 ... =n7# + 11 ...${}_3C$ = 2111 (11–n)7# + 11	..	.C_B = 223 =n7# + 13 ${}_BC$ = 2113. (11–n)7# + 13
for the fractal −*13#* = 30,030, mod(*13#*), maxR$_3$ = 2*13, at the P.I. section (C_B–C_A) = 2*17, n = 3								
.C_A = 6913. =n11#–*17* .${}_AC$ = 23,083. =13#–C_B	. ...	..C_1 = 6917. =n11#–*13* ${}_1C$ = 23,087. =13#–C_3	11, 3,7, 5,3	(13*533),C_2 = 6931 =n11#. ± .*1* ${}_2C$ = 23,099,(13*1777) =(13–n)11# ± 1	3,5, 7,3, 11,	C_3 = 6943 =n11# + 13 ...${}_3C$ = 23,113 (13–n)11# + 13	..	.C_B = 6947 =n11# + 17 ${}_BC$ = 23,117. (13–n)11# + 17
for the fractal −*17#* = 510,510,mod(*17#*), maxR$_3$ = 2*17, at the P.I. section (C_B–C_A) = 2*19,n = 2								
.C_A = 60,041. =n13#–*19* .${}_AC$ = 450,431. =17#–C_B	. ...	..C_1 = 60,043. =n13#–*17* ${}_1C$ = 450,433. =17#–C_3	13, 11, 3,7, 5,3	C_2 = 60,059,(17*3533) =n13#. ± .*1* (17*26497), ${}_2C$ = 450,451 =(17–n)13# ± 1	3,5, 7,3, 11, 13,	C_3 = 60,077.. =n13# + 17 ...${}_3C$ = 450,467 (17-n)13# + 17	..	.C_B = 60,079 =n13# + 19 ${}_BC$ = 450,469 (17–n)13# + 19
for the fractal −*19#* = 9,699,690,mod(*19#*), maxR$_3$ = 2*19, at the P.I. section (C_B –C_A) = 2*23, n = 1								
C_A = 510,487 ${}_AC$ = 9,189,157	...	C_1 = 510,491 ${}_1C$ = 9,189,161	17.. 5,3	C_2 = 510,509 и кр.19. Кр.19 и ${}_2C$ = 9,189,181	3,5, ...17,	..C_3 = 510,529 ...${}_3C$ = 9,189,199	..	.C_B = 510,533 ${}_BC$ = 9,189,203

Table 15.
The numerical examples of the two line-symmetrical maximally long subgroups No 3 (containing 3 residues = $C_{1\text{-}2\text{-}3}$), according to the increasing modulus.

"eliminates," that is rearranges to the mod($P_2\#$) one time every of P_2 repeated numbers of the P.I. section from 1 to $P_1\#$ (1 number at every P_2 line of every column No 4 and No 3), "eliminating" the residues C_1-C_2-C_3-C_4 in the groups No 4 (consult Section 8.1), and ALL numbers, besides the residues C_1-C_2-C_3 in the groups No 3 (consult Section 8.3).

8.1

It is quite obvious, that after "elimination" from every $\varphi(P_1\#)$ column of the group No 4 of mod($P_1\#$) one time every residue C_2 and C_3, we'll get at P_2 lines of every column of the No 4 groups of mod($P_1\#$), –TWO groups No 3 of mod($P_2\#$)

That is, we'll get $2\varphi(P_1\#)$ subgroups No 3 of mod($P_2\#$) with invariance length of the previous groups, that is R_4 $\varphi(P_1\#)$ groups of No 4 mod($P_1\#$), would become = R_3 for $2\varphi(P_1\#)$ groups No 3 of mod($P_2\#$), with changing the alternance structure from $\le P_1$ to $\le P_2$, that are situated at the P.I section from 1 to $P_2\#$ that is at the fractal-$P_2\#$.

