Chapter

New Topology on Symmetrized Omega Algebra

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Abstract

The purpose of this paper is to define a new topology called symmetrized omega algebra topology and discuss some of its topological properties. Two different examples from an ordered infinite set of symmetrized omega topology are introduced. Furthermore, we study the relationship between symmetrized omega topology and weaker kinds of normality.

Keywords: tropical geometry, idempotent semiring, topological space, topological properties, omega algebra and symmetrized omega algebra

1. Introduction

Tropical geometry is the most recent but fast growing branch of mathematical sciences, which is analytically based on idempotent analysis and algebraically on idempotent semirings also known as tropical semirings. These are basically extended sets of real numbers $\mathbb{R}_{\infty} : \mathbb{R} \cup \{\infty\}$ and $\mathbb{R}_{-\infty} : \mathbb{R} \cup \{-\infty\}$ which are given monoidal structures by using min and max operations for addition, respectively. In order to adhere to the semiring structure, the additive operation of $\mathbb{R}$ is used as the multiplication operation. By these choices, both $\mathbb{R}_{\infty}$ and $\mathbb{R}_{-\infty}$ become idempotent semirings. The literature, they are also termed as min and max plus algebras, respectively. In both cases, 0 of $\mathbb{R}$ becomes a multiplicative identity and ∞ and $-\infty$ become additive identities of these semirings, respectively. Interestingly, some authors associate $\mathbb{R}_{-\infty}$ to tropical geometry, while other authors associate $\mathbb{R}_{\infty}$ to tropical geometry (see [1–4]). Omega algebra or "$\omega-$ algebra" for short, unifies the different terms and introduces an original structure, which, in fact, is an "abstract tropical algebra". The $\mathbb{R}_{-\infty}$ and $\mathbb{R}_{\infty}$ and their nearby structures, like $\min - \max$ and $\max -$ times algebras, etc., are all subsumed under omega algebra. All these are idempotent semirings, which are also called dioids. In previous studies, for the construction of all such semirings, an ordered infinite abelian group is mandatory. In $\omega-$ algebra, the definition is extended to cyclically ordered abelian groups and also to finite sets under some suitable ordering. Note that cyclically ordered abelian groups are more general than that of ordered abelian groups [5]. The aim of this paper is to define a new topology on symmetrized omega algebra, and discus some of its topological properties. Two different examples from an ordered infinite sets of symmetrized omega topology are introduced. Furthermore, we study the relationship between symmetrized omega topology and weaker kinds of normality. Our paper is organized as follows. In Section 2, we review an abstract definition for some basic facts about abstract omega algebras. In addition, we give a brief of symmetrized omega algebra and rules of calculation in omega. In Section 3, we define a new topology on symmetrized omega algebra and discuss some of its topological properties. In Section 4, we provide two different examples of symmetrized omega topology: the first and second examples are from an ordered infinite set. Finally, we study the relationship between symmetrized omega topology and weaker kinds of normality in Section 5. Throughout this paper, we do not assume T_2 in the definition of compactness. We also do not assume regularity in the definition of Lindelöfness.

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2. Preliminaries

In this section, we provide an abstract definition for review some basic facts about abstract omega algebra. Furthermore, we also provide a brief of symmetrized omega algebra and rules of calculation in omega. For more details, see [6].

2.1 Omega algebra

Let $(G, \circ, e)$ be an abelian group. Let A be a closed subset of G and $e \in A$. Then $(A, \circ, e)$ is a submonoid of G. Assume that ω is an indeterminate (may belong to A or G, as we will see in Examples 1 and 2). Obviously, in this case ω is no longer an indeterminate. Because the terms are generated from tropical geometry, this indeterminate can be called a tropical indeterminate.

