Chapter

Soliton and Rogue-Wave Solutions of Derivative Nonlinear Schrödinger Equation - Part 1

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Abstract

Based upon different methods such as a newly revised version of inverse scattering transform, Marchenko formalism, and Hirota's bilinear derivative transform, this chapter aims to study and solve the derivative nonlinear Schrödinger (DNLS for brevity) equation under vanishing boundary condition (VBC for brevity). The explicit one-soliton and multi-soliton solutions had been derived by some algebra techniques for the VBC case. Meanwhile, the asymptotic behaviors of those multi-soliton solutions had been analyzed and discussed in detail.

Keywords: soliton, nonlinear equation, derivative nonlinear Schrödinger equation, inverse scattering transform, Zakharov-Shabat equation, Marchenko formalism, Hirota's bilinear derivative transform, rogue wave

1. Introduction

Derivative nonlinear Schrödinger (DNLS for brevity) equation is one of the several rare kinds of integrable nonlinear models. Research of DNLS equation has not only mathematic interest and significance, but also important physical application background. It was first found that the Alfven waves in space plasma [1–3] can be modeled with DNLS equation. The modified nonlinear Schrödinger (MNLS for brevity) equation, which is used to describe the sub-picosecond pulses in single mode optical fibers [4–6], is actually a transformed version of DNLS equation. The weak nonlinear electromagnetic waves in ferromagnetic, anti-ferromagnetic, or dielectric systems [5–9] under external magnetic fields can also be modeled by DNLS equation.

Although DNLS equation is similar to NLS equation in form, it does not belong to the famous AKNS hierarchy at all. As is well known, a nonlinear integrable equation can be transformed to a pair of Lax equation satisfied by its Jost functions, the original nonlinear equation is only the compatibility condition of the Lax pair, that is, the so-called zero-curvature condition. Another fact had been found by some scholars that those nonlinear integrable equations which have the same first operator of the Lax pair belong to the same hierarchy and can deal with the same inverse scattering transform (IST for brevity). As a matter of fact, the DNLS equation has a squared spectral parameter of λ^2 in the first operator of its Lax pair, while the famous NLS equation, one typical example in AKNS hierarchy, has a spectral parameter of λ. Thus, the IST of the DNLS equation is greatly different from that of the NLS equation which is familiar to us. In a word, it deserves us to demonstrate several different approaches of solving it as a typical integrable nonlinear equation.

In this chapter, we will solve the DNLS equation under two kinds of boundary condition, that is, the vanishing boundary condition (VBC for brevity) and the non-vanishing boundary condition (NVBC for brevity), by means of three different methods – the revised IST method, the Marchenko formalism, and the Hirota's bilinear derivative method. Meanwhile, we will search for different types of special soliton solution to the DNLS equation, such as the light/dark solitons, the pure solitons, the breather-type solitons, and the rogue wave solution, in one- or multi-soliton form.

2. An N-soliton solution to the DNLS equation based on a revised inverse scattering transform

For the VBC case of DNLS equation, which is just the concerned theme of the section, some attempts and progress have been made to solve the DNLS equation. Since Kaup and Newell proposed an IST with a revision in their pioneer works [10, 11], one-soliton solution was firstly attained and several versions of raw or explicit multi-soliton solutions were also obtained by means of different approaches [12–20]. Huang and Chen have got a $N-$ soliton solution by means of Darboux transformation [15]. Steudel has derived a formula for $N-$ soliton solution in terms of Vandermonde-like determinants by means of Bäcklund transformation [13]; but just as Chen points out in Ref. [16], Steudel's multi-soliton solution is difficult to demonstrate collisions among solitons and still has a too complicate form to be used in the soliton perturbation theory of DNLS equation, although it can easily generate compute pictures. Since the integral kernel in Zakharov-Shabat (Z-S for brevity) equation does not tend to zero in the limit of spectral parameter λ with $|\lambda| \to \infty$, the contribution of the path integral along the big circle (the out contour) is also nonvanishing, the usual procedure to perform inverse scattering transform encounters difficulty and is invalid. Kaup thus proposed a revised IST by multiplying an additional weighing factor before the Jost solution $E(x, \lambda)$, so that it tends to zero as $|\lambda| \to \infty$, thus the modified *Z-S* kernel should lead to vanishing contribution of the integral along the big circle of Cauchy contour. Though the one-soliton solution has been found by the obtained Z-S equation of their IST, it is very difficult to derive directly its multi-soliton solution by their IST due to the existence of a complicated phase factor which is related to the solution itself [11]. We thus consider proposing a new revised IST to avoid the excessive complexity. Our N-soliton solution obviously has a standard multi-soliton form. It can be easily used to discuss its asymptotic behaviors and then develop its direct perturbation theory. On the other hand, in solving Z-S equation for DNLS with VBC, unavoidably we will encounter a problem of calculating determinant $\det(\mathrm{I} + \mathrm{Q}_1\mathrm{Q}_2)$, for two $N \times N$ matrices Q_1 and Q_2, where I is a $N \times N$ identity matrix. Our work also shows Binet-Cauchy formula and some other linear algebra techniques, (**Appendices** A.1–4 in Part 2), play important roles in the whole process, and actually also effective for some other nonlinear integrable models [21].

2.1 The revised inverse scattering transform and the Zakharov-Shabat equation for DNLS equation with VBC

2.1.1 The fundamental concepts for the IST theory of DNLS equation

DNLS equation for the one-dimension wave function $u(x, t)$ is usually expressed as

$$iu_t + u_{xx} + i\left(|u|^2 u\right)_x = 0 \tag{1}$$

with VBC, where the subscripts stand for partial derivative. Eq. (1) is also called Kaup-Newell (KN for brevity) equation. Its Lax pair is given by

$$L = -i\lambda^2\sigma_3 + \lambda U, U = \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix} \tag{2}$$

and

$$M = -i2\lambda^4\sigma_3 + 2\lambda^3 U - i\lambda^2 U^2\sigma_3 - \lambda\left(-U^3 + iU_x\sigma_3\right) \tag{3}$$

where λ is a spectral parameter, and σ_3 is the third one of Pauli matrices σ_1, σ_2, σ_3, and a bar over a letter, (e.g., $\overline{u}$ in (2)), represents complex conjugate. The first Lax equation is

$$\partial_x f(x,\ \lambda) = L(x,\ \lambda) f(x,\ \lambda) \tag{4}$$

In the limit of $|x| \to \infty$, $u \to 0$, and

$$L \to L_0 = -i\lambda^2\sigma_3;\ M \to M_0 = -i2\lambda^4\sigma_3 \tag{5}$$

The free Jost solution is a 2×2 matrix.

$$E(x,\ \lambda) = e^{-i\lambda^2 x\sigma_3}; E_{\bullet 1}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda^2 x}, E_{\bullet 2}(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda^2 x} \tag{6}$$

The Jost solutions of (4) are defined by their asymptotic behaviors as $x \to \pm\infty$.

$$\Psi(x,\lambda) = (\tilde{\psi}(x,\lambda), \psi(x,\lambda)) \to E(x,\lambda), \text{as } x \to \infty \tag{7}$$

$$\Phi(x,\lambda) = \left(\phi(x,\lambda), \tilde{\phi}(x,\lambda)\right) \to E(x,\lambda), \text{as } x \to -\infty \tag{8}$$

where $\psi(x,\lambda) = (\psi_1(x,\lambda), \psi_2(x,\lambda))^{\mathrm{T}}$, $\tilde{\psi}(x,\lambda) = (\tilde{\psi}_1(x,\lambda), \tilde{\psi}_2(x,\lambda))^{\mathrm{T}}$, etc., and superscript "T" represents transposing of a matrix here and afterwards.

Since the first Lax equation of DNLS is similar to that of NLS, there are some similar properties of the Jost solutions. The monodromy matrix $\mathrm{T}(\lambda)$ is defined as

$$\Phi(x,\lambda) = \Psi(x,\lambda)\ \ \mathrm{T}(\lambda), \tag{9}$$

where

$$\mathrm{T}(\lambda) = \begin{pmatrix} a(t,\lambda) & -\tilde{b}(t,\lambda) \\ b(t,\lambda) & \tilde{a}(t,\lambda) \end{pmatrix} \tag{10}$$

It is easy to find from (2) and (9) that

$$\sigma_2\ \overline{L(\overline{\lambda})}\ \sigma_2 = L(\lambda),\ \ \sigma_2\ \overline{\mathrm{T}(\overline{\lambda})}\ \sigma_2 = \mathrm{T}(\lambda) \tag{11}$$

$$\sigma_2\ \overline{\Psi(x,\overline{\lambda})}\ \sigma_2 = \Psi(x,\lambda),\ \ \sigma_2\ \overline{\Phi(x,\overline{\lambda})}\ \sigma_2 = \Phi(x,\lambda) \tag{12}$$

and

$$\sigma_3\Psi(x,\lambda)\sigma_3 = \Psi(x,-\lambda),\ \ \sigma_3\Phi(x,\lambda)\sigma_3 = \Phi(x,-\lambda) \tag{13}$$

$$\sigma_3 L(\lambda)\sigma_3 = L(-\lambda),\ \ \sigma_3\mathrm{T}(\lambda)\sigma_3 = \mathrm{T}(-\lambda) \tag{14}$$

Then we can get the following reduction relation and symmetry properties

$$i\sigma_2\overline{\psi(x,\overline{\lambda})} = \tilde{\psi}(x,\lambda) \tag{15}$$

$$-i\sigma_2\overline{\varphi}(x,\overline{\lambda}) = \tilde{\varphi}(x,\lambda) \tag{16}$$

$$\overline{\tilde{a}}\left(\overline{\lambda}\right) = a(\lambda);\ \ \overline{\tilde{b}}\left(\overline{\lambda}\right) = b(\lambda) \tag{17}$$

and

$$\psi(x,-\lambda) = -\sigma_3\psi(x,\lambda) \tag{18}$$

$$\tilde{\psi}(x,-\lambda) = \sigma_3\tilde{\psi}(x,\lambda) \tag{19}$$

$$\begin{aligned} a(-\lambda) &= a(\lambda);\ b(-\lambda) = -b(\lambda) \\ \tilde{a}(-\lambda) &= \tilde{a}(\lambda);\ \tilde{b}(-\lambda) = -\tilde{b}(\lambda) \end{aligned} \tag{20}$$

2.1.2 Relation between Jost functions and the solutions to the DNLS equation

The asymptotic behaviors of the Jost solutions in the limit of $|\lambda| \to \infty$ can be obtained by simple derivation. Let $v = (v_1, v_2)^T \equiv \tilde{\psi}(x,\lambda)$; Eq. (4) can be rewritten as

$$v_{1x} + i\lambda^2 v_1 = \lambda u v_2, v_{2x} - i\lambda^2 v_2 = -\lambda\overline{u}v_1 \tag{21}$$

Then we have

$$v_{1xx} - u_x\left(v_{1x} + i\lambda^2 v_1\right)/u + \lambda^4 v_1 + \lambda^2|u|^2 v_1 = 0 \tag{22}$$

In the limit $|\lambda| \to \infty$, we assume $\tilde{\psi}_1(x,\lambda) = e^{-i\lambda^2 x + g}$, substituting it into Eq. (22), then we have

$$\left(-i\lambda^2 + g_x\right)^2 + g_{xx} - u_x g_x/u + \lambda^4 + \lambda^2|u|^2 = 0 \tag{23}$$

In the limit $|\lambda| \to \infty, g_x$ can be expanded as series of $\left(\lambda^{-2}\right)^j, j = 1, 2, \cdots$.

$$ig_x \equiv \mu = \mu_0 + \mu_2\left(2\lambda^2\right)^{-1} + \cdots \tag{24}$$

and

$$\mu_0 = |u|^2/2, \mu_2 = -i\overline{u}_x u/2 - |u|^4/4, \cdots \tag{25}$$

Eq. (21) leads to $g_x v_1 = \lambda u v_2$. Considering (25), in the limit of $|\lambda| \to \infty$, we find a useful formula

$$\overline{u} = i2 \lim_{|\lambda|\to\infty} \lambda \tilde{\psi}_2(x,\lambda)/\tilde{\psi}_1(x,\lambda) \tag{26}$$

which expresses the conjugate of solution u in terms of the Jost solutions as $|\lambda| \to \infty$.

On the other hand, the zeros of $a(\lambda)$ appear in pairs and can be designed by λ_n, $n = 1, 2, \cdots, N$ in the I quadrant, and $\lambda_{n+N} = -\lambda_n$ in the III quadrant. The discrete part of $a(\lambda)$ is [21–23].

$$a(\lambda) = \prod_{n=1}^{N} \frac{\lambda^2 - \lambda_n^2}{\lambda^2 - \overline{\lambda_n^2}} \bullet \frac{\overline{\lambda_n^2}}{\lambda_n^2} \tag{27}$$

where $a(0) = 1$. It comes from our consideration of the fact that, from the sum of two Cauchy integrals

$$\frac{\ln a(\lambda)}{\lambda} + 0 = \frac{1}{2\pi i}\int_\Gamma d\lambda' \frac{\ln a(\lambda')\tilde{a}(\lambda')}{\lambda'(\lambda' - \lambda)}, \Gamma = (0,\infty)\cup(i\infty, i0)\cup(0,-\infty)\cup(-i\infty, i0),$$

in order to maintain that $\ln a(\lambda) \to 0$, as $\lambda \to 0$, and $\ln a(\lambda)$ is finite as $|\lambda| \to \infty$, we then have to introduce a factor $\overline{\lambda}_n^2/\lambda_n^2$ in (27). At the zeros of $a(\lambda)$, we have

$$\phi(x,\lambda_n) = b_n \psi(x,\lambda_n), \dot{a}(-\lambda_n) = -\dot{a}(\lambda_n), b_{n+N} = -b_n \tag{28}$$

Due to $\mu_0 \neq 0$ in (24) and (25), the Jost solutions do not tend to free Jost solutions $E(x,\lambda)$ in the limit of $|\lambda| \to \infty$. This is their most typical property which means that the usual procedure of constructing the equation of IST by a Cauchy contour integral must be invalid and abortive, thus a newly revised procedure to derive a suitable IST and the corresponding Z-S equation is proposed in our group.

2.1.3 The revised IST and Zakharov-Shabat equation for DNLS equation with VBC

The 2×1 column function $\Theta(x,\lambda)$ can be introduced as usual

$$\Theta(x,\lambda) = \begin{cases} \phi(x,\lambda)/a(\lambda), & \text{as } \lambda \text{ in I, III quadrants.} \\ \tilde{\psi}(x,\lambda), & \text{as } \lambda \text{ in II, IV quadrants.} \end{cases} \tag{29}$$

An alternative form of IST equation is proposed as

$$\frac{1}{\lambda^2}\{\Theta_1(x,\lambda) - E_{11}(x,\lambda)\}e^{i\lambda^2 x} = \frac{1}{2\pi i}\int_\Gamma d\lambda' \frac{1}{\lambda' - \lambda}\frac{1}{\lambda'^2}\{\Theta_1(x,\lambda') - E_{11}(x,\lambda')\}e^{i\lambda'^2 x} \tag{30}$$

Because in the limit of $|\lambda| \to \infty$, $\lim_{|\lambda|\to\infty} e^{i\lambda^2 x} = 0$,

as $\begin{cases} x > 0, \ \operatorname{Im}\lambda^2 > 0, \ (\lambda \text{ in the I, III quadrants}), \\ x < 0, \ \operatorname{Im}\lambda^2 < 0, \ (\lambda \text{ in the II, IV quadrants}), \end{cases}$ then the integral path Γ should be chosen as shown in **Figure 1**, where the radius of big circle tends to infinite, while the radius of small circle tends to zero. And the factor λ^{-2} is introduced to ensure the contribution of the integral along the big arc is vanishing. Meanwhile,

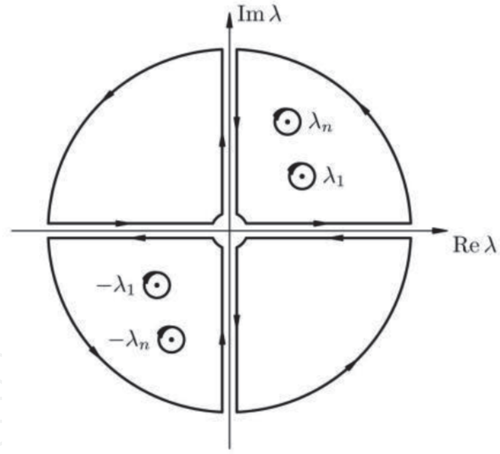

Figure 1.
The integral path for IST of the DNLS.