Herewith within the $2\varphi(P_1\#)$ groups No 3 of mod($P_2\#$), there are accounted all residues = C_1 and = C_4 of mod($P_1\#$) and alternances $\le P_2$ of such rearranged groups from No 4 of mod($P_1\#$) to No 3 of mod($P_2\#$). As along of P_2 duplication of the three adjoined groups No 4 of mod($P_1\#$), represented in **Table 3**, we'll get **Table 17**, with every residue = C_2 or = C_3, situated at one of the 3 lines of **Table 3** (for example, at line a of **Table 17**), is accounted as residue C_1 or C_4 at the other two adjoined groups No 4 (at lines: b or c of **Table 17**), where they are situated within 4 consequent residues, going as the second or the third, that is they are "excluded" as $C_{2\ \mathrm{or}\ 3}$ in these adjoined groups (lines):

- for line (b) C_2, C_3. (C_4 = multiple to P_2), C_3 we'll get the alternance $\le P_2$. $R_3 < 2P_2$.
- for line (c) C_2, (C_1 = multiple to P_2), C_2, C_3 we'll get the alternance $\le P_2$. $R_3 < 2P_2$.

Thus, "eliminating" that means transition to mod($P_2\#$), one time every 4 residue in $\varphi(P_1\#)$ groups No 4 of mod($P_1\#$), at the P.I. sections from 1 to $P_2\#$, that is at the fractal -$P_2\#$, we'll, get the loopback, represented as $2\varphi(P_1\#)$ groups No 3 of mod ($P_2\#$) type: $C_{1\ pp}C_{2\ pp}(C_{2-3}$ = multiple $P_2)$ $_{pp}C_3$, with the alternances $\le P_2$, length $R_3 \le 2P_2$. Including, pure 2 subgroups No 3 of mod($P_2\#$) with length maximal $R_3 = 2P_2$.

8.2

Herewith, it is quite obvious that every three consequent residues of every subgroup No 3 according to the increasing group of mod($P_n\#$), represented in **Table 17**, are still within the P.I. section with length not exceeding - $2P_{n+1}$ of the whole numbers, as "eliminated" residues $C_{2;3}$ and $C_{1;4}$ of mod($P_n\#$) doesn't change the location of every subgroup No 3. That is, we'll get at the three adjoined groups No 3 lines of **Table 17** for the mod($P_2\#$): type (a) =(C_B–C_A) < $2P_3$; type (b) = (C_2–C_1) < $2P_3$; type (c) =(C_4–C_3) < 2_3.

Including pure two subgroups No 3 max $R_3 = 2P_2$, located within the maximally long section with length = $2P_3$ of the whole numbers. (The rearrangement is studied in Section 6).

8.3

Within the P_2 duplications $\varphi(P_1\#)$ of the groups No 3 of mod($P_1\#$), number P_2 "eliminated," that is rearranges to the mod($P_2\#$) 1 time every of all previously eliminated numbers of group No 3, except three residues C_1–C_2–C_3. That is, transits to the mod($P_2\#$) (P_2–3) of No 3 groups in every $\varphi(P_1\#)$ column of No 3 groups.

Then at the P.I. section from 1 to $P_2\#$, (that is included into fractal -$P_2\#$), we'll get the loop back, represented as (P_2–3) $\varphi(P_1\#)$ of No 3 groups repetition for the mod(P_2#) with "changing" the alternance from $\leq P_1$ to $\leq P_2$. Without changes the groups No 3 length for the mod(P_2#), R_3 = const and numbers composition within the alternances $\leq P_2$; that is the previously "eliminated" $\leq P_1$, according to the 1 least >1 from the number are accounted. Type: $C_{1\ pp}C_{2\ pp}C_3$.

With: R_3 = const =? (with to mod(P_2#) $R_3 << R_4 = 2P_2$).

8.4

Total at the fractal -P_2# at the P.I. section from 1 to P_2# we'll get the loopback of the rearranged groups No 3 for the mod(P_2#) represented in Sections 8.1 and 8.3.

That is: $2\varphi(P_1\#)$ (Section 8.1 with the length = $R_3 \leq 2P_2$) + $\varphi(P_1\#)(P_2–3)$ (Section 8.3 with the length $R_3 << R_4 = 2P_2$) = $P_2\varphi(P_1\#)–3\varphi(P_1\#) + 2\varphi(P_1\#) = P_2\ \varphi(P_1\#)–\varphi(P_1\#) = \varphi(P_1\#)(P_2–1) = \varphi(P_2\#)$ of the subgroups No 3 for the mod(P_2#), included at the alternances $\leq P_2$ with length $R_3 \leq 2P_2$, including two maximally long subgroups No 3 max $R_3 = 2P_2$.