Definition 1. [6]

We say that $A_\omega = A \cup \{\omega\}$ is an *omega algebra* (in short $\omega-$ *algebra*) over the group G in case A_ω is closed under two binary operations,

$$\oplus, \otimes : A_\omega \times A_\omega \to A_\omega, \tag{1}$$

such that $\forall a_1, a_2, a_3 \in A$, the following axioms are satisfied:

i. $a_1 \oplus a_2 = a_1 \; or \; a_2$;

ii. $a_1 \oplus \omega = a_1 = \omega \oplus a_1$;

iii. $\omega \oplus \omega = \omega$;

iv. $a_1 \otimes a_2 = a_2 \otimes a_1 \in A$;

v. $(a_1 \otimes a_2) \otimes a_3 = a_1 \otimes (a_2 \otimes a_3)$;

vi. $a_1 \otimes e = a_1$;

vii. $a_1 \otimes \omega = \omega \otimes a_1 = \begin{cases} \omega & if \quad \omega \neq e \\ a_1 & if \quad \omega = e \end{cases}$;

viii. $\omega \otimes \omega = \omega$;

ix. $a_1 \otimes (a_2 \oplus a_3) = (a_1 \otimes a_2) \oplus (a_1 \otimes a_3)$.

Remark 2. [6]

1. $\oplus$ is a pairwise comparison operation such as max, min, inf, sup, up, down, lexicographic ordering, or anything else that compairs two elements of A_ω.

Obviously, it is associative and commutative and the tropical indeterminate ω plays the role of the identity. Hence $(A_\omega, \oplus, \omega)$ is a commutative monoid.

2. $\otimes$ is also associative and commutative on A_ω, and e plays the role of the multiplicative identity of A_ω. Hence, $(A_\omega, \otimes, e)$ is also a commutative monoid.

3. The left distributive law (ix) also gives the right distributive law.

4. Every element of A_ω is an idempotent under $\oplus$.

5. Altogether, we write both structures as: $A_\omega = (A_\omega, \oplus, \otimes, \omega, e)$. This is an idempotent semiring.

Remark 3. [6] A $\omega-$ algebra can similarly be defined over a commutative monoid, ring, or even a semiring. More generally, one may construct analogously such algebras on other weaker structures. In this note, we confined ourselves to only $\omega-$ algebras over abelian groups and rings.

Example 4. [6] *Max-plus algebra, min-plus algebra* and all such "so called" *algebras* are particular cases of the *$\omega-$ algebra* over the ring $\mathbb{R}$ or its associated subrings. A simpler example is the following. In the abelian group $(\mathbb{Z}, +)$, for any integer m, we have $W(m) = \{0, m, 2m, \cdots\}$. This is an additive submonoid of $(\mathbb{Z},+)$. Let $\omega = -\infty$, $a_1 \oplus a_2 = \max(a_1, a_2)$ and $a_1 \otimes a_2 = a_1 + a_2, \forall a_1, a_2 \in W(m)$. Then,

$$W(m)_{-\infty} = \left(W(m)_{-\infty}, \oplus, \otimes, -\infty, 0\right) \tag{2}$$

is $-\infty$ $-$ algebra over the abelian group of integers $\mathbb{Z}$. Hence, we have a sequence of $\omega-$ subalgebras

$$W(m) \geq W(2m) \geq \cdots.$$

Example 5. [6] Cartesian products of omega algebras. In this example, we explain a construction of an omega algebra from other given omega algebras. Let $\{(G_i, \circ_i, e_i) : i = 1, \cdots, n\}$ be abelian groups and $\{(A_{\omega_i}, \oplus_i, \otimes_i, \omega_i, e_i) : i = 1, \cdots, n\}$ be a respective family of omega algebras, where ω_i are tropical indeterminate. As usual, we define the Cartesian product as

$$\mathcal{X}_\omega = A_{\omega_1} \times \cdots \times A_{\omega_n} = \{(a_1, \cdots a_n) : a_i \in A_{\omega_i}; i = 1, \cdots, n\}. \tag{3}$$

In order to provide a convenient technique to give an additive structure to $\mathcal{X}_\omega$, we assume that the $n-tuples$ $\mathbf{a} = (a_1, \cdots a_n)$, $\mathbf{b} = (b_1, \cdots b_n) \in \mathcal{X}_\omega$ are in lexicographic ordering. Then, to define the sum