our modification produces no new poles since Lax operator $L \to 0$, as $\lambda \to 0$. In the reflectionless case, the revised IST equation gives

$$\tilde{\psi}_1(x,\lambda) = e^{-i\lambda^2 x} + \sum_{n=1}^{2N} \frac{1}{\lambda_n^2} \frac{\lambda^2}{\lambda - \lambda_n} \frac{b_n}{\dot{a}(\lambda_n)} \psi_1(x,\lambda_n) e^{i\lambda_n^2 x} e^{-i\lambda^2 x} \tag{31}$$

where $\dot{a}(\lambda_n) = \mathrm{d}a(\lambda)/\mathrm{d}\lambda|_{\lambda=\lambda_n}$. Similarly, an alternative form of IST equation is proposed as follows:

$$\frac{1}{\lambda}\{\Theta_2(x,\lambda)\}e^{i\lambda^2 x} = \frac{1}{2\pi i}\int_\Gamma d\lambda' \frac{1}{\lambda' - \lambda}\frac{1}{\lambda'}\{\Theta_2(x,\lambda')\}e^{i\lambda'^2 x} \tag{32}$$

where a factor λ^{-1} is introduced for the same reason as λ^{-2} in Eq. (30). Then in the reflectionless case, we can attain

$$\tilde{\psi}_2(x,\lambda) = \sum_{n=1}^{2N} \frac{1}{\lambda_n} \frac{\lambda}{\lambda - \lambda_n} \frac{b_n}{\dot{a}(\lambda_n)} \psi_2(x,\lambda_n) e^{i\lambda_n^2 x} e^{-i\lambda^2 x} \tag{33}$$

Taking the symmetry and reduction relation (18) and (28) into consideration, from (31) and (33), we can obtain the revised Zakharov-Shabat equation for DNLS equation with VBC, that is,

$$\tilde{\psi}_1(x,\lambda) = e^{-i\lambda^2 x} + \sum_{n=1}^{N} \frac{2\lambda^2}{\lambda_n\left(\lambda^2 - \lambda_n^2\right)} \frac{b_n}{\dot{a}(\lambda_n)} \psi_1(x_1,\lambda_n) e^{i\lambda_n^2 x} e^{-i\lambda^2 x} \tag{34}$$

$$\tilde{\psi}_2(x,\lambda) = \sum_{n=1}^{N} \frac{2\lambda}{\lambda^2 - \lambda_n^2} \frac{b_n}{\dot{a}(\lambda_n)} \psi_2(x_s,\lambda_n) e^{i\lambda_n^2 x} e^{-i\lambda^2 x} \tag{35}$$

2.2 The raw expression of N-soliton solution

Substituting Eqs. (34) and (35) into formula (26), we thus attain the N-soliton solution

$$\overline{u}_N = -i2U_N/V_N \tag{36}$$

where

$$U_N = \sum_{n=1}^{N} \frac{2b_n}{\dot{a}(\lambda_n)} \psi_2(x,\lambda_n) e^{i\lambda_n^2 x} \tag{37}$$

$$V_N = 1 + \sum_{n=1}^{N} \frac{2b_n}{\lambda_n \dot{a}(\lambda_n)} \psi_1(x,\lambda_n) e^{i\lambda_n^2 x} \tag{38}$$

Let $\lambda = \overline{\lambda}_m$, m = 1, 2, … , N, respectively, in Eqs. (34) and (35), and make use of the symmetry and reduction relation (15), we can attain

$$\overline{\psi_2}(x,\lambda_m) = e^{-i\overline{\lambda_m^2}x} + \sum_{n=1}^{N} \frac{2\overline{\lambda_m^2}}{\lambda_n\left(\overline{\lambda_m^2} - \lambda_n^2\right)} c_n \psi_1(x,\lambda_n) e^{i\lambda_n^2 x} e^{-i\overline{\lambda_m^2}x} \tag{39}$$

$$\overline{\psi}_1(x,\lambda_m) = -\sum_{n=1}^{N} \frac{2\overline{\lambda_m^2}}{\overline{\lambda_m^2} - \lambda_n^2} c_n \psi_2(x,\lambda_n) e^{i\lambda_n^2 x} e^{-i\overline{\lambda_m^2}x};\ m = 1,2,\dots,N. \tag{40}$$

where $c_n = b_n/\dot{a}(\lambda_n)$. We also define

$$f_n = \sqrt{2c_n} e^{i\lambda_n^2 x},\quad (w_j)_n = \sqrt{2c_n}\psi_j(\lambda_n)\, j = 1,2; \text{and } n = 1,2,\dots N. \tag{41}$$

$$(B_1)_{mn} = \overline{f}_m \frac{\overline{\lambda_m^2}}{\left(\overline{\lambda_m^2} - \lambda_n^2\right)\lambda_n} f_n,\ (B_2)_{mn} = \overline{f}_m \frac{\overline{\lambda_m}}{\overline{\lambda}_m^2 - \lambda_n^2} f_n; m,n = 1,2,\dots,N \tag{42}$$

$$\begin{gathered} W_1 = \left((w_1)_1, (w_1)_2, \cdots, (w_1)_N\right)^{\mathrm{T}}, W_2 = \left((w_2)_1, (w_2)_2, \cdots, (w_2)_N\right)^{\mathrm{T}}, \\ F = \left(f_1, f_2, \cdots, f_N\right)^{\mathrm{T}}, G = \left(f_1/\lambda_1, f_2/\lambda_2, \cdots, f_N/\lambda_N\right)^{\mathrm{T}} \end{gathered} \tag{43}$$

where superscript "T" represents transposition of a matrix. Then Eqs. (39) and (40) can be rewritten as

$$(\overline{w_2})_m = \overline{f}_m + \sum_{n=1}^{N} (B_1)_{mn} (w_1)_n \tag{44}$$

$$(\overline{w_1})_m = -\sum_{n=1}^{N} (B_2)_{mn} (w_2)_n \tag{45}$$

where m = 1, 2, … , N. They can be rewritten in a more compact matrix form.

$$\overline{W}_2 = \overline{F} + B_1 \cdot W_1 \tag{46}$$

$$\overline{W}_1 = -B_2 \cdot W_2 \tag{47}$$

Then

$$W_2 = \left(I + \overline{B}_1 B_2\right)^{-1} F \tag{48}$$

$$W_1 = -\overline{B}_2\left(I + B_1 \overline{B}_2\right)^{-1} \overline{F} \tag{49}$$

where I is the $N \times N$ identity matrix. On the other hand, from (37) and (38), we know

$$U_N = \sum_{n=1}^{N} f_n w_{2n} = F^{\mathrm{T}} W_2 \tag{50}$$

$$V_N = 1 + \sum_{n=1}^{N} \left(f_n / \lambda_n \right) w_{1n} = 1 + G^{\mathrm{T}} W_1 \tag{51}$$

Substituting Eqs. (48), (49) into (50) and (51) and then substituting (50) and (51) into formula (36), we thus attain

$$\begin{aligned} \overline{u}_N &= -i2 \frac{F^{\mathrm{T}} W_2}{1 + G^{\mathrm{T}} W_1} = -i2 \frac{F^{\mathrm{T}} \left(I + \overline{B}_1 B_2 \right)^{-1} F}{1 - G^T \overline{B}_2 \left(I + B_1 \overline{B}_2 \right)^{-1} \overline{F}} \\ &= -i2 \frac{\det\left(I + \overline{B}_1 B_2 + F F^{\mathrm{T}} \right) - \det\left(I + \overline{B}_1 B_2 \right)}{\det\left[I + \left(B_1 - \overline{F} G^{\mathrm{T}} \right) \overline{B}_2 \right]} \cdot \frac{\det\left(I + B_1 \overline{B}_2 \right)}{\det\left(I + \overline{B}_1 B_2 \right)} \equiv -2i \frac{A \cdot D}{\overline{D}^2} \end{aligned} \tag{52}$$

where

$$A \equiv \det\left(I + \overline{B}_1 B_2 + F F^{\mathrm{T}} \right) - \det\left(I + \overline{B}_1 B_2 \right) \tag{53}$$

$$D \equiv \det\left(I + B_1 \overline{B}_2 \right) \tag{54}$$

In the subsequent chapter, we will prove that

$$\det\left[I + \left(B_1 - \overline{F} G^{\mathrm{T}} \right) \overline{B}_2 \right] = \det\left(I + \overline{B}_1 B_2 \right) \tag{55}$$

It is obvious that formula (52) has the usual standard form of soliton solution. Here in formula (52), some algebra techniques have been used and can be found in Appendix A.1 in Part 2.

2.3 Explicit expression of N-soliton solution

2.3.1 Verification of standard form for the N-soliton solution

We only need to prove that Eq. (55) holds. Firstly, we define $N \times N$ matrices P_1, P_2, Q_1, Q_2, respectively, as

$$(P_1)_{nm} \equiv \left(B_1 - \overline{F} G^{\mathrm{T}} \right)_{nm} = \overline{f}_n \frac{\lambda_m}{\overline{\lambda}_n^2 - \lambda_m^2} f_m ; (P_2)_{mn} \equiv \left(\overline{B}_2 \right)_{mn} = f_m \frac{\lambda_m}{\lambda_m^2 - \overline{\lambda}_n^2} \overline{f}_n \tag{56}$$

$$(Q_1)_{nm} \equiv \left(\overline{B}_1 \right)_{nm} = f_n \frac{\lambda_n^2}{\lambda_n^2 - \overline{\lambda}_m^2} \left(\frac{\overline{f}_m}{\overline{\lambda}_m} \right) ; (Q_2)_{mn} \equiv (B_2)_{mn} = \overline{f}_m \frac{\overline{\lambda}_m}{\overline{\lambda}_m^2 - \lambda_n^2} f_n \tag{57}$$

Then

$$\begin{aligned} \overline{D} &= \det(I + Q_1 Q_2) = 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < n_2 < \cdots < n_r \le N} \overline{D}_r(n_1, n_2, \cdots n_r) \\ &= 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < \cdots < n_r \le N} \sum_{1 \le m_1 < \cdots < m_r \le N} Q_1(n_1, n_2, \cdots n_r; m_1, m_2, \cdots, m_r) Q_2(m_1, m_2, \cdots m_r; n_1, n_2 \cdots, n_r) \end{aligned} \tag{58}$$

where $Q_1(n_1, n_2, \cdots n_r; m_1, m_2, \cdots, m_r)$ denotes a minor, which is the determinant of a submatrix of Q_1 consisting of elements belonging to not only rows $(n_1, n_2, \dots, n_r)$ but also columns $(m_1, m_2, \dots, m_r)$. Here use is made of Binet-Cauchy formula in the Appendices A.2–4 in Part 2. Then

$$\begin{aligned}
&Q_1(n_1, n_2, \cdots n_r; m_1, m_2, \cdots, m_r) Q_2(m_1, m_2, \cdots m_r; n_1, n_2 \cdots, n_r) \\
&= \prod_{n,m} \frac{f_n \overline{f}_m}{\lambda_n^2 - \overline{\lambda}_m^2} \frac{\lambda_n^2}{\overline{\lambda}_m} \prod_{n<n', m<m'} (\lambda_n^2 - \lambda_{n'}^2)(\lambda_{m'}^2 - \lambda_m^2) \prod_{m,n} \frac{\overline{f}_m f_n}{\overline{\lambda}_m^2 - \lambda_n^2} \overline{\lambda}_m \prod_{n<n', m<m'} \left(\overline{\lambda}_m^2 - \lambda_{m'}^2\right)(\lambda_{n'}^2 - \lambda_n^2) \\
&= (-1)^r \prod_{m,n} \frac{\lambda_n^2 f_n^2 \overline{f_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2} \prod_{n<n', m<m'} (\lambda_n^2 - \lambda_{n\prime}^2)^2 \left(\overline{\lambda_m^2} - \overline{\lambda_{m\prime}^2}\right)^2
\end{aligned} \tag{59}$$

where

$$n, n' \in \{n_1, n_2, \cdots, n_r\}, m, m' \in \{m_1, m_2, \cdots, m_r\} \tag{60}$$

Similarly,

$$\begin{aligned}
&P_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) P_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \\
&\quad = (-1)^r \prod_{n,m} \frac{\overline{f}_n^2 f_m^2 \lambda_m^2}{\left(\overline{\lambda}_n^2 - \lambda_m^2\right)^2} \prod_{n<n', m<m'} (\lambda_m^2 - \lambda_{m\prime}^2)^2 \left(\overline{\lambda_n^2} - \overline{\lambda_{n\prime}^2}\right)^2
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
&n, n' \in \{n_1, n_2, \cdots, n_r\}; m, m' \in \{m_1, m_2, \cdots, m_r\}, \text{and} \\
&\det\left[I + \left(B_1 - \overline{F} G^{\mathrm{T}}\right) \overline{B}_2\right] = \det(I + P_1 P_2) \\
&= 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < \cdots < n_r \le N} \sum_{1 \le m_1 < \cdots < m_r \le N} P_1(n_1, \cdots, n_r; m_1, \cdots, m_r) P_2(m_1, \cdots, m_r; n_1, \cdots, n_r)
\end{aligned} \tag{62}$$

It is easy to find a kind of permutation symmetry existed between expressions (59) and (61), that is,

$$\begin{aligned}
&P_1(n_1, \cdots, n_r; m_1, \cdots, m_r) P_2(m_1, \cdots, m_r; n_1, \cdots, n_r) \\
&= Q_1(m_1, \cdots, m_r; n_1, \cdots, n_r) Q_2(n_1, \cdots, n_r; m_1, \cdots, m_r)
\end{aligned} \tag{63}$$

Comparing (58) with (62) and making use of (63), we thus complete verification of Eq. (55). The soliton solution is surely of a typical form as that in NLS equation and can be expressed as formula (52).

2.3.2 Introduction of time evolution function

The time evolution factor of the scattering data can be introduced by standard procedure [21]. Due to the fact that the second Lax operator $M \to -i2\lambda^4 \sigma_3$ in the limit of $|x| \to \infty$, it is easy to derive the time dependence of scattering date.

$$d\lambda_n/dt = 0, da(\lambda_n)/dt = 0; c_n(t) = c_{n0} e^{i4\lambda_n^4 t}, c_{n0} = b_{n0}/\dot{a}(\lambda_n), b_n(t) = b_{n0} e^{i4\lambda_n^4 t} \tag{64}$$

Then the typical soliton arguments θ_n and φ_n can be defined according to

$$f_n^2 = 2c_{n0}e^{i2\lambda_n^2 x}e^{i4\lambda_n^4 t} \equiv 2c_{n0}e^{-\theta_n}e^{i\varphi_n} \tag{65}$$

where $\lambda \equiv \mu_n + i\nu_n$, and $\theta_n = 4\mu_n\nu_n\left[x + 4\left(\mu_n^2 - \nu_n^2\right)t\right] = 4\kappa_n(x - V_n t)$;
$\varphi_n = 2\left(\mu_n^2 - \nu_n^2\right)x + \left[4\left(\mu_n^2 - \nu_n^2\right)^2 - 16\mu_n^2\nu_n^2\right] \cdot t$

$$V_n = -4\left(\mu_n^2 - \nu_n^2\right), \kappa_n = 4\mu_n\nu_n \tag{66}$$

2.3.3 Calculation of determinant of $\overline{D}$ and A

Substituting expression (64) and (65) into formula (59) and then into (58), we have

$$\begin{aligned}&Q_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r)Q_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r)\\ &= (-1)^r \prod_{n,m}(2c_n)(2\overline{c}_m)e^{-\theta_n}e^{i\varphi_n}e^{-\theta_m}e^{-i\varphi_m}\frac{\lambda_n^2}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2}\prod_{n<n', m<m'}\left(\lambda_n^2 - \lambda_{n'}^2\right)^2\left(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\right)^2\end{aligned} \tag{67}$$

with $n, n' \in \{n_1, n_2, \cdots, n_r\}$ and $m, m' \in \{m_1, m_2, \cdots, m_r\}$. Where use is made of Binet-Cauchy formula which is numerated in Appendix A. 3–4 in Part 2. Substituting expression (67) into formula (58) thus complete the calculation of determinant $\overline{D}$.

About the calculation of the most complicate determinant A in (52), we introduce a $N\times(N+1)$ matrix Ω_1 and a $(N+1)\times N$ matrix Ω_2 defined as

$$\begin{aligned}(\Omega_1)_{nm} = \left(\overline{B}_1\right)_{nm} = (Q_1)_{nm}, (\Omega_1)_{n0} = f_n,\\ (\Omega_2)_{mn} = (B_2)_{mn} = (Q_2)_{mn}, (\Omega_2)_{0n} = f_n\end{aligned} \tag{68}$$

with $n, m = 1, 2, \cdots, N$. We thus have

$$\begin{aligned}&\det\left(I + \overline{B}_1 B_2 + FF^{\mathrm{T}}\right) = \det(I + \Omega_1\Omega_2)\\ &= 1 + \sum_{r=1}^{N}\sum_{1\le n_1<\cdots<n_r\le N}\sum_{0\le m_1<\cdots<m_r\le N}\Omega_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r)\Omega_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r)\end{aligned} \tag{69}$$

The above summation obviously can be decomposed into two parts: one is extended to $m_1 = 0$, the other is extended to $m_1 \ge 1$. Subtracted from (69), the part that is extended to $m_1 \ge 1$, the remaining parts of (69) is just A in (52) (with $m_1 = 0$ and $m_2 \ge 1$). Due to (68), we thus have

$$\begin{aligned}A &= \det(I + \Omega_1\Omega_2) - \det(I + Q_1 Q_2)\\ &= \sum_{r=1}^{N}\sum_{1\le n_1<\cdots<n_r\le N}\sum_{1\le m_2<\cdots<m_r\le N} A_r(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r)\\ &= \sum_{r=1}^{N}\sum_{1\le n_1<n_2<n_r\le N}^{N}\sum_{1\le m_2<m_3<\cdots<m_r\le N}\Omega_1(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r)\Omega_2(0, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r)\end{aligned} \tag{70}$$

with

$$
\begin{aligned}
&\Omega_1(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r)\Omega_2(0, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \\
&= (-1)^{r-1}\prod_{n,m}\frac{f_n^2\overline{f_m^2}\overline{\lambda_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2}\prod_{n<n', m<m'}\left(\lambda_n^2 - \lambda_{n\prime}^2\right)^2\left(\overline{\lambda_m^2} - \overline{\lambda_{m\prime}^2}\right)^2 \\
&= (-1)^{r-1}\prod_{n,m}(2c_n)(2\overline{c_m})e^{-\theta_n}e^{i\varphi_n}e^{-\theta_m}e^{-i\varphi_m}\frac{\overline{\lambda_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2}\prod_{n<n', m<m'}\left(\lambda_n^2 - \lambda_{n\prime}^2\right)^2\left(\overline{\lambda_m^2} - \overline{\lambda_{m\prime}^2}\right)^2
\end{aligned} \tag{71}
$$

Here $n, n' \in \{n_1, n_2, \cdots, n_r\}$ and especially $m, m' \in \{m_2, \cdots, m_r\}$, which completes the calculation of determinant A in formula (52). Substituting the explicit expressions of D, $\overline{D}$, and A into (52), we finally attain the explicit expression of N-soliton solution to the DNLS equation under VBC and reflectionless case, based upon a newly revised IST technique.

