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Chapter

Soliton and Rogue-Wave Solutions of Derivative Nonlinear Schrödinger Equation - Part 2

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Abstract

A revised and rigorously proved inverse scattering transform (IST for brevity) for DNLS⁺ equation, with a constant nonvanishing boundary condition (NVBC) and normal group velocity dispersion, is proposed by introducing a suitable affine parameter in the Zakharov-Shabat IST integral; the explicit breather-type and pure *N*-soliton solutions had been derived by some algebra techniques. On the other hand, DNLS equation with a non-vanishing background of harmonic plane wave is also solved by means of Hirota's bilinear formalism. Its space periodic solutions are determined, and its rogue wave solution is derived as a long-wave limit of this space periodic solution.

Keywords: soliton, nonlinear equation, derivative nonlinear Schrödinger equation, inverse scattering transform, Zakharov-Shabat equation, Hirota's bilinear derivative method, DNLS equation, space periodic solution, rogue wave

1. Breather-type and pure *N*-soliton solution to DNLS⁺ equation with NVBC based on revised IST

DNLS⁺ equation with NVBC, the concerned theme of this section, is only a transformed version of modified nonlinear Schrödinger equation with normal group velocity dispersion and a nonlinear self-steepen term and can be expressed as

$$iu_t - u_{xx} + i\left(|u|^2 u\right)_x = 0, \tag{1}$$

here the subscripts represent partial derivatives.

Some progress have been made by several researchers to solve the DNLS equation for DNLS equation with NVBC, many heuristic and interesting results have been attained [1–14]. An early proposed IST worked on the Riemann sheets can only determine the modulus of the one-soliton solution [3, 15]. References [4, 5, 16] had attained a pure single dark/bright soliton solution. Reference [6] had derived a formula for $N-$ soliton solution in terms of Vandermonde-like determinants by means of Bäcklund transformation; but just as reference [7, 9] pointed out, this multi-soliton solution is still difficult to exhibit the internal elastic collisions among solitons and the typical asymptotic behaviors of multi-soliton of DNLS equation. By introducing an affine parameter in the integral of Zakharov-Shabat IST, reference [7] had found their pure $N-$ soliton solution for a special case that all the simple

poles (zeros of $a(\lambda)$) were located on a circle of radius ρ centered at the origin, while reference [8] also found its multi-soliton solution for some extended case with N poles on a circle and M poles out of the circle, and further developed its perturbation theory based on IST. Reference [7] constructed their theory by introducing an damping factor in the integral of Zakharov-Shabat IST, to make it convergent, and further adopted a good idea of introducing an affine parameter to avoid the trouble of multi-value problem in Riemann sheets, but both of their results are assumed $N-$soliton solutions and the soliton solution gotten from their IST had a self-dependent and complicated phase factor [7–9], hence reference [8] had to verify an identity demanded by the standard form of a soliton solution (see expression (52) in Ref. [8]). Such kind of an identity is rather difficult to prove for $N \geq 2$ case even by the use of computer techniques and Mathematica. On the other hand, author of reference [8] also admits his soliton solution is short of a rigorous verification of standard form. Then questions naturally generates – whether the traditional IST for DNLS equation with NVBC can be avoided and further improved? And whether a rigorous manifestation of soliton standard form can be given and a more reasonable and natural IST can be constructed?

A newly revised IST is thus proposed in this section to avoid the dual difficulty and the excessive complexity. An additional affine factor $1/\lambda$, $\lambda = (z + \rho^2 z^{-1})/2$, is introduced in the Z-S IST integral to make the contour integral convergent in the big circle [7, 10–13]. Meanwhile, the additional two poles on the imaginary axis caused by $\lambda = 0$ are removable poles due to the fact that the first Lax operator $L(\lambda) \to 0$, as $\lambda \to 0$. What is more different from reference [7] is that we locate the N simple poles off the circle of radius ρ centered at origin O, which corresponds to the general case of N breather-type solitons. When part of the poles approach the circle, the corresponding part of the breathers must tend to the pure solitons, which is the case described in Ref. [8]. The resulted one soliton solution can naturally tend to the well-established conclusion of VBC case as $\rho \to 0$ [17–20] and the pure one soliton solution in the degenerate case. The result of this section appears to be strict and reliable.

1.1 The fundamental concepts for the IST theory of DNLS equation

Under a Galileo transformation $(x, t) \to (x + \rho^2 t, t)$, DNLS$^+$ Eq. (1) can be changed into

$$iu_t - u_{xx} + i\left[\left(|u|^2 - \rho^2\right)u\right]_x = 0, \tag{2}$$

with nonvanishing boundary condition:

$$u \to \rho, \quad \text{as} \quad |\mathrm{x}| \to \infty. \tag{3}$$

According to references [7–9], the phase difference between the two infinite ends should be zero. The Lax pair of DNLS$^+$ Eq. (3) is

$$\boldsymbol{L} = -i\lambda^2 \sigma_3 + \lambda \boldsymbol{U}, \ \boldsymbol{U} = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} = u\boldsymbol{\sigma}_+ + \overline{u}\sigma_- \tag{4}$$

$$\boldsymbol{M} = i2\lambda^4 \sigma_3 - 2\lambda^3 \boldsymbol{U} + i\lambda^2(\boldsymbol{U}^2 - \rho^2)\sigma_3 - \lambda(\boldsymbol{U}^2 - \rho^2)\boldsymbol{U} + i\lambda \boldsymbol{U}_x \sigma_3 \tag{5}$$

where σ_3, σ_+, and $\boldsymbol{\sigma}_-$ involve in standard Pauli's matrices and their linear combination. Here and hereafter a bar over a variable represents complex

conjugate. An affine parameter z and two aided parameters η, λ are introduced to avoid the trouble of dealing with double-valued functions on Riemann sheets

$$\lambda \equiv \left(z + \rho^2 z^{-1}\right)/2, \eta \equiv \left(z - \rho^2 z^{-1}\right)/2 \tag{6}$$

The Jost functions satisfy first Lax equation

$$\partial_x F(x, \ z) = L(x, \ z) F(x, \ z), \tag{7}$$

here Jost functions $F(x, \ z) \in \{\Psi(x,z), \Phi(x,z)\}$.

$$\Psi(x,z) = (\tilde{\psi}(x,z), \ \psi(x,z)) \rightarrow E(x,z), \text{as } x \rightarrow \infty \tag{8}$$

$$\Phi(x,z) = (\phi(x,z), \tilde{\phi}(x,z)) \rightarrow E(x,z), \text{as } x \rightarrow -\infty \tag{9}$$

The free Jost function $E(x,z)$ can be easily attained as follows:

$$E(x,z) = \left(I + \rho z^{-1} \sigma_2\right) \exp\left(-i\lambda \eta x \sigma_3\right), \tag{10}$$

which can be verified satisfying Eq. (7). The monodramy matrix is

$$T(z) = \begin{pmatrix} a(z) & \tilde{b}(z) \\ b(z) & \tilde{a}(z) \end{pmatrix}, \tag{11}$$

which is defined by

$$\Phi(x,z) = \Psi(x,z) \ T(z) \tag{12}$$

Some useful and important symmetry properties can be found

$$\sigma_1 \overline{L}(z) \sigma_1 = L(\bar{z}), \sigma_3 L(-z) \sigma_3 = L(z) \tag{13}$$

Symmetry relations in (13) lead to

$$\tilde{\psi}(x,z) = \sigma_1 \overline{\psi(x,\bar{z})}, \tilde{\phi}(x,z) = \sigma_1 \overline{\phi(x,\bar{z})} \tag{14}$$

$$\tilde{\psi}(x, -z) = \sigma_3 \tilde{\psi}(x,z), \psi(x, -z) = -\sigma_3 \psi(x,z) \tag{15}$$

$$\sigma_1 \overline{T(x,\bar{z})} \sigma_1 = T(z), \sigma_3 T(-z) \sigma_3 = T(z) \tag{16}$$

The above symmetry relations further result in

$$\tilde{a}(\bar{z}) = \overline{a(z)}, \quad \tilde{b}(z) = \overline{b(z)} \tag{17}$$

$$a(-z) = a(z), b(-z) = -b(z) \tag{18}$$

Other important symmetry properties called reduction relations can also be easily found

$$\lambda\left(\rho^2 z^{-1}\right) = \lambda(z), \quad \eta\left(\rho^2 z^{-1}\right) = -\eta(z), L\left(x, \rho^2 z^{-1}\right) = L(x,z) \tag{19}$$

$$E\left(x, \rho^2 z^{-1}\right) = \rho^{-1} z \left(I + \rho z^{-1} \sigma_2\right) e^{-i\eta \lambda x \sigma_3} \sigma_2 \tag{20}$$

The above symmetry properties lead to following reduction relations among Jost functions

$$\tilde{\psi}(x, \rho^2 z^{-1}) = i\rho^{-1}z\psi(x, z), \psi(x, \rho^2 z^{-1}) = -i\rho^{-1}z\tilde{\psi}(x, z) \qquad (21)$$

$$\phi(x, \rho^2 z^{-1}) = i\rho^{-1}z\tilde{\phi}(x, z), \tilde{\phi}(x, \rho^2 z^{-1}) = -i\rho^{-1}z\phi(x, z) \qquad (22)$$

The important symmetries among the transition coefficients further resulted from (12), (21), and (22):

$$\sigma_2 T(\rho^2 z^{-1})\sigma_2 = T(z), \tilde{a}(\rho^2 z^{-1}) = a(z), \tilde{b}(\rho^2 z^{-1}) = -b(z) \qquad (23)$$

On the other hand, the simple poles, or zeros of $a(z)$, appear in quadruplet, and can be designated by z_n, $(n = 1, 2, \cdots, 2N)$, in the I quadrant, and $z_{n+2N} = -z_n$ in the III quadrant. Due to symmetry (17), (18) and (23), the n' th subset of zero points is

$$\left\{ z_{2n-1}, z_{2n} = \rho^2 \bar{z}_{2n-1}^{-1}, -z_{2n-1}, -z_{2n} = -\rho^2 \bar{z}_{2n-1}^{-1} \right\} \qquad (24)$$

And we arrange the $2N$ zeros in the first quadrant in following sequence

$$z_1, z_2; z_3, z_4; \cdots; z_{2N-1}, z_{2N} \qquad (25)$$

According to the standard procedure [21, 22], the discrete part of $a(z)$ can be deduced

$$a(z) = \prod_{n=1}^{2N} \frac{z^2 - z_n^2}{z^2 - \bar{z}_n^2} \frac{\bar{z}_n}{z_n} \qquad (26)$$

At the zeros of $a(z)$, or poles z_n, $(n = 1, 2, \cdots, 2N - 1, 2N)$, we have

$$\phi(x, z_n) = b_n\psi(x, z_n), \dot{a}(-z_n) = -\dot{a}(z_n) \qquad (27)$$

Using symmetry relation in (14), (17), (21)–(24), (27), we can prove that

$$\bar{b}_{2n} = -b_{2n-1}, \bar{c}_{2n-1} = \rho^2 z_{2n}^{-2} c_{2n} \qquad (28)$$

2. Relation between the solution and Jost functions of DNLS⁺ equation

The asymptotic behaviors of the Jost solutions in the limit of $|\lambda| \to \infty$ can be obtained by simple derivation. Let $F_{.1} = \tilde{\psi}(x, \lambda)$, then Eq. (7) can be rewritten as

$$\tilde{\psi}_{1x} + i\lambda^2\tilde{\psi}_1 = \lambda u\tilde{\psi}_2, \tilde{\psi}_{2x} - i\lambda^2\tilde{\psi}_2 = -\lambda\bar{u}\tilde{\psi}_1, \qquad (29)$$

then we have

$$\tilde{\psi}_{1xx} - (\tilde{\psi}_{1x} + i\lambda^2\tilde{\psi}_1)u_x/u + \lambda^4\tilde{\psi}_1 - \lambda^2|u|^2\tilde{\psi}_1 = 0 \qquad (30)$$

We assume a function g to satisfy the following equation

$$\tilde{\psi}_1(x, \lambda) = e^{-i\lambda\eta x + g} \qquad (31)$$

$$\tilde{\psi}_{1x} = (-i\lambda\eta + g_x)\tilde{\psi}_1, \tilde{\psi}_{1xx} = \left[(-i\lambda\eta + g_x)^2 + g_{xx} \right]\tilde{\psi}_1 \qquad (32)$$

Substituting (31)–(32) into Eq. (30), we have

$$g_{xx} + g_x^2 - (2i\lambda\eta + u_x/u)g_x - i\lambda(\lambda - \eta)u_x/u + \lambda^2(\lambda^2 - \eta^2) - \lambda^2|u|^2 = 0 \qquad (33)$$

In the limit of $|z| \to \infty$, g_x can be expanded as sum of series of $(z^{-2})^j$, $j = 0, 1, \ 2, \ \cdots$

$$g_x \equiv \mu = \mu_0 + \mu_2 z^{-2} + \mu_4 z^{-4} + \cdots, \qquad (34)$$

where

$$\mu_0 = -i\left(\rho^2 - |u|^2\right)/2 = O(1) \qquad (35)$$

Inserting formula (34) and (35) in Eq. (33) at $|z| \to \infty$ leads to
$u = \frac{\tilde{\psi}_1}{\lambda\tilde{\psi}_2}\left[i\lambda(\lambda - \eta) - \frac{1}{2}i\left(\rho^2 - |u|^2\right)\right] \to i\frac{\tilde{\psi}_1}{\tilde{\psi}_2}\frac{|u|^2}{z}, \quad (as \ |z| \to \infty)$
Then we find a useful formula

$$\bar{u} = -i \lim_{|z|\to\infty} z\tilde{\psi}_2(x,z)/\tilde{\psi}_1(x,z), \qquad (36)$$

which expresses the conjugate of the solution u in terms of Jost functions as $|z| \to \infty$.