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{a} \text{ or } \mathbf{b} \tag{4}$$

by using the following rules:

$$\text{If } a_1 \oplus_1 b_1 = a_1 \text{ then } \mathbf{a} \oplus \mathbf{b} = \mathbf{a}. \tag{5}$$

$$\text{If } a_i = b_i \text{ for } 1 \leq i \leq k \leq n, \quad \text{and } a_{k+1} \oplus_{k+1} b_{k+1} = a_{k+1}, \text{then, } \mathbf{a} \oplus \mathbf{b} = \mathbf{a}. \tag{6}$$

Similarly, rules for $\mathbf{a} \oplus \mathbf{b} = \mathbf{b}$ can be determined. Multiplication can be to define component wise. Thus,

$$\mathbf{a} \otimes \mathbf{b} = (a_1 \otimes_1 b_1, \cdots, a_n \otimes_n b_n). \tag{7}$$

The other rules of the Definition 1 can straightforwardly be verified. Hence, $(\mathcal{X}_\omega, \oplus, \otimes, \omega, e)$, where $\omega = (\omega_1, \cdots, \omega_n)$ is the additive identity and $e = (e_1, \cdots, e_n)$ is the multiplicative identity of $\mathcal{X}_\omega$, is an omega algebra over the Cartesian product of abelian groups $G_1 \times \cdots \times G_n$..

2.2 The symmetrized omega algebra

Let $(G, \circ, e)$ be an abelian group and $(A_\omega, \oplus, \otimes, \omega, e)$ an $\omega-$algebra over the group G. Following the method used in constructing integers from the natural numbers, we consider the set of ordered pairs $\mathcal{P}_\omega = A_\omega^2$ with component wise addition $\oplus$, for all $(a,b), (c,d) \in \mathcal{P}_\omega$,

$$(a,b) \oplus (c,d) = (a \oplus c, b \oplus d) \tag{8}$$

Because of the four possibilities (a,b), (a,d), (c,d) or (c,b) for the result, the addition in (8), is ambiguous. As our goal from constructing the algebra of pairs is the construction of the symmetrized omega algebra of A_ω, we are in front of two possibilities: One is to use –for $n = 2$, and define an equivalence relation $\sim$ on the $\omega-$algebra of pairs which is compatible with relevant operations, and the other is to define an equivalence relation on the set $\mathcal{P}_\omega$ that allows the component wise addition to be defined in the quotient set.

First Construction, let $\leq$ be the ordering defined on A_ω by the relation

$$a \leq b \Leftrightarrow a \oplus b = b \tag{9}$$

which gives a total order on A_ω and for all $a \in A_\omega$, we have $\omega \leq a$. For $a \neq b$, such that $a \oplus b = b$, we denote by $a < b$. Under the ordering $\leq$, rules (5) and (6) defined in Example 5, are satisfied on $\mathcal{P}_\omega = A_\omega^2$ and so $\mathcal{P}_\omega$ is an $\omega-$algebra under the addition defined in 1 and the component wise multiplication. Let ∇ be the relation defined on $\mathcal{P}_\omega$ as follows: for all $(a,b), (c,d) \in \mathcal{P}_\omega$

$$(a,b) \nabla (c,d) \Leftrightarrow a \oplus d = b \oplus c. \tag{10}$$

Then ∇ is reflexive and symmetric but not transitive for A_ω contains more than 4 elements. In fact, let $a, b, c, d \in A_\omega$ such that $a < b < c < d$, then we have

$$a \oplus d = d = b \oplus d = c \oplus d \text{ and } a \oplus c = c \neq b = b \oplus b$$

which give $(a,b) \nabla (d,d)$ and $(d,d) \nabla (b,c)$, but there is no relation between (a,b) and (b,c). As ∇ is not an equivalence relation, we cannot use it to obtain the quotient $\omega-$algebra $\frac{\mathcal{P}_\omega}{\nabla}$ (like the one to obtain integers from the natural numbers).

Definition 6. [6] Let $\sim$ be the equivalence relation close to ∇ defined as follows: for all $(a,b), (c,d) \in \mathcal{P}_\omega$,

$$(a,b) \sim (c,d) \Leftrightarrow \begin{cases} (a,b) \nabla (c,d) & \text{if } a \neq b \text{ and } c \neq d \\ (a,b) = (c,d) & \text{otherwise} \end{cases} \tag{11}$$

In addition to the class element $\overline{\omega} = \overline{(\omega, \omega)}$; for all $a \in A_\omega$, with $a \neq \omega$, we have three kinds of equivalence classes:

1. $\overline{(a, \omega)} = \{(a,b) \in \mathcal{P}_\omega, b < a\}$, called positive $\omega-$element.