8.5

As all $\varphi(P_2\#)$ of the subgroups No 3 of mod(P_2#) are examined in Sections 8.1 and 8.3, with all eliminated one time residues C_2 or C_3, examined in Section 8.1, cannot change the length = $R_3 \leq 2P_2$, none of the $2\varphi(P_1\#)$ groups, rearranged from No 4 to No 3 for the mod(P_2#). Herewith the residues C_1 and C_4 are also accounted in two adjoining groups of **Table 17** as C_2 or C_3. And indiscriminately $\varphi(P_1\#)(P_2–3)$ groups No 3 for the mod(P_2#), examined in Section 8.3 are shorter than limit $R_3 = 2P_2$.

So, there are no other ways to make groups No 3 of mod(P_2#) with length $R_3 > 2P_2$, besides the way to form the maximally long subgroups No 3 with length max $R_3 = 2P_2$, represented in Sections 6 and 8.1.

8.6

Thus we got, that for every recurrent = P_2, $\varphi(P_2\#)$ residues of mod(P_2#) situated in the fractal -P_2# (P_2 lines of **Table 1**), represented as loop back $\varphi(P_2\#)$ of subgroups No 3 (3 residues of mod(P_2#), represented in Section 8.4), that is pure THREE consequent prime ($_{A,B,C}C$) type: $P_2 < (_AC\text{-}_BC\text{-}_CC) < P_3^2$ at the P.I. section from P_2 to P_3^2. And further, from P_3^2 to P_2#, pure THREE consequent residue of mod(P_2#) at every P.I. section, with length not exceeding $2P_3$ of the whole numbers (see Section 8.2), with length of every subgroup No 3 at every section is $R_3 = (_CC–_AC) \leq \mathbf{2}P_2$ (consult Sections 8.1 and 8.4).

And so on, for every of all eventual primes = P_n, represented as the loopback of groups distribution of the residues No 3 at the increasing fractals -P_n# according to the increasing meanings of modulus-mod(P_n#) that proves the validity of section (b) of the Theorem 1 (loopback of groups No 3 is represented in column No 3 of **Table 14**).

9. The loopback of rearrangement for φ(P_n#) groups No 2 (2 residues) from the mod(P_{n-1}#) to mod(P_n#)

The looped back order of rearrangement $\varphi(P_n\#)$ of No 2 groups, according to the increasing modulus, that are represented in column No 4 of **Table 14**, in Section 9 are examined by steps for every recurrent increasing fractal -P_n#:

Representing the first P_2 lines in **Table 1** as one line, we'll get the fractal -P_2# according to mod(P_2#)-I.R.S of mod(P_2#) at the P.I. sections from 1 to P_2# (1 line of **Table 2**).

With $\varphi(P_2\#)$ line-symmetrical the least residues of mod(P_2#), which according to Section 2 are indexed according to $\varphi(P_2\#)$ groups of residues No 4-3-2 for the mod(P_2#).