2.1 Introduction of time evolution factor

In order to make the Jost functions satisfy the second Lax equation, a time evolution factor $h(t,z)$ should be introduced by a standard procedure [21, 22] in the Jost functions and the scattering data. Considering the asymptotic behavior of the second Lax operator

$$M(x,t;z) \to i2\lambda^4\sigma_3 - 2\lambda^3\rho\sigma_1, \quad as \ x \to \infty, \qquad (37)$$

we let $h(t,k)$ to satisfy

$$[\partial/\partial t - M(x,t;z)]h^{-1}(t,z)\psi(x,t;z) = 0, \text{ as } x \to \infty \qquad (38)$$

then

$$[\partial/\partial t - i2\lambda^4\sigma_3 - 2\lambda^3\rho\sigma_1]h^{-1}(t,z)E_{\cdot 2}(x,z) = 0. \qquad (39)$$

Due to

$$E_{\cdot 2}(x,z) = \left(-i\rho z^{-1}e^{i\lambda\eta x}, \ e^{i\lambda\eta x}\right)^{\mathrm{T}}, \qquad (40)$$

from (39) and (40), we have

$$h(t,z) = e^{i2\lambda^3\eta t}. \qquad (41)$$

Therefore, the complete Jost functions should depend on time as follows

$$h(t,z)\tilde{\psi}(x,z), \quad h^{-1}(t,z)\psi(x,z); h(t,z)\phi(x,z), \quad h^{-1}(t,z)\tilde{\phi}(x,z) \qquad (42)$$

Nevertheless, hereafter the time variable in Jost functions will be suppressed because it has no influence on the treatment of Z-S equation. By a similar procedure [9, 15], the scattering data has following time dependences

$$a(z,t) = a(z,0), b(t,z) = b(0)e^{-i4\lambda^3\eta t} \tag{43}$$

2.2 Zakharov-Shabat equations and breather-type N-soliton solution

A 2×1 column function $\boldsymbol{\Pi}(x,z)$ is introduced as usual

$$\boldsymbol{\Pi}(x,z) \equiv \begin{cases} \boldsymbol{\phi}(x,z)/a(z), & as\ z\ in\ \mathrm{I,\ III}\ quadrants. \\ \tilde{\boldsymbol{\psi}}(x,z), & as\ z\ in\ \mathrm{II,\ IV}\ quadrants. \end{cases} \tag{44}$$

here and hereafter note "$\equiv$" represents definition. There is an abrupt jump for $\boldsymbol{\Pi}(x,z)$ across both real and imaginary axes

$$\boldsymbol{\phi}(x,z)/a(z) - \tilde{\boldsymbol{\psi}}(x,z) = r(z)\boldsymbol{\psi}(x,z), \tag{45}$$

where

$$r(z) = b(z)/a(z) \tag{46}$$

is called the reflection coefficient. Due to $\mu_0 \neq 0$ in (34), Jost solutions do not tend to the free Jost solutions $\boldsymbol{E}(x,z)$ in the limit of $|z| \to \infty$. This is their most typical property which means that the usual procedure of constructing the equation of IST by a Cauchy contour integral must be invalid. In view of these abortive experiences, we proposed a revised method to derive a suitable IST and the corresponding Z-S equation by multiplying an inverse spectral parameter $1/\lambda$, $\lambda = (z + \rho^2 z^{-1})/2$, before the Z-S integrand. Meanwhile, our modification produces no new poles since the Lax operator $\boldsymbol{L}(\lambda) \to 0$, as $\lambda \to 0$. In another word, the both additional poles $z_0 = \pm i\rho$ generated by $\lambda = 0$ are removable. Under reflectionless case, that is, $r(z) = 0$, the Cauchy integral along with contour Γ shown in **Figure 1** gives

$$\frac{1}{\lambda}\{\boldsymbol{\Pi}(x,z) - \boldsymbol{E}_{\cdot 1}(x,z)\}e^{i\lambda\eta x} = \frac{1}{2\pi i}\oint_\Gamma dz' \frac{1}{z'-z}\frac{1}{\lambda'}\{\boldsymbol{\Pi}(x,z') - \boldsymbol{E}_{\cdot 1}(x,z')\}e^{i\lambda'\eta'x} \tag{47}$$
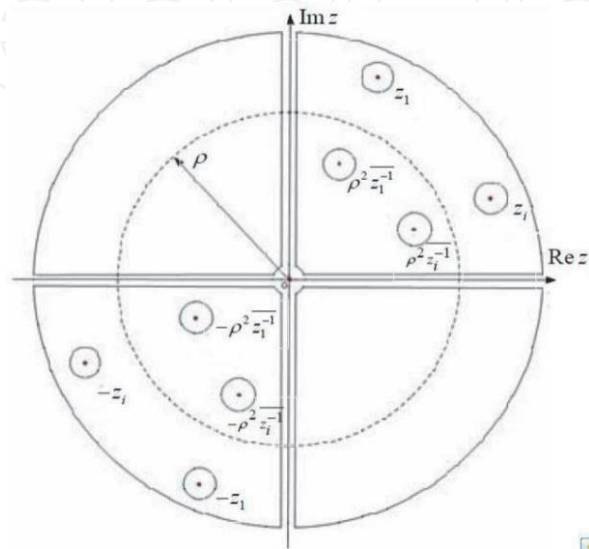

Figure 1.
The integral path for IST of the DNLS$^+$.

or

$$\tilde{\psi}(x,z) = e^{-i\lambda\eta x} + \lambda \left\{ -\sum_{n=1}^{4N} \frac{1}{\lambda_n} \frac{1}{z_n - z} c_n \psi(x,z_n) e^{i\lambda_n \eta_n x} \right\} e^{-i\lambda\eta x} \tag{48}$$

where

$$c_n \equiv b_n / \dot{a}(z_n), \dot{a}(z_n) = da(z)/dz|_{z=z_n}; n = 1, 2, \cdots, 4N \tag{49}$$

Note that (27), (45), and (46) have been used in (48). The minus sign before the sum of residue number in (48) comes from the clock-wise contour integrals around the $4N$ simple poles when the residue theorem is used, shown in **Figure 1**. By a standard procedure, the time dependences of b_n and c_n similar to (43) can be derived

$$b_n(t) = b_{n0} e^{-i4\lambda_n^3 \eta_n t}, c_n = c_n(0) e^{-i4\lambda_n^3 \eta t} \tag{50}$$

$$c_n(0) = b_{n0} / \dot{a}_n; n = 1, 2, \cdots, 4N \tag{51}$$

In the reflectionless case, the Zakharov-Shabat equations for DNLS⁺ equation can be derived immediately from (48) as follows

$$\tilde{\psi}_1(x,z) = e^{-i\Lambda x} + \lambda \left[\sum_{n=1}^{2N} \frac{2z}{\lambda_n} \frac{1}{z^2 - z_n^2} c_n \psi_1(x,z_n) e^{i\Lambda_n x} \right] e^{-i\Lambda x} \tag{52}$$

$$\tilde{\psi}_2(x,z) = i\rho z^{-1} e^{-i\Lambda x} + \lambda \left[\sum_{n=1}^{2N} \frac{2z_n}{\lambda_n} \frac{1}{z^2 - z_n^2} c_n \psi_2(x,z_n) e^{i\Lambda_n x} \right] e^{-i\Lambda x} \tag{53}$$

here $\Lambda \equiv \lambda\eta$, $\Lambda_n \equiv \lambda(z_n)\eta(z_n) = \lambda_n \eta_n$, and in Eqs. (52) and (53), the terms corresponding to poles z_n, $(n = 1, 2, \cdots, 2N)$, have been combined with the terms corresponding to poles $z_{n+2N} = -z_n$. Substituting Eqs. (52) and (53) into formula (36) and letting $z \to \infty$, we attain the conjugate of the raw N-soliton solution (the time dependence is suppressed).

$$\overline{u}_N = U_N / V_N \tag{54}$$

$$U_N \equiv \rho \left[1 - \sum_{n=1}^{2N} i \frac{z_n}{\rho\lambda_n} c_n \psi_2(x,z_n) e^{i\Lambda_n x} \right] \tag{55}$$

$$V_N \equiv 1 + \sum_{n=1}^{2N} \frac{c_n}{\lambda_n} \psi_1(x,z_n) e^{i\Lambda_n x} \tag{56}$$

Letting $z = \rho^2 z_m^{-1}$, $(m = 1, 2, \cdots, 2N)$, respectively, in Eqs. (52) and (53), by use of reduction relations (19), (21), and (22), we can further change Eqs. (52) and (53) into the following form

$$\psi_1(x,z_m) = -i\rho z_m^{-1} e^{i\Lambda_m x} + \sum_{n=1}^{2N} \frac{\lambda_m c_n}{\lambda_n z_m^2} \frac{2\rho^3}{i(\rho^4 z_m^{-2} - z_n^2)} \psi_1(x,z_n) e^{i(\Lambda_n + \Lambda_m)x} \tag{57}$$

$$\psi_2(x,z_m) = e^{i\Lambda_m x} + \sum_{n=1}^{2N} \frac{\lambda_m z_n c_n}{\lambda_n z_m} \cdot \frac{2\rho}{i(\rho^4 z_m^{-2} - z_n^2)} \psi_2(x,z_n) e^{i(\Lambda_n + \Lambda_m)x} \tag{58}$$

$m = 1, 2, \cdots, 2N$. An implicit time dependence of the complete Jost functions ψ_1 and ψ_2 besides c_n should be understood. To solve Eq. (58), we define that

$$(\boldsymbol{\varphi_2})_n \equiv i\frac{z_n}{\rho\lambda_n}\sqrt{\frac{c_n}{2}}\psi_2(x,z_n), \boldsymbol{\varphi_2} \equiv \left((\varphi_2)_1, (\varphi_2)_2, ..., (\varphi_2)_{2N}\right) \tag{59}$$

$$f_n \equiv \sqrt{2c_n}e^{i\Lambda_n x}, g_n \equiv i\sqrt{\frac{c_n}{2}\frac{z_n}{\rho\lambda_n}}e^{i\Lambda_n x} = \frac{iz_n}{2\rho\lambda_n}f_n, n = 1, 2, \cdots, 2N \tag{60}$$

$$\boldsymbol{f} \equiv \left(f_1, f_2, ...f_{2N}\right), \boldsymbol{g} \equiv \left(g_1, g_2, ...g_{2N}\right) \tag{61}$$

$\boldsymbol{B} \equiv$ Matrix $(B_{nm})_{2N\times 2N}$, with

$$B_{nm} \equiv f_n \frac{\rho}{i\left(z_n^2 - \rho^4 z_m^{-2}\right)} f_m, m, n = 1, 2, \cdots, 2N. \tag{62}$$

Then Eq. (58) can be rewritten as

$$(\boldsymbol{\varphi_2})_m = g_m - \sum_{n=1}^{2N}(\varphi_2)_n B_{nm}, m = 1, 2, \cdots, 2N \tag{63}$$

or in a more compact form

$$\boldsymbol{\varphi_2} = \boldsymbol{g} - \boldsymbol{\varphi_2}\boldsymbol{B}. \tag{64}$$

The above equation gives

$$\boldsymbol{\varphi_2} = \boldsymbol{g}(\boldsymbol{I} + \boldsymbol{B})^{-1}. \tag{65}$$

Note that the choice of poles, z_n, ($n = 1, 2, \cdots, N$), should make $\det(\boldsymbol{I} + \boldsymbol{B})$ nonzero and $(\boldsymbol{I} + \boldsymbol{B})$ an invertible matrix. On the other hand, Eq. (55) can be rewritten as

$$U_N = \rho\left[1 - \boldsymbol{\varphi_2}\boldsymbol{f}^{\mathrm{T}}\right], \tag{66}$$

hereafter a superscript "T" represents transposing of a matrix. Substituting Eq. (65) into (66) leads to

$$U_N = \rho\left[1 - \boldsymbol{g}(\boldsymbol{I} + \boldsymbol{B})^{-1}\boldsymbol{f}^{\mathrm{T}}\right] = \rho\frac{\det\left(\boldsymbol{I} + \boldsymbol{B} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{g}\right)}{\det(\boldsymbol{I} + \boldsymbol{B})} = \rho\frac{\det(\boldsymbol{I} + \boldsymbol{A})}{\det(\boldsymbol{I} + \boldsymbol{B})} \tag{67}$$

where

$$\boldsymbol{A} \equiv \boldsymbol{B} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{g} \tag{68}$$

with

$$A_{nm} \equiv B_{nm} - f_n g_m = \left(z_m z_n \lambda_n / \rho^2 \lambda_m\right)B_{nm}. \tag{69}$$

To solve Eq. (57), we define that

$$(\boldsymbol{\varphi_1})_m \equiv i\sqrt{\frac{c_m}{2}\frac{z_m}{\rho\lambda_m}}\psi_1(x,z_m), \boldsymbol{\varphi_1} \equiv \left((\varphi_1)_1, (\varphi_1)_2, \cdots, (\varphi_1)_{2N}\right) \tag{70}$$

$$f'_m \equiv i\sqrt{2c_m}\cdot\frac{\rho}{z_m}e^{i\Lambda_m x} = i\frac{\rho}{z_m}f_m; g'_m = \sqrt{\frac{c_m}{2}}\cdot\frac{1}{\lambda_m}e^{i\Lambda_m x} = \frac{-iz_m}{2\rho\lambda_m}f'_m \tag{71}$$

$$f' = \left(f'_1, f'_2, \ldots f'_{2N} \right); g' = \left(g'_1, g'_2, \ldots g'_{2N} \right); \tag{72}$$

$$D'_{nm} \equiv f'_n \left[\frac{-\rho}{i\left(z_n^2 - \rho^4 z_m^{-2}\right)} \right] f'_m = \frac{\rho^2}{z_n z_m} B_{nm} \quad (\text{c.f.}(1.70) \text{ and } (1.61)) \tag{73}$$

with $n, m = 1, 2, \cdots, 2N$. Then Eq. (57) can be rewritten as

$$(\boldsymbol{\varphi}_1)_m = g'_m - \sum_{n=1}^{2N} (\boldsymbol{\varphi}_1)_n D'_{nm}, m = 1, 2, \cdots, 2N \tag{74}$$

or in a more compact form

$$\boldsymbol{\varphi}_1 = g' - \boldsymbol{\varphi}_1 \boldsymbol{D}' \tag{75}$$

The above equation givess

$$\boldsymbol{\varphi}_1 = g'(\boldsymbol{I} + \boldsymbol{D}')^{-1} \tag{76}$$

Note that the choice of poles, z_n, ($n = 1, 2, \cdots, N$), should make $\det(\boldsymbol{I} + \boldsymbol{D}')$ non-zero and $(\boldsymbol{I} + \boldsymbol{D}')$ an invertible matrix. On the other hand, Eq. (56) can be rewritten as

$$V_N = 1 - \sum_{n=1}^{2N} (\boldsymbol{\varphi}_1)_n f'_n = 1 - \boldsymbol{\varphi}_1 f'^{\mathrm{T}} \tag{77}$$

Substituting Eq. (76) into (77), we thus attain

$$V_N = 1 - g'(\boldsymbol{I} + \boldsymbol{D}')^{-1} f'^{\mathrm{T}} = \frac{\det\left(\boldsymbol{I} + \boldsymbol{D}' - f'^{\mathrm{T}} g'\right)}{\det(\boldsymbol{I} + \boldsymbol{D}')} \equiv \frac{\det(\boldsymbol{I} + \boldsymbol{B}')}{\det(\boldsymbol{I} + \boldsymbol{D}')} \tag{78}$$

where use is made of Appendix A.1 and

$$B'_{nm} \equiv \left(\boldsymbol{D}' - f'^{\mathrm{T}} g'\right)_{nm} = \frac{z_m z_n \lambda_n}{\rho^2 \lambda_m} D'_{nm} = \frac{\lambda_n}{\lambda_m} B_{nm} \tag{79}$$

In the end, by substituting (67) and (78) into (54), we attain the N-soliton solution to the DNLS$^+$ Eq. (3) under NVBC and reflectionless case (note that the time dependence of soliton solution naturally emerges in $c_n(t)$):

$$\bar{u}(x,t) = \frac{U_N}{V_N} = \rho \frac{\det(\boldsymbol{I} + \boldsymbol{A})\det(\boldsymbol{I} + \boldsymbol{D}')}{\det(\boldsymbol{I} + \boldsymbol{B})\det(\boldsymbol{I} + \boldsymbol{B}')} \equiv \rho \frac{C_N D_N}{\overline{D}_N^2}, \tag{80}$$

here

$$C_N \equiv \det(\boldsymbol{I} + \boldsymbol{A}), \overline{D}_N \equiv \det(\boldsymbol{I} + \boldsymbol{B}) \tag{81}$$

The solution has a standard form as (80), that is

$$\det(\boldsymbol{I} + \boldsymbol{B}') = \det(\boldsymbol{I} + \boldsymbol{B}) = \overline{\det(\boldsymbol{I} + \boldsymbol{D}')} \equiv \overline{D}, \tag{82}$$

which can be proved by direct calculation for the $N = 1$ case and by some special algebra techniques for the $N > 1$ case.