An interesting conclusion is found that, besides a permitted well-known constant global phase factor, there is also an undetermined constant complex parameter b_{n0} before each of the typical soliton factor $e^{-\theta_n}e^{i\varphi_n}$, $(n = 1,2, \dots, N)$. It can be absorbed into $e^{-\theta_n}e^{i\varphi_n}$ by redefinition of soliton center and its initial phase factor. This kind of arbitrariness is in correspondence with the unfixed initial conditions of the DNLS equation.

2.4 The typical examples for one- and two-soliton solutions

We give two concrete examples – the one- and two-soliton solutions as illustrations of the general explicit soliton solution.

In the case of one-soliton solution, $N = 1$, $\lambda_2 = -\lambda_1$, $\lambda_1 = \rho_1 e^{i\beta_1} = \mu_1 + i\nu_1$, and

$$A_1 = \Omega_1(n_1 = 1; m_1 = 0)\Omega_2(m_1 = 0; n_1 = 1) = f_1^2 \tag{72}$$

$$\overline{D}_1 = Q_1(n_1 = 1; m_1 = 1)Q_2(m_1 = 1; n_2 = 1) = 1 - |f_1|^4\lambda_1^2/\left(\lambda_1^2 - \overline{\lambda_1^2}\right)^2 \tag{73}$$

$$
\begin{aligned}
&f_1^2 = 2c_{10}e^{i2\lambda_1^2 x}e^{i4\lambda_1^4 t}; c_{10} = b_{10}/\dot{a}(\lambda_1); b_{10} = e^{4\mu_1\nu_1 x_{10}}e^{i\alpha_{10}}; b_{10}e^{i2\lambda_1^2 x}e^{i4\lambda_1^4 t} \equiv e^{-\theta_1}e^{i\varphi_1}; \\
&\theta_1 = 4\mu_1\nu_1\left[x - x_{10} + 4\left(\mu_1^2 - \nu_1^2\right)t\right]; \varphi_1 = 2\left(\mu_1^2 - \nu_1^2\right)x + \left[4\left(\mu_1^2 - \nu_1^2\right)^2 - 16\mu_1\nu_1^2\right]t + \alpha_{10}
\end{aligned} \tag{74}
$$

It is different slightly from the definition in (66) for that here b_{10} has been absorbed into the soliton center and initial phase. Then

$$A_1 = \lambda_1\left(\lambda_1^2 - \overline{\lambda_1^2}\right)e^{-\theta_1}e^{i\varphi_1}/\overline{\lambda}_1^2 = i2\rho_1 \sin 2\beta_1 e^{i3\beta_1}e^{-\theta_1}e^{i\varphi_1}$$

$$\overline{D}_1 = 1 - \frac{\left|\lambda_1^2 - \overline{\lambda_1^2}\right|^2}{|\lambda_1|^2}\frac{\lambda_1^2}{\left(\lambda_1^2 - \overline{\lambda_1^2}\right)^2}e^{-2\theta_1} = 1 + e^{i2\beta_1}e^{-2\theta_1}$$

and

$$\overline{u}_1(x,t) = -i2A_1D_1/\overline{D}_1^2 = \frac{4\rho_1 \sin 2\beta_1 e^{i3\beta_1}\left(1 + e^{-i2\beta_1}e^{-2\theta_1}\right)}{\left(1 + e^{i2\beta_1}e^{-2\theta_1}\right)^2}e^{-\theta_1}e^{i\varphi_1} \tag{75}$$

The complex conjugate of one-soliton solution $\overline{u}_1(x,t)$ in (75) is $u_1(x,t)$, which is just in conformity with that gotten from pure Marchenko formalism [24] (see the next section), up to a permitted global constant phase factor. In the case of two-soliton solution, $N = 2$, $\lambda_3 = -\lambda_1$, $\lambda_4 = -\lambda_2$ and

$$\lambda_1 = \rho_1 e^{i\beta_1} = \mu_1 + i\nu_1;\ \lambda_2 = \rho_2 e^{i\beta_2} = \mu_2 + i\nu_2 \tag{76}$$

$$\begin{aligned} c_{10} &= \frac{b_{10}}{\dot{a}(\lambda_1)} = b_{10}\frac{\lambda_1^2 - \overline{\lambda_1^2}}{2\lambda_1}\cdot\frac{\lambda_1^2 - \overline{\lambda_2^2}}{\lambda_1^2 - \lambda_2^2}\cdot\frac{\lambda_1^2}{\overline{\lambda_1^2}}\cdot\frac{\lambda_2^2}{\overline{\lambda_2^2}} \\ c_{20} &= \frac{b_{20}}{\dot{a}(\lambda_2)} = b_{20}\frac{\lambda_2^2 - \overline{\lambda_2}}{2\lambda_2}\cdot\frac{\lambda_2^2 - \overline{\lambda_1^2}}{\lambda_2^2 - \lambda_1^2}\cdot\frac{\lambda_1^2}{\overline{\lambda_1^2}}\cdot\frac{\lambda_2^2}{\overline{\lambda_2^2}} \end{aligned} \tag{77}$$

$$f_j^2 = 2c_{j0}e^{i2\lambda_j^2 x}e^{i4\lambda_j^4 t}, j = 1,2\ (C.f.(1.62)), b_{j0}e^{i2\lambda_j^2 x + i4\lambda_j^4 t} \equiv e^{-\theta_j}e^{i\varphi_j}, j = 1,2 \tag{78}$$

where $\theta_j = 4\mu_j\nu_j\left[x - x_{j0} + 4\left(\mu_j^2 - \nu_j^2\right)t\right]$

$$\varphi_j = 2\left(\mu_j^2 - \nu_j^2\right)x + \left[4\left(\mu_j^2 - \nu_j^2\right)^2 - 16\mu_j^2\nu_j^2\right]\cdot t + \alpha_{j0} \tag{79}$$

and b_{j0} is absorbed into the soliton center and the initial phase by

$$b_{j0} = e^{4\mu_j\nu_j x_{j0}}e^{i\alpha_{j0}}; j = 1,2 \tag{80}$$

And we get

$$\begin{aligned} A_2 &= \sum_{\substack{n_1 = 1,2 \\ m_1 = 0}} \Omega_1(n_1, 0)\Omega_2(0, n_1) + \sum_{\substack{n_1 = 1, n_2 = 2 \\ m_1 = 0, m_2 = 1,2}} \Omega_1(n_1, n_2; 0, m_2)\Omega_2(0, m_2; n_1, n_2) \\ &= \Omega_1(1;0)\Omega_2(0;1) + \Omega_1(2;0)\Omega_2(0;2) + \Omega_1(1,2;0,1)\Omega_2(0,1;1,2) + \Omega_1(1,2;0,2)\Omega_2(0,2;1,2) \\ &= f_1^2 + f_2^2 - |f_1|^4 f_2^2\frac{\left(\lambda_1^2 - \lambda_2^2\right)^2\overline{\lambda_1^2}}{\left(\lambda_1^2 - \overline{\lambda_1^2}\right)^2\left(\overline{\lambda_1^2} - \lambda_2^2\right)^2} - |f_2|^4 f_1^2\frac{\left(\lambda_1^2 - \lambda_2^2\right)^2\overline{\lambda_2^2}}{\left(\overline{\lambda_2^2} - \lambda_1^2\right)^2\left(\lambda_2^2 - \overline{\lambda_2^2}\right)^2} \\ &= \lambda_1\left(1 - e^{-i4\beta_1}\right)\frac{\lambda_1^2 - \overline{\lambda_2^2}}{\lambda_1^2 - \lambda_2^2}e^{i4(\beta_1+\beta_2)}e^{-\theta_1}e^{i\varphi_1} + \lambda_2\left(1 - e^{-i4\beta_2}\right)\frac{\overline{\lambda_1^2} - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}e^{i4(\beta_1+\beta_2)}e^{-\theta_2}e^{i\varphi_2} \\ &+ \left[\lambda_1\left(1 - e^{-i4\beta_1}\right)e^{-i2\beta_2}\frac{\overline{\lambda_1^2} - \lambda_2^2}{\overline{\lambda_1^2} - \overline{\lambda_2^2}}e^{-\theta_2 - i\varphi_2} + \lambda_2\left(1 - e^{-i4\beta_2}\right)e^{-i2\beta_1}\frac{\lambda_1^2 - \overline{\lambda_2^2}}{\overline{\lambda_1^2} - \overline{\lambda_2^2}}e^{-\theta_1 - i\varphi_1}\right]\bullet e^{-(\theta_1+\theta_2)}e^{i(\varphi_1+\varphi_2)}e^{i4(\beta_1+\beta_2)} \\ &= i2\left|\frac{\lambda_1^2 - \overline{\lambda_2^2}}{\lambda_1^2 - \lambda_2^2}\right|\left[\rho_1\sin 2\beta_1 e^{i(\varphi-\alpha)}e^{i(3\beta_1+4\beta_2)}e^{-\theta_1+i\varphi_1} + \rho_2\sin 2\beta_2 e^{-i(\varphi+\alpha)}e^{i(4\beta_1+3\beta_2)}e^{-\theta_2+i\phi_2}\right. \\ &\left.+\rho_1\sin 2\beta_1 e^{-i(\varphi-\alpha)}e^{i(3\beta_1+2\beta_2)}e^{-2\theta_2-\theta_1}\cdot e^{i\phi_1} + \rho_2\sin 2\beta_2 e^{i(\varphi+\alpha)}e^{i(2\beta_1+3\beta_2)}e^{-2\theta_1-\theta_2}\cdot e^{i\phi_2}\right] \end{aligned} \tag{81}$$

where

$$\begin{aligned} \varphi &= \arg\left(\lambda_1^2 - \overline{\lambda_2^2}\right) = \arctan\left(\rho_1^2\sin 2\beta_1 + \rho_2^2\sin 2\beta_2\right)/\left(\rho_1^2\cos 2\beta_1 - \rho_2^2\cos 2\beta_2\right) \\ \alpha &= \arg\left(\lambda_1^2 - \lambda_2^2\right) = \arctan\left(\rho_1^2\sin 2\beta_1 - \rho_2^2\sin 2\beta_2\right)/\left(\rho_1^2\cos 2\beta_1 - \rho_2^2\cos 2\beta_2\right) \end{aligned} \tag{82}$$

and

$$\begin{aligned}
\overline{D}_2 &= 1+\sum_{r=1}^{2}\sum_{1\le n_1<n_2\le 2}\sum_{1\le m_1<m_2\le 2} Q_1(n_1,\cdots,n_r;m_1,\cdots,m_r)Q_2(m_1,\cdots,m_r;n_1,\cdots,n_r)\\
&= 1+Q_1(n_1=1;m_1=1)Q_2(m_1=1,n_1=1)+Q_1(n_1=1;m_1=2)Q_2(m_1=2,n_1=1)\\
&+Q_1(n_1=2;m_1=1)Q_2(m_1=1,n_1=2)+Q_1(n_1=2;m_1=2)Q_2(m_1=2,n_1=2)\\
&+Q_1(n_1=1,n_2=2;m_1=1,m_2=2)Q_2(m_1=1,m_2=2;n_1=1,n_2=2)\\
&= 1-|f_1|^4\frac{\lambda_1^2}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2}-|f_2|^4\frac{\lambda_2^2}{\left(\lambda_2^2-\overline{\lambda_2^2}\right)^2}-f_1^2\overline{f_2^2}\frac{\lambda_1^2}{\left(\lambda_1^2-\overline{\lambda_2^2}\right)^2}-\overline{f_1^2}f_2^2\frac{\lambda_2^2}{\left(\overline{\lambda_1^2}-\lambda_2^2\right)^2}\\
&+|f_1f_2|^4\cdot\frac{\lambda_1^2\lambda_2^2\left(\overline{\lambda_1^2}-\overline{\lambda_2^2}\right)^2\left(\lambda_1^2-\lambda_2^2\right)^2}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2\left(\lambda_1^2-\overline{\lambda_2^2}\right)^2\left(\lambda_2^2-\overline{\lambda_1^2}\right)^2\left(\lambda_2^2-\overline{\lambda_2^2}\right)^2}
\end{aligned}\tag{83}$$

$$\begin{aligned}
D_2 = 1 &+ \left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right|^2\left(e^{-i2\beta_1}e^{-2\theta_1}+e^{-i2\beta_2}e^{-2\theta_2}\right)\\
&+\left(1-\left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right|^2\right)e^{-(\theta_1+\theta_2)}e^{-i(\beta_1+\beta_2)}\left[\frac{\rho_1}{\rho_2}e^{i(\varphi_2-\varphi_1)}+\frac{\rho_2}{\rho_1}e^{i(\varphi_1-\varphi_2)}\right]\\
&+e^{-i2(\beta_1+\beta_2)}e^{-2(\theta_1+\theta_2)}
\end{aligned}\tag{84}$$

where

$$\left|\frac{\lambda_1^2-\overline{\lambda_2^2}}{\lambda_1^2-\lambda_2^2}\right|^2=\frac{(\rho_1/\rho_2-\rho_2/\rho_1)^2+4\sin^2(\beta_1+\beta_2)}{(\rho_1/\rho_2-\rho_2/\rho_1)^2+4\sin^2(\beta_1-\beta_2)}\tag{85}$$

Substituting (81) and (84) into formula (52), we thus get the two-soliton solution to the DNLS equation with VBC

$$\overline{u}_2=-i2A_2D_2/\overline{D}_2^2\tag{86}$$

Once again we find that, up to a permitted global constant phase factor, the above two-soliton solution is equivalent to that gotten in Ref. [23, 24], verifying the validity of our formula of N-soliton solution and the reliability of those linear algebra techniques. As a matter of fact, a general and strict demonstration of our revised IST for DNLS equation with VBC has been given in one paper by use of Liouville theorem [25].

2.5 The asymptotic behaviors of *N*-soliton solution

The complex conjugate of expression (52) gives the explicit expression of N-soliton solution as

$$u_N=i2\overline{A}_N\overline{D}_N/D_N^2\tag{87}$$

Without the loss of generality, for $\lambda_n=\mu_n+i\nu_n$, $V_\mathrm{n}=-4\left(\mu_n^2-\nu_n^2\right)$, $n=1,2,\cdots,N,$ we assume $V_1<V_2<\cdots<V_n<\cdots V_N$ and define the n'th vicinity area as $\Gamma_n: x-x_{no}-V_nt\sim 0, (n=1.2,\cdots,N)$.

As $t\to-\infty$, N vicinity areas $\Gamma_n, n=1,2\ldots,N$, queue up in a descending series

$$\Gamma_N,\Gamma_{N-1},\cdots,\Gamma_1\tag{88}$$

and in the vicinity of Γ_n, we have (note that $\kappa_j>0$)

$$\theta_j = 4\kappa_j\left(x - x_{j0} - V_j t\right) \rightarrow \begin{cases} +\infty, & \text{for } j > n \\ -\infty, & \text{for } j < n \end{cases} \tag{89}$$

Here the complex constant $2c_{n0}$ in expression (65) has been absorbed into $e^{-\theta_n}e^{i\varphi_n}$ by redefinition of the soliton center x_{n0} and the initial phase α_{n0}.