2. $\overline{(\omega, a)} = \{(b, a) \in \mathcal{P}_\omega, b < a\}$, called negative $\omega-$element.

3. $\overline{(a, a)}$ called balanced $\omega-$element.

Unfortunately, the addition defined by (7) and rules (8) and (9) in Example 5 is not compatible with the equivalence relation in $\mathcal{P}_\omega$, because for (a, ω), (a, b), (ω, c), $(d, c) \in \mathcal{P}_\omega$, such that

$$\begin{cases} (a, \omega) \sim (a, b) \\ (\omega, c) \sim (d, c), \end{cases} \tag{12}$$

we have

$$(a, \omega) \oplus (\omega, c) \sim (a, b) \oplus (d, c) \text{ iff } (a, b) \oplus (d, c) = (a, b) \tag{13}$$

$$\text{and if } (a, b) \oplus (d, c) = (d, c), \tag{14}$$

then there is no compatibility. So the omega algebra of pairs cannot produce the symmetrized omega algebra.

Second Construction

Proposition 7. [6]

The addition operation $\overline{\oplus}$ defined by

$$\overline{(a, b)}\,\overline{\oplus}\,\overline{(c, d)} = \overline{(a \oplus c, b \oplus d)}$$

on the quotient set $\frac{\mathcal{P}_\omega}{\sim}$ is well defined and satisfies the axioms (i), (ii) and (iii) of Definition 1, with the zero class element $\overline{w} = \overline{(\omega, \omega)}$, except this case $\overline{(a, \omega)}\,\overline{\oplus}\,\overline{(\omega, a)} = \overline{(\omega, a)}\,\overline{\oplus}\,\overline{(a, \omega)} = \overline{(a, a)}$, where $a \in A_\omega \backslash \{\omega\}$ does not satisfy the axiom (i).

Proposition 8. [6]

i. The set $\frac{\mathcal{P}_\omega}{\sim}$ is closed under the binary multiplication operation $\overline{\otimes}$ defined as follows: for all $\overline{(a, b)}$, $\overline{(c, d)} \in \frac{\mathcal{P}_\omega}{\sim}$;

$$\overline{(a, b)}\,\overline{\otimes}\,\overline{(c, d)} = \overline{((a \otimes c) \oplus (b \otimes d), (a \otimes d) \oplus (b \otimes c))} \tag{15}$$

and satisfies axioms from (iv) to (ix) of Definition 1, with the unit class element $\overline{e} = \overline{(e, \omega)}$.

ii. In addition, we have for all $a, b \in A_\omega$

a. $\overline{(a, \omega)}\,\overline{\otimes}\,\overline{(b, \omega)} = \overline{(a \otimes b, \omega)}$;

b. $\overline{(a, \omega)}\,\overline{\otimes}\,\overline{(\omega, b)} = \overline{(\omega, a \otimes b)}$;

c. $\overline{(a, \omega)}\,\overline{\otimes}\,\overline{(b, b)} = \overline{(a \otimes b, a \otimes b)}$;

d. $\overline{(\omega, a)}\,\overline{\otimes}\,\overline{(b, b)} = \overline{(a \otimes b, a \otimes b)}$.

Definition 9. [6] The structure $\left(\frac{\mathcal{P}_\omega}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{\omega}, \overline{e}\right)$ is called the symmetrized $\omega-$algebra over the abelian group $G \times G$ and we denote it by $\mathbb{S}_\omega$.

In the coming sections just for simplicity we will only use $\oplus$ and $\otimes$ instead the operations $\overline{\oplus}$ and $\overline{\otimes}$, respectively.