3. Verification of standard form and the explicit breather-type multi-soliton solution

3.1 Verification of $\det(I + B') = \det(I + B)$

In order to prove the first identity in (82), we firstly calculate $\overline{D}_N = \det(I + B)$. By use of (60)–(62), Binet-Cauchy formula, (Appendix (A.2)) and an important determinant formula, (Appendix (A.3)), we have

$$\overline{D}_N \equiv \det(I + B) = 1 + \sum_{r=1}^{2N} \sum_{1 \le n_1 < \cdots < n_r \le 2N} B(n_1, n_2, \cdots, n_r), \tag{83}$$

here $B(n_1, n_2, \cdots, n_r)$ is a rth-order principal minor of B consisting of elements belonging to not only rows $(n_1, n_2, \cdots, n_r)$ but also columns $(n_1, n_2, \cdots, n_r)$. Due to (62),

$$B(n_1, n_2, \cdots, n_r) = \prod_{n,m} f_n \left[\frac{\rho}{i(z_n^2 - \rho^4 z_m^{-2})} \right] f_m \prod_{n<m} i(z_n^2 - z_m^2) \cdot i\left[(-\rho^4 z_n^{-2}) - (-\rho^4 z_m^{-2}) \right] \tag{84}$$

in (84), $n, m \in (n_1, n_2, \cdots, n_r)$. The technique of calculating $B(n_1, n_2, \cdots, n_r)$ is to couple term $i(z_n^2 - \rho^4 z_m^{-2})^{-1}$ with term $i(z_m^2 - \rho^4 z_n^{-2})$ into pair, $(n \neq m)$, in the denominator of $\prod_{n,m}(\cdots)$, (with totally $r(r-1)/2$ pairs), and transplant them into the denominator of $\prod_{n<m}(\cdots)$, and combine with $i(z_n^2 - z_m^2) i(\rho^4 z_m^{-2} - \rho^4 z_n^{-2})$ in $\prod_{n<m}(\cdots)$ to form a typical factor as a whole, (with just totally $r(r-1)/2$ pairs). Note that if we define

$$z_n \equiv \rho e^{\delta_n + i\beta_n}, \text{ with } \delta_n > 0, \beta_n \in (0, \pi/2), \tag{85}$$

and further define that

$$z_n^2 / \rho^2 = e^{2\delta_n + i2\beta_n} \equiv \tanh \Theta_n, \tag{86}$$

then the typical factor is

$$\frac{i(z_n^2 - z_m^2) i(\rho^4 z_m^{-2} - \rho^4 z_n^{-2})}{i(z_n^2 - \rho^4 z_m^{-2}) i(z_m^2 - \rho^4 z_n^{-2})} = \left(\frac{z_n^2/\rho^2 - z_m^2/\rho^2}{1 - z_n^2 z_m^2/\rho^4} \right)^2 = \tanh^2(\Theta_n - \Theta_m) \tag{87}$$

and

$$\begin{aligned} B(n_1, n_2, \cdots, n_r) &= \prod_n f_n^2 \left[\frac{\rho}{i(z_n^2 - \rho^4 z_n^{-2})} \right] \prod_{n<m} \tanh^2(\Theta_n - \Theta_m) \\ &= \prod_n F_n \prod_{n<m} \tanh^2(\Theta_n - \Theta_m) \end{aligned} \tag{88}$$

here $n, m \in (n_1, n_2, \cdots, n_r)$, and a typical function F_n is defined as

$$F_n \equiv B_{n\,n} = f_n^2 \frac{\rho}{i(z_n^2 - \rho^4 z_n^{-2})} = \frac{2\rho}{i(z_n^2 - \rho^4 z_n^{-2})} c_n(t) e^{i2\Lambda_n x} \tag{89}$$

where use is made of formula (60) and (62), the time dependence of the solution naturally emerged in $c_n(t)$. Substituting Eq. (88) into (83) thus completes the computation of $\overline{D}_N$.

Secondly, let us calculate $\det(I + D')$. By use of (72) and (73), Binet-Cauchy formula, (Appendix A.2) and an important matrix formula, (Appendix A.3), we have

$$\det(I + D') = 1 + \sum_{r=1}^{2N} \sum_{1 \leq n_1 < n_2 < \cdots < n_r \leq 2N} D'(n_1, n_2, \cdots, n_r), \qquad (90)$$

where $D'(n_1, n_2, \cdots, n_r)$ is the principal minor of a rth-order submatrix of D' consisting of elements belonging to not only rows $(n_1, n_2, \cdots, n_r)$ but also columns $(n_1, n_2, \cdots, n_r)$, and

$$D'(n_1, n_2, \cdots, n_r) = \prod_{n,m} f'_n \left[\frac{i\rho}{z_n^2 - \rho^4 z_m^{-2}} \right] f'_m \prod_{n<m} (z_n^2 - z_m^2)(\rho^4 z_m^{-2} - \rho^4 z_n^{-2}) \qquad (91)$$

$n, m \in (n_1, n_2, \cdots, n_r)$. Using the same tricks as used in dealing with (84) leads to

$$D'(n_1, n_2, \cdots, n_r) = \prod_n (-1)^r f_n'^2 \left[\frac{\rho}{i(z_n^2 - \rho^4 z_n^{-2})} \right] \prod_{n<m} \frac{(z_n^2 - z_m^2)(\rho^4 z_m^{-2} - \rho^4 z_n^{-2})}{(z_n^2 - \rho^4 z_m^{-2})(z_m^2 - \rho^4 z_n^{-2})}$$

$$= \prod_n \left(\frac{\rho}{z_n} \right)^2 F_n \prod_{n<m} \tanh^2(\Theta_n - \Theta_m) \qquad (92)$$

Thirdly, let us calculate $\det\left(I + D' - f'^{\mathrm{T}} g'\right) \equiv \det(I + B')$ with $B' \equiv D' - f'^{\mathrm{T}} g'$.

According to (79) and Binet-Cauchy formula (Appendix (A.2)), similarly we have

$$\det\left(I + D' - f'^{\mathrm{T}} g'\right) = \det(I + B') = 1 + \sum_{r=1}^{2N} \sum_{1 \leq n_1 < n_2 < \cdots < n_r \leq 2N} B'(n_1, n_2, \cdots, n_r) \qquad (93)$$

$$B'(n_1, n_2, \cdots, n_r) = \prod_{n,m} \left(\frac{z_n z_m \lambda_n}{\rho^2 \lambda_m} \right) f'_n \left[\frac{-\rho}{i(z_n^2 - \rho^4 z_m^{-2})} \right] f'_m \prod_{n<m} i(z_n^2 - z_m^2) i(\rho^4 z_m^{-2} - \rho^4 z_n^{-2}) \qquad (94)$$

$n, m \in (n_1, n_2, \cdots, n_r)$. Using the same tricks as that used in treating (84) leads to

$$B'(n_1, n_2, \cdots, n_r) = \prod_n \left(\frac{z_n}{\rho} \right)^2 \left(\frac{i\rho}{z_n} \right)^2 \cdot f_n^2 \left(\frac{-\rho}{i(z_n^2 - \rho^4 z_n^{-2})} \right) \prod_{n<m} \tanh^2(\Theta_n - \Theta_m)$$

$$= \prod_n f_n^2 \left(\frac{\rho}{i(z_n^2 - \rho^4 z_n^{-2})} \right) \prod_{n<m} \tanh^2(\Theta_n - \Theta_m) = \prod_n F_n \prod_{n<m} \tanh^2(\Theta_n - \Theta_m)$$

$$\equiv B(n_1, n_2, \cdots, n_r) \qquad (95)$$

Due to (95), comparing (83) and (93) results in the expected identity and completes the verification of the first identity in (82).

3.2 Verification of $\overline{\det(I + D')} = \det(I + B)$

Our most difficult and challenging task is to prove the second identity in (82). For convenience of discussion, we define that

$$z_{\hat{n}} \equiv \rho^2 \overline{z_n^{-1}} \tag{96}$$

then

$$z_{2n} = z_{\wedge 2n-1} = \rho^2 \overline{z_{2n-1}^{-1}}, z_{2n-1} = z_{\hat{2}n} = \rho^2 \overline{z_{2n}^{-1}} \tag{97}$$

or

$$\hat{2n} = 2n - 1, 2n \; \hat{-} \; 1 = 2n, (n = 1, 2, \cdots, N) \tag{98}$$

Then the sequence of poles (25) is just in the same order as follows

$$z_{\hat{2}}, z_{\hat{1}}; z_{\hat{4}}, z_{\hat{3}}; \cdots; z_{\wedge 2N}, z_{\wedge 2N-1} \tag{99}$$

On the other hand, due to (28), (62), and (73), we have

$$D'_{nm} = \frac{\rho^2}{z_n z_m} \cdot f_n \frac{\rho}{i\left(z_n^2 - \rho^4 z_m^{-2}\right)} f_m \tag{100}$$

Then

$$\overline{D'_{nm}} = \frac{\rho^2}{\overline{z_n} \overline{z_m}} \cdot \sqrt{4 \overline{c}_n \overline{c}_m} \cdot \frac{\rho}{-i\left(\overline{z_n^2} - \rho^4 \overline{z_m^{-2}}\right)} \cdot e^{-i\left(\overline{\eta_n \lambda_n} + \overline{\eta_m \lambda_m}\right)x} \tag{101}$$

Substituting $\overline{z}_n = \rho^2 z_{\hat{n}}^{-1}, \overline{z}_m = \rho^2 z_{\hat{m}}^{-1}$ into above formula and using following relation

$$\overline{\eta_n \lambda_n} = -\overline{\eta_{\hat{n}} \lambda_{\hat{n}}}, \overline{c}_n = \rho^2 z_{\hat{n}}^{-2} c_{\hat{n}} \tag{102}$$

We can get an important relation between D'_{nm} and $B_{\hat{m}\hat{n}}$

$$\overline{D}'_{nm} = f_{\hat{m}} \frac{\rho}{i\left(z_{\hat{m}}^2 - \rho^4 z_{\hat{n}}^{-2}\right)} f_{\hat{n}} = B_{\hat{m}\hat{n}} = B_{\hat{n}\hat{m}}^{\mathrm{T}} \tag{103}$$

On the other hand, an unobvious symmetry between matrices $(B_{nm})_{2N \times 2N}$ and $(B_{\hat{n}\hat{m}})_{2N \times 2N}$ is found

$$diag(\sigma_1, \cdots, \sigma_1)_{2N \times 2N}(B_{nm})_{2N \times 2N} diag(\sigma_1, \cdots, \sigma_1)_{2N \times 2N} = (B_{\hat{n}\hat{m}})_{2N \times 2N} \tag{104}$$

It can be rewritten in a more explicit form

$$
\begin{pmatrix} \sigma_1 & 0 & \cdots & \cdots & 0 \\ 0 & \sigma_1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \sigma_1 \end{pmatrix}
\begin{pmatrix} B_{11} & B_{12} & \cdots & \cdots & B_{1,2N} \\ B_{21} & B_{22} & \cdots & \cdots & B_{2,2N} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ B_{2N,1} & B_{2N,2} & \cdots & \cdots & B_{2N,2N} \end{pmatrix}
\begin{pmatrix} \sigma_1 & 0 & \cdots & \cdots & 0 \\ 0 & \sigma_1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \sigma_1 \end{pmatrix}
$$

$$
= \begin{pmatrix} B_{22} & B_{21} & \cdots & B_{2,2N} & B_{2,2N-1} \\ B_{12} & B_{11} & \cdots & B_{1,2N} & B_{1,2N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{2N,2} & B_{2N,1} & \cdots & B_{2N,2N} & B_{2N,2N-1} \\ B_{2N-1,2} & B_{2N-1,1} & \cdots & B_{2N-1,2N} & B_{2N-1,2N-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} B_{\hat{1}\hat{1}} & B_{\hat{1}\hat{2}} & \cdots & \cdots & B_{\widehat{1}\widehat{2N}} \\ B_{\hat{2}\hat{1}} & B_{\hat{2}\hat{2}} & \cdots & \cdots & B_{\hat{2},\widehat{2N}} \\ \vdots & \vdots & & & \vdots \\ B_{\widehat{2N-1},\hat{1}} & B_{\widehat{2N-1},\hat{2}} & & & B_{\widehat{2N-1},\widehat{2N}} \\ B_{\widehat{2N},\hat{1}} & B_{\widehat{2N},\hat{2}} & \cdots & \cdots & B_{\widehat{2N},\widehat{2N}} \end{pmatrix}
$$

(105)

The last equation in (105) is due to (97) and (99), thus from (103) and (104), we have

$$
\begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix} (I + B)_{2N \times 2N} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_1 \end{pmatrix} = \left(I + \overline{D'}^{\mathrm{T}} \right)
$$

(106)

The determinants of matrices at both sides of (106) are equal to each other

$$
(\det \sigma_1)^N \det(I + B)(\det \sigma_1)^N = \det\left(I + \overline{D'}^{\mathrm{T}} \right) = \overline{\det(I + D')}
$$

(107)

The left hand of (107) is just $\det(I + B)$, and this completes verification of identity (82). From the verified (82), we know multi-soliton solution (80) is surely of a typical form as expected.

3.3 The explicit N-soliton solution to the DNLS$^+$ equation with NVBC

In order to get an explicit N-soliton solution to the DNLS$^+$ Eq. (1) with NVBC, firstly we need to make an inverse Galileo transformation of (2) by $(x, t) \rightarrow (x - \rho^2 t, t)$ in $F_n(x, t)$ in (89). Due to (51) and (85)–(87), the typical soliton kernel function F_n can be rewritten as

$$
F_n = \frac{2\rho}{i\left(z_n^2 - \rho^4 z_n^{-2}\right)} \frac{b_n(0)}{\dot{a}(z_n)} \exp i2\Lambda_n \left[x - \left(2\lambda_n^2 + \rho^2\right)t \right]
$$

(108)

$$
F_n \equiv \exp\left(-\theta_n + i\varphi_n\right)
$$

(109)

$$
\theta_n(x, t) = \left(\rho^2 \sin 2\beta_n ch 2\delta_n\right) \left[(x - x_{n0}) - \rho^2 \left(2 + \frac{\cos 2\beta_n ch 4\delta_n}{ch 2\delta_n}\right)t \right] \equiv \nu_n (x - \upsilon_n t - x_{n0})
$$

(110)

$$\varphi_n(x,t) = \left(\rho^2 \cos 2\beta_n sh 2\delta_n\right)\left[x - \rho^2\left(2 + \frac{\cos 4\beta_n ch 2\delta_n}{\cos 2\beta_n}\right)t\right] + \varphi_{n0} \equiv \mu_n(x - \xi_n t) + \varphi_{n0}$$

(111)

$$\mu_n = \rho^2 \cos 2\beta_n sh 2\delta_n, \nu_n = \rho^2 \sin 2\beta_n ch 2\delta_n$$

(112)

$$\upsilon_n = \rho^2\left(2 + \frac{ch 4\delta_n \cos 2\beta_n}{ch 2\delta_n}\right), \xi_n = \rho^2\left(2 + \frac{ch 2\delta_n \cos 4\beta_n}{\cos 2\beta_n}\right)$$

(113)

$$\frac{2\rho}{i\left(z_n^2 - \rho^4 z_n^{-2}\right)}\frac{b_n(0)}{\dot{a}(z_n)} \equiv \exp\left(\nu_n x_{n0}\right)\exp\left(i\varphi_{n0}\right)$$

(114)

$$\tanh\left(\Theta_n - \Theta_m\right) = -\frac{sh(\delta_n - \delta_m)\cos\left(\beta_n - \beta_m\right) + ich(\delta_n - \delta_m)\sin\left(\beta_n - \beta_m\right)}{sh(\delta_n + \delta_m)\cos\left(\beta_n + \beta_m\right) + ich(\delta_n + \delta_m)\sin\left(\beta_n + \beta_m\right)}$$

(115)

where in (114) the n'th pole-dependent constant factor has been absorbed by redefinition of the n'th soliton center and initial phase in (110)–(111).