Introducing a typical factor $F_n = -e^{-2\theta_n}/\left(\lambda_n^2 - \overline{\lambda}_n^2\right)^2 > 0,\ n = 1, 2, \cdots, N$; then

$$D_n(1, 2, \cdots, n) = \prod_{j=1}^{n} \overline{\lambda}_j^2 F_j \prod_{l<m} \left| \frac{\lambda_l^2 - \lambda_m^2}{\lambda_l^2 - \overline{\lambda}_m^2} \right|^4 \tag{90}$$

where $l, m \in \{1, 2, \cdots, n\}$. Thus

$$D \simeq D_{n-1}(1, 2, \cdots, n-1) + D_n(1, 2, \cdots, n) = \left(1 + \overline{\lambda}_n^2 F_n \prod_{j=1}^{n-1} \left| \frac{\lambda_j^2 - \lambda_n^2}{\lambda_j^2 - \overline{\lambda}_n^2} \right|^4 \right) D_{n-1}(1, 2, \cdots, n-1) \tag{91}$$

and

$$\overline{A} \simeq \overline{A}_n(1, 2, \cdots, n; 0, 1, 2, \cdots n-1) = D_{n-1} e^{-\theta_n} e^{-i\varphi_n} \prod_{j=1}^{n-1} \frac{\left(\overline{\lambda}_j^2 - \overline{\lambda}_n^2\right)^2}{\left(\lambda_j^2 - \overline{\lambda}_n^2\right)^2} e^{i4\beta_j} \tag{92}$$

In the vicinity of Γ_n,

$$u(x, t) = i2\overline{A}\overline{D}/D^2 \simeq u_1\left(\theta_n + \Delta\theta_n^{(-)}, \varphi_n + \Delta\varphi_n^{(-)}\right) \tag{93}$$

Here

$$\Delta\theta_n^{(-)} = 2\sum_{j=1}^{n-1} \ln \left| \frac{\lambda_j^2 - \overline{\lambda}_n^2}{\lambda_j^2 - \lambda_n^2} \right|^2 \tag{94}$$

$$\begin{aligned} \Delta\varphi_n^{(-)} &= -\sum_{j=1}^{n-1} \left\{ \arg \left[\frac{\left(\overline{\lambda}_j^2 - \overline{\lambda}_n^2\right)^2}{\left(\lambda_j^2 - \overline{\lambda}_n^2\right)^2} \right] + 4\beta_j \right\} \\ &= 2\sum_{j=1}^{n-1} \left[\arg\left(\lambda_j^2 - \overline{\lambda}_n^2\right) - \arg\left(\overline{\lambda}_j^2 - \overline{\lambda}_n^2\right) - 2\beta_j \right] \end{aligned} \tag{95}$$

then

$$u_N \simeq \sum_{n=1}^{N} u_1\left(\theta_n + \Delta\theta_n^{(-)}, \varphi_n + \Delta\varphi_n^{(-)}\right) \tag{96}$$

Each $u_1(\theta_n, \varphi_n)$, $(1, 2, \cdots, n)$ is a one-soliton solution characterized by one parameter λ_n, moving along the positive direction of the x-axis, queuing up in a series with descending order number n as in series (88). As $t \rightarrow \infty$, in the vicinity of Γ_n, we have (note that $\kappa_j > 0$)

$$\theta_j = 4\kappa_j(x - x_{j0} - V_j t) \to \begin{cases} -\infty, & \text{for } j > n \\ +\infty, & \text{for } j < n \end{cases} \quad D \simeq D_{N-n}(n+1, n+2, \cdots, N) + D_{N-n+1}(n, n+1, \cdots, N)$$

$$= \left(1 + \overline{\lambda}_n^2 F_n \prod_{j=n+1}^{N} \left|\frac{\lambda_j^2 - \lambda_n^2}{\lambda_j^2 - \overline{\lambda}_n^2}\right|^4\right) D_{N-n}(n+1, n+2, \cdots, N) \tag{97}$$

$$\overline{A} \simeq \overline{A}_{N-n+1}(n, n+1, \cdots, N; 0, n+1, n+2, \cdots, N)$$
$$= D_{N-n}(n+1, n+2, \cdots, N) e^{-\theta_n} e^{-i\varphi_n} \prod_{j=n+1}^{N} \frac{\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right)^2}{\left(\lambda_j^2 - \overline{\lambda_n^2}\right)^2} \frac{\lambda_j^2}{\overline{\lambda}_j^2}, \tag{98}$$

So as $t \to \infty$, in the vicinity of Γ_n,

$$u = i2\overline{AD}/D^2 \simeq u_1\left(\theta_n + \Delta\theta_n^{(+)}, \varphi_n + \Delta\varphi_n^{(+)}\right) \tag{99}$$

$$\Delta\theta_n^{(+)} = 2 \sum_{j=n+1}^{N} \ln\left|\frac{\lambda_j^2 - \overline{\lambda_n^2}}{\lambda_j^2 - \lambda_n^2}\right|^2 \tag{100}$$

$$\Delta\varphi_n^{(+)} = -\sum_{j=n+1}^{N} \left\{ \arg\left[\frac{\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right)^2}{\left(\lambda_j^2 - \overline{\lambda_n^2}\right)^2}\right] + 4\beta_j \right\}$$
$$= 2 \sum_{j=n+1}^{N} \left[\arg\left(\lambda_j^2 - \overline{\lambda_n^2}\right) - \arg\left(\overline{\lambda_j^2} - \overline{\lambda_n^2}\right) - 2\beta_j\right], \tag{101}$$

then as $t \to \infty$,

$$u_N \simeq \sum_{n=1}^{N} u_1\left(\theta_n + \Delta\theta_n^{(+)}, \varphi_n + \Delta\varphi_n^{(+)}\right) \tag{102}$$

That is to say, the N-soliton solution can be viewed as N well-separated exact one- solitons, queuing up in a series with ascending order number n: $\Gamma_1, \Gamma_2, \cdots, \Gamma_N$. In the course going from $t \to -\infty$ to $t \to \infty$, the n'th one-soliton overtakes the solitons from the first to $n-1$'th and is overtaken by the solitons from $n+1$'th to N'th. In the meantime, due to collisions, the n'th soliton got a total forward shift $\Delta\theta_n^{(-)}/\kappa_n$ from exceeding those slower soliton from the first to $n-1$'th, and got a total backward shift $\Delta\theta_n^{(+)}/\kappa_n$ from being exceeded by those faster solitons from $n+1$'th to N'th, and just equals to the summation of shifts due to each collision between two solitons, together with a total phase shift $\Delta\varphi_n$, that is,

$$\Delta x_n = \left|\Delta\theta_n^{(+)} - \Delta\theta_n^{(-)}\right|/\kappa_n \tag{103}$$

$$\Delta\varphi_n = \Delta\varphi_n^{(+)} - \Delta\varphi_n^{(-)} \tag{104}$$

2.6 N-soliton solution to MNLS equation

Finally, we indicate that the exact N-soliton solution to the DNLS equation can be converted to that of MNLS equation by a gauge-like transformation. A nonlinear Schrödinger equation including the nonlinear dispersion term expressed as

$$i\partial_t \upsilon + \partial_{xx}\upsilon + i\alpha\partial_x\left(|\upsilon|^2\upsilon\right) + 2\beta|\upsilon|^2\upsilon = 0 \tag{105}$$

is also integrable [23] and called modified nonlinear Schrödinger (MNLS for brevity) equation. It is well known that MNLS equation well describes transmission of femtosecond pulses in optical fibers [4–6] and is related to DNLS equation by a gauge-like transformation [23] formulated as

$$\upsilon(x,t) = u(X,T)e^{i2\rho X + i4\rho^2 T} \tag{106}$$

with $x = \alpha^{-1}(X + 4\rho T)$, $t = \alpha^{-2}T$; $X = \alpha x - 4\beta t$, $T = \alpha^2 t$; $\rho = \beta\alpha^{-2}$. Using a method that is analogous to reference [16], and applying above gauge-like transformation to Eq. (105), the MNLS equation with VBC can be transformed into DNLS equation with VBC.

$$i\partial_T u + \partial_{XX} u + i\partial_X\left(|u|^2 u\right) = 0 \tag{107}$$

with $u = u(X,T)$. So according to (106), the N-soliton solution to MNLS equation can also be attained by a gauge-like transformation from that of DNLS equation.

The N-soliton solution to the DNLS equation with VBC has been derived by means of a IST considered anew and some special linear algebra techniques. The one- and two-soliton solutions have been given as two typical examples in illustration of the general formula of the N-soliton solution. It is found to be perfectly in agreement with that gotten in the following section based on a pure Marchenko formalism or Hirota's Bilinear derivative transformation [24, 26, 27]. The demonstration of the revised IST considered anew for DNLS equation with VBC has also been given by use of Liouville theorem [25].

The newly revised IST technique for DNLS equation with VBC supplies substantial foundation for its direct perturbation theory.

3. A simple method to derive and solve Marchenko equation for DNLS equation

Gel'fand-Levitan-Marchenko (GLM for brevity) equations can be viewed as an integral-transformed version of IST for those integrable nonlinear equations [21, 24, 28].

In this section, a simple method is used to derive and solve Marchenko equation (or GLM equation) for DNLS E with VBC [28]. Firstly, starting from the first Lax equation, we derive two conditions to be satisfied by the kernel matrix $N(x,y)$ of GLM by applying the Lax operator $\partial_x - L$ upon the integral representation of Jost function for DNLSE. Secondly, based on Lax equation, a strict demonstration has been given for the validness of Marchenko formalism. At last, the Marchenko formalism is determined by choosing a suitable $F(x+y)$ and $G(x+y)$, and their relation (135) has been constructed. The one and multi-soliton solution in the reflectionless case is attained based upon a pure Marchenko formalism by avoiding direct use of inverse scattering data and verified by using direct substitution method with Mathematica.

3.1 The lax pair and its Jost functions of DNLS equation

DNLS equation is usual expressed as

$$iu_t + u_{xx} + i\left(|u|^2 u\right)_x = 0 \tag{108}$$

with vanishing boundary, $|x| \to \infty$, $u \to 0$. Here the subscript denotes partial derivative. Its Lax pair is given by

$$L = -i\lambda^2\sigma_3 + \lambda U, U = \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix} \tag{109}$$

$$M = -i2\lambda^4\sigma_3 + 2\lambda^3 U - i\lambda^2 U^2\sigma_3 - \lambda\left(-U^3 + iU_x\sigma_3\right) \tag{110}$$

The first Lax equation is

$$\partial_x f(x,\lambda) = L(x,\lambda)f(x,\lambda) \tag{111}$$

In the case of $|x| \to \infty$, $u \to 0$, $L \to L_0 = -i\lambda^2\sigma_3$, the free Jost solution is

$$E(x,\lambda) = e^{-i\lambda^2 x\sigma_3}, E_{\bullet 1}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda^2 x}; E_{\bullet 2}(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda^2 x} \tag{112}$$

where λ^2 is a real squared parameter, $E(x,\lambda)$ expresses two independent solutions with two components. The Jost solutions of (4) are defined by their asymptotic properties at $x \to \pm\infty$,

$$\Psi(x,\lambda) = (\tilde{\psi}(x,\lambda), \psi(x,\lambda)) \to E(x,\lambda), \quad as \;\; x \to \infty \tag{113}$$

$$\Phi(x,\lambda) = (\varphi(x,\lambda), \tilde{\varphi}(x,\lambda)) \to E(x,\lambda), \quad as \;\; x \to -\infty \tag{114}$$

3.2 The integral representation of Jost function

As usual, we introduce the integral representation,

$$\Psi(x,\lambda) = E(x,\lambda) + \int_x^\infty \mathrm{d}y \left\{\lambda^2 N^d(x,y) + \lambda N^o(x,y)\right\} E(y,\lambda) \tag{115}$$

where the superscripts d and o mean the diagonal and off-diagonal elements, respectively. According to the conventional operation in IST, the time variable is suppressed temporarily. Here

$$N^d(x,y) = \begin{pmatrix} N(x,y)_{11} & 0 \\ 0 & N(x,y)_{22} \end{pmatrix}, N^o(x,y) = \begin{pmatrix} 0 & N(x,y)_{12} \\ N(x,y)_{21} & 0 \end{pmatrix}$$

Due to the symmetry of the first Lax operator $\lambda^2(-i\sigma_3)_{11} = \lambda^2\overline{(-i\sigma_3)}_{22}$ and $\lambda U_{21} = -\lambda\overline{U}_{12}$, the kernel matrix $N(x,y)$ of the integral representation of Jost function should have the same symmetry as follows:

$$\lambda^2 N^d(x,y)_{11} = \lambda^2\overline{N^d(x,y)_{22}}; \lambda N^o(x,y)_{21} = -\lambda\overline{N^o(x,y)_{12}} \tag{116}$$

Substitute Eq. (115) into the first Lax Eq. (111). By simply partial integration, we have the following terms:

$$\{\partial_x - L\}E(x,\lambda) = -(L - L_0)E(x,\lambda) = -\lambda U(x)E(x,\lambda) \tag{117}$$

$$\begin{aligned}&\partial_x \int_x^\infty \mathrm{d}y\{\lambda^2 N^d(x,y) + \lambda N^o(x,y)\}E(y,\lambda)\\ &= -[\lambda^2 N^d(x,x) + \lambda N^o(x,x)]E(x,\lambda) + \int_x^\infty \mathrm{d}y[\lambda^2 N_x^d(x,y) + \lambda N_x^o(x,y)]E(y,\lambda)\end{aligned} \tag{118}$$

$$\begin{aligned}&-(-i\lambda^2\sigma_3)\int_x \mathrm{d}y[\lambda^2 N^d(x,y) + \lambda N^\circ(x,y)]E(y,\lambda) = -\int_x \mathrm{d}y\sigma_3[\lambda^2 N^d(x,y) + \lambda N^o(x,y)]\sigma_3 E'(y,\lambda)\\ &= \sigma_3[\lambda^2 N^d(x,x) + \lambda N^o(x,x)]\sigma_3 E(x,\lambda) + \int_x^\infty \mathrm{d}y\sigma_3\left[\lambda^2 N_y^d(x,y) + \lambda N_y^o(x,y)\right]\sigma_3 E(y,\lambda)\end{aligned} \tag{119}$$

and

$$\begin{aligned}&-\lambda U(x)\int_x^\infty \mathrm{d}y\{\lambda^2 N^d(x,y)\}E(y,\lambda) = -\int_x^\infty \mathrm{d}y\lambda U(x)N^d(x,y)i\sigma_3 E'(y,\lambda)\\ &= i\lambda U(x)N^d(x,x)\sigma_3 E(x,\lambda) + i\int_x^\infty \mathrm{d}y\lambda U(x)N_y^d(x,y)\sigma_3 E(y,\lambda)\end{aligned} \tag{120}$$

Use is made of that $-i\lambda^2\sigma_3 E(y,\lambda) = E'(y,\lambda)$, then

$$-\lambda U(x)\int_x^\infty \mathrm{d}yN^o(x,y) \equiv -\int_x^\infty \mathrm{d}y\lambda U(x)N^o(x,y) \tag{121}$$

According to equation $(\partial_x - L)\Psi(x,\lambda) = 0$, adding up the l.h.s. and r.h.s., respectively, of Eq. (117)–(120), (121). We obtain two equations involving with terms λ^2 and λ outside of the integral $\int \mathrm{d}y\cdots$ as follows:

$$\lambda^2 : -N^d(x,x) + \sigma_3 N^d(x,x)\sigma_3 = 0 \tag{122}$$

$$\lambda^1 : -U(x) - N^o(x,x) + \sigma_3 N^o(x,x)\sigma_3 + iU(x)N^d(x,x)\sigma_3 = 0 \tag{123}$$

Or

$$U_{12} = u(x) = -2\frac{N_{12}(x,x)}{1 + i\overline{N}_{11}(x,x)}, \tag{124}$$

and the equations in the integral $\int \mathrm{d}y\{S\} = 0$, where $\{S\}$ is equal to

$$[\lambda^2 N_x^d(x,y) + \lambda N_x^o(x,y)] + \sigma_3\left[\lambda^2 N_y^d(x,y) + \lambda N_y^o(x,y)\right]\sigma_3 - \lambda U(x)\left[-iN_y^d(x,y)\sigma_3 + \lambda N^o(x,y)\right] = 0 \tag{125}$$

Therefore, Eq. (125) gives two conditions to be satisfied by the kernel matrix $N(x,y)$ in the integral representation of Jost solution

$$\lambda^2 \text{ terms} : A(x,y) \equiv N_x^d(x,y) + \sigma_3 N_y^d(x,y)\sigma_3 - U(x)N^o(x,y) = 0 \tag{126}$$

$$\lambda^1 \text{ terms} : B(x,y) \equiv N_x^o(x,y) + \sigma_3 N_y^o(x,y)\sigma_3 + iU(x)N_y^d(x,y)\sigma_3 = 0 \tag{127}$$

Since (122) is an identity, Eq. (123) or (124) gives the solution $U(x)$ or $u(x)$ in terms of $N(x,x)$, thus the first Lax equation gives two conditions (126) and (127) which should be satisfied by the integral kernel $N(x,y)$. Note that the time variable of $u(x)$ in (124) is suppressed temporarily.

3.3 Marchenko equation for DNLSE and its demonstration

In Eq. (115), the $N^d(x,y)$ and $N^o(x,y)$ appear in different manner, we assume the form of Marchenko equation for DNLSE with VBC is

$$N^d(x,y) + \int_x^\infty \mathrm{d}z N^o(x,z)F(z+y) = 0 \tag{128}$$

$$N^o(x,y) + F(x+y) + \int_x^\infty \mathrm{d}z N^d(x,z)G(z+y) = 0 \tag{129}$$

where $F(x+y)$ is only with off-diagonal terms. $G(x+y)$ is considered as another function with only off-diagonal terms. We notice that the Marchenko equation needn't involve obviously the function of spectral parameter λ.

We now show the kernel $N(x,y)$ determined by (128) and (129) indeed satisfy the conditions (126) and (127) as long as we choose a suitable form of expression for $G(x+y)$.