Remark 10. [6]

1. Despite the nature of the positive and the negative $\omega-$elements, they are not the inverses of each other for the additive operation $\overline{\oplus}$,

2. We have three symmetrized $\omega-$subalgebras of $\mathbb{S}_\omega$,

$$\mathbb{S}_\omega^{(+)} = \left\{\overline{(a,\omega)}, a \in A_\omega\right\},$$
$$\mathbb{S}_\omega^{(-)} = \left\{\overline{(\omega,a)}, a \in A_\omega\right\},$$
$$\mathbb{S}_\omega^{(0)} = \left\{\overline{(a,a)}, a \in A_\omega\right\}.$$

3. The three symmetrized $\omega-$subalgebras of $\mathbb{S}_\omega$ are connected by the zero class element $\overline{\omega}$.

4. The positive $\omega-$elements, the negative $\omega-$elements and the balanced elements are called signed and denoted by $\mathbb{S}_\omega^\vee = \mathbb{S}_\omega^{(+)} \cup \mathbb{S}_\omega^{(-)}$, where the zero class $\overline{(\omega,\omega)}$ corresponds to ω.

2.3 Rules of calculation in omega

Notation 11. [6] *Let $a \in \mathbb{A}_\omega$. Then we admit the following notations:*

$$+a = \overline{(a,\omega)},\ -a = \overline{(\omega,a)},\ \cdot a = \overline{(a,a)}. \tag{16}$$

By results in Proposition 7 and Proposition 8 and the above notation, it is easy to verify the rules of calculation in the following proposition:

Proposition 12. [6] *For all $a, b \in A_\omega$, we have*

i. $(+a) \oplus (+b) = +(a \oplus b)$;

ii. $(+a) \oplus (-b) = \begin{cases} +a & \text{if } b < a \\ -b & \text{if } b > a; \\ \cdot a & \text{if } b - a \end{cases}$

iii. $(\pm a) \oplus (\cdot b) = \begin{cases} \pm a & \text{if } b < a \\ \cdot b & \text{if } b > a \end{cases}$:

iv. $(-a) \oplus (-b) = -(a \oplus b)$;

v. $(+a) \otimes (+b) = +(a \otimes b)$;

vi. $(+a) \otimes (-b) = -(a \otimes b)$;

vii. $(\pm a) \otimes (\cdot b) = \cdot(a \otimes b)$;

viii. $(-a) \otimes (-b) = +(a \otimes b)$.

From the previous rules, we can notice that the sign of the result in the addition operation follows the greater element in A_ω. While in the multiplication operation, the balance sign is the strong one (has priority).

3. Symmetrized omega topology

In this section, we define a new topology on symmetrized omega algebra and discuss some of its topological properties.

Throughout this paper, we assume that $\otimes|A = \circ$.

Proposition 13. Let $\mathbb{S}_\omega = \left(\frac{\mathcal{P}_\omega}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{\omega}, \overline{e}\right)$ be a symmetrized $\omega-$algebra over the abelian group $G \times G$, where $\mathcal{P}_\omega = A_\omega \times A_\omega$ and $\otimes|A = \circ$. We define a new topology on $\mathbb{S}_\omega$ called a symmetrized omega topology, denoted by τ_ω as follow:

$\tau_\omega = \{\emptyset, \mathbb{S}_\omega\} \cup \{U \subseteq \mathbb{S}_\omega : \mathbb{S}_\omega^{(0)} \subseteq U$ *and for any* $+a, -a \in U$, *their multiplicative inverses exists in U, where* $a \in A_\omega \backslash \{\omega\}\}$.