Secondly, we need to calculate determinant $C_N = \det(I + A) \equiv \det\left(I + B - f^T g\right)$. According to the definition of A in (68)–(69), using Binet-Cauchy formula, (Appendix (A.2)), leads to

$$C_N = \det(I + A) = \det\left(I + B - f^T g\right) = 1 + \sum_{r=1}^{2N}\sum_{1 \le n_1 < n_2 < \cdots < n_r \le 2N} A(n_1, n_2, \cdots, n_r)$$

(116)

where $A(n_1, n_2, \cdots, n_r)$ is the determinant of a rth-order minor of A consisting of elements belonging to not only rows $(n_1, n_2, \cdots, n_r)$ but also columns $(n_1, n_2, \cdots, n_r)$.

$$A(n_1, n_2, \cdots, n_f) = \prod_{n,m}\left(\frac{z_n z_m \lambda_n}{\rho^2 \lambda_m}\right)f_n\left[\frac{\rho}{i\left(z_n^2 - \rho^4 z_m^{-2}\right)}\right]f_m \prod_{n<m} i\left(z_n^2 - z_m^2\right)i\left(\rho^4 z_m^{-2} - \rho^4 z_n^{-2}\right)$$

(117)

$n, m \in (n_1, n_2, \cdots, n_r)$. Using the same tricks as used in dealing with (84) leads to

$$A(n_1, n_2, \cdots, n_r) = \prod_n \frac{z_n^2}{\rho^2}f_n^2\frac{\rho}{i\left(z_n^2 - \rho^4 z_n^{-2}\right)}\prod_{n<m}\tanh^2(\Theta_n - \Theta_m)$$
$$= \prod_n F_n \tanh\Theta_n \prod_{n<m}\tanh^2(\Theta_n - \Theta_m)$$

(118)

$n, m \in (n_1, n_2, \cdots, n_r)$. Substituting (108)–(115) into (88) and (83) gives the explicit values of $\overline{D_N} \equiv \det(I + B)$ and D_N. Substituting (118) into (116) then completes calculation of C_N in (81). In the end, by substituting (83) and (116) into (80), we thus attain an explicit breather-type N-soliton solution of the DNLS⁺ Eq. (1) with NVBC under reflectionless case, based upon a revised and improved inverse scattering transform. Due to the limitation of space, the asymptotic behaviors of the N-soliton solution are just similar to that of the pure N-soliton solution in Ref. [7] and thus not discussed here, but it should be emphasized that in the limit of $t \to \pm\infty$, the N-soliton solution surely can be viewed as summation of N single solitons with a definite displacement and phase shift of each soliton in the whole process of elastic collisions.

4. The one and two-soliton solutions to DNLS⁺ equation with NVBC

We give two concrete examples – the one and two breather-type soliton solutions in illustration of the general explicit $N-$ soliton formula.

In the case of one-soliton solution, $N = 1$, $z_1 \equiv \rho e^{\delta_1} e^{i\beta_1}$, $z_2 = \rho^2 \bar{z}_1^{-1} = \rho e^{-\delta_1} e^{i\beta_1}$, and $\delta_1 > 0$, $\beta_1 \in (0, \pi/2)$, using formula (82), (88), (116), (108)–(115), and

$$c_1(0) = \frac{b_{10}}{\dot{a}(z_1)} = b_{10} \frac{z_1^2 - \bar{z}_1^2}{2z_1} \frac{z_1^2 - \bar{z}_2^2}{z_1^2 - z_2^2} \frac{z_1 z_2}{\bar{z}_1 \bar{z}_2} \tag{119}$$

$$c_2(0) = \frac{b_{20}}{\dot{a}(z_2)} = b_{20} \frac{z_2^2 - \bar{z}_2^2}{2z_2} \frac{z_2^2 - \bar{z}_1^2}{z_2^2 - z_1^2} \frac{z_1 z_2}{\bar{z}_1 \bar{z}_2} \tag{120}$$

we have

$$\begin{aligned}
\overline{D}_1 &= 1 + B(n_1 = 1) + B(n_1 = 2) + B(n_1 = 1, n_2 = 2) \\
&= 1 + F_1 + F_2 + F_1 F_2 \tanh^2(\Theta_1 - \Theta_2) \\
&= 1 + \frac{\sin 2\beta_1}{\sinh 2\delta_1} e^{i\beta_1} e^{-\theta_1} \left(e^{\delta_1} e^{i\varphi_1} + e^{-\delta_1} e^{-i\varphi_1} \right) - e^{i2\beta_1} e^{-2\theta_1}
\end{aligned} \tag{121}$$

$$\begin{aligned}
C_1 &= 1 + A(n_1 = 1) + A(n_1 = 2) + A(n_1 = 1, n_2 = 2) \\
&= 1 + F_1 \tanh \Theta_1 + F_2 \tanh \Theta_2 + F_1 F_2 \tanh \Theta_1 \tanh \Theta_2 \tanh^2(\Theta_1 - \Theta_2) \\
&= 1 + \frac{\sin 2\beta_1}{\sinh 2\delta_1} e^{i3\beta_1} e^{-\theta_1} \left(e^{3\delta_1} e^{i\varphi_1} + e^{-3\delta_1} e^{-i\varphi_1} \right) - e^{i6\beta_1} e^{-2\theta_1}
\end{aligned} \tag{122}$$

where not as that in (114), we define

$$b_{10} e^{i2\lambda_1 \eta_1 \left(x - 2\lambda_1^2 t\right)} \equiv e^{-\theta_1} e^{i\varphi_1}, b_{10} = e^{\nu_1 x_{10}} e^{i\varphi_{10}} \tag{123}$$

$$-b_{20} e^{i2\lambda_2 \eta_2 \left(x - 2\lambda_2^2 t\right)} \equiv e^{-\theta_2} e^{i\varphi_2} \tag{124}$$

$$\theta_1(x, t) \equiv \nu_1(x - \upsilon_1 t - x_{10}) \tag{125}$$

$$\varphi_1(x, t) \equiv \mu_1(x - \xi_1 t) + \varphi_{10}, \tag{126}$$

$$\text{with } \mu_1 = \rho^2 \cos 2\beta_1 sh 2\delta_1, \nu_1 = \rho^2 \sin 2\beta_1 ch 2\delta_1, \text{ and} \tag{127}$$

$$\upsilon_1 = \rho^2 \left(2 + \frac{ch 4\delta_1 \cos 2\beta_1}{ch 2\delta_1} \right), \xi_1 = \rho^2 \left(2 + \frac{ch 2\delta_1 \cos 4\beta_1}{\cos 2\beta_1} \right) \tag{128}$$

$$\theta_2 = \theta_1, \varphi_2 = -\varphi_1 \tag{129}$$

It is different slightly from the definition in Eq. (114) for the reason that an additional minus sign "$-$" before b_{20} can support (131)–(133) due to $-b_{20} = \overline{b}_{10}$. Substituting (121)–(122) into the following formula gives the one-soliton solution of DNLS⁺ Eq. (1) with NVBC.

$$\overline{u}_1(x, t) = \rho C_1 D_1 / \overline{D}_1^2, \text{ or } u_1(x, t) = \rho \overline{C}_1 \overline{D}_1 / D_1^2, \tag{130}$$

which is generally called a breather solution and shown as **Figure 2**.

Formula (130) includes the one-soliton solution of the DNLS equation with VBC as its limit case. In the limit of $\rho \to 0$, $\delta_1 \to \infty$ but an invariant ρe^{δ_1}, we have

$$\rho C_1 \to 4|\lambda_1| \sin 2\beta_1 e^{i3\beta_1} e^{-\theta_1} e^{i\varphi_1} \tag{131}$$

$$\overline{D}_1 \to 1 - e^{i2\beta_1} e^{-2\theta_1}, \text{ and } D_1 \to 1 - e^{-i2\beta_1} e^{-2\theta_1} \tag{132}$$
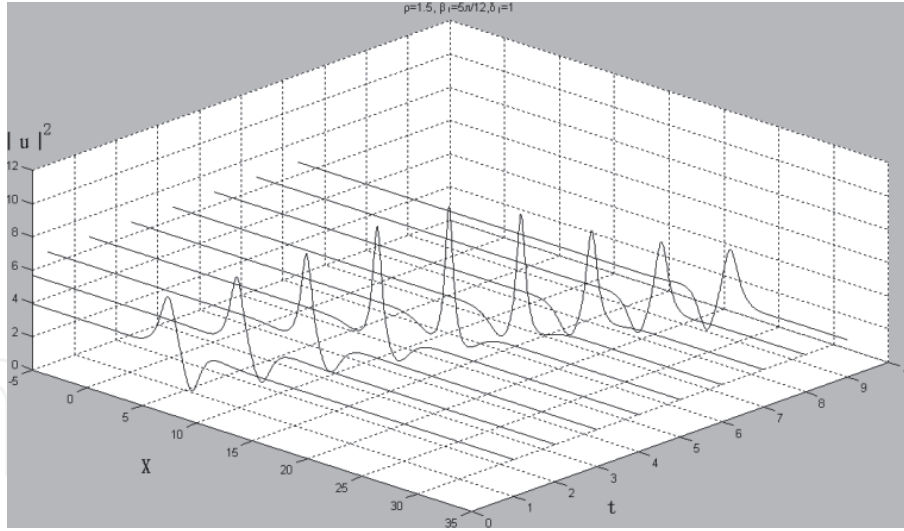
Figure 2.
The evolution of one-breather solution in time and space.

Substituting (131) and (132) into (130), we can attain

$$u_1(x,t) = 4|\lambda_1| \sin 2\beta_1 e^{-i3\beta_1} e^{-\theta_1} e^{-i\varphi_1} \left(1 - e^{i2\beta_1} e^{-2\theta_1}\right) / \left(1 - e^{-i2\beta_1} e^{-2\theta_1}\right)^2 \qquad (133)$$

If we redefine $z_1 \equiv \rho e^{\delta_1} e^{i(\pi/2 - \beta_1')}$, $z_2 \equiv \rho e^{-\delta_1} e^{i(\pi/2 - \beta_1')}$, then $\overline{u}_1(x,t) = q_1(x,t)$, the complex conjugate of one-soliton solution (133), completely reproduce the one-soliton solution that gotten in [17–20, 23], under the VBC limit with $\rho \to 0$, $\delta_1 \to \infty$, but $\rho e^{\delta_1} = 2|\lambda_1'|$ invariant, up to a permitted global constant phase factor. This verifies the validity of our formula of N-Soliton solution and the reliability of the newly revised inverse scattering transform.

The degenerate case for $N = 1$, or the so-called pure one soliton solution, is also a typical illustration of the present improved IST. It can be dealt with by letting $\delta_1 \to 0$. The simple poles $z_1(= \rho e^{i\beta_1})$ and $z_2(= \rho^2 \overline{z}_1^{-1} = \rho e^{i\beta_1})$ are coincident, so do $z_3(= -z_1)$ and $z_4(= -\rho e^{i\beta_1})$. Meanwhile $\mu_1 \to 0$, $\varphi_1 \to 0$, $\nu_1 = \rho^2 \sin 2\beta_1$, $-ib_{10} \in \mathbb{R}$. Especially for the degenerate case, we have

$$a(z) = \frac{z^2 - z_1^2 \overline{z}_1}{z^2 - \overline{z}_1^2 z_1}, c_1(0) = \frac{b_{10}}{\dot{a}(z_1)} = b_{10} \frac{z_1^2 - \overline{z}_1^2 z_1}{2z_1 \overline{z}_1} \qquad (134)$$

$$-ib_{10} e^{i2\lambda_1 \eta_1 (x - 2\lambda_1^2 t)} \equiv \varepsilon e^{-\theta_1}, \theta_1(x,t) \equiv \nu_1(x - \upsilon_1 t - x_{10}) \qquad (135)$$

with $\nu_1 = \rho^2 \sin 2\beta_1$, $\upsilon_1 = \rho^2(1 + 2\cos^2 \beta_1)$, $\varepsilon = \text{sgn}(-ib_{10})$. Then we have

$$\overline{D} = 1 + \varepsilon e^{i\beta_1} e^{-\theta_1}, \text{ or } D = 1 + \varepsilon e^{-i\beta_1} e^{-\theta_1}; C = 1 + z_1^2 F_1/\rho_1^2 = 1 + \varepsilon e^{i3\beta_1} e^{-\theta_1} \qquad (136)$$

$$\overline{u}_1(x,t) = \rho \frac{C_1 D_1}{\overline{D}_1^2} = \rho \frac{\left(1 + \varepsilon e^{i3\beta_1} e^{-\theta_1}\right)\left(1 + \varepsilon e^{-i\beta_1} e^{-\theta_1}\right)}{\left(1 + \varepsilon e^{i\beta_1} e^{-\theta_1}\right)^2} = \rho \left[1 - \frac{4\varepsilon \sin^2 \beta_1}{e^{\theta_1} e^{-i\beta_1} + e^{-\theta_1} e^{i\beta_1} + 2\varepsilon}\right]$$

$$(137)$$

where $\varepsilon = 1(-1)$ corresponds to dark (bright) soliton. Similarly if we redefine that $\beta_1 \equiv \pi/2 - \beta_1'$, then solution (137) is just the same as that gotten in [4, 5, 11, 12, 16] and called one-parameter pure soliton. This further convinces us of the validity and reliability of the newly revised IST for NVBC.

In the case of breather-type two-soliton solution, $N = 2$, we define that

$$z_1 \equiv \rho e^{\delta_1} e^{i\beta_1}, z_2 = \rho e^{-\delta_1} e^{i\beta_1}, z_3 \equiv \rho e^{\delta_3} e^{i\beta_3}, z_4 = \rho e^{-\delta_3} e^{i\beta_3} \tag{138}$$

$$F_j \equiv e^{-\theta_j} e^{i\varphi_j}, j = 1, 2, 3, 4 \tag{139}$$

which is just the same as that defined in (108)–(115), the pole z_j-related constant complex factor is absorbed into the j'th soliton center and the initial phase. Using formula (80)–(82), (88), (116), (108)–(115), we have

$$\overline{u}_2(x, t) = \rho C_2 D_2 / \overline{D}_2^2, \text{ or } u_2(x, t) = \rho \overline{C}_2 \overline{D}_2 / D_2^2 \tag{140}$$

$$
\begin{aligned}
\overline{D}_2 &= \det(\boldsymbol{I} + \boldsymbol{B}) = 1 + B(n_1 = 1) + B(n_1 = 2) + B(n_1 = 3) + B(n_1 = 4) \\
&\quad + B(n_1 = 1, n_2 = 2) + B(n_1 = 1, n_2 = 3) + B(n_1 = 1, n_2 = 4) + B(n_1 = 2, n_2 = 3) \\
&\quad + B(n_1 = 2, n_2 = 4) + B(n_1 = 3, n_2 = 4) + B(n_1 = 1, n_2 = 2, n_3 = 3) \\
&\quad + B(n_1 = 1, n_2 = 2, n_3 = 4) + B(n_1 = 1, n_2 = 3, n_3 = 4) \\
&\quad + B(n_1 = 2, n_2 = 3, n_4 = 4) + B(n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4) \\
&= 1 + F_1 + F_2 + F_3 + F_4 + F_1 F_2 \tanh^2(\Theta_1 - \Theta_2) + F_1 F_3 \tanh^2(\Theta_1 - \Theta_3) \\
&\quad + F_1 F_4 \tanh^2(\Theta_1 - \Theta_4) + F_2 F_3 \tanh^2(\Theta_2 - \Theta_3) \\
&\quad + F_2 F_4 \tanh^2(\Theta_2 - \Theta_4) + F_3 F_4 \tanh^2(\Theta_3 - \Theta_4) \\
&\quad + F_1 F_2 F_3 \tanh^2(\Theta_1 - \Theta_2) \tanh^2(\Theta_1 - \Theta_3) \tanh^2(\Theta_2 - \Theta_3) \\
&\quad + F_1 F_2 F_4 \tanh^2(\Theta_1 - \Theta_2) \tanh^2(\Theta_1 - \Theta_4) \tanh^2(\Theta_2 - \Theta_4) \\
&\quad + F_1 F_3 F_4 \tanh^2(\Theta_1 - \Theta_3) \tanh^2(\Theta_1 - \Theta_4) \tanh^2(\Theta_3 - \Theta_4) \\
&\quad + F_2 F_3 F_4 \tanh^2(\Theta_2 - \Theta_3) \tanh^2(\Theta_2 - \Theta_4) \tanh^2(\Theta_3 - \Theta_4) \\
&\quad + F_1 F_2 F_3 F_4 \tanh^2(\Theta_1 - \Theta_2) \tanh^2(\Theta_1 - \Theta_3) \tanh^2(\Theta_1 - \Theta_4) \\
&\quad \cdot \tanh^2(\Theta_2 - \Theta_3) \tanh^2(\Theta_2 - \Theta_4) \tanh^2(\Theta_3 - \Theta_4)
\end{aligned}
\tag{141}
$$