Making partial derivation in (128) with respect to x and y, respectively, we obtain

$$N_x^d(x,y) - N^o(x,x)F(x+y) + \int_x^\infty \mathrm{d}z N_x^o(x,z)F(z+y) = 0 \tag{130}$$

$$N_y^d(x,y) + \int_x^\infty \mathrm{d}z N^o(x,z)F'(z+y) = 0 \tag{131}$$

By partial integrating, Eq. (131) becomes

$$N_y^d(x,y) - N^o(x,x)F(x+y) - \int_x^\infty \mathrm{d}z N_z^o(x,z)F(z+y) = 0 \tag{132}$$

Use is made of the fact that $F'(z+y) = F_y(z+y) = F_z(z+y)$ in (131). Making a weighing summation as follows:

$$\{l.h.s. \quad of \quad (20)\} + \sigma_3\{l.h.s. \quad of \quad (22)\}\sigma_3 - U(x)\cdot\{l.h.s. \quad of \quad (19)\} = 0$$

We find

$$\begin{aligned} &N_x^d(x,y) + \sigma_3 N_y^d(x,y)\sigma_3 - U(x)N^o(x,y) - N^o(x,x)F(x+y) - \sigma_3 N^o(x,x)F(x+y)\sigma_3 - U(x)F(x+y) \\ &+ \textstyle\int_x^\infty \mathrm{d}z\{N_x^o(x,z)F(z+y) - \sigma_3 N_z^o(x,z)F(z+y)\sigma_3 - U(x)N^d(x,z)G(z+y)\} = 0 \end{aligned} \tag{133}$$

Since $F(x)$ is off-diagonal, $F(x)\sigma_3 = -\sigma_3 F(x)$. Thus the terms involving with $F(x+y)$ outside of integral are equal to $-iU(x)N^d(x,x)\sigma_3 F(x+y)$ by use of Eq. (123). Then (133) can be rewritten as

$$\begin{aligned} &A(x,y) - iU(x)N^d(x,x)\sigma_3 F(x+y) \\ &+ \textstyle\int_x^\infty \mathrm{d}z\left[B(x,z)F(z+y) - iU(x)N_z^d(x,z)\sigma_3 F(z+y) - U(x)N^d(x,z)G(z+y)\right] = 0 \end{aligned} \tag{134}$$

If we choose

$$G(z+y) = i\sigma_3 F'(z+y), \tag{135}$$

then

$$iU(x)N^d(x,x)\sigma_3F(x+y) + \int_z^\infty dz\{iU(x)N_z^d(x,z)\sigma_3F(z+y) + U(x)N^d(x,z)i\sigma_3F'(z+y)\} = 0 \tag{136}$$

Thus, Eq. (134) becomes

$$A(x,y) + \int_x^\infty dzB(x,z)F(z+y) = 0 \tag{137}$$

Now substituting (135) into (129), we find

$$N^o(x,y) + F(x+y) + \int_x^\infty dzN^d(x,z)i\sigma_3F'(z+y) = 0 \tag{138}$$

Making partial derivation with respect to x and y, respectively, on the l.h.s. of Eq. (138), we have

$$N_x^o(x,y) + F'(x+y) - N^d(x,x)i\sigma_3F'(x+y) + \int_x^\infty dzN_x^d(x,z)i\sigma_3F'(z+y) = 0 \tag{139}$$

$$N_y^o(x,y) + F'(x+y) + \int_x^\infty dzN^d(x,z)i\sigma_3F''(z+y) = 0 \tag{140}$$

or

$$N_y^o(x,y) + F'(x+y) - N^d(x,x)i\sigma_3F'(x+y) - \int_x^\infty dzN_z^d(x,x)i\sigma_3F'(z+y) = 0 \tag{141}$$

Now we make a weighing summation as

$$\{l.h.s.\ \ of\ \ (29)\} + \sigma_3\{l.h.s.\ \ of\ \ (31)\}\sigma_3 + iU(x)\{l.h.s.\ \ of\ \ (21)\}\sigma_3 = 0$$

Hence, we have

$$\begin{aligned}
&N_x^o(x,y) + \sigma_3N_y^o(x,y)\sigma_3 + iU(x)N_y^d(x,y)\sigma_3 \\
&+F'(x+y) + \sigma_3F'(x+y)\sigma_3 - N^d(x,x)i\sigma_3F'(x+y) - \sigma_3N^d(x,x)i\sigma_3F'(x+y)\sigma_3 \\
&+\int_x^\infty dz\{N_x^d(x,z)i\sigma_3F'(z+y) - \sigma_3N_z^d(x,z)i\sigma_3F'(z+y)\sigma_3 + iU(x)N^o(x,z)F'(z+y)\sigma_3\} = 0
\end{aligned} \tag{142}$$

Noticing $F(x)\sigma_3 = -\sigma_3F(x)$, Eq. (142) becomes

$$B(x,y) + \int_x^\infty dzA(x,z)i\sigma_3F'(z+y) = 0 \tag{143}$$

We find that, as long as we choose a suitable form for $G(x+y)$ as well as $F(x+y)$ according to Eq. (135), Eq. (128) and (129) will just satisfy the two conditions (126) and (127) derived from the first Lax Eq. (111). On the other hand, owing to the symmetry properties of $N^o(x,y)$ and $N^d(x,y)$, the function $f(x+y)$ in (128) and (129) can only has off-diagonal elements, we write

$$F(x+y) = \begin{pmatrix} 0 & \overline{-f(x+y)} \\ f(x+y) & 0 \end{pmatrix}; G(z+y) = \begin{pmatrix} 0 & -\overline{h(z+y)} \\ h(z+y) & 0 \end{pmatrix} = i\sigma_3 F'(z+y) \tag{144}$$

Considering the dependence of the Jost solutions on the squared spectral parameter λ^2, in the reflectionless case, we choose

$$f(x+y) = \sum_{n=1}^{N} C_n(t) e^{i\lambda_n^2(x+y)} \tag{145}$$

where $C_n(t)$ contains a time-dependent factor $e^{i4\lambda_n^4 t}$, which can be introduced by a standard procedure [29], due to a fact of the Lax operator $M \to -i2\lambda^4\sigma_3$ as $x \to \pm\infty$.

As is well known, Lax equations are linear equation so that a constant factor can be introduced in its solution, that is, $C_n = e^{\beta_n + i\alpha_n} e^{i4\lambda_n^4 t}$. It means that β_n is related to the center of soliton and α_n expresses the initial phase up to a constant factor. Thus, the time-independent part of C_n is inessential and can be absorbed or normalized only by redefinition of the soliton center and initial phase. On the other hand, notice the terms generated by partial integral in (133)–(142), in order to ensure the convergence of the partial integral, we must let $\lim_{x\to\infty} e^{i\lambda_n^2 x} = 0$, so we only consider the N zero points of $a(\lambda)$ in the first quadrant of complex plane of λ (also in the upper half part of the complex plane of λ^2), that is, the discrete spectrum for $\lambda_1,\ \lambda_2$, $\cdots\cdots\lambda_N$, although $-\lambda_n,\ \ (n = 12, \cdots, N)$ in the third quadrant of the complex plane of λ are also the zero points of $a(\lambda)$ due to symmetry of Lax operator and transition matrix. Then Eq. (145) corresponds to the N-soliton solution in the reflectionless case, and we have completed the derivation and manifestation of Marchenko equation (128) and (129), (144), and (145) for DNLSE with VBC.

3.4 A multi-soliton solution of the DNLS equation based upon pure Marchenko formalism

When there are N simple poles $\lambda_1, \lambda_2, \cdots, \lambda_N$ in the first quadrant of the complex plane of λ, the Marchenko equation will give a N-solition solution to the DNLS equation with VBC in the reflectionless case. We can assume that

$$f(x+y) = F_{21}(x+y) = \sum_{n=1}^{N} g_n(x,t) h_n(y) \equiv G(x,t) H(y)^{\mathrm{T}} \tag{146}$$

where $g_n(x,t) \equiv C_n(t) e^{i\lambda_n^2 x},\ \ h_n(y) \equiv e^{i\lambda_n^2 y},\ \ \ n = 1, 2, \cdots, N$, and

$$G(x,t) \equiv \left(g_1(x,t), g_2(x,t), \cdots, g_N(x,t)\right)_;\ H(y)^{\mathrm{T}} \equiv \left(h_1(y), h_2(y), \cdots, h_N(y)\right)^{\mathrm{T}} \tag{147}$$

Here and hereafter the superscript T represents transposing of a matrix. On the other hand, we assume that

$$N_{11}(x,y) = N_{11}(x) H(y)^{\mathrm{T}}, N_{12}(x,y) = N_{12}(x) \overline{H}(y)^{\mathrm{T}} \tag{148}$$

Then

$$F_{12}(x+y) = -\overline{F}_{12}(x+y) = -\overline{G}(x)\overline{H}(y)^{\mathrm{T}};\ F'_{12}(x+y) = i\overline{\lambda_n^2}\overline{C}_n e^{-i\overline{\lambda_n^2}(x+y)}$$
$$= -\overline{G'(x)}\overline{H}(y)^{\mathrm{T}} \tag{149}$$

Substituting (146)–(149) into the Marchenko equation (128) and (129), we have

$$\begin{cases} N_{11}(x)\mathrm{H}(y)^{\mathrm{T}} + N_{12}(x)\int_x^\infty \mathrm{d}z\overline{\mathrm{H}}(z)^{\mathrm{T}}\mathrm{G}(z)\mathrm{H}(y)^{\mathrm{T}} = 0 \\ N_{12}(x)\overline{\mathrm{H}}(y)^{\mathrm{T}} - \overline{\mathrm{G}}(x)\overline{\mathrm{H}}(y)^{\mathrm{T}} - iN_{11}(x)\int_x^\infty \mathrm{d}z\mathrm{H}(z)^{\mathrm{T}}\overline{\mathrm{G}}'(z)\overline{\mathrm{H}}(y)^{\mathrm{T}} = 0 \end{cases} \tag{150}$$

or

$$N_{11}(x) + N_{12}(x)\Delta_1(x) = 0, \tag{151}$$

$$N_{12}(x) - iN_{11}(x)\Delta_2(x) = \overline{G}(x) \tag{152}$$

here

$$\Delta_1(x) = \int_x^\infty \overline{\mathrm{H}}(z)^T\mathrm{G}(z)\mathrm{d}z,\ \Delta_2(x) = \int_x^\infty \mathrm{H}(z)^{\mathrm{T}}\overline{\mathrm{G}}'(z)\mathrm{d}z \tag{153}$$

Both of them are $N \times N$ matrices and their matrix element are, respectively, expressed as

$$\Delta_1(x)_{mn} = \int_x^\infty e^{-i\overline{\lambda_m^2}z}C_n e^{i\lambda_n^2 z}\mathrm{d}z = \overline{h}_m(x)\frac{-i}{\overline{\lambda_m^2} - \lambda_n^2}g_n(x) \tag{154}$$

$$\Delta_2(x)_{mn} = \int_x^\infty e^{-i\lambda_m^2 z}\overline{C}_n\left(-i\overline{\lambda_n^2}\right)e^{-i\overline{\lambda_n^2}z}\mathrm{d}z = h_m(x)\frac{\overline{\lambda_n^2}}{\lambda_m^2 - \overline{\lambda_n^2}}\overline{g}_n(x) \tag{155}$$

From (151) and (152), we immediately get

$$N_{11}(x) = -\overline{\mathrm{G}}(x)[1 + i\Delta_1(x)\Delta_2(x)]^{-1}\Delta_1(x) \tag{156}$$

$$N_{12}(x) = \overline{\mathrm{G}}(x)[1 + i\Delta_1(x)\Delta_2(x)]^{-1} \tag{157}$$

from (148), (156), and (157), we have

$$N_{11}(x,y) = -\overline{G}(x)[\mathrm{I} + i\Delta_1(x)\Delta_2(x)]^{-1}\Delta_1(x)H(y)^{\mathrm{T}} \tag{158}$$

$$N_{12}(x,y) = \overline{\mathrm{G}}(x)[\mathrm{I} + i\Delta_1(x)\Delta_2(x)]^{-1}\overline{\mathrm{H}}(y)^{\mathrm{T}} \tag{159}$$

then

$$N_{11}(x,x) = i\mathrm{Tr}\left\{i\Delta_1(x)\mathrm{H}(x)^{\mathrm{T}}\overline{\mathrm{G}}(x)[\mathrm{I} + i\Delta_1(x)\Delta_2(x)]^{-1}\right\}$$
$$= i\left\{\frac{\det\left[\mathrm{I} + i\Delta_1(x)\Delta_2(x) + i\Delta_1(x)\mathrm{H}(x)^{\mathrm{T}}\overline{\mathrm{G}}(x)\right]}{\det[\mathrm{I} + i\Delta_1(x)\Delta_2(x)]} - 1\right\} \tag{160}$$

and

$$N_{12}(x,x) = \mathrm{Tr}\left\{\overline{\mathrm{H}}(x)^{\mathrm{T}}\overline{\mathrm{G}}(x)[\mathrm{I} + i\Delta_1(x)\Delta_2(x)]^{-1}\right\} = \frac{\det\left[\mathrm{I} + \mathrm{i}\Delta_1(\mathrm{x})\Delta_2(\mathrm{x}) + \overline{\mathrm{H}}^{\mathrm{T}}(\mathrm{x})\overline{\mathrm{G}}(\mathrm{x})\right]}{\det[\mathrm{I} + \mathrm{i}\Delta_1(\mathrm{x})\Delta_2(\mathrm{x})]} - 1 \tag{161}$$

Substituting (160) and (161) into Eq. (124), we thus attain the N-soliton solution as follows in a pure Marchenko formalism.

$$u_N(x,t) = -2\frac{\det\left(\mathrm{I} + i\Delta_{\mathrm{i}}\Delta_2 + \overline{\mathrm{H}}^{\mathrm{T}}\overline{\mathrm{G}}\right) - \det(\mathrm{I} + i\Delta_1\Delta_2)}{\det\left(\mathrm{I} - i\overline{\Delta}_1\overline{\Delta}_2 - i\overline{\Delta}_1\overline{\mathrm{H}}^{\mathrm{T}}\mathrm{G}\right)} \cdot \frac{\det(\mathrm{I} - i\overline{\Delta}_1\overline{\Delta}_2)}{\det(\mathrm{I} + i\Delta_1\Delta_2)} = -2C\overline{D}/D^2 \tag{162}$$

where

$$D \equiv \det(\mathrm{I} + i\Delta_1\Delta_2), C \equiv \det\left(\mathrm{I} + i\Delta_1\Delta_2 + \overline{\mathrm{H}}^{\mathrm{T}}\overline{\mathrm{G}}\right) - \det(\mathrm{I} + i\Delta_1\Delta_2), \tag{163}$$

and we will prove that in (136)

$$\det\left(\mathrm{I} - i\overline{\Delta}_1\overline{\Delta}_2 - i\overline{\Delta}_1\overline{\mathrm{H}}^{\mathrm{T}}\mathrm{G}\right) = \det(\mathrm{I} + i\Delta_1\Delta_2) \equiv D \tag{164}$$

By means of some linear algebraic techniques, especially the Binet-Cauchy formula for some special matrices (see the Appendices 2–3 in Part2), the determinant D and C can be expanded explicitly as a summation of all possible principal minors. Firstly, we can prove identity (164) by means of Binet-Cauchy formula.

$$\det\left(I - i\overline{\Delta}_1\overline{\Delta}_2 - i\overline{\Delta}_1\overline{H}^{\mathrm{T}}G\right) = \det\left[I + \left(-i\overline{\Delta}_1\right)\left(\overline{\Delta}_2 + \overline{H}^{\mathrm{T}}G\right)\right] := \det(I + \mathrm{M}_1\mathrm{M}_2) \tag{165}$$

where

$$(\mathrm{M}_1)_{nm} \equiv \left(-i\overline{\Delta}_1\right)_{mn} = h_n\frac{1}{\lambda_n^2 - \overline{\lambda_m^2}}\overline{g}_m, (\mathrm{M}_2)_{mn} \equiv \left(\overline{\Delta}_2 + \overline{H}^{\mathrm{T}}G\right)_{mn} = \overline{h}_m\frac{\overline{\lambda_m^2}}{\overline{\lambda_m^2} - \lambda_n^2}g_n \tag{166}$$