Proof. Condition $\emptyset, \mathbb{S}_\omega \in \tau_\omega$ is satisfied from the definition of τ_ω. Now let $V_1, V_2 \in \tau_\omega$ be arbitrary. If either V_1 or V_2 is equal $\emptyset$, then $V_1 \cap V_2 = \emptyset \in \tau_\omega$. Assume now, $V_1 \neq \emptyset \neq V_2$. If either V_1 or V_2 is equal $\mathbb{S}_\omega$, then $V_1 \cap V_2 = V_1 \, or \, V_2 \in \tau_\omega$. So assume that, $V_1 \neq \mathbb{S}_\omega \neq V_2$, then $V_1 \cap V_2 \in \tau_\omega$, because $\mathbb{S}_\omega^{(0)} \subseteq V_1$ and $\mathbb{S}_\omega^{(0)} \subseteq V_2$, hence $\mathbb{S}_\omega^{(0)} \subseteq V_1 \cap V_2$, also for any element $+a, -a \in V_1 \cap V_2$, where $a \neq \omega$, then we have $+a, -a \in V_1$ and $+a, -a \in V_2$, then the multiplicative inverse of $+a, -a$ must belong to V_1 and V_2. Hence, the multiplicative inverse of $+a, -a$ belong to $V_1 \cap V_2$, then $V_1 \cap V_2 \in \tau_\omega$. For the third condition let $S_\gamma \in \tau_\omega$ for any $\gamma \in \Lambda$. If $S_\gamma = \emptyset$ for all $\gamma \in \Lambda$, then $\cup_{\gamma \in \Lambda} S_\gamma = \emptyset \in \tau_\omega$. So, assume that some member is nonempty, but since the empty set does not affect any union, we may assume, without loss of generality, that $S_\gamma \neq \emptyset$ for all $\gamma \in \Lambda$. If there exist $\gamma_1 \in \Lambda$ such that $S_{\gamma_1} = \mathbb{S}_\omega$, then $\cup_{\gamma \in \Lambda} S_\gamma = \mathbb{S}_\omega \in \tau_\omega$. So, assume now that $S_\gamma \neq \mathbb{S}_\omega$ for all $\gamma \in \Lambda$. Then $\cup_{\gamma \in \Lambda} S_\gamma \in \tau_\omega$, because $\mathbb{S}_\omega^{(0)} \subseteq S_\gamma$ for all $\gamma \in \Lambda$. Hence $\mathbb{S}_\omega^{(0)} \subseteq \cup_{\gamma \in \Lambda} S_\gamma$. Also for any $+a, -a \in \cup_{\gamma \in \Lambda} S_\gamma$, where $a \neq \omega$, there exists $\gamma_1, \gamma_2 \in \Lambda$ such that $+a \in S_{\gamma_1}$ and $-a \in S_{\gamma_2}$. Hence the multiplicative inverse of $+a, -a$ belong to S_{γ_1} and S_{γ_2} respectively, then the multiplicative inverse of $+a, -a$ belong to $\cup_{\gamma \in \Lambda} S_\gamma$. Hence $\cup_{\gamma \in \Lambda} S_\gamma \in \tau_\omega$.

Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is topological space.

Proposition 14. *If* $\mathbb{S}_\omega = \left(\frac{\mathcal{P}_\omega}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{\omega}, \overline{e}\right)$ *be a symmetrized* $\omega-$*algebra over the abelian group* $G \times G$, *where* $\mathcal{P}_\omega = A_\omega \times A_\omega$, *and* $\otimes|A = \circ$. *Then an element a has a multiplicative inverse in* A_ω *if and only if the elements* $+a, -a$ *have a multiplicative inverses in* $\mathbb{S}_\omega$.

Proof. Let $a \in A_\omega$ be arbitrary, which has a multiplicative inverse, denoted by a^{-1}, then

$$\begin{aligned}(+a) \otimes \left(+a^{-1}\right) = \overline{(a, \omega)} \otimes \overline{(a^{-1}, \omega)} &= \overline{((a \otimes a^{-1}) \oplus (\omega \otimes \omega), (a \otimes \omega) \oplus (\omega \otimes a^{-1}))} \\ &= \overline{(a \otimes a^{-1}, \omega)} \\ &= \overline{(a \circ a^{-1}, \omega)} = \overline{(e, \omega)} = \overline{e},\end{aligned}$$

then $+a^{-1}$ is a multiplicative inverse of $+a$ in $\mathbb{S}_\omega$. Also,

$$\begin{aligned}(-a) \otimes \left(-a^{-1}\right) = \overline{(\omega, a)} \otimes \overline{(\omega, a^{-1})} &= \overline{((\omega \otimes \omega) \oplus (a \otimes a^{-1}), (\omega \otimes a^{-1}) \oplus (a \otimes \omega))} \\ &= \overline{(a \otimes a^{-1}, \omega)} \\ &= \overline{(a \circ a^{-1}, \omega)} = \overline{(e, \omega)} = \overline{e},\end{aligned}$$

then $-a^{-1}$ is a multiplicative inverse of $-a$ in $\mathbb{S}_\omega$.