for the fractal - *7#* = 210, mod(*7#*), maxR_2 = 2*5, section (C_B–C_A) = 2*7, n = 1								
C_A = –1 ${}_AC$ = 197	...	C_1 = 1. ${}_1C$ = 199	3.	(5) и (7) (7*29) и (5*41)	3.	C_2 = 11 ${}_2C$ = 209	..	.C_B = 13 ${}_BC$ = 211
for the fractal - *11#* = 2310, mod(*11#*), maxR_2 = 2*7, section (C_B–C_A) = 2*11, n = 4								
.C_A = 109. =n5#–*11* .${}_AC$ = 2179. =11#–C_B	. ...	..C_1 = 113. =n5#–*7* ${}_1C$ = 2183. =11#–C_2	5, 3.	(7*17),(11*11) =n5#. ± *.1* = 120 ± 1 (11*191),(7*313) =(7*11–n)5# ± 1	3, 5.	..C_2 = 127... =n5# + 7 ...${}_2C$ = 2197... (7*11–n)5# + 7	..	.C_B = 131 =n5# + 11 ${}_BC$ = 2201. (7*11–n)5# + 11
for the fractal - *13#* = 30,030, mod(*13#*), maxR_2 = 2*11, section (C_B–C_A) = 2*13, n = 45								
.C_A = 9437. =n7#–*13* .${}_AC$ = 20,567. =13#–C_B	. ...	..C_1 = 9439. =n7#–*11* ${}_1C$ = 20,569. =13#–C_2	3, 7, 5. 3.	(11*859),(13*727) =n7#. ± *.1* = *9450 ± 1* (13*1583),(11*1871) =(11*13–n)7# ± 1	3, 5, 7, 3.	... C_2 = 9461 =n7# + 11 ...${}_2C$ = 20,591.. (11*13-n)7# + 11	..	.C_B = 9463 =n7# + 13 ${}_BC$ = 20,593.. (11*13–n)7# + 13
for the fractal - *17#* = 510,510,mod(*17#*), maxR_2 = 2*13, section (C_B–C_A) = 2*17,n = 94								
.C_A = 217,123. =n11#–*17* .${}_AC$ = 293,353. =17#–C_B	. ...	..C_1 = 217,127. =n11#–*13* ${}_1C$ = 293,357. =17#–C_2	11, 3, 7, 5, 3.	(13*16703),(17*12773) =n11#. ± *.1* = *217,140± 1* (17*17257),(13*22567) =(13*17–n)11# ± 1	3 5, 7, 3, 11,	..C_2 = 217,153.. =n11# + 13 ...${}_2C$ = 293,383 (13*17–n)11# + 13	..	.C_B = 217,157 =n11# + 17 ${}_BC$ = 293,387 (13*17–n)11# + 17
for the fractal - *19#* = 9,699,690,mod(*19#*), maxR_2 = 2*17, section (C_B –C_A) = 2*19, n = 2								
C_A = 60,041 ${}_AC$ = 9,639,611	...	C_1 = 60,043 ${}_1C$ = 9,639,613	13 3	кр.19., кр.17 = 60,061 Кр.17.,кр.19 = 9,639,631	3, .13.	..C_2 = 60,077 ${}_2C$ = 9,639,647	..	.C_B = 60,079 ${}_BC$ = 9,639,649

Table 16.
The numerical examples of the two line-symmetrical maximally long subgroups No 2 (containing 2 residues), according to the increasing modulus.

Type-(a) group from 4 to No 3 (C_1–C_4)	C_A	**C_1 (C_2 = mtp.P_2) *или* (C_3 = mtp.P_2), C_4**	C_B	**with R_4 is R_3 (C_4–C_1) < $2P_2$**	**at the P.I. section (C_B–C_A) < $2P_3$**
Type-(b) group from 4 to No 3 (C_2–C_3)	C_1	C_2, C_3, (C_4 = multiple to P_2),C_3	C_2	with R_4 is R_3 (C_3–C_2) < $2P_2$	at the P.I. section (C_2–C_1) < $2P_3$
Type-(c) group from 4 to No 3 (C_2–C_3)	C_3	C_2, (C_1 = multiple to P_2) C_2,C_3	C_4	with R_4 is R_3 (C_3–C_2) < $2P_2$	at the P.I. section (C_4 – C_3) < $2P_3$

Table 17.
*The representation of rearrangement of every 3 adjoined subgroups No 4 of mod(P_1#) (represented in **Table 3**), while P_2 duplication of fractal -P_1# from No 4 groups of mod(P_1#) to No 3 groups of mod(P_2#).*

At the transition from mod(P_2#) to mod(P_3#), the fractal -P_2# and every from the φ(P_2#) groups No 4-3-2 mod(P_2#) are repeated P_3 times. Then at the P.I. section from 1 to P_3# we'll get φ(P_2#) columns of No 4-3-2 groups (P_3 lines at the column). Number P_3 according to diagonals P_3 lines "eliminates," that is transits for the mod (P_3#) one time every P_3 of the duplicated numbers of the P.I. section from 1 to P_2#. (One number at every P_3 line of every column No 3 and No 2), "eliminating" the residues C_1-C_2-C_3 at the groups No 3 (consult Section 9.1) and ALL numbers, except the residues C_1-C_2 at the groups No 2 (consult Section 9.3).