Similarly we can attain C_2 from (116) and (118) as follows

$$
\begin{aligned}
C_2 &= \det(\boldsymbol{I} + A) \\
&= 1 + F_1 \tan \Theta_1 + F_2 \tan \Theta_2 + F_3 \tan \Theta_3 + F_4 \tan \Theta_4 \\
&\quad + F_1 F_2 \tanh \Theta_1 \tanh \Theta_2 \tanh^2(\Theta_1 - \Theta_2) + \tanh \Theta_1 \tanh \Theta_3 F_1 F_3 \tanh^2(\Theta_1 - \Theta_3) \\
&\quad + F_1 F_4 \tanh \Theta_1 \tanh \Theta_4 \tanh^2(\Theta_1 - \Theta_4) + F_2 F_3 \tanh \Theta_2 \tanh \Theta_3 \tanh^2(\Theta_2 - \Theta_3) \\
&\quad + F_2 F_4 \tanh \Theta_2 \tanh \Theta_4 \tanh^2(\Theta_2 - \Theta_4) + F_3 F_4 \tanh \Theta_3 \tanh \Theta_4 \tanh^2(\Theta_3 - \Theta_4) \\
&\quad + F_1 F_2 F_3 \tanh \Theta_1 \tanh \Theta_2 \tanh \Theta_3 \tanh^2(\Theta_1 - \Theta_2) \tanh^2(\Theta_1 - \Theta_3) \tanh^2(\Theta_2 - \Theta_3) \\
&\quad + F_1 F_2 F_4 \tanh \Theta_1 \tanh \Theta_2 \tanh \Theta_4 \tanh^2(\Theta_1 - \Theta_2) \tanh^2(\Theta_1 - \Theta_4) \tanh^2(\Theta_2 - \Theta_4) \\
&\quad + F_1 F_3 F_4 \tanh \Theta_1 \tanh \Theta_3 \tanh \Theta_3 \tanh^2(\Theta_1 - \Theta_3) \tanh^2(\Theta_1 - \Theta_4) \tanh^2(\Theta_3 - \Theta_4) \\
&\quad + F_2 F_3 F_4 \tanh \Theta_2 \tanh \Theta_3 \tanh \Theta_4 \tanh^2(\Theta_2 - \Theta_3) \tanh^2(\Theta_2 - \Theta_4) \tanh^2(\Theta_3 - \Theta_4) \\
&\quad + F_1 F_2 F_3 F_4 \tanh \Theta_1 \tanh \Theta_2 \tanh \Theta_3 \tanh \Theta_4 \tanh^2(\Theta_1 - \Theta_2) \tanh^2(\Theta_1 - \Theta_3) \tanh^2(\Theta_1 - \Theta_4) \\
&\quad \cdot \tanh^2(\Theta_2 - \Theta_3) \tanh^2(\Theta_2 - \Theta_4) \tanh^2(\Theta_3 - \Theta_4)
\end{aligned}
\tag{142}
$$

Substituting (141)–(142) into (140) completes the calculation of breather-type two-soliton solution. The evolution of breather-type two-soliton solution with respect to time and space is given in **Figure 3**. It clearly display the whole process of the elastic collision between two breather solitons, and in the limit of infinite time $t \to \pm\infty$, the breather-type two-soliton is asymptotically decomposed into two breather-type 1-solitons.

4.1 Explicit pure N-soliton solution to the DNLS⁺ equation with NVBC

When all the simple poles are on the circle (O, ρ) centered at the origin O, just as shown in **Figure 4**, our revised IST for DNLS⁺ equation with NVBC will give a typical pure N-soliton solution. The discrete part of $a(z)$ is of a slightly different form from that of the case for breather-type solution, and it can be expressed as

$$a(z) = \prod_{n=1}^{N} \frac{z^2 - z_n^2 \overline{z}_n}{z^2 - \overline{z}_n^2 z_n}; \dot{a}(z_n) = \frac{2z_n \overline{z}_n}{z_n^2 - \overline{z}_n^2 z_n} \prod_{m=1, m\neq n}^{N} \frac{z_n^2 - z_m^2 \overline{z}_m}{z_n^2 - \overline{z}_m^2 z_m} \tag{143}$$
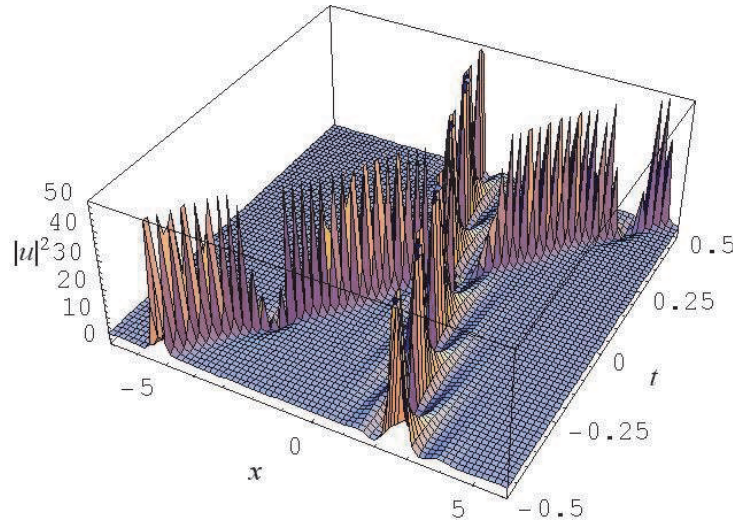

Figure 3.
Evolution of the square amplitude of a breather-type two-soliton with respect to time and space $\rho = 2; \delta_1 = 0.4; \delta_3 = 0.6; \beta_1 = \pi/5.0; \beta_3 = \pi/2.2; x_{10} = 0; x_{20} = 0 = 0; x_{30} = 0; x_{40} = 0; \varphi_{10} = 0; \varphi_{20} = 0; \varphi_{30} = 0; \varphi_{40} = 0.
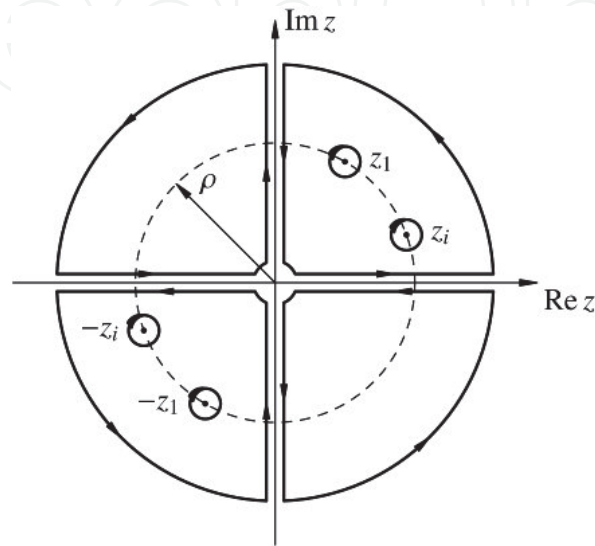

Figure 4.
Integral contour as all poles are on the circle of radiusρ.

At the zeros of $a(z)$, we have

$$\phi(x, z_n) = b_n \psi(x, z_n), \dot{a}(-z_n) = -\dot{a}(z_n), \overline{b}_n = -b_n \tag{144}$$

On the other hand, the zeros of $a(z)$ appear in pairs and can be designed by z_n, $(n = 1, 2, \cdots, N)$, in the I quadrant, and $z_{n+N} = -z_n$ in the III quadrant. The Zakharov-Shabat equation for pure soliton case of DNLS+ equation under reflectionless case can be derived immediately

$$\tilde{\psi}_1(x, z) = e^{-i\Lambda x} + \lambda \left[\sum_{n=1}^{N} \frac{2z}{\lambda_n} \frac{1}{z^2 - z_n^2} c_n \psi_1(x, z_n) e^{i\Lambda_n x} \right] e^{-i\Lambda x} \tag{145}$$

$$\tilde{\psi}_2(x, z) = i\rho z^{-1} e^{-i\Lambda x} + \lambda \left[\sum_{n=1}^{N} \frac{2z_n}{\lambda_n} \frac{1}{z^2 - z_n^2} c_n \psi_2(x, z_n) e^{i\Lambda_n x} \right] e^{-i\Lambda x} \tag{146}$$

Here $\Lambda = \kappa\lambda$, $\Lambda_n = \kappa_n \lambda_n$; Letting $z = \rho^2 z_m^{-1}$, $m = 1, 2, \cdots, N$, then

$$\psi_1(x, z_m) = -i\rho z_m^{-1} e^{i\Lambda_m x} + \sum_{n=1}^{N} \frac{\lambda_m c_n}{\lambda_n z_m^2} \cdot \frac{2\rho^3}{i(\rho^4 z_m^{-2} - z_n^2)} \psi_1(x, z_n) e^{i(\Lambda_n + \Lambda_m)x} \tag{147}$$

$$\psi_2(x, z_m) = e^{i\Lambda_m x} + \sum_{n=1}^{N} \frac{\lambda_m z_n c_n}{\lambda_n z_m} \cdot \frac{2\rho}{i(\rho^4 z_m^{-2} - z_n^2)} \psi_2(x, z_n) e^{i(\Lambda_n + \Lambda_m)x} \tag{148}$$

Different from that in breather-type case, we define $z_n \equiv \rho e^{\delta_n} e^{i\beta_n} = \rho e^{i\beta_n}$, with $\beta_n \in (0, \pi/2), \delta_n = 0, (i = 1, 2, \cdots, N)$, specially we have

$$c_{n0} = b_{n0}/\dot{a}(z_n) = i b_{n0} \rho \sin 2\beta_n e^{i\beta_n} \prod_{k=1; k\neq n}^{N} \frac{\sin(\beta_n + \beta_k)}{\sin(\beta_n - \beta_k)} \tag{149}$$

$$\tanh^2(\Theta_n - \Theta_m) = \sin^2(\beta_n - \beta_m)/\sin^2(\beta_n + \beta_m) \tag{150}$$

An inverse Galileo transformation $(x, t) \to (x - \rho^2 t, t)$ changes $F_n(x, t)$ into

$$F_n \equiv f_n^2 \frac{\rho}{i(z_n^2 - \rho^4 z_n^{-2})} = \frac{2\rho}{i(z_n^2 - \rho^4 z_n^{-2})} c_{n0} e^{-i4\lambda_n^3 \kappa_n t} e^{i2\Lambda_n x}$$

$$= (-i b_{n0}) \prod_{k=1; k\neq n}^{N} \frac{\sin(\beta_n + \beta_k)}{\sin(\beta_n - \beta_k)} e^{i2\Lambda_n [x - (2\lambda^2 + \rho^2)t] + i\beta_n} \tag{151}$$

Due to $\overline{b}_{n0} = -b_{n0}$, $-i b_{n0} \in \mathbb{R}$, following equations hold:

$$F_n = \left(e^{i\beta_n} \prod_{k=1; k\neq n}^{N} \frac{\sin(\beta_n + \beta_k)}{\sin(\beta_n - \beta_k)} \right) \left(-i b_{n0} e^{i2\Lambda_n [x - (2\lambda_n^2 + \rho^2)t]} \right) \equiv \varepsilon_n E_n e^{i\beta_n} \exp(-\theta_n + i\varphi_n) \tag{152}$$

where

$$-i b_{n0} e^{i2\Lambda_n [x - (2\lambda_n^2 + \rho^2)t]} \equiv \varepsilon_n \exp(-\theta_n + i\varphi_n) \tag{153}$$

$$\theta_n(x, t) \equiv \nu_n(x - \upsilon_n t - x_{n0}), \varphi_n = 0, \tag{154}$$

$$\varepsilon_n \equiv \text{sgn}(-i b_{n0}); |-i b_{n0}| \equiv e^{\nu_n x_{n0}} \tag{155}$$

$$\nu_n = \rho^2 \sin 2\beta_n, \upsilon_n = \rho^2 \left(1 + 2\cos^2\beta_n\right) \qquad (156)$$

$$E_n \equiv \prod_{k=1; k \neq n}^{N} \frac{\sin(\beta_n + \beta_k)}{\sin(\beta_n - \beta_k)} \qquad (157)$$

where E_n is also a real constant which is only dependent upon the order number n. The constant and positive real number $|-ib_{n0}|$ has been absorbed by redefinition of the n' th soliton center x_{n0} in (155). Thus the determinants in formula (83) for pure soliton solution can be calculated as follows

$$\overline{D}_N \equiv \det(I + B) = 1 + \sum_{r=1}^{N} \sum_{1 \leq n_1 < \cdots < n_r \leq N} B(n_1, n_2, \cdots, n_r) \qquad (158)$$

$$B(n_1, n_2, \cdots, n_r) = \prod_n f_n^2 \left[\frac{\rho}{i\left(z_n^2 - \rho^4 z_n^{-2}\right)}\right] \prod_{n<m} \frac{i\left(z_n^2 - z_m^2\right)i\left(\rho^4 z_m^{-2} - \rho^4 z_n^{-2}\right)}{i\left(z_n^2 - \rho^4 z_m^{-2}\right)i\left(z_m^2 - \rho^4 z_n^{-2}\right)}$$

$$= \prod_n F_n \prod_{n<m} \tanh^2(\Theta_n - \Theta_m) \qquad (159)$$

$$= \prod_n \varepsilon_n E_n e^{i\beta_n} e^{-\theta_n} \prod_{n<m} \frac{\sin^2(\beta_n - \beta_m)}{\sin^2(\beta_n + \beta_m)}; n, m \in (n_1, n_2, \cdots, n_r)$$

$$C_N = \det(I + A) = 1 + \sum_{r=1}^{N} \sum_{1 \leq n_1 < n_2 < \cdots < n_r \leq N} A(n_1, n_2, \cdots, n_r) \qquad (160)$$

$$A(n_1, n_2, \cdots, n_f) = \prod_n \frac{z_n^2}{\rho^2} F_n \prod_{n<m} \tanh^2(\Theta_n - \Theta_m)$$

$$= \prod_n \varepsilon_n E_n e^{i3\beta_n} e^{-\theta_n} \prod_{n<m} \frac{\sin^2(\beta_n - \beta_m)}{\sin^2(\beta_n + \beta_m)}; n, m \in (n_1, n_2, \cdots, n_r) \qquad (161)$$

Substituting (149)–(157) into (158)–(161), and substituting (158)–(161) into the following formula, we attain the explicit pure N-soliton solution

$$\overline{u}_N \equiv \rho C_N D_N / \overline{D}_N^2 \text{ or } u_N \equiv \rho \overline{C}_N \overline{D}_N / D_N^2 \qquad (162)$$

The $N = 2$ case, that is, the pure two-soliton is also a typical illustration of the general explicit N-soliton formula. According to (158)–(162), it can be calculated as follows

$$\overline{D}_2 = 1 + B(1) + B(2) + B(1, 2)$$

$$= 1 + \varepsilon_1 E_1 e^{i\beta_1} e^{-\theta_1} + \varepsilon_2 E_2 e^{i\beta_2} e^{-\theta_2} + \varepsilon_1 \varepsilon_2 E_1 E_2 e^{i(\beta_1+\beta_2)} e^{-(\theta_1+\theta_2)} \sin^2(\beta_1 - \beta_2) / \sin^2(\beta_1 + \beta_2)$$

$$= 1 + \varepsilon_1 \frac{\sin(\beta_1 + \beta_2)}{\sin(\beta_1 - \beta_2)} e^{i\beta_1} e^{-\theta_1} - \varepsilon_2 \frac{\sin(\beta_1 + \beta_2)}{\sin(\beta_1 - \beta_2)} e^{i\beta_2} e^{-\theta_2} - \varepsilon_1 \varepsilon_2 e^{i(\beta_1+\beta_2)} e^{-(\theta_1+\theta_2)}$$

$$(163)$$

$$C_2 = 1 + A(1) + A(2) + A(1, 2)$$

$$= 1 + \varepsilon_1 \frac{\sin(\beta_1 + \beta_2)}{\sin(\beta_1 - \beta_2)} e^{i3\beta_1} e^{-\theta_1} - \varepsilon_2 \frac{\sin(\beta_1 + \beta_2)}{\sin(\beta_1 - \beta_2)} e^{i3\beta_2} e^{-\theta_2} - \varepsilon_1 \varepsilon_2 e^{i3(\beta_1+\beta_2)} e^{-(\theta_1+\theta_2)}$$

$$(164)$$

$$\overline{u}_2(x,t) = \rho C_2 D_2 / \overline{D}_2^2 \tag{165}$$

The evolution of pure two-soliton solution with respect to time and space is given in **Figure 5**. It clearly demonstrates the whole process of the elastic collision between pure two solitons. If $0 < \beta_2 < \beta_1 < \pi/2$, then $\varepsilon_1 = 1$, $\text{sgn}\, E_1 = 1$ and $\varepsilon_2 = -1$, $\text{sgn}\, E_2 = -1$ correspond to double-dark pure 2-soliton solution as in **Figure 5a**; $\varepsilon_1 = -1$, $\text{sgn}\, E_1 = 1$ and $\varepsilon_2 = 1$, $\text{sgn}\, E_2 = -1$ correspond to a double-bright pure 2-soliton solution in **Figure 5c**; $\varepsilon_1 = 1$, $\text{sgn}\, E_1 = 1$ and $\varepsilon_2 = 1$, $\text{sgn}\, E_2 = -1$ correspond to a dark-bright-mixed pure 2-soliton solution in **Figure 5b**. In the limit of infinite time $t \to \pm\infty$, the pure 2-soliton solution is asymptotically decomposed into two pure 1-solitons.