The complex constant factor c_{n0} can be absorbed into the soliton center and initial phase by redefining

$$g_n(x,t)h_n(x) = C_n(t)e^{i2\lambda_n^2x} = c_{no}e^{i4\lambda_n^4t}e^{i2\lambda_n^2x} \equiv e^{-\theta_n}e^{i\varphi_n} \tag{167}$$

here $\lambda_n = \mu_n + iv_n$, and

$$\begin{aligned}\theta_n &\equiv 4\mu_n v_n\left[x - x_{n0} + 4\left(\mu_n^2 - v_n^2\right)t\right] = 4\kappa_n(x - x_{n0} - \upsilon_n t); \kappa_n = 4\mu_n\nu_n; \upsilon_{\mathrm{n}} \\ &= -4\left(\mu_n^2 - v_n^2\right);\ \varphi_n \equiv 2\left(\mu_n^2 - v_n^2\right)x + \left[4\left(\mu_n^2 - v_n^2\right)^2 - 16\mu_n^2v_n^2\right]\bullet t + \alpha_{no}; c_{n0} \\ &\equiv e^{4\kappa_n x_{no}}e^{i\alpha_{no}}; n = 1, 2, \cdots, N\end{aligned} \tag{168}$$

$$\begin{aligned}&\det\left(I - i\overline{\Delta}_1\overline{\Delta}_2 - i\overline{\Delta}_1\overline{H}^{\mathrm{T}}G\right) = 1 + \sum_{r=1}^{N}\sum_{1\le n_1<n_2<\cdots<n_r\le N}\ \sum_{1\le m_1<m_2<\cdots<m_r\le N} \\ &\mathrm{M}_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r)\mathrm{M}_2(m_1, m_2, \cdots, m_r, n_1, n_2, \cdots, n_r)\end{aligned} \tag{169}$$

where $\mathrm{M}_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r)$ denotes a minor, which is the determinant of a submatrix of M_1, consisting of elements belonging to not only $(n_1, n_2, \ldots, n_r)$ rows but also columns $(m_1, m_2, \ldots, m_r)$.

$$\mathrm{M}_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r) = \prod_{n,m} \frac{h_n \overline{g}_m}{\lambda_n^2 - \overline{\lambda_m^2}} \prod_{n<n', m<m'} \left(\lambda_n^2 - \lambda_{n'}^2\right)\left(\overline{\lambda_{m'}^2} - \overline{\lambda_m^2}\right) \tag{170}$$

$$\mathrm{M}_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots n_r) = \prod_{n,m} \frac{\overline{h}_m g_n \overline{\lambda_m^2}}{\overline{\lambda_m^2} - \lambda_n^2} \prod_{n<n', m<m'} \left(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\right)\left(\lambda_{n'}^2 - \lambda_n^2\right) \tag{171}$$

where $n, n' \in \{n_1, n_2, \cdots, n_r\}$, $m, m' \in \{m_1, m_2, \cdots, m_r\}$, then $\mathrm{M}_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r)\mathrm{M}_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots n_r)$

$$= (-1)^r \prod_{n,m} \frac{e^{-\theta_n} e^{i\varphi_n} e^{-\theta_m} e^{-i\varphi_m} \overline{\lambda_m^2}}{\left(\lambda_n^2 - \overline{\lambda_m^2}\right)^2} \prod_{n<n', m<m'} \left(\overline{\lambda_m^2} - \overline{\lambda_{m'}^2}\right)^2 \left(\lambda_n^2 - \lambda_{n'}^2\right)^2 \tag{172}$$

If we define matrices $Q_1 = i\Delta_1$ and $Q_2 = \Delta_2$, then we can similarly attain

$$D = \det(\mathrm{I} + i\Delta_1\Delta_2) = \det(\mathrm{I} + \mathrm{Q}_1\mathrm{Q}_2) = 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < n_2 < \cdots < n_r \le N} \sum_{1 \le m_1 < m_2 < \cdots < m_r \le N} Q_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r) Q_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \tag{173}$$

and

$$\begin{aligned} &\mathrm{Q}_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r)\mathrm{Q}_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \\ &= (-1)^r \prod_{n,m} \frac{e^{-\theta_m} e^{i\varphi_m} e^{-\theta_n} e^{-i\varphi_n} \overline{\lambda_n^2}}{\left(\lambda_m^2 - \overline{\lambda_n^2}\right)^2} \prod_{n<n', m<m'} \left(\lambda_m^2 - \lambda_{m'}^2\right)^2 \left(\overline{\lambda_n^2} - \overline{\lambda_{n'}^2}\right)^2 \end{aligned} \tag{174}$$

where $n, n' \in \{n_1, n_2, \cdots, n_r\}$, $\;m, m' \in \{m_1, m_2, \cdots, m_r\}$. Comparing (172) and (174), we find the following permutation symmetry between them

$$\begin{aligned} &\mathrm{M}_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots m_r)\mathrm{M}_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \\ &= \mathrm{Q}_1(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots n_r)\mathrm{Q}_2(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) \end{aligned}$$

Using above identity, comparing (169), (172), (173), and (174), we find that identity (164) holds and complete the computation of D.

Secondly, we compute the most complicate determinant C in (163). In order to calculate $\det\left(I + i\Delta_1\Delta_2 + \overline{H}^T\overline{G}\right)$, we introduce an $N \times (N+1)$ matrix Ω_1 and an $(N+1) \times N$ matrix Ω_2

$$\begin{aligned} &(\Omega_1)_{nm} = (i\Delta_1)_{nm}, (\Omega_1)_{n0} = \overline{h}_n = \frac{\overline{h}_n \overline{\lambda_n^2}}{\overline{\lambda_n^2} - 0^2}; (\Omega_2)_{mn} = (\Delta_2)_{mn}, \\ &(\Omega_2)_{0n} = \overline{g}_n = \frac{-\overline{\lambda_n^2}\overline{g}_n}{0^2 - \overline{\lambda_n^2}} \end{aligned} \tag{175}$$

with n, m = 1, 2, … , N. We thus have

$$\det(I+\Omega_1\Omega_2)=1+\sum_{r=1}^{N}\sum_{1\le n_1\le n_2<\cdots<n_r\le N}\sum_{0\le m_1<m_2<\cdots<m_r\le N} \Omega_1(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots m_r)\Omega_2(m_1,m_2,\cdots,m_r;n_1,n_2,\cdots,n_r) \tag{176}$$

The above summation obviously can be decomposed into two parts: one is extended to $m_1 = 0$ and the other extended to $m_1 \ge 1$. Subtracted from (176), the part that is extended to $m_1 \ge 1$, the remaining parts of (176) is just C in Eq. (163) (with $m_1 = 0$, $m_2 \ge 1$). Due to (175), we have

$$\begin{aligned} C &= \det(I+\Omega_1\Omega_2)-\det(I+i\Delta_1\Delta_2) \\ &= \sum_{r=1}^{N}\sum_{1\le n_1<n_2<\cdots<n_r\le N}\sum_{1\le m_2<m_3<\cdots<m_r\le N} \Omega_1(n_1,n_2,\cdots,n_r;0,m_2,\cdots,m_r)\Omega_2(0,m_2,\cdots,m_r;n_1,n_2,\cdots,n_r) \end{aligned} \tag{177}$$

$$\Omega_1(n_1,n_2,\cdots,n_r;0,m_2,\cdots m_r)=\prod_n \overline{h}_n\prod_m g_m\prod_{n<n',m<m'}\left(\overline{\lambda_n^2}-\overline{\lambda_{n'}^2}\right)\left(\lambda_{m'}^2-\lambda_m^2\right)\prod_{n,m}\frac{1}{\overline{\lambda_n^2}-\lambda_m^2} \tag{178}$$

$$\begin{aligned}\Omega_2(0,m_2,\cdots,m_r;n_1,n_2,\cdots n_r)=&\prod_n \overline{g}_n\prod_m h_m\prod_{n<n',m<m'}\left(\overline{\lambda_n^2}-\overline{\lambda_{n'}^2}\right)\left(\lambda_{m'}^2-\lambda_m^2\right)\prod_{n,m}\frac{1}{\lambda_m^2-\overline{\lambda_n^2}} \\ &\cdot(-1)^{r+1}\prod_m \lambda_m^2,\end{aligned} \tag{179}$$

which leads to

$$\begin{aligned}&\Omega_1(n_1,n_2,\cdots,n_r;0,m_2,\cdots m_r)\Omega_2(0,m_2,\cdots,m_r;n_1,n_2,\cdots,n_r) \\ &=(-1)^{r+1}\prod_n e^{-\theta_n}e^{-i\varphi_n}\prod_m e^{-\theta_m}e^{i\varphi_m}\prod_{n<n',m<m'}\left(\overline{\lambda_n^2}-\overline{\lambda_{n'}^2}\right)^2\left(\lambda_{m'}^2-\lambda_m^2\right)^2\prod_{n,m}\frac{1}{\left(\lambda_m^2-\overline{\lambda_n^2}\right)^2}\prod_m \lambda_m^2,\end{aligned} \tag{180}$$

here $n, n' \in (n_1, n_2, \cdots, n_r)$, $m, m' \in (m_2, \cdots, m_r)$ in (178)–(180). Finally, substituting (174) into (173), (180) into (177), and (173 and 177) into (162), we thus attain the explicit N-soliton solution to the DNLS equation with VBC under the reflectionless case, based on a pure Marchenko formalism and in no need of the concrete spectrum expression of $a(\lambda)$. Obviously, the N-soliton solution permits uncertain complex constants c_{n0} $(n = 1, 2, \cdots, N)$ as well as an arbitrary global constant phase factor.

3.5 The special examples for one- and two-soliton solutions

In the case of one simple pole and one-soliton solution as $N = 1$, according to (173), (177), (174), and (180), we have

$$C_1=\Omega_1(n_1=1;m_1=0)\Omega_2(m_1=0;n_1=1)=\overline{g}_1\overline{h}_1(-1)^{1+1}=\overline{g}_1\overline{h}_1 \tag{181}$$

$$D_1=Q_1(n_1=1;m_1=1)Q_2(m_1=1;n_2=1)=1-\frac{g_1h_1\overline{g}_1\overline{h}_1\overline{\lambda_1^2}}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2}=1-\left|g_1h_1\right|^2\frac{\overline{\lambda_1^2}}{\left(\lambda_1^2-\overline{\lambda_1^2}\right)^2} \tag{182}$$

From (167) and (168), we have (suppose $\lambda_1=\rho_1e^{i\beta_1}=\mu_1+i\nu_1$ and $c_{10}=e^{4\kappa_1x_{10}}e^{i\alpha_{10}}$)

$$g_1 h_1 = C_1(t) e^{i2\lambda_1^2 x} = c_{10} e^{i4\lambda_1^4 t} e^{i2\lambda_1^2 x} = e^{-\theta_1} e^{i\varphi_1} \tag{183}$$

$$\begin{aligned} \theta_1 &= 4\mu_1 v_1 \left[x - x_{10} + 4\left(\mu_1^2 - v_1^2\right)t\right] \\ \varphi_1 &= 2\left(\mu_1^2 - v_1^2\right)x + \left[4\left(\mu_1^2 - v_1^2\right)^2 - 16\mu_1^2 v_1^2\right] \cdot t + \alpha_{10} \end{aligned} \tag{184}$$

Then from (181) and (182), we attain the one-soliton solution

$$u_1(x,t) = -2\frac{C_1 \overline{D}_1}{D_1^2} = -2\left(1 - \frac{\lambda_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2} e^{-2\theta_1}\right) e^{-\theta_1} e^{-i\varphi_1} \Big/ \left(1 - \frac{\overline{\lambda}_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2} e^{-2\theta_1}\right)^2 \tag{185}$$

By further redefinition of its soliton center and initial phase, the single soliton solution can be further rewritten as usual standard form. It is easy to find, up to a permitted well-known constant global phase factor, the one-soliton solution to DNLS equation gotten in the pure Marchenko formalism is in perfectly agreement with that gotten from other approaches [23, 24, 26, 27].

As $N = 2$ in the case of two-soliton solution corresponding to double simple poles, we have

$$u_2(x,t) = -2C_2 \overline{D}_2 / D_2^2 \tag{186}$$

$$\begin{aligned} C_2 &= \sum_{\substack{n_1 = 1,2 \\ m_1 = 0}} \Omega_1(n_1, 0)\Omega_2(0; n_1) + \sum_{\substack{n_1 = 1, n_2 = 2 \\ m_1 = 0, m_2 = 1,2}} \Omega_1(n_1, n_2; 0, m_2)\Omega_2(0, m_2; \mathrm{n}_1, n_2) \\ &= \Omega_1(n_1 = 1; m_1 = 0)\Omega_2(m_1 = 0; n_1 = 1) + \Omega_1(n_1 = 2; m_1 = 0)\Omega_2(m_1 = 0; n_1 = 2) \\ &+ \Omega_1(n_1 = 1, n_2 = 2; m_1 = 0, m_2 = 1)\Omega_2(m_1 = 0, m_2 = 1; n_2 = 1, n_2 = 2) \\ &+ \Omega_1(n_1 = 1, n_2 = 2; m_1 = 0, m_2 = 2)\Omega_2(m_1 = 0, m_2 = 2; n_1 = 1, n_2 = 2) \\ &= \overline{g}_1 \overline{h}_1 + \overline{g}_2 \overline{h}_2 - \left|\overline{g}_1 \overline{h}_1\right|^2 \overline{g}_2 \overline{h}_2 \frac{\left(\overline{\lambda}_1^2 - \overline{\lambda}_2^2\right)^2 \cdot \lambda_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_1^2 - \overline{\lambda}_2^2\right)^2} - \left|\overline{g}_2 \overline{h}_2\right|^2 \overline{g}_1 \overline{h}_1 \frac{\left(\overline{\lambda}_1^2 - \overline{\lambda}_2^2\right)^2 \cdot \lambda_2^2}{\left(\lambda_2^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_2^2 - \overline{\lambda}_2^2\right)^2} \\ &= e^{-\theta} e^{-i\varphi_1} + e^{-\theta_2} e^{-i\varphi_2} - \frac{\left(\overline{\lambda}_1^2 - \overline{\lambda}_2^2\right)^2 \cdot \lambda_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_1^2 - \overline{\lambda}_2^2\right)^2} e^{-2\theta_1 - \theta_2} e^{-i\varphi_2} - \frac{\left(\overline{\lambda}_1^2 - \overline{\lambda}_2^2\right)^2 \cdot \lambda_2^2}{\left(\lambda_2^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_2^2 - \overline{\lambda}_2^2\right)^2} e^{-2\theta_2 - \theta_1} e^{-i\varphi_1} \end{aligned} \tag{187}$$

$$\begin{aligned} D_2 &= 1 + \sum_{r=1}^{2} \sum_{1 \le n_1 < n_2 \le 2} \sum_{1 \le m_1 < m_2 \le 2} Q_1(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) Q_2(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \\ &= 1 + Q_1(n_1 = 1; m_1 = 1)Q_2(m_1 = 1, n_1 = 1) + Q_1(n_1 = 1; m_1 = 2)Q_2(m_1 = 2; n_1 = 1) \\ &+ Q_1(n_1 = 2; m_1 = 1)Q_2(m_1 = 1, n_1 = 2) + Q_1(n_1 = 2, m_1 = 2)Q_2(m_1 = 2; n_1 = 2) \\ &+ Q_1(n_1 = 1, n_2 = 2; m_1 = 1, m_2 = 2)Q_2(m_1 = 1, m_2 = 2; n_1 = 1, n_2 = 2) \\ &= 1 - \left|g_1 h_1\right|^2 \frac{\overline{\lambda}_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2} - \left|g_2 h_2\right|^2 \frac{\overline{\lambda}_2^2}{\left(\lambda_2^2 - \overline{\lambda}_2^2\right)^2} - \overline{g}_1 \overline{h}_1 g_2 h_2 \frac{\overline{\lambda}_1^2}{\left(\overline{\lambda}_1^2 - \lambda_2^2\right)^2} \\ &- \overline{g}_2 \overline{h}_2 g_1 h_1 \frac{\overline{\lambda}_2^2}{\left(\lambda_1^2 - \overline{\lambda}_2^2\right)^2} + \left|g_1 h_1\right|^2 \left|g_2 h_2\right|^2 \frac{\overline{\lambda_1^2 \lambda_2^2} \left(\lambda_1^2 - \lambda_2^2\right)^2 \left(\overline{\lambda}_1^2 - \overline{\lambda}_2^2\right)^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_1^2 - \overline{\lambda}_2^2\right)^2 \left(\lambda_2^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_2^2 - \overline{\lambda}_2^2\right)^2} \\ &= 1 - \frac{\overline{\lambda}_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2} e^{-2\theta_1} - \frac{\overline{\lambda}_2^2}{\left(\lambda_2^2 - \overline{\lambda}_2^2\right)^2} e^{-2\theta_2} - \frac{\overline{\lambda}_1^2}{\left(\overline{\lambda}_1^2 - \lambda_2^2\right)^2} e^{-\theta_1 - \theta_2} e^{i(\varphi_2 - \varphi_1)} \\ &- \frac{\overline{\lambda}_2^2}{\left(\lambda_1^2 - \overline{\lambda}_2^2\right)^2} e^{-\theta_1 - \theta_2} e^{i(\varphi_1 - \varphi_2)} + \frac{\overline{\lambda_1^2 \lambda_2^2} \left(\lambda_1^2 - \lambda_2^2\right)^2 \left(\overline{\lambda}_1^2 - \overline{\lambda}_2^2\right)^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_1^2 - \overline{\lambda}_2^2\right)^2 \left(\lambda_2^2 - \overline{\lambda}_1^2\right)^2 \left(\lambda_2^2 - \overline{\lambda}_2^2\right)^2} e^{-2(\theta_1 + \theta_2)} \end{aligned} \tag{188}$$

Up to a permitted constant global phase factor, the two-soliton solution gotten above is actually equivalent to that gotten from both IST and Hirota's method [23, 24, 26, 27], verifying the validity of the algebraic techniques that is used and our formula of the generalized multi-soliton solution. Because Marchenko equations (128), (129), (144), and (145) had been strictly proved, the multi-soliton solution is certainly right as long as we correctly use the algebraic techniques, especially Binet-Cauchy formula for the principal minor expansion of some special matrices.