9.1

It is quite obvious, that after "elimination" in every φ(P_2#) column of group No 3 of mod(P_2#), 1 time the residue - C_2, we'll get in P_3 line of every column of groups No 3 of mod(P_2#), one group No 2 of mod(P_3#), that is totally we'll get φ(P_2#) subgroups No 2 of mod(P_3#) with invariance length of previous groups, that is R_3 φ(P_2#) groups No 3 of mod(P_2#) would become = R_2 for φ(P_2#) groups No 2 of mod(P_3#) with changing the alternance composition from $\leq P_2$ to $\leq P_3$, that are situated at P.I. section from 1 to P_3# that is at the fractal -P_3#

Herewith in φ(P_2#) groups No 2 of mod(P_3#) are accounted all residues = C_1 and = C_3 of mod(P_2#) and alternances $\leq P_3$ of such "rearranged" groups from No 3 of mod(P_2#) to No 2 of mod(P_3#). As in the course of P_3 duplication of two adjoined groups No 3 of mod(P_2#), represented in **Table 5**, we'll get the table No 11 with each of the residues = C_2, situated on one of two lines of **Table 5** (for example, the line (a) **Table 18**, is accounted as the residue C_1 or C_3 at the other adjoined group No 3 (in line b). of **Table 12**), where it is represented in 3 consequent residues as the second one, that is "excluded" as C_2 at this adjoined group (line):

-for line (b) ${}_2$C., (${}_3$C = multiple P_3)., C_2, we'll get the alternance $\leq P_3$. $R_2 < 2P_2$

Thus, after "elimination" that is transferring to mod(P_3#), one time for every of 3 residues at φ(P_2#) groups No 3 of mod(P_2#), at the P.I. sections from 1 to P_3#, that is in fractal -P_3#, we'll get the loop back in the form of φ(P_2#) groups No 2 of mod(P_3#), type: C_1 ${}_{pp}$(C_2 = multiple P_2) ${}_{pp}C_3$, with the alternances $\leq P_3$, with length $R_2 \leq 2P_2$. Including pure 2 subgroups No 2 of mod(P_3#) with length maximal $R_2 = 2P_2$.

9.2

Herewith it is quite obvious that any two consequent residues of any subgroup No 2 according to the increasing mod(P_n#), represented in **Table 18** are still within the P.I. section with length not exceeding - $2P_{n+1}$ of the whole numbers, as the "eliminated" residues C_2 and $C_{1;3}$ of mod(P_n#) doesn't change the location of any subgroup No 2. That is, in two adjoined groups No 2 lines of **Table 18** for mod(P_3#) we get: type (a) =(C_B–C_A) < $2P_3$; type (b) =(C_1–${}_1$C) < $2P_3$.

Including pure two subgroups No 2 max $R_2 = 2P_2$, located within the maximally long section with length = $2P_3$ of the whole numbers, two rearrangement is studies in Section 7.

9.3

Along with P_3 duplications $\varphi(P_2\#)$ of groups No 2 of mod($P_2\#$), the number P_3 "eliminates" that is transits to the mod($P_3\#$) 1 time every of all previously eliminated numbers of every group No 2, except two residues C_1–C_2. That it, it transits to mod($P_3\#$) (P_3–2) of groups No 2 in every $\varphi(P_2\#)$ column of No 2 groups.

Then at the P.I. section from 1 to $P_3\#$ that is within the fractal -$P_3\#$ we'll get the loopback, represented as $(P_3–2)\varphi(P_2\#)$ duplications of No 2 groups for the mod ($P_3\#$) with the alternance "changes" from $\leq P_2$ to $\leq P_3$. Without changing the length of groups No 2 for mod($P_3\#$), R_2 = const and numbers composition at the alternances $\leq P_3$ (as previously eliminated $\leq P_2$, for the 1 least >1 from the number is accounted). Type: $C_{1\ pppp}C_2$. With: R_2 = const =? (with to mod($P_3\#$) $R_2 < <$ $R_3 = 2P_2$).