By the way, it should be point out, although our method and solution have different forms from that of Refs. [7, 16], they are actually equivalent to each other. In fact if the constant E_n, $(n = 1, 2, \cdots, N)$, is also absorbed into the n'th soliton center x_{n0} just like $-ib_{n0}$ does in (152)–(154), and replace β_n with $\beta'_n = \pi/2 - \beta_n \in (0, \pi/2)$, the result for the pure soliton case in this section will reproduce the solution gotten in Refs. [7, 9, 16].

On the other hand, letting only part of the poles converge in pairs on the circle in **Figure 1** and rewriting the expression of $a_n(z)$ as in Ref. [7, 8, 12], our result can
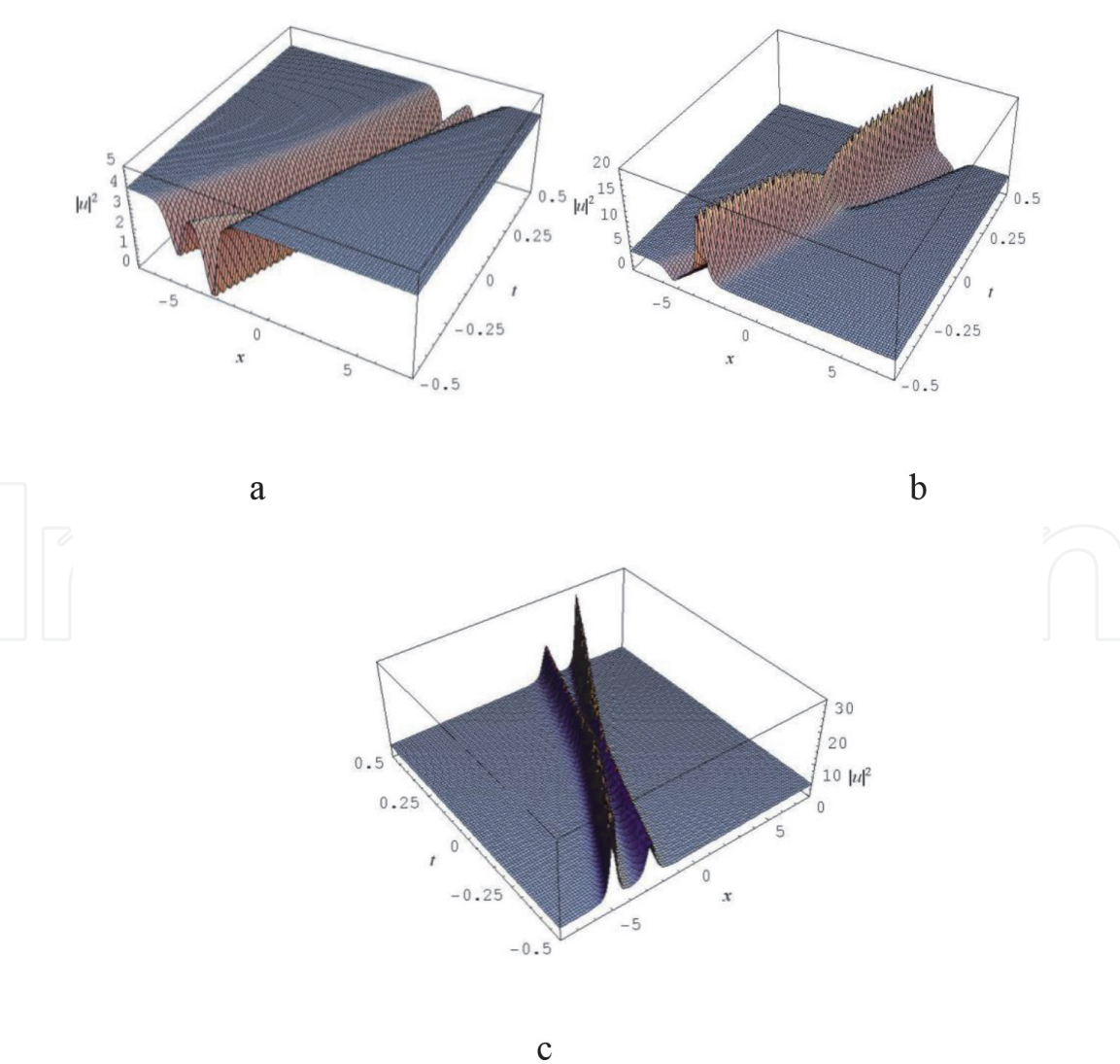

a

b

c

Figure 5.
Evolution of pure two soliton solution in time and space. (a) dark-dark pure 2-soilton, (b) dark-bright pure 2-soilton, and (c) bright-bright pure 2-soilton.

naturally generate the mixed case with both pure and breather-type multi-soliton solution.

4.2 The asymptotic behaviors of the N-soliton solution

Without loss of generality, we assume $\beta_1 > \beta_2 > \cdots > \beta_n > \cdots > \beta_N$; $v_1 < v_2 < \cdots < v_n < \cdots < v_N$ in (156), and define the n'th neighboring area as Υ_n: $x - x_{no} - v_n t \sim 0, (n = 1.2, \cdots, N)$. In the neighboring area of Υ_n,

$$\theta_j = \nu_j(x - x_{j0} - v_j t) \rightarrow \begin{cases} +\infty, & \text{for } j > n \\ -\infty, & \text{for } j < n \end{cases} \tag{166}$$

$$\overline{D} \approx B(1, 2, ..., n-1) + B(1, 2, ..., n-1, n) \tag{167}$$

$$C \approx A(1, 2, ..., n-1) + A(1, 2, ..., n-1, n) \tag{168}$$

where

$$B(1, 2, ..., n) = \varepsilon_n E_n e^{-\theta_n - i\beta_n} \prod_{j=1}^{n-1} \frac{\sin^2\left(\beta_j - \beta_n\right)}{\sin^2\left(\beta_j + \beta_n\right)} B(1, 2, ..., n-1) \tag{169}$$

$$A(1, 2, ..., n-1, n) = \varepsilon_n E_n e^{-\theta_n + i3\beta_n} \prod_{j=1}^{n-1} \frac{\sin^2\left(\beta_j - \beta_n\right)}{\sin^2\left(\beta_j + \beta_n\right)} A(1, 2, ..., n-1) \tag{170}$$

In the neighboring area of Υ_n, we have

$$u \simeq u_1\left(\theta_n + \Delta\theta_n^{(-)}\right) \tag{171}$$

With

$$\Delta\theta_n^{(-)} = 2 \sum_{j=1}^{n-1} \ln \left| \frac{\sin\left(\beta_j - \beta_n\right)}{\sin\left(\beta_j + \beta_n\right)} \right| \tag{172}$$

As $t \rightarrow -\infty$, the N neighboring areas queue up in a descending series $\Upsilon_N, \Upsilon_{N-1}, \cdots, \Upsilon_1$, then

$$u_N \simeq \sum_{n=1}^{N} u_1\left(\theta_n + \Delta\theta_n^{(-)}\right) \tag{173}$$

the N-soliton solution can be viewed as N well-separated exact pure one solitons, each $u_1\left(\theta_n + \Delta\theta_n^{(-)}\right), (1, 2, \cdots, n)$ is a single pure soliton characterized by one parameter β_n, moving to the positive direction of the x-axis, queuing up in a series with descending order number n.

As $t \rightarrow \infty$, in the neighboring area of Υ_n we have

$$\theta_j = \nu_j(x - x_{j0} - v_j t) \rightarrow \begin{cases} -\infty, & \text{for } j > n \\ +\infty, & \text{for } j < n \end{cases} \tag{174}$$

$$\overline{D} \approx B(n, n+1, ..., N) + B(n+1, n+2, ..., N) \tag{175}$$

$$C \approx A(n, n+1, ..., N) + A(n+1, n+2, ..., N) \tag{176}$$

where

$$B(n, n+1, ..., N) = \varepsilon_n E_n e^{-\theta_n - i\beta_n} \prod_{j=n+1}^{N} \frac{\sin^2\left(\beta_j - \beta_n\right)}{\sin^2\left(\beta_j + \beta_n\right)} B(n+1, n+2, ..., N) \quad (177)$$

$$A(n, n+1, ..., N) = \varepsilon_n E_n e^{-\theta_n + i3\beta_n} \prod_{j=n+1}^{N} \frac{\sin^2\left(\beta_j - \beta_n\right)}{\sin^2\left(\beta_j + \beta_n\right)} A(n+1, n+2, ..., N)$$

$$(178)$$

$$u \simeq u_1\left(\theta_n + \Delta\theta_n^{(+)}\right) \quad (179)$$

$$\Delta\theta_n^{(+)} = 2 \sum_{n+1}^{N} \ln \left| \frac{\sin\left(\beta_j + \beta_n\right)}{\sin\left(\beta_j - \beta_n\right)} \right| \quad (180)$$

$$u_N \simeq \sum_{n=1}^{N} u_1\left(\theta_n + \Delta\theta_n^{(+)}\right) \quad (181)$$

That is, the N-soliton solution can be viewed as N well-separated exact pure one solitons, queuing up in a series with ascending order number n such as $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_N$..

In the process of going from $t \to -\infty$ to $t \to \infty$, the n'th pure single soliton overtakes the solitons from the 1'th to $n-1$'th and is overtaken by the solitons from $n+1$'th to N'th. In the meantime, due to collisions, the n'th soliton got a total forward shift $\Delta\theta_n^{(-)}/\nu_n$ from exceeding those slower soliton from the 1'th to $n-1$'th, got a total backward shift $\Delta\theta_n^{(+)}/\nu_n$ from being exceeded by those faster solitons from $n+1$'th to N'th, and just equals to the summation of shifts due to each collision between two solitons, that is,

$$\Delta x_n = \left|\Delta\theta_n^{(+)} - \Delta\theta_n^{(-)}\right|/\nu_n \quad (182)$$

By introducing an suitable affine parameter in the IST and based upon a newly revised and improved inverse scattering transform and the Z-S equation for the DNLS$^+$ equation with NVBC and normal dispersion, the rigorously proved breather-type N-soliton solution to the DNLS$^+$ equation with NVBC has been derived by use of some special linear algebra techniques. The one- and two-soliton solutions have been given as two typical examples in illustration of the unified formula of the N-soliton solution and the general computation procedures. It can perfectly reproduce the well-established conclusions for the special limit case. On the other hand, letting part/all of the poles converge in pairs on the circle in **Figure 4** and rewriting the expression of $a_n(z)$ as in [7, 12, 13], can naturally generate the partly/wholly pure multi-soliton solution. Moreover, the exact breather-type multi-soliton solution to the DNLS$^+$equation can be converted to that of the MNLS equation by a gauge-like transformation [17].

Finally, the elastic collision among the breathers of the above multi-soliton solution has been demonstrated by the case of a breather-type 2-soliton solution. The newly revised IST for DNLS$^+$ equation with NVBC and normal dispersion makes corresponding Jost functions be of regular properties and asymptotic behaviors, and thus supplies substantial foundation for its direct perturbation theory.

5. Space periodic solutions and rogue wave solution of DNLS equation

DNLS equation is one of the most important nonlinear integrable equations in mathematical physics, which can describe many physical phenomena in different application fields, especially in space plasma physics and nonlinear optics [1, 2, 16, 24–29]. We have found that DNLS equation can generate not only some usual soliton solutions such as dark/bright solitons and pure/breather-type solitons, but also some special solutions – space periodic solutions and rogue wave solution [14].

There are two celebrated models of the DNLS equations. One equation is called Kaup-Newell (KN) equation [15]:

$$iu_t + u_{xx} + i\left(u^2\bar{u}\right)_x = 0 \tag{183}$$

and the other is called Chen-Lee-Liu (CLL) equation [30]:

$$iv_t + v_{xx} + iv\bar{v}v_x = 0 \tag{184}$$

Actually, there is a gauge transformation between these two Eqs. (183) and (184) [14, 30, 31]. Supposing u is one of the solutions of the KN Eq. (183), then

$$v = u \cdot \exp\left(\frac{i}{2}\int^x |u|^2 \mathrm{d}x\right) \tag{185}$$

will be the solution of the CLL equation.

This section focuses on the KN Eq. (183) with NVBC – periodic plane-wave background. The first soliton solution of (183) was derived by Kaup and Newell via inverse scattering transformation (IST) [3, 15, 32]. Whereafter, the multi-soliton solution was gotten by Nakamura and Chen by virtue of the Hirota method [30, 31]. The determinant expression of the N-soliton solution was found by Huang and Chen on the basis of the Darboux transformation (DT for brevity) [33], and by Zhou et al., by use of a newly revised IST [7, 11–13, 17].

Recently, rogue waves which seem to appear from nowhere and disappear without a trace have drawn much attention [34, 35]. The most significant feature of rogue wave is its extremely large wave amplitude and space-time locality [35]. The simplest way to derive the lowest order of rogue wave, that is, the Peregrine solution [35, 36], is to take the long-wave limit of an Akhmediev breather [37] or a Ma breather [38], both of which are special cases of the periodic solution. Thus, the key procedure of generating a rogue wave is to obtain an Akhmediev breather or a Ma breather. As far as we know, DT plays an irreplaceable role in deriving the rogue wave solution [39–41]. Because both Akhmediev breather and Ma breather can exist only on a plane-wave background; Darboux transformation has the special privilege that a specific background or, in other words, a specific boundary condition can be chosen as the seed solution used in DT. For instance, if we choose $q_0 = 0$ as the seed solution of the DT of the KN Eq. (183), then after 2-fold DT, a new solution will be gotten under VBC:

$$q^{[2]} = 4i\alpha\beta \frac{\left(-i\alpha_1 \cosh\left(2\Gamma\right) + \beta_1 \sinh\left(2\Gamma\right)\right)^3}{\left(\left(-\alpha_1^2 - \beta_1^2\right)\cosh\left(2\Gamma\right)^2 + \beta_1^2\right)^2} \tag{186}$$

(where all the parameters are defined in Ref. [38]). Similarly, setting a seed solution $q_0 = c\exp i[ax + (-c^2 + a)at]$, a plane-wave solution to Eq. (183), will generate a new solution after 2-fold DT under a plane-wave background. Therefore,

there is no need to discuss the boundary conditions or background when applying DT to solve those nonlinear integrable equations. This makes DT the most effective and prevailing method in obtaining a rogue wave solution.