4. Soliton solution of the DNLS equation based on Hirota's bilinear derivative transform

Bilinear derivative operator D had been found and defined in the early 1970s by Hirota R., a Japanese mathematical scientist [30–33]. Hirota's bilinear-derivative transform (HBDT for brevity) can be used to deal with some partial differential equation and to find some special solutions, such as soliton solutions and rogue wave solutions [26, 27, 32]. In this section, we use HBDT to solve DNLS equation with VBC and search for its soliton solution. The DNLS equation with VBC, that is,

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}\left(|u|^2 u\right)_x = 0, \tag{189}$$

is one of the typical integrable nonlinear models, which is of a different form from the following equation:

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}2|u|^2 u_x = 0, \tag{190}$$

which had been solved in Ref. [14] by using HBDT. We have paid special attention to the following solution form in it [14]:

$$u = g/f, \tag{191}$$

where f, g are usually complex functions. Solution (191) is suitable for Eq. (190) and NLS equation, and so on, but not suitable for the DNLS equation. Just due to this fact, their work cannot deal with Eq. (189) at the same time. As is well known, rightly selecting an appropriate solution form is an important and key step to apply Hirota's bilinear derivative transform to an integrable equation like Eq. (189). Refs. [13, 16, 17, 23], etc., have proved the soliton solution of the DNLS equation must has following standard form

$$u = g\bar{f}/f^2 \tag{192}$$

here and henceforth a bar over a letter represents complex conjugation.

In view of the existing experiences of dealing with the DNLS equation, in the present section, we attempt to use the solution form (192) and HBDT to solve the DNLS equation. We demonstrate our solving approach step by step, and naturally extend our conclusion to the n-soliton case in the end.

4.1 Fundamental concepts and general properties of bilinear derivative transform

For two differentiable functions $A(x, t), B(x, t)$ of two variables x and t, Hirota's bilinear derivative operator, D, is defined as

$$D_t^n D_x^m A \cdot B = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m A\left(x, t\right) B(x', t')|_{t'=t, x'=x} \tag{193}$$

which is different from the usual derivative, for example,

$$\begin{aligned} D_x A \cdot B &= A_x B - A B_x \\ D_x^2 A \cdot B &= A_{xx} B - 2A_x B_x + A B_{xx} \\ D_x^3 A \cdot B &= A_{xxx} B - 3A_{xx} B_x + 3A_x B_{xx} - A B_{xxx} \end{aligned} \tag{194}$$

where $A(x, t), B(x, t)$ are two functions derivable for an arbitrary order, and the dot $\cdot$ between them represents a kind of ordered product. Hirota's bilinear derivative has many interesting properties. Some important properties to be used afterwards are listed as follows:

$$①D_t^n D_x^m A \cdot B = (-1)^{n+m} D_t^n D_x^m B \cdot A \tag{195}$$

for example, $D_x A \cdot B = -D_x B \cdot A$; $D_x A \cdot A = 0$; $D_x^2 A \cdot B = D_x^2 B \cdot A$; $D_x^n A \cdot 1 = \partial_x^n A$; $D_x^n 1 \cdot A = (-1)^n \partial_x^n A$

$$②D_x^n A \cdot B = D_x^{n-m} D_x^m A \cdot B, (m < n) \tag{196}$$

③ Suppose $\eta_i = \Omega_i t + \Lambda_i x + \eta_{0i}$, $i = 1, 2$, Ω_i, Λ_i, η_{0i} are complex constants, then

$$D_t^n D_x^m \exp(\eta_1) \cdot \exp(\eta_2) = (\Omega_1 - \Omega_2)^n (\Lambda_1 - \Lambda_2)^m \exp(\eta_1 + \eta_2) \tag{197}$$

Especially, we have $D_t^n D_x^m \exp(\eta_1) \cdot \exp(\eta_2) = 0$ as $\Omega_1 = \Omega_2$ or $\Lambda_1 = \Lambda_2$. Some other important properties are listed in the Appendix.

4.2 Bilinear derivative transform of DNLS equation

After a suitable solution form, for example, (192) has been selected, under the Hirota's bilinear derivative transform, a partial differential equation like (189) can be generally changed into [20, 26, 27].

$$F_1(D_t, D_x \cdots) g_1 \cdot f_1 + F_2(D_t, D_x \cdots) g_2 \cdot f_2 = 0 \tag{198}$$

where $F_i(D_t, D_x \cdots)$, $i = 1, 2$ are the polynomial functions of D_t, D_x $\cdots$; and g_i, f_i, $i = 1, 2$, are the differentiable functions of two variables x and t . Using formulae in the Appendix and properties ①–③ of bilinear derivative transform numerated in the last chapter, with respect to (192), we have

$$u_t = D_t g\bar{f} \cdot f^2 / f^4 = \left[f\bar{f} D_t g \cdot f - g f D_t f \cdot \bar{f}\right] / f^4 \tag{199}$$

$$u_x = D_x g\bar{f} \cdot f^2 / f^4 = \left[f\bar{f} D_x g \cdot f - g f D_x f \cdot \bar{f}\right] / f^4 \tag{200}$$

$$u_{xx} = \left[f\bar{f} D_x^2 g \cdot f - 2(D_x g \cdot f)\left(D_x f \cdot \bar{f}\right) + g f D_x^2 f \cdot \bar{f}\right] / f^4 - 2g\bar{f} D_x^2 f \cdot f / f^4 \tag{201}$$

$$\left(|u|^2 u\right)_x = \left[2g\bar{g} D_x g \cdot f - g^2 (\bar{g} f)_x\right] / f^4 \tag{202}$$

Substituting the above expressions (199)–(202) into Eq. (189), the latter can be reduced to [26, 27].

$$f\bar{f}\left(\mathrm{i}D_t + D_x^2\right)g\cdot f - gf\left(\mathrm{i}D_t + D_x^2\right)f\cdot\bar{f} + f^{-2}D_x f^3\cdot\left[g\left(2D_x f\cdot\bar{f} - \mathrm{i}g\bar{g}\right)\right] = 0 \tag{203}$$

We can extract the needed bilinear derivative equations from Eq. (203) as follows:

$$\left(\mathrm{i}D_t + D_x^2\right)g\cdot f = 0 \tag{204}$$

$$\left(\mathrm{i}D_t + D_x^2\right)f\cdot\bar{f} = 0 \tag{205}$$

$$D_x f\cdot\bar{f} = \mathrm{i}g\bar{g}/2 \tag{206}$$

Functions $g(x,t), f(x,t)$ can be expanded, respectively, as series of a small parameter ε

$$g = \sum_i \varepsilon^i g^{(i)} \tag{207}$$

$$f = 1 + \sum_i \varepsilon^i f^{(i)} \tag{208}$$

Substituting (207) and (208) into (204)–(206) and equating the sum of the terms with the same orders of ε at two sides of (204)–(206), we attain

$$\left(\mathrm{i}\partial_t + \partial_x^2\right)g^{(1)} = 0 \tag{209}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(f^{(1)}\cdot 1 + 1\cdot\bar{f}^{(1)}\right) = 0 \tag{210}$$

$$D_x\left(f^{(1)}\cdot 1 + 1\cdot\bar{f}^{(1)}\right) = 0 \tag{211}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(g^{(2)}\cdot 1 + g^{(1)}\cdot f^{(1)}\right) = 0 \tag{212}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(f^{(2)}\cdot 1 + 1\cdot\bar{f}^{(2)} + f^{(1)}\cdot\bar{f}^{(1)}\right) = 0 \tag{213}$$

$$D_x\left(f^{(2)}\cdot 1 + 1\cdot\bar{f}^{(2)} + f^{(1)}\cdot\bar{f}^{(1)}\right) = \mathrm{i}g^{(1)}\cdot\bar{g}^{(1)}/2 \tag{214}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(g^{(3)}\cdot 1 + g^{(2)}\cdot f^{(1)} + g^{(1)}\cdot f^{(2)}\right) = 0 \tag{215}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(f^{(3)}\cdot 1 + 1\cdot\bar{f}^{(3)} + f^{(2)}\cdot\bar{f}^{(1)} + f^{(1)}\cdot\bar{f}^{(2)}\right) = 0 \tag{216}$$

$$D_x\left(f^{(3)}\cdot 1 + 1\cdot\bar{f}^{(3)} + f^{(2)}\cdot\bar{f}^{(1)} + f^{(1)}\cdot\bar{f}^{(2)}\right) = \mathrm{i}\left(g^{(2)}\bar{g}^{(1)} + g^{(1)}\bar{g}^{(2)}\right)/2 \tag{217}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(g^{(4)}\cdot 1 + g^{(3)}\cdot f^{(1)} + g^{(2)}\cdot f^{(2)} + g^{(1)}\cdot f^{(3)}\right) = 0 \tag{218}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(f^{(4)}\cdot 1 + 1\cdot\bar{f}^{(4)} + f^{(3)}\cdot\bar{f}^{(1)} + f^{(1)}\cdot\bar{f}^{(3)} + f^{(2)}\cdot\bar{f}^{(2)}\right) = 0 \tag{219}$$

$$\begin{aligned} &D_x\left(f^{(4)}\cdot 1 + 1\cdot\bar{f}^{(4)} + f^{(3)}\cdot\bar{f}^{(1)} + f^{(1)}\cdot\bar{f}^{(3)} + f^{(2)}\cdot\bar{f}^{(2)}\right) \\ &\quad = \mathrm{i}\left(g^{(3)}\bar{g}^{(1)} + g^{(2)}\bar{g}^{(2)} + g^{(1)}\bar{g}^{(3)}\right)/2 \end{aligned} \tag{220}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(g^{(5)}\cdot 1 + g^{(4)}\cdot f^{(1)} + g^{(3)}\cdot f^{(2)} + g^{(2)}\cdot f^{(3)}\right) = 0 \tag{221}$$

$$\left(\mathrm{i}D_t + D_x^2\right)\left(f^{(5)} \cdot 1 + 1 \cdot \overline{f}^{(5)} + f^{(4)} \cdot \overline{f}^{(1)} + f^{(1)} \cdot \overline{f}^{(4)} + f^{(3)} \cdot \overline{f}^{(2)} + f^{(2)} \cdot \overline{f}^{(3)}\right) = 0 \tag{222}$$

$$\begin{aligned} &D_x\left(f^{(5)} \cdot 1 + 1 \cdot \overline{f}^{(5)} + f^{(4)} \cdot \overline{f}^{(1)} + f^{(1)} \cdot \overline{f}^{(4)} + f^{(3)} \cdot \overline{f}^{(2)} + f^{(2)} \cdot \overline{f}^{(3)}\right) \\ &\quad = \mathrm{i}\left(g^{(4)}\overline{g}^{(1)} + g^{(3)}\overline{g}^{(2)} + g^{(2)}\overline{g}^{(3)} + g^{(1)}\overline{g}^{(4)}\right)/2 \end{aligned} \tag{223}$$

The above equations, (209)–(223), contain the whole information needed to search for a soliton solution of the DNLS equation with VBC.

4.3 Soliton solution of the DNLS equation with VBC based on HBDT

4.3.1 One-soliton solution

For the one-soliton case, due to (209)–(211) and considering the transform property ③, we can select $g^{(1)}$ and $f^{(1)}$ respectively as

$$g^{(1)} = e^{\eta_1}, \eta_1 = \Omega_1 t + \Lambda_1 x + \eta_{10}, \Omega_1 = \mathrm{i}\Lambda_1^2, \tag{224}$$

$$f^{(1)} = 0 \tag{225}$$

From (212), one can select $g^{(2)} = 0$. From (214), we can attain

$$f^{(2)} - \overline{f}^{(2)} = \frac{\mathrm{i}}{2}\frac{1}{\Lambda_1 + \overline{\Lambda}_1}\mathrm{e}^{\eta_1 + \overline{\eta}_1} \tag{226}$$

where the vanishing boundary condition, $u \to 0$ as $|x| \to \infty$, is used. Then

$$\partial_t\left(f^{(2)} - \overline{f}^{(2)}\right) = -\frac{1}{2}\frac{\Lambda_1^2 - \overline{\Lambda}_1^2}{\Lambda_1 + \overline{\Lambda}_1}\mathrm{e}^{\eta_1 + \overline{\eta}_1} \tag{227}$$

Substituting (226) and (227) into Eq. (213), we can attain

$$f^{(2)} + \overline{f}^{(2)} = \frac{\mathrm{i}}{2}\frac{\Lambda_1 - \overline{\Lambda}_1}{\left(\Lambda_1 + \overline{\Lambda}_1\right)^2}\mathrm{e}^{\eta_1 + \overline{\eta}_1} \tag{228}$$

From (226) and (228), we can get an expression of $f^{(2)}$

$$f^{(2)} = \frac{\mathrm{i}}{2}\frac{\Lambda_1}{\left(\Lambda_1 + \overline{\Lambda}_1\right)^2}\mathrm{e}^{\eta_1 + \overline{\eta}_1} \tag{229}$$

Due to (224) and (229), we can also easily verify that

$$\left(\mathrm{i}D_t + D_x^2\right)g^{(1)} \cdot f^{(2)} = 0 \tag{230}$$

which immediately leads to

$$\left(\mathrm{i}\partial_t + \partial_x^2\right)g^{(3)} = 0 \tag{231}$$

in Eq. (215). Then from (215), we can select $g^{(3)} = 0$. For the same reason, from (216)–(223), we can select $f^{(3)}$, $g^{(4)}, g^{(5)}$, … ; $f^{(4)}, f^{(5)}$, … all to be zero. Thus the series (207) and (208) have been successfully cut off to have limited terms as follows:

$$g_1 = e^{\eta_1} \tag{232}$$

$$f_1 = 1 + \frac{\mathrm{i}}{2}\frac{\Lambda_1}{\left(\Lambda_1 + \overline{\Lambda}_1\right)^2}\mathrm{e}^{\eta_1+\overline{\eta}_1} \tag{233}$$

where ε^i has been absorbed into the constant $e^{\eta_{10}}$ by redefiniing η_{10}. In the end, we attain the one-soliton solution to the DNLS equation with VBC

$$u_1(x,t) = g_1\overline{f}_1/f_1^2 \tag{234}$$

which is characterized with two complex parameters Λ_1 and η_{10} and shown in **Figure 1**. If we redefine the parameter Λ_1 as $\Lambda_1 \equiv -\mathrm{i}2\overline{\lambda}_1^2$ and $\lambda_1 \equiv \mu_1 + \mathrm{i}v_1$, then

$$\begin{aligned}\mathrm{e}^{\mathrm{i}2\lambda_1^2 x+\mathrm{i}4\lambda_1^4 t+\overline{\eta}_{10}}/2 &\equiv \mathrm{e}^{-\Theta_1}\mathrm{e}^{\mathrm{i}\Phi_1}; \mathrm{e}^{\overline{\eta}_{10}}/2 \equiv \mathrm{e}^{4\mu_1\nu_1 x_{10}}\mathrm{e}^{\mathrm{i}\alpha_{10}}; \Theta_1\\ &\equiv 4\mu_1 v_1\left[x - x_{10} + 4\left(\mu_1^2 - v_1^2\right)t\right]; \Phi_1\\ &\equiv 2\left(\mu_1^2 - v_1^2\right)x + \left[4\left(\mu_1^2 - v_1^2\right)^2 - 16\mu_1^2 v_1^2\right]t + \alpha_{10}\end{aligned} \tag{235}$$

Then

$$\begin{aligned}g_1 &= \mathrm{e}^{\eta_1} \equiv \mathrm{e}^{-\mathrm{i}2\overline{\lambda}^2 x - \mathrm{i}4\overline{\lambda}^4 t + \eta_{10}} = 2\mathrm{e}^{-\Theta_1}\mathrm{e}^{-\mathrm{i}\Phi_1}\\ f_1 &= 1 + \frac{\mathrm{i}}{2}\frac{\Lambda_1}{\left(\Lambda_1 + \overline{\Lambda}_1\right)^2}\mathrm{e}^{\eta_1+\overline{\eta}_1}\\ &= 1 - \frac{\overline{\lambda}^2}{\left(\lambda^2 - \overline{\lambda}^2\right)^2}\mathrm{e}^{-2\Theta}\end{aligned} \tag{236}$$

$$u_1(x,t) = \frac{g_1\overline{f}_1}{f_1^2} = 2\left(1 - \frac{\lambda_1^2}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2}\mathrm{e}^{-2\Theta_1}\right)\mathrm{e}^{-\Theta_1}\mathrm{e}^{-\mathrm{i}\Phi_1}\Bigg/\left(1 - \frac{\overline{\lambda_1^2}}{\left(\lambda_1^2 - \overline{\lambda}_1^2\right)^2}\mathrm{e}^{-2\Theta_1}\right)^2 \tag{237}$$

It is easy to find, up to a permitted constant global phase factor $\mathrm{e}^{\mathrm{i}\pi} = -1$, the one-soliton solution (234) or (237) gotten in this paper is in perfect agreement with that gotten from other approaches [16]. By further redefining its soliton center, initial phase and $\lambda_1 = \rho_1\mathrm{e}^{\mathrm{i}\beta_1}$, the one-soliton solution can be changed into the usual typical form [16, 23, 26, 27].

$$u_1(x,t) = 4|\lambda_1|\sin 2\beta_1\left(1 + \mathrm{e}^{\mathrm{i}2\beta_1}\mathrm{e}^{-2\Theta_1}\right)\mathrm{e}^{-\Theta_1}\mathrm{e}^{-\mathrm{i}\Phi_1}/\left(1 + \mathrm{e}^{-\mathrm{i}2\beta_1}\mathrm{e}^{-2\Theta_1}\right)^2 \tag{238}$$

On the other hand, just like in Ref. [13], we can rewrite g_1 and f_1 in a more appropriate or "standard" form

$$g_1 = \mathrm{e}^{\eta_1+\varphi_1} \tag{239}$$

$$f_1 = 1 + \mathrm{e}^{(\eta_1+\varphi_1)+(\overline{\eta}_1+\varphi_{1'})+\theta_{11'}} \tag{240}$$

Here

$$e^{\varphi_1} = 1, e^{\varphi_{1'}} = i/\overline{\Lambda}_1, e^{\theta_{11'}} = i\Lambda_1\left(-i\overline{\Lambda}_1\right)/2\left(\Lambda_1 + \overline{\Lambda}_1\right)^2, \tag{241}$$

which makes us easily extend the solution form to the case of n-soliton solution.