9.4

Totally at the fractal $P_3\#$ at P.I. section from 1 to $P_3\#$ we'll get the loopback of the rearranged groups No 2 for mod($P_3\#$) represented in Sections 9.1 and 9.3. That is: $\varphi(P_2\#)$ (Section 9.1 with length = $R_2 \leq 2P_2$) + $\varphi(P_2\#)(P_3–2)$ (Section 9.3 with length $R_2 < <R_3 = 2P_2$) = $P_3\varphi(P_2\#)–2\varphi(P_2\#) + \varphi(P_2\#) = P_3\ \varphi(P_2\#)–\varphi(P_2\#) = \varphi(P_2\#)(P_3–1) = \varphi(P_3\#)$ of the subgroups No 2 for mod($P_3\#$), located within the alternances $\leq P_3$ with length $R_2 \leq 2P_2$, including two maximally long subgroups No 2 max $R_2 = 2P_2$.

9.5

As all $\varphi(P_3\#)$ of the subgroups No 2 of mod($P_3\#$) are examines in Sections 9.1 and 9.3, with every eliminated one time in the column residue C_2 examined in Section 9.1, can change the length = $R_2 \leq 2P_2$, of none of $\varphi(P_2\#)$ groups, rearranged from No 3 to No 2 for mod($P_3\#$), herewith residues C_1 and C_3 are excluded at the adjoined group of **Table 18** as C_2 and indiscriminately $(P_3–2)\varphi(P_2\#)$ groups No 2 of mod($P_2\#$), examined in Section 9.3 are shorter than limit $R = 2P_2$.

Thus, there are no other ways of making groups No 2 of mod($P_3\#$) with length $R_2 > 2P_2$, but constructing two maximally long subgroups No 2 with length max $R_2 = 2P_2$, as examined in Sections 7 and 9.1.

9.6

And so we get, that for every recurrent prime = P_3, $\varphi(P_3\#)$ residues of mod ($P_3\#$) are in the fractal -$P_3\#$ (in P_3 lines of **Table 2**), represented as loopback $\varphi(P_3\#)$ of the subgroups No 2 (2 residues of mod($P_3\#$) are indicated in Section 9.4). Thus, pure TWO consequent primes ($_{A,B}C$) of type: $P_1, P_2 < (_AC\text{-}_BC) < P_3^2$ at the P.I. section from P_2 to P_3^2, and further, from P_3^2 to $P_3\#$ pure TWO consequent residues of mod ($P_3\#$), at every P.I. sections with length not exceeding $2P_3$ of the whole numbers (consult Section 9.2), with length of every subgroup No 2 at every section is: $R_2 = (_BC–_AC) \leq \mathbf{2}P_2$ (consult Sections 9.1 and 9.4).

And so on, for every from all eventual primes = P_n, in the form of loopback of residues of groups No 2 distribution at the increasing fractals -$P_n\#$, with the increasing values of modulus of mod($P_n\#$), that proofs the validity of section (c) of Theorem 1 (loopback of groups No 2 is represented in column No 4 of **Table 14**).

Type-(a) group from 3 to No 2 (C_1–C_3)	C_A	C_1, (C_2 = multiple P_3)., C_3	C_B	with R_3 is R_2 (C_3–C_1) < $2P_2$	at the P.I. section (C_B–C_A) < $2P_3$
Type (a) group from 3 to No 2 ($_2C$–C_2)	$_1C$	$_2C$., ($_3C_1$ = multiple P_3).,C_2	C_1	with R_3 is R_2 (C_2–$_2C$) < $2P_2$	At the P.I. section (C_1–$_1C$) < $2P_3$

Table 18.
Representation of rearrangement of any 2 adjoined subgroups No 3 of mod($P_2\#$) (represented in ***Table 5****), within P_3 duplications of fractal -$P_2\#$. From groups No 3 of mod ($P_2\#$) to the groups No 2 of mod ($P_3\#$).*

10. Proof of section (b) and section (c) of Theorem 1

While examining the P.I., represented in the form of alternance (array) of primes (according to the 1 least prime factor > 1 from every whole number), we'll get that for every recurrent prime -P_n; the P.I. is the line-symmetrical primary-repeated fractal -P_n#, located at the P.I. section from 1 to P_n#, represented as φ(P_n#) of the I.R.S. residue of mod(P_n#), between which the P.I. sections are situated (with different length), represented as the alternances (array) of different amounts of different first primes ≤P_n – NOT residues mod(P_n#).