Compared with DT, the IST has its fatal flaw that the difficulty of dealing with the boundary condition is unavoidable, which limits the possible application of the IST. Although the KN equation has been solved theoretically by means of an improved IST for both VBC and the NVBC [7–9, 17–20], there is no report that the KN equation could be solved under a plane-wave background by means of IST. And consequently, it appears that rogue wave solutions cannot be obtained through the IST method. This major problem is caused by the difficulty of finding appropriate Jost solutions under the plane-wave background.

On the other hand, the Hirota's bilinear-derivative transform (HBDT for brevity) [42–46], though not as a prevalent method as DT, has its particular advantages. The core of this method is a bilinear operator D which is defined by:

$$D_t^n D_x^m A \bullet B \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m A(x,t)B(x',t')|_{t'=t; x'=x} \tag{187}$$

where, at the left side of the above formula, a dot • between two functions $A(x,t)$ and $B(x',t')$ represents an ordered product. The HBDT method is very useful in dealing with periodic solutions for its convenience in computing the bilinear derivatives of an exponential function [44]:

$$\begin{aligned} &F(D_x, D_y, \cdots, D_t) \exp(kx + ly + \cdots + \omega t) \bullet \exp(k'x + l'y + \cdots + \omega't) \\ &= F(k - k', l - l', \cdots, \omega - \omega') \exp\left[(k + k')x + (l + l')y + \cdots + (\omega + \omega')t\right] \end{aligned} \tag{188}$$

Here, F represents general function expressed by the finite or infinite power series expansion of the Hirota's bilinear differential operators. Formula (188) is the generalization of Appendix 5.5. Thus using HBDT method to find space periodic solutions of KN equation is practicable. The space periodic solutions possess the characters that they approach the plane-wave solution when $|t| \to \infty$ and are periodic in space. The first space periodic solution was found by Akhmediev with one parameter [45]. Actually, we can regard the space periodic solution as a special Akhmediev breather with a pure complex-valued wave number. Further, through a space periodic solution, a rogue wave solution can be constructed. This means besides DT, HBDT method is also an alternative and effective way to find rogue wave solution of KN equation.

5.1 Bilinear derivative transformation of DNLS equation

The Hirota bilinear transformation is an effective method which could help to solve the KN equation. Due to the similarity of the first equation of Lax pairs between that of DNLS equation and AKNS system, there is a direct inference and manifestation that $u(x,t)$ has a typical standard form [6, 7]:

$$u(x,t) = g\bar{f}/f^2 \tag{189}$$

where f and g are complex auxiliary functions needed to be determined. Applying the bilinear derivative transform to (189), we can rewrite the derivatives of $u(x,t)$ in the bilinear form [19, 20, 42–46]:

$$u_t = \left(f\bar{f}D_t g \cdot f - gf D_t f \cdot \bar{f}\right)/f^4 \tag{190}$$

$$u_{xx} = \left[f\overline{f}D_x^2 g \cdot f - 2(D_x g \cdot f)\left(D_x f \cdot \overline{f}\right) + gf D_x^2 f \cdot \overline{f} - 2g\overline{f}D_x^2 f \cdot f \right]/f^4 \quad (191)$$

$$\left(|u|^2 u\right)_x = \left(2g\overline{g}D_x g \cdot f + g^2 D_x \overline{g} \cdot f\right)/f^4 \quad (192)$$

Directly substituting the above Eqs. (190)–(192) into (183) gives:

$$f\overline{f}(iD_t + D_x^2)g \cdot f - gf(iD_t + D_x^2)f \cdot \overline{f} + f^{-2}D_x\left[f^3 \cdot g\left(2D_x f \cdot \overline{f} - ig\overline{g}\right)\right] = 0 \quad (193)$$

Then the above transformed KN equation can be decomposed into the following bilinear equations:

$$\left(iD_t + D_x^2 - \lambda\right)g \cdot f = 0 \quad (194)$$

$$\left(iD_t + D_x^2 - \lambda\right)f \cdot \overline{f} = 0 \quad (195)$$

$$D_x f \cdot \overline{f} = ig\overline{g}/2 \quad (196)$$

where λ is a constant which needs to be determined. Notice that if $\lambda = 0$ then the above bilinear equations are overdetermined because we have only two variables but three equations. Actually, setting $\lambda = 0$ is the approach to search for the soliton solution of the DNLS equation under vanishing boundary condition [19, 20]. Here, we set λ as a nonzero constant to find solutions under a different boundary condition – a plane-wave background.

5.2 Solution of bilinear equations

5.2.1 First order space periodic solution and rogue wave solution

Let us assume that the series expansion of the complex functions f and g in (189) are cut off, up to the 2'th power order of ϵ, and have the following formal form:

$$f = f_0(1 + \epsilon f_1 + \epsilon^2 f_2); g = g_0(1 + \epsilon g_1 + \epsilon^2 g_2) \quad (197)$$

Substituting f and g into Eqs. (194)–(196) yields a system of equations at the ascending power orders of ϵ, which allows for determination of its coefficients [14, 19, 20]. We have 15 equations [14, 19, 20] corresponding to the different orders of ϵ. After solving all the equations, then we can obtain the solution of the DNLS equation:

$$u^{[1]}(x,t) = \overline{f}^{[1]}g^{[1]}/f^{[1]2} \quad (198)$$

with

$$g^{[1]} = \rho e^{i\omega t}\left(1 + a_1 e^{px+\Omega t+\phi_0} + a_2 e^{-px+\overline{\Omega}t+\overline{\phi}_0} + Ma_1 a_2 e^{(\Omega+\overline{\Omega})t+\phi_0+\overline{\phi}_0}\right) \quad (199)$$

$$f^{[1]} = e^{i\beta x}\left(1 + b_1 e^{px+\Omega t+\phi_0} + b_2 e^{-px+\overline{\Omega}t+\overline{\phi}_0} + Mb_1 b_2 e^{(\Omega+\overline{\Omega})t+\phi_0+\overline{\phi}_0}\right) \quad (200)$$

where

$$\omega = 3\rho^4/16; \beta = \rho^2/4 \quad (201)$$

$$a_1 = b_1 \frac{2\Omega + 2ip^2 - p\rho^2}{2\Omega - 2ip^2 - p\rho^2} ; a_2 = b_2 \frac{2\overline{\Omega} + 2ip^2 + p\rho^2}{2\overline{\Omega} - 2ip^2 + p\rho^2} \tag{202}$$

$$b_2 = \frac{\overline{b}_1 (\overline{\Omega} + ip^2 - p\rho^2)}{\overline{\Omega} - ip^2 - p\rho^2} ; M = 1 + \frac{4p^4}{(\Omega + \overline{\Omega})^2} \tag{203}$$

Notice that ρ and M are real; b_1 and φ_0 are complex constants, so there are two restrictions for a valid calculation: (1) the wave number p must be a pure imaginary number; (2) the angular frequency Ω must not be purely imaginary number and must furthermore satisfy the quadratic dispersion relation:

$$4\Omega^2 + 4p\rho^2\Omega + 4p^4 + 3p^2\rho^4 = 0 \tag{204}$$

According to the test rule for a one-variable quadratic, there is a threshold condition under which Ω will not be a pure imaginary number:

$$2p^4 + p^2\rho^4 < 0 \tag{205}$$

The asymptotic behavior of this breather is apparent. Because the wave number p is a pure imaginary number, the breather is a periodic function of x. The quadratic dispersion relation (204) permits the angular frequency Ω to have two solutions:

$$\Omega_+ = \left[-p\rho^2 + \sqrt{-2(2p^4 + p^2\rho^4)} \right] / 2 \tag{206}$$

$$\Omega_- = \left[-p\rho^2 - \sqrt{-2(2p^4 + p^2\rho^4)} \right] / 2 \tag{207}$$

If we set $\Omega = \Omega_+$, because $\sqrt{-(2p^4 + p^2\rho^4)} > 0$, then $t \to -\infty$ will lead to:

$$g^{[1]} \to \rho \exp(i\omega t) \tag{208}$$

$$f^{[1]} \to \exp(i\beta x) \tag{209}$$

$$u^{[1]} \to \rho \exp i(-3\beta x + \omega t) \tag{210}$$

And $t \to \infty$ will lead to:

$$g^{[1]} \to \rho M a_1 a_2 \exp \left[\sqrt{-2(2p^4 + p^2\rho^4)} + \phi_0 + \overline{\phi}_0 + i\omega t \right] \tag{211}$$

$$f^{[1]} \to M b_1 b_2 \exp \left[\sqrt{-2(2p^4 + p^2\rho^4)} + \phi_0 + \overline{\phi}_0 + i\beta x \right] \tag{212}$$

$$u^{[1]} \to \rho \exp i(-3\beta x + \omega t + \varphi) \tag{213}$$

where φ is the phase shift across the breather:

$$\exp(i\varphi) = a_1 a_2 / b_1 b_2 \tag{214}$$

and due to $|a_1 a_2| = |b_1 b_2|$, thus the above phase shift φ is real and does not affect the module of the breather $u^{[1]}$ when $t \to \infty$. As for the other choice $\Omega = \Omega_-$, further algebra computation shows the antithetical asymptotic behavior of $g^{[1]}$, $f^{[1]}$, and $u^{[1]}$ when $|t| \to \infty$. In a nutshell, $u^{[1]}$ will degenerate into a plane wave.
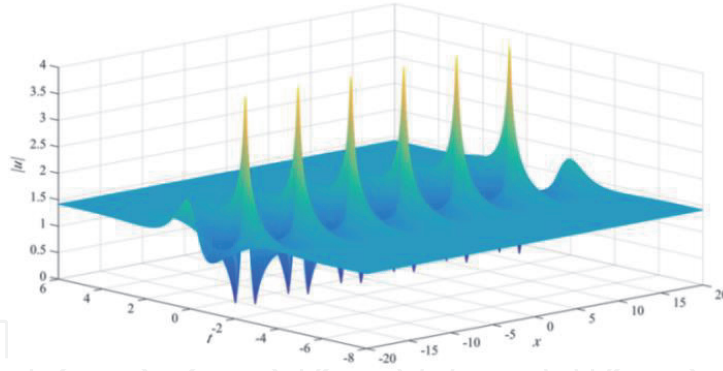
Figure 6.
The space-time evolution of the module of the 1st order space periodic solution in (198) with $p = i, \rho = \sqrt{2}, b_1 = i$ and $\Omega = \Omega_+$, complex constant φ_o is set to zero.

Hereto, we have completed the computation of the 1st-order space periodic solution, the space-time evolution of its module is depicted in **Figure 6**. In what follows, we will take the long-wave limit, that is, $p \to 0$, to construct a rogue wave solution. Supposing $p = iq$, here q is a real value and $q \to 0$, then the asymptotic expansion of the angular frequency Ω is:

$$\Omega = q\rho^2(-i + \sigma)/2 + O(q^3) \tag{215}$$

where $\sigma = \pm\sqrt{2}$. For the sake of a valid form of the rogue wave solution, we need to set $b_1 = 1$ and $\varphi_0 = 0$ (of course, setting $b_1 = 1$ and $e^{\varphi_0} = -1$ is alright, all we need is to make sure that the coefficients of the q_0 and q_1 in the expansions of $f^{[1]}$ and $g^{[1]}$ are annihilated). Therefore, the expansions of $g^{[1]}$ and $f^{[1]}$ in terms of q are given by:

$$g^{[1]} = q^2 e^{i\omega t} \frac{-8(7i + 5\sigma) + 16x(1 - 2i\sigma)\rho^2 + 3(-i + \sigma)\rho^4(4x^2 - 4\rho^2tx - 8it + 3\rho^4t^2)}{12(-i + \sigma)\rho^3} + O(q^3) \tag{216}$$

$$f^{[1]} = q^2 e^{i\beta x} \frac{8(-i + \sigma) + 16x\rho^2 + (-i + \sigma)\rho^4(4x^2 - 4\rho^2tx - 8it + 3\rho^4t^2)}{4(-i + \sigma)\rho^4} + O(q^3) \tag{217}$$

Consequently, the rogue wave solution can be derived according to Eq. (198):

$$u_{RW} = \rho \, e^{i(-3\beta x + \omega t)} \left(g' \, \overline{f'} \right) / f'^2 \tag{218}$$

where

$$g' = -8(7i + 5\sigma) + 16x(1 - 2i\sigma)\rho^2 + 3(-i + \sigma)\rho^4(4x^2 - 4\rho^2tx - 8it + 3\rho^4t^2);$$

$$f' = 24(-i + \sigma) + 48x\rho^2 + 3(-i + \sigma)\rho^4(4x^2 - 4\rho^2tx - 8it + 3\rho^4t^2).$$

Here ω and β are given by Eq. (201), ρ is an arbitrary real constant. The module of rogue wave solution Eq. (218) is shown in **Figure 7**.

As we discussed in the Introduction section, there is a gauge transformation between KN Eq. (183) and CLL Eq. (184). Thus, it is instructive to use the integral transformation Eq. (185) to construct a solution of Eq. (184). Substituting the solution (198) into (185), further algebra computation will lead to a space periodic solution of the CLL equation:

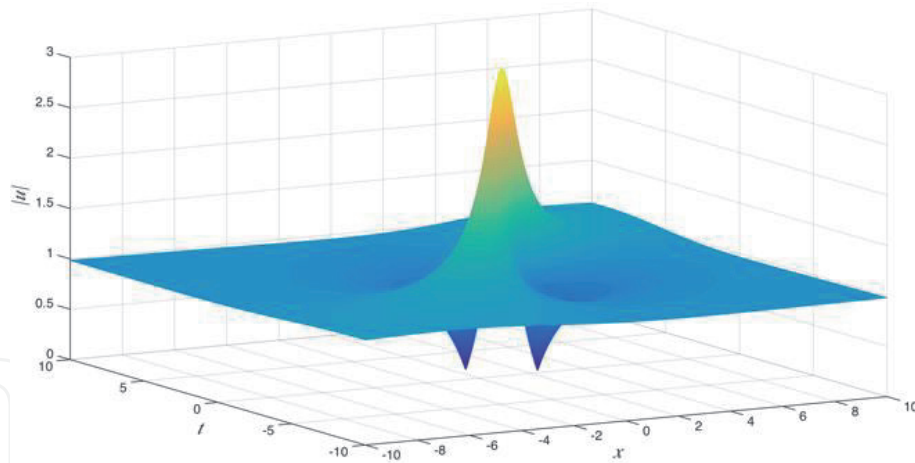$$v_c(x, t) = g^{[1]}/f^{[1]} \tag{219}$$

Figure 7.
The space-time evolution of the module of the rogue wave solution with $\rho = 1$ and $\sigma = \sqrt{2}$. The max amplitude is equal to 3 at the point $(x = -\sqrt{2}, t = -2\sqrt{2}/3)$.

where, $g^{[1]}$, $f^{[1]}$, and other auxiliary parameters are invariant and given by Eqs. (199)–(203). The same procedures which are used to derive the rogue wave solution of the KN equation can be used to turn v_c into a rogue wave solution of the CLL equation:

$$v_{c,RW} = \rho\, e^{i(-\beta x + \omega t)} g' / f' \tag{220}$$

which has the same parameters as u_{RW}. And this solution $v_{c,RW}$ has exactly the same form as the result given by ref. [46].