4.3.2 The two-soliton solution

For the two-soliton case, again from (209), we can select $g_2^{(1)}$ as

$$g_2^{(1)} = e^{\eta_1} + e^{\eta_2}, \eta_i = \Omega_i t + \Lambda_i x + \eta_{i0}, \Omega_i = i\Lambda_i^2, i = 1, 2. \tag{242}$$

The similar procedures to that used in the one-soliton case can be used to deduce g_2 and f_2. From (210) and (211), we can select $f_2^{(1)} = 0$, then from (212), we has to select $g_2^{(2)} = 0$. From (213) and (214), we can get the expressions of $f_2^{(2)} - \overline{f}_2^{(2)}$ and $f_2^{(2)} + \overline{f}_2^{(2)}$, then attain $f_2^{(2)}$ to be

$$\begin{aligned} f_2^{(2)} = {} & \frac{i\Lambda_1}{2\left(\Lambda_1 + \overline{\Lambda}_1\right)^2} e^{\eta_1 + \overline{\eta}_1} + \frac{i\Lambda_2}{2\left(\Lambda_2 + \overline{\Lambda}_2\right)^2} e^{\eta_2 + \overline{\eta}_2} + \frac{i\Lambda_1}{2\left(\Lambda_1 + \overline{\Lambda}_2\right)^2} e^{\eta_1 + \overline{\eta}_2} \\ & + \frac{i\Lambda_2}{2\left(\Lambda_2 + \overline{\Lambda}_1\right)^2} e^{\eta_2 + \overline{\eta}_1} \end{aligned} \tag{243}$$

Substituting (242) and (243) into (215), one can attain $g_2^{(3)}$ to be

$$g_2^{(3)} = \frac{-i(\Lambda_1 - \Lambda_2)^2 e^{\eta_1 + \eta_2}}{2} \left[\frac{\overline{\Lambda}_1 e^{\overline{\eta}_1}}{\left(\Lambda_1 + \overline{\Lambda}_1\right)^2 \left(\Lambda_2 + \overline{\Lambda}_1\right)^2} + \frac{\overline{\Lambda}_2 e^{\overline{\eta}_2}}{\left(\Lambda_2 + \overline{\Lambda}_2\right)^2 \left(\Lambda_1 + \overline{\Lambda}_2\right)^2} \right] \tag{244}$$

Substituting the expressions of $g_2^{(1)}, g_2^{(2)}, g_2^{(3)}, f_2^{(1)}, f_2^{(2)}$ into (216) and (217), we can select that $f_2^{(3)} = 0$. Then from the expressions of $g_2^{(1)}, g_2^{(2)}, g_2^{(3)}, f_2^{(1)}, f_2^{(2)}, f_2^{(3)}$ and (218), we can select $g_2^{(4)} = 0$. From (219) and (220), we can get the expressions of $f_2^{(4)} - \overline{f}_2^{(4)}$ and $f_2^{(4)} + \overline{f}_2^{(4)}$, then get $f_2^{(4)}$ to be

$$f_2^{(4)} = -\frac{\Lambda_1\Lambda_2|\Lambda_1 - \Lambda_2|^4}{4\left(\Lambda_1 + \overline{\Lambda}_1\right)^2 \left(\Lambda_2 + \overline{\Lambda}_2\right)^2 \left(\Lambda_1 + \overline{\Lambda}_2\right)^2 \left(\Lambda_2 + \overline{\Lambda}_1\right)^2} e^{\eta_1 + \eta_2 + \overline{\eta}_1 + \overline{\eta}_2} \tag{245}$$

Due to (243) and (244), we can also easily verify that

$$\left(iD_t + D_x^2\right) g_2^{(3)} \cdot f_2^{(2)} = 0 \tag{246}$$

Then from (244), (245), (246), and (221), we can select $g_2^{(5)} = 0$. From (222)–(223) and so on, we find that the series of (207) and (208) can be cut off by selecting $g_2^{(5)}, f_2^{(5)}; g_2^{(6)}, f_2^{(6)} \cdots$, all to be zero. We thus attain the last result of g_2, f_2 to be

$$\begin{aligned} g_2 = {} & e^{\eta_1} + e^{\eta_2} \\ & - \frac{i(\Lambda_1 - \Lambda_2)^2}{2} e^{\eta_1 + \eta_2} \bullet \left[\frac{\overline{\Lambda}_1}{\left(\Lambda_1 + \overline{\Lambda}_1\right)^2 \left(\Lambda_2 + \overline{\Lambda}_1\right)^2} e^{\overline{\eta}_1} + \frac{\overline{\Lambda}_2}{\left(\Lambda_2 + \overline{\Lambda}_2\right)^2 \left(\Lambda_1 + \overline{\Lambda}_2\right)^2} e^{\overline{\eta}_2} \right] \end{aligned} \tag{247}$$

$$f_2 = 1 + \frac{i\Lambda_1}{2(\Lambda_1+\overline{\Lambda}_1)^2} e^{\eta_1+\overline{\eta}_1} + \frac{i\Lambda_2}{2(\Lambda_2+\overline{\Lambda}_2)^2} e^{\eta_2+\overline{\eta}_2} + \frac{i\Lambda_1}{2(\Lambda_1+\overline{\Lambda}_2)^2} e^{\eta_1+\overline{\eta}_2} + \frac{i\Lambda_2}{2(\Lambda_2+\overline{\Lambda}_1)^2} e^{\eta_2+\overline{\eta}_1}$$
$$- \frac{\Lambda_1\Lambda_2|\Lambda_1-\Lambda_2|^4}{4(\Lambda_1+\overline{\Lambda}_1)^2(\Lambda_2+\overline{\Lambda}_2)^2(\Lambda_1+\overline{\Lambda}_2)^2(\Lambda_2+\overline{\Lambda}_1)^2} e^{\eta_1+\eta_2+\overline{\eta}_1+\overline{\eta}_2} \tag{248}$$

It can also be rewritten in a standard form as follows:

$$\begin{aligned} g_2 &= e^{\eta_1+\varphi_1} + e^{\eta_2+\varphi_2} + e^{(\eta_1+\varphi_1)+(\eta_2+\varphi_2)+(\overline{\eta}_1+\varphi_{1'})+\theta_{12}+\theta_{11'}+\theta_{21'}} \\ &\quad + e^{(\eta_1+\varphi_1)+(\eta_2+\varphi_2)+(\overline{\eta}_2+\varphi_{2'})+\theta_{12}+\theta_{12'}+\theta_{22'}} \end{aligned} \tag{249}$$

$$\begin{aligned} f_2 &= 1 + e^{(\eta_1+\varphi_1)+(\overline{\eta}_1+\varphi_{1'})+\theta_{11'}} + e^{(\eta_1+\varphi_1)+(\overline{\eta}_2+\varphi_{2'})+\theta_{12'}} + e^{(\eta_2+\varphi_2)+(\overline{\eta}_1+\varphi_{1'})+\theta_{21'}} \\ &\quad + e^{(\eta_2+\varphi_2)+(\overline{\eta}_2+\varphi_{2'})+\theta_{22'}} + e^{(\eta_1+\varphi_1)+(\eta_2+\varphi_2)+(\overline{\eta}_1+\varphi_{1'})+(\overline{\eta}_2+\varphi_{2'})+\theta_{12}+\theta_{11'}+\theta_{12'}+\theta_{21'}+\theta_{22'}+\theta_{1'2'}} \end{aligned} \tag{250}$$

where φ_i, θ_{ij} in (249) and (250) are defined afterwards in (256). We then attain the two-soliton solution as

$$u_2(x,t) = g_2\overline{f}_2/f_2^2 \tag{251}$$

which is characterized with four complex parameters Λ_1, Λ_2, η_{10}, and η_{20} and shown in **Figure 3**. By redefining parameters η_{i0} and

$$\Lambda_k = -i2\overline{\lambda}_k^2, k = 1, 2, \tag{252}$$

we can easily transform it to a two-soliton form given in Ref. [23], up to a permitted constant global phase factor.

4.3.3 Extension to the N*-soliton solution*

Generally for the case of N-soliton solution, if we select $g^{(1)}$ in (209) to be

$$\begin{aligned} g_N^{(1)} &= e^{\eta_1} + e^{\eta_2} + \dots + e^{\eta_N}, \\ \eta_i &= \Omega_i t + \Lambda_i x + \eta_{0i}, \Omega_i = i\Lambda_i^2, i = 1, 2, \cdots, N \end{aligned} \tag{253}$$

then using an induction method, we can write the N-soliton solution as

$$g_N = \sum_{\kappa_j=0,1}^{(1)} \exp\left\{\sum_{j=1}^{2N}\kappa_j\left(\eta_j+\varphi_j\right) + \sum_{1\le j<k}^{2N}\kappa_j\kappa_k\theta_{jk}\right\} \tag{254}$$

$$f_N = \sum_{\kappa_j=0,1}^{(0)} \exp\left\{\sum_{j=1}^{2N}\kappa_j\left(\eta_j+\varphi_j\right) + \sum_{1\le j<k}^{2N}\kappa_j\kappa_k\theta_{jk}\right\} \tag{255}$$

where $\eta_{N+j} = \overline{\eta}_j$ ($j \le N$)

$$\begin{aligned} &e^{\phi_j} = 1, e^{\varphi_{j+N}} = e^{\varphi_{j'}} = \left(-i\overline{\Lambda}_j\right)^{-1} \\ &e^{\theta_{jk}} = 2(\Lambda_j-\Lambda_k)^2/i\Lambda_j\cdot i\Lambda_k, e^{\theta_{j,k+N}} = e^{\theta_{j,k'}} = i\Lambda_j\left(-i\overline{\Lambda}_k\right)/2\left(\Lambda_j+\overline{\Lambda}_k\right)^2 \\ &e^{\theta_{j+N,k+N}} = e^{\theta_{j'k'}} = 2\left(\overline{\Lambda}_j-\overline{\Lambda}_k\right)^2/\left(-i\overline{\Lambda}_j\right)\left(-i\overline{\Lambda}_k\right), (1\le j\le N, 1\le k\le N), \end{aligned} \tag{256}$$

therein $\sum_{\kappa_j=0,1}^{(l)}$ represents a summation over $\kappa_j = 0, 1$ under the condition $\sum_{j=1}^{N}\kappa_j = l + \sum_{j=1}^{N}\kappa_{j+N}$.

Here, we have some discussion in order. Because what concerns us only is the soliton solutions, our soliton solution of DNLS equation with VBC is only a subset of the whole solution set. Actually in the whole process of deriving the bilinear-form equations and searching for the one and two-soliton solutions, some of the latter results are only the sufficient but not the necessary conditions of the former equations. Thereby some possible modes might have been missing. For example, the solutions of Eqs. (209)–(211) are not as unique as in (224) and (225), some other possibilities thus get lost here. This is also why we use a term "select" to determine a solution of an equation. In another word, we have selected a soliton solution. Meanwhile, we have demonstrated in **Figures 2** and **3**, the three-dimensional evolution of the one- and two-soliton amplitude with time and space, respectively. The elastic collision of two solitons in the two-soliton case has been demonstrated in **Figure 4(a–d)** too. It can be found that each soliton keeps the same form and characteristic after the collision as that before the collision. In this section, by means of introducing HBDT and employing an appropriate solution form (192),

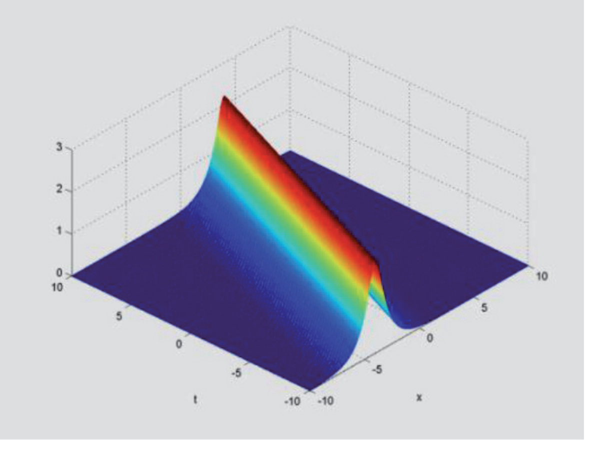

Figure 2.
The evolution of one-soliton solution with time and space under parameter $\Lambda_1 = -1 + 0.2i$, $\eta_{10} = 1$ in (234).

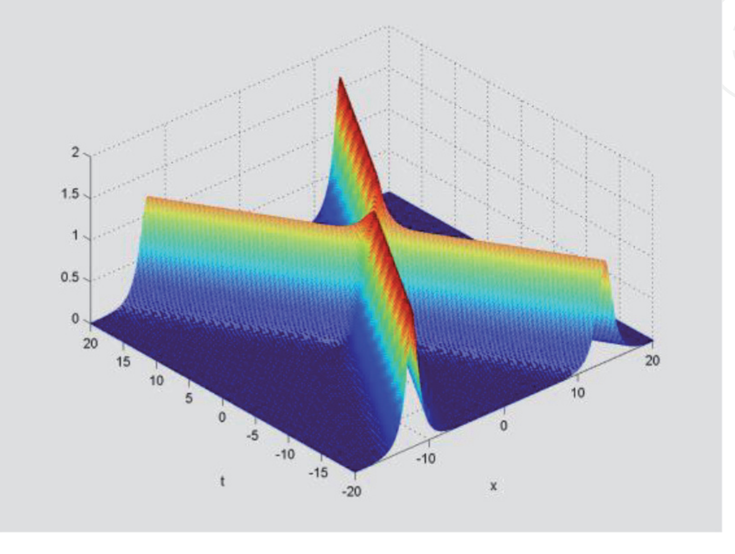

Figure 3.
The evolution of two-soliton solution with time and space under parameter $\Lambda_1 = 1 + 0.3i$, $\Lambda_2 = 1 - 0.3i$, $\eta_{10} = \eta_{20} = 1$ in (251).

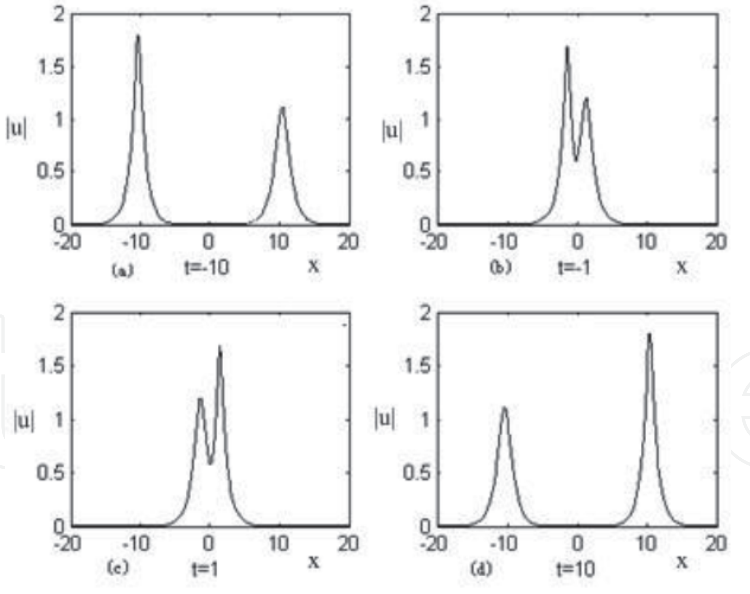

Figure 4.
The elastic collision between two solitons at 4 typical moments: (a) $t = -10$(normalized time); (b) $t = -1$; (c) $t = 1$; (d) $t = 10$, from -10 before collision to 10 after collision.

we successfully solve the derivative nonlinear Schrödinger equation with VBC. The one- and two-soliton solutions are derived and their equivalence to the existing results is manifested. The N-soliton solution has been given by an induction method. On the other hand, by using simple parameter transformations (e.g., (235) and (252)), the soliton solutions attained here can be changed into or equivalent to that gotten based on IST, up to a permitted global constant phase factor. This section impresses us so greatly for a fact that, ranked with the extensively used IST [23] and other methods, the HBDT is another effective and important tool to deal with a partial differential equation. It is especially suitable for some integrable nonlinear models.

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