By indexing φ(P_n#) of the least residues of every recurrent fractal -P_n#, groups pure with 4, 3, and 2 consequent residue of mod(P_n#) (analogously as in Section 2). We'll get that every recurrent fractal -P_n# has got three types of arrays of the subgroups of residues of mod(P_n#): with φ(P_n#) groups: No 4 (with 4 residues), No 3 (with 3 residues), and No 2 (with 2 residues, repeated without changes with the period = P_n#).

At every recurrent transition, for example, from mod(P_2#) to mod(P_3#), the first line of fractal -P_2# and group No 4-3-2 of mod(P_2#) are repeated P_3 times (consult P_3 lines of **Table 2**). At the P.I. section from 1 to P_3# we'll get φ(P_2#) columns of groups No 4-3-2 of mod(P_2#) within the alternances ≤P_2 (P_3 lines in column), "eliminating" 1 number multiple to -P_3 (in line of every column No 4-3-2), that is by transition of this group to mod(P_3#) with changing of its length: from $R_{(4,3)}$ to $R_{3,2}$ and the alternance composition from ≤P_2 to ≤P_3, we'll get φ(P_3#) groups No 4-3-2 of mod(P_3#).

The rearrangement order of these subgroups No 4-3-2 at the increasing modulus is proved in Sections 5, 6, 7, 8, and 9. Representing as one line of the P.I. section from 1 to P_3# we'll get the fractal -P_3# of mod(P_3#), with φ(P_3#) groups No 4-3-2 of mod(P_3#).

And so on for every recurrent prime = P_n, the results of proof are demonstrated in Sections from 5 to 9 and for visualization they are grouped together in **Table 14**.

As far as we know that for every recurrent prime -P_n = $P_{(1)}$, the P.I. is the line-symmetrical primordial repeated fractal -$P_{(1)}$#, located at the P.I. section from 1 to $P_{(1)}$#. There are φ($\boldsymbol{P_{(1)}}$#) residues of mod($\boldsymbol{P_{(1)}}$#) between which are located the alternances (arrays) of primes ≤ $P_{(1)}$ (1 the least>1 from every NOT residue of mod ($P_{(1)}$#)).

According to Section 5, there is the only maximally long subgroup No 4 (with 4 residues) of mod($P_{(1)}$#) with length maxR_4 = 2$P_{(2)}$.

Let us assume that there are such primes $P_{(2)}$ or $P_{(3)}$, for which at the P.I., represented as alternance ≤ $P_{(2)}$, in the form of fractal –$P_{(2)}$# we can form more than two maximally long subgroups No 3 of mod($P_{(2)}$#), with: (or) R_3 > 2$P_{(2)}$ or at the fractal –$P_{(3)}$# with P.I. is represented by alternances ≤ $P_{(3)}$, we can compare more than maximally long subgroups No 2 of mod($P_{(3)}$#), with: (or) R_2 > 2$P_{(2)}$.

Then, in the course of the opposite reduction of the modulus, that is at the result of $P_{(2)}$*$P_{(3)}$ repetition of such subgroups as No 3 or No 2 with the repetition period $P_{(1)}$# (for the downward meanings of numbers) and backing up the $P_{(2)}$ and $P_{(3)}$ numbers as residues (according to the decreased moduli)**, that are situated in $P_{(2)}$*$P_{(3)}$ lines analogues to **Table 1**. At the upper lines of such columns, consisting of $P_{(2)}$*$P_{(3)}$ lines, we'll get the P.I as fractal-$P_{(1)}$#, represented by alternances ≤ $P_{(1)}$, where at the P.I. section from 1 to $P_{(1)}$# would be located more than one subgroup No 4 (with 4 residues) of mod($P_{(1)}$#) or subgroups No 4, with R_4 > 2$P_{(2)}$. It is quite obvious, that any such group No 4 according to the reestablished mod($P_{(1)}$#) would be line-symmetrical to the left and to the right from the symmetry center of number = $P_{(1)}$#/2, that means formed by two different ways, that contradicts to the axiom set.

**Number $P_{(3)}$ of the fractal -$P_{(3)}$# -NOT residue of mod(P_3#), located in the group No 2 of mod($P_{(3)}$#) within the alternance $\leq P_{(3)}$ with length $R_{(2)} > 2P_{(2)}$, and for mod($P_{(2)}$#) would be accounted as the third residue in the group No 3 within the alternance $\leq P_{(2)}$, without changing the length of this group No 3 with $R_{(2)}$ is $R_3 > 2P_{(2)}$.