5.2.2 Second-order periodic solution

Taking the similar procedures described previously could help us to derive the 2nd-order space periodic solution. Assume the auxiliary functions f and g to have higher order expansions in terms of ϵ:

$$g = g_0\left(1 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 + \epsilon^4 g_4\right) \tag{221}$$

$$f = f_0\left(1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \epsilon^4 f_4\right) \tag{222}$$

Similarly, substituting f and g into the bilinear Eqs. (194)–(196) leads to the 27 equations [14, 19, 20] corresponding to different orders of ϵ. Solving these equations is tedious and troublesome but worthy and fruitful. The results are expressed in the following form:

$$u^{[2]}(x,t) = \overline{f}^{[2]} g^{[2]} / f^{[2]2} \tag{223}$$

with

$$g^{[2]} = \rho e^{i\omega t}\left(1 + g_1 + g_2 + g_3 + g_4\right) \tag{224}$$

$$f^{[2]} = e^{i\beta x}\left(1 + f_1 + f_2 + f_3 + f_4\right) \tag{225}$$

$$\beta = \rho^2/4;\ \omega = 3\rho^4/16;\ \lambda = \rho^4/16 \tag{226}$$

$$g_1 = \sum_i a_i e^{\phi_i};\ \ f_1 = \sum_i b_i e^{\phi_i} \tag{227}$$

$$g_2 = \sum_{i<j} M_{ij} a_i a_j e^{\phi_i + \phi_j}; f_2 = \sum_{i<j} M_{ij} b_i b_j e^{\phi_i + \phi_j} \qquad (228)$$

$$g_3 = \sum_{i<j<k} T_{ijk} a_i a_j a_k e^{\phi_i + \phi_j + \phi_k}; f_3 = \sum_{i<j<k} T_{ijk} b_i b_j b_k e^{\phi_i + \phi_j + \phi_k} \qquad (229)$$

$$g_4 = A a_1 a_2 a_3 a_4 e^{\phi_1 + \phi_2 + \phi_3 + \phi_4}; f_4 = A b_1 b_2 b_3 b_4 e^{\phi_1 + \phi_2 + \phi_3 + \phi_4} \qquad (230)$$

where $i, j, k = 1, 2, 3, 4$, and the above parameters and coefficients are given respectively by:

$$p_2 = \bar{p}_1; p_4 = \bar{p}_3; \Omega_2 = \bar{\Omega}_1; \Omega_4 = \bar{\Omega}_3 \qquad (231)$$

$$\phi_i = p_i x + \Omega_i t + \phi_{0i}; a_i = b_i 2\Omega_i + 2ip_i^2 - p_i \rho^2 / 2\Omega_i - 2ip_i^2 - p_i \rho^2 \qquad (232)$$

$$b_2 = \bar{b}_1 \frac{\Omega_2 + ip_2^2 + p_2 \rho^2}{\Omega_2 - ip_2^2 + p_2 \rho^2}; b_4 = \bar{b}_3 \frac{\Omega_4 + ip_4^2 + p_4 \rho^2}{\Omega_4 - ip_4^2 + p_4 \rho^2} \qquad (233)$$

$$M_{ij} = \frac{\left(\Omega_i p_j - \Omega_j p_i\right)^2 + p_i^2 p_j^2 \left(p_i - p_j\right)^2}{\left(\Omega_i p_j - \Omega_j p_i\right)^2 + p_i^2 p_j^2 \left(p_i + p_j\right)^2} \qquad (234)$$

$$T_{ijk} = M_{ij} M_{jk} M_{ki}; A = \prod_{i<j} M_{ij} \qquad (235)$$

Of course, for a valid and complete calculation, we are faced with the same situation as the 1st-order breather: ρ is real, b_1, b_3 and all φ_{0i} are complex constants. Certainly, each wave number p_i must be a pure imaginary number and each angular frequency Ω_i has to satisfy the quadratic dispersion relation:

$$4\Omega_i^2 + 4p_i \rho^2 \Omega_i + 4p_i^4 + 3p_i^2 \rho^4 = 0, (i = 1, 2, 3, 4) \qquad (236)$$

And the threshold conditions for each complex-valued Ω_i share the same form as Eq. (205):
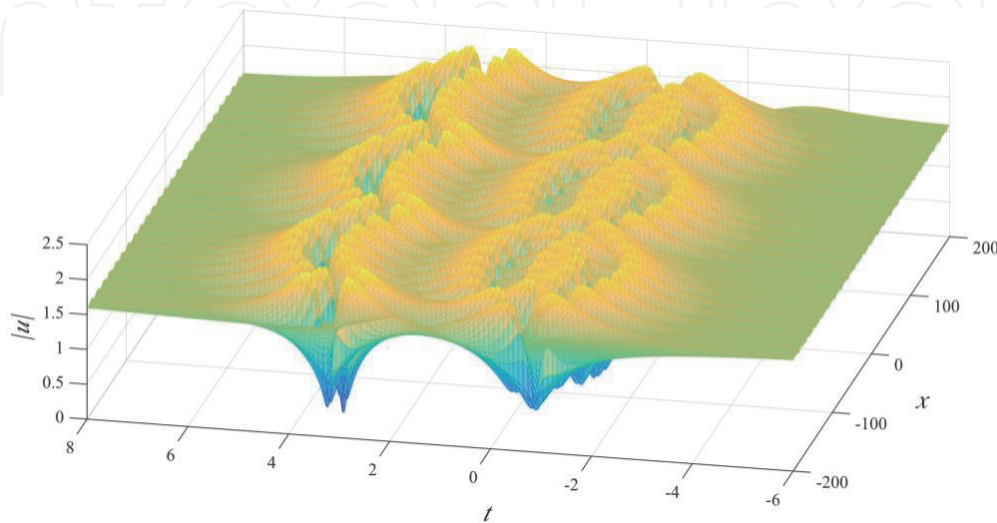
$$2p_i^4 + p_i^2 \rho^4 < 0 \qquad (237)$$

Figure 8.
The space-time evolution of the module of the 2nd order space periodic solution with
$p_1 = 0.4i, p_3 = 0.75i, b_1 = i, b_3 = 1$ and $\rho = 1.6$. Other phase factors φ_1 and φ_3 are set to zero.

The space-time evolution of the module of the 2nd order space periodic solution (223) is shown in **Figure 8**. Paying attention to the form of this breather and the previous one, we will notice that this breather can exactly degenerate into the 1st-order breather if we take $p_3 = p_1$. Under this condition, $M_{13} = M_{24} = 0$, thus the higher order interaction coefficients T_{ijk} and A will vanish. Therefore, $g^{[2]}$ and $f^{[2]}$ will degenerate into the forms of $g^{[1]}$ and $f^{[1]}$, respectively:

$$g^{[2]}_{p_3=p_1} = g'^{[1]} = \rho e^{i\omega t}\left(1 + a_1' e^{\phi_1} + a_2' e^{\phi_2} + M_{12} a_1' a_2' e^{\phi_1+\phi_2}\right) \tag{238}$$

$$f^{[2]}_{p_3=p_1} = f'^{[1]} = e^{i\beta x}\left(1 + b_1' e^{\phi_1} + b_2' e^{\phi_2} + M_{12} b_1' b_2' e^{\phi_1+\phi_2}\right) \tag{239}$$

where $b_1' = \chi b_1, b_2' = \overline{\chi} b_2, a_1' = \chi a_1$ and $a_2' = \overline{\chi} a_2$ with $\chi = (b_1 + b_3)/b_1$. That is how $u^{[2]}$ can be reduced to $u^{[1]}$. Given to this reduction, a generalized form of these two breathers arises:

$$u^{[N]} = \overline{f}^{[N]} g^{[N]} / f^{[N]2}; (N = 1, 2) \tag{240}$$

$$g^{[N]} = \rho e^{i\omega t}\left(1 + \sum_{r=1}^{2N} \sum_{1 \leq n_1 < \cdots < n_r \leq 2N} M(n_1, \cdots, n_r) \prod_{i=n_1}^{n_r} a_i e^{\phi_i}\right) \tag{241}$$

$$f^{[N]} = e^{i\beta x}\left(1 + \sum_{r=1}^{2N} \sum_{1 \leq n_1 < \cdots < n_r \leq 2N} M(n_1, \cdots, n_r) \prod_{i=n_1}^{n_r} b_i e^{\phi_i}\right) \tag{242}$$

where the coefficient M is defined by:

$$M(i) = 1 \tag{243}$$

$$M(n_1, \cdots, n_r) = \prod_{i<j} M_{ij}; i,j \in (n_1, \cdots, n_r) \tag{244}$$

On the other hand, this breather possesses the same feature as the former one that it is periodic with respect to variable x due to the pure imaginary numbers p_1 and p_3. In addition, its asymptotic behaviors are analogical to the 1st-order space periodic solution. Each quadratic dispersion equation has two roots, respectively:

$$\Omega_{1\pm} = \left[-p_1\rho^2 \pm \sqrt{-2(2p_1^4 + p_1^2\rho^4)}\right]/2 \tag{245}$$

$$\Omega_{3\pm} = \left[-p_3\rho^2 \pm \sqrt{-2(2p_3^4 + p_3^2\rho^4)}\right]/2 \tag{246}$$

Thus, we will have four combinations of Ω_1 and Ω_2. Details are numerated in **Table 1**. The parameters φ_0, φ and φ' in **Table 1** are the phase shifts which are all real so that they will not change the module of $u^{[2]}$ when $t \to \infty$. And φ is given in Eq. (214), and others are determined by:

$$\exp(i\varphi_0) = a_1 a_2 a_3 a_4 / b_1 b_2 b_3 b_4 \tag{247}$$

$$\exp(i\varphi') = a_3 a_4 / b_3 b_4 \tag{248}$$

From **Table 1**, we could draw the conclusion that this breather will also degenerate into the background plane wave as $|t| \to \infty$. Furthermore, there is a phase shift across the breather from $t = -\infty$ to $t = \infty$, which depended on the choice of Ω_1 and Ω_2.

Choice of Ω_i	Ω_{1+}, Ω_{2+}	Ω_{1+}, Ω_{2-}	Ω_{1-}, Ω_{2+}	Ω_{1-}, Ω_{2-}
$t \to -\infty$	$\rho e^{-3i\beta x + i\omega t}$	$\rho e^{-3i\beta x + i\omega t + i\varphi'}$	$\rho e^{-3i\beta x + i\omega t + i\varphi}$	$\rho e^{-3i\beta x + i\omega t + i\varphi_0}$
$t \to \infty$	$\rho e^{-3i\beta x + i\omega t + i\varphi_0}$	$\rho e^{-3i\beta x + i\omega t + i\varphi}$	$\rho e^{-3i\beta x + i\omega t + i\varphi'}$	$\rho e^{-3i\beta x + i\omega t}$

Table 1.
Asymptotic behaviors of $u^{[2]}$.

In this section, the 1st order and the 2nd order space periodic solutions of KN equation have been derived by means of HBDT. And after an integral transformation, these two breathers can be transferred into the solutions of CLL equation. Meanwhile, based on the long-wave limit, the simplest rogue wave model has been obtained according to the 1st order space periodic solution. Furthermore, the asymptotic behaviors of these breathers have been discussed in detail. As $|t| \to \infty$, both breathers will regress into the plane wave with a phase shift.

In addition, the generalized form of these two breathers is obtained, which gives us an instinctive speculation that higher order space periodic solutions may hold this generalized form, but a precise demonstration is needed. Moreover, higher order rogue wave models cannot be constructed directly by the long-wave limit of a higher order space periodic solution because the higher order space periodic solution has multiple wave numbers p_i, we are also interested in seeking an alternative method besides DT that could help us to determine the higher order rogue wave solutions.

6. Concluding remarks

In the end, as the author of the above two parts, part 1 and 2, I want to give some concluding remarks. As a whole, the two parts had taken the DNLS equation as a reference, systematically introduced several principal methods, such as IST, GLM (Marchenko) method, HBDT, to solve an integrable nonlinear equation under VBC and NVBC. We had gotten different kinds of soliton solutions, such as the light/ dark soliton, the breather-type soliton, the pure soliton, the mixed breather-type and pure soliton, and especially the rogue-wave solution. We had also gotten soliton solutions in a different numbers, such as the one-soliton solution, the two-soliton solution, and the N-soliton solution. Nevertheless, I regret most that I had not introduced the Bäcklund transform or Darboux transform to search for a rogue wave solution or a soliton solution to the DNLS equation, just like professor Huang N.N., one of my guiders in my academic research career, had done in his paper [33]. Another regretful thing is that, limited to the size of this chapter, I had not introduced an important part of soliton studies, the perturbation theory for the nearly-integrable perturbed DNLS equation. Meanwhile, this chapter have not yet involved in the cutting-edge research of the higher-order soliton and rogue wave solution to the DNLS equation, which remain to be studied and concluded in the future.

A. Appendices

Some useful formulae.
A1, If A_1 and A_2 are $N \times 1$ matrices, A is a regular $N \times N$ matrix, then

$$A_1^T A^{-1} A_2 = \det(A + A_2 A_1^T) / \det(A) - 1 \qquad (A1)$$

A2, Binet-Cauchy formula: For a squared $N \times N$ matrix B

$$\det(I + B) = 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < n_2 < \cdots < n_r \le N} B(n_1, n_2, \cdots, n_r) \tag{A2}$$

where $B(n_1, n_2, \cdots, n_r)$ is a r'th-order principal minor of B.

A3, For a $N \times N$ matrix Q_1 and a $N \times N$ matrix Q_2,

$$\det(I + Q_1 Q_2) = 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < n_2 < \cdots < n_r \le N} \Omega_r(n_1, n_2, \cdots n_r)$$

$$= 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < \cdots < n_r \le N} \sum_{1 \le m_1 < \cdots < m_r \le N} Q_1(n_1, n_2, \cdots n_r; m_1, m_2, \cdots, m_r) Q_2(m_1, m_2, \cdots m_r; n_1, n_2 \cdots, n_r) \tag{A3}$$

where $Q_1(n_1, n_2, \cdots n_r; m_1, m_2, \cdots, m_r)$ denotes a minor, which is the determinant of a submatrix of Q_1 consisting of elements belonging to not only rows $(n_1, n_2, \cdots n_r)$ but also columns $(m_1, m_2, \cdots, m_r)$.

The above formula also holds for the case of $\det(I + \Omega_1 \Omega_2)$ With Ω_1 to be a $N \times (N+1)$ matrix and Ω_2 a $(N+1) \times N$ matrix.

A4, For a squared matrix C with elements $C_{jk-} = f_j g_k (x_j - y_k)^{-1}$,

$$\det(C) = \prod_j f_j g_j \prod_{j < j', k < k'} (x_j - x_{j'})(y_{k'} - y_k) \prod_{j,k} (x_j - y_k)^{-1} \tag{A4}$$

A5, Some useful blinear derivative formulae.

$$\left(\frac{A}{B} \right)_x = \frac{D_x A \cdot B}{B^2} \tag{A5}$$

$$\left(\frac{A}{B} \right)_{xx} = \frac{D_x^2 A \cdot B}{B^2} - \frac{A}{B} \frac{D_x^2 B \cdot B}{B^2} \tag{A5.1}$$

$$D_x ab \cdot cd = bd D_x a \cdot c + ac D_x b \cdot d = bc D_x a \cdot d + ad D_x b \cdot c \tag{A5.2}$$

$$D_x^2 ab \cdot cd = bd D_x^2 a \cdot c + 2(D_x a \cdot c)(D_x b \cdot d) + ac D_x^2 b \cdot d \tag{A5.3}$$

$$D_t^n D_x^m \exp(\eta_1) \cdot \exp(\eta_2) = (\Omega_1 - \Omega_2)^n (\Lambda_1 - \Lambda_2)^m \exp(\eta_1 + \eta_2), \tag{A5.4}$$

where $\eta_i = \Omega_i t + \Lambda_i x + \eta_{0i}$, $i = 1, 2$; $\Omega_i, \Lambda_i, \eta_{0i}$ are complex constants.

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