Number $P_{(2)}$ of the fractal -$P_{(2)}$# -NOT residue of mod(P_2#) is located in the group No 3 of mod($P_{(2)}$#) within the alternance $\leq P_{(2)}$, with length $R_3 > 2P_{(2)}$, and for mod

($P_{(1)}$#), would be accounted as the fourth residue in the group No 4 within the alternance $\leq P_{(1)}$, without changing the length of this group No 4 with $R_{(3)}$ is $R_4 > 2P_{(2)}$.

11. Conclusion

Thus, Theorem 1 allowed us to prove the existence of a new law in mathematics – "on the plume distribution of Prime numbers." Since the methods used in number theory do not allow us to approach the problem of the distribution of prime numbers, it means that further expansion of the method proposed in the article for studying the natural series of numbers will simplify and solve many other problems that are not solved in mathematics.

So from Theorem 1 "Loopback of primes distribution" follows:

Theorem No 2. For every whole number = N at the P.I. section from 1 to N + $2\sqrt{N}$:

1. Primes are located as groups, pure three consequent primes of (P_1-P_2-P_3) type. Herewith the distance from the first to the third prime of every group is less than $2\sqrt{N}$ of the whole numbers, that is $(P_3–P_1) < 2\sqrt{N}$ whole numbers.

2. The same primes are distributed as the loopback, pure two consequent primes at every P.I. sections, shorter than $2\sqrt{N}$ whole numbers.

Proof. Every whole number = N is located within the squared two consequent primes: $P_1^2 < N \leq P_2^2$ with: $2\sqrt{N} > 2P_1$.

That means every N is located within the fractal -P_1#. Then:

1. From the section (b) of the theorem "Loopback of primes distribution" follows, that at fractal -P_1# of mod(P_1#) at the P.I. section from 1 to $P_2^2 \geq (N + 2\sqrt{N})$, at every P.I. section with length not exceeding $2P_2$ of the whole numbers is located at the subgroup from three consequent primes of (P_1-P_2-P_3) type, with length of every subgroup, that is distance from the first to the third prime of every subgroup doesn't exceed $2P_1$ whole numbers, that is $(P_3–P_1) < \mathbf{2}P_1$ whole numbers. As length of every section $2\sqrt{N} >$ length of the section = $2P_1$. Then from 1 to N + $2\sqrt{N}$ – every:

$(P_3–P_1) < 2\sqrt{N}$ of the whole numbers.

2. From the section (c) of the Theorem "Loopback of primes distribution" it follows, that at the fractal -P_1# of mod(P_1#) at the P.I. section from 1 to $P_2^2 \geq (N + 2\sqrt{N})$ at every I.P. section with length not exceeding $2P_1$ of the

whole numbers, there is the subgroup from two consequent primes P_1 and P_2. As $2\sqrt{N}$ whole numbers $>2P_1$ whole numbers, that means, that at the P.I. section from 1 to N + $2\sqrt{N}$ at every P.I. section with length not exceeding $2\sqrt{N}$ whole numbers, there is the loopback of primes, represented as the subgroup for two consequent primes.

Genuinely: Every P.I. section with length = $2\sqrt{N}$ of the whole numbers is located at the fractal -$P_1\#$ at the P.I. section. $<P_2^2$ as, $P_1^2 < (\mathrm{N} + 2\sqrt{N}) \le P_2^2$, with: $2\sqrt{N} > 2P_1$.

It is feasible, that there is a P.I. section with length = $2\sqrt{N}$ of the whole numbers, where there are no two primes, that is, two consequent primes are located at the P.I. section with length exceeding – $2\sqrt{N}$ of the whole numbers $>2P_1$, but this contradicts to section (c) of the Theorem "Loopback of primes distribution," that states, that is every fractal -$P_1\#$ according to mod($P_1\#$), on every P.I. sections with length not exceeding **2**P_1 of the whole numbers, there is a subgroup No 2 with 2 residues of mod($P_1\#$), that is two primes $<P_2^2$.

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