Conversely, let $+a \in \mathbb{S}_\omega$ be arbitrary, which has a multiplicative inverse $\overline{(x, y)}$, where $x, y \in A_\omega$, then we have:

$$\begin{aligned}(+a) \otimes \overline{(x,y)} = \overline{(a,\omega)} \otimes \overline{(x,y)} &= \overline{((a \otimes x) \oplus (\omega \otimes y), (a \otimes y) \oplus (\omega \otimes x))} \\ &= \overline{(a \otimes x, a \otimes y)} \\ &= \overline{(a \circ x, a \circ y)} = \overline{(e,\omega)} = \bar{e},\end{aligned} \tag{17}$$

then $a \circ x = e$ and $a \circ y = \omega$. Hence, $x = a^{-1}$ is the multiplicative of a in A_ω.

Let $-a \in \mathbb{S}_\omega$ be arbitrary, which has a multiplicative inverse $\overline{(x, y)}$, where $x, y \in A_\omega$, then we have:

$$\begin{aligned}(-a) \otimes \overline{(x,y)} = \overline{(\omega,a)} \otimes \overline{(x,y)} &= \overline{((\omega \otimes x) \oplus (a \otimes y), (\omega \otimes y) \oplus (a \otimes x))} \\ &= \overline{(a \otimes y, a \otimes x)} \\ &= \overline{(a \circ y, a \circ x)} = \overline{(e,\omega)} = \bar{e},\end{aligned} \tag{18}$$

then $a \circ y = e$ and $a \circ x = \omega$. Hence, $y = a^{-1}$ is the multiplicative inverse of a in A_ω.

Proposition 15. *For any* $\cdot a \in \mathbb{S}_\omega^{(0)}$*, where* $\omega \neq e$*, then* $\cdot a$ *has no multiplicative inverse.*

Proof. Suppose that, $\cdot a \in \mathbb{S}_\omega^{(0)}$ has a multiplicative inverse $\overline{(x, y)}$, where $x, y \in A_\omega$, then

$$\begin{aligned}(\cdot a) \otimes \overline{(x,y)} = \overline{(a,a)} \otimes \overline{(x,y)} &= \overline{((a \otimes x) \oplus (a \otimes y), (a \otimes y) \oplus (a \otimes x))} \\ &= \overline{((a \circ x) \oplus (a \circ y), (a \circ y) \oplus (a \circ x))} = \overline{(e,\omega)}.\end{aligned} \tag{19}$$

Hence, $(a \circ x) \oplus (a \circ y) = e$ and $(a \circ y) \oplus (a \circ x) = \omega$, thus a contradiction.

Corollary 16. *If* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse, then* $\mathbb{S}_\omega$ *is the only open set in* $(\mathbb{S}_\omega, \tau_\omega)$ *containing* $+a$ *and* $-a$*..*

Remark 17.

1. We denote for any element $a \in \mathbb{S}_\omega$, by $sign(.)a$ or $sign(a)a$, where $sign(.), sign(a) \in \{+, -, \cdot\}$;

2. If $a = \omega$, then $\cdot a = +a = -a$;

3. If a^{-1} is the multiplicative inverse of a in A_ω, then $+a^{-1}$ and $-a^{-1}$ are the multiplicative inverses of $+a$ and $-a$, respectively in $\mathbb{S}_\omega$ (vice versa);

4. If a has no multiplicative inverse in A_ω, then $+a$ and $-a$ have no multiplicative inverses in $\mathbb{S}_\omega$ (vice versa).

Proposition 18. *A symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *has a base*

$$\begin{aligned}\mathfrak{B} = \Big\{&\mathbb{S}_\omega, \mathbb{S}_\omega^{(0)}, \mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\}, \mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\} : a \in A_\omega \backslash \{\omega\} \\ &\textit{has a multiplicative inverse}\}.\end{aligned} \tag{20}$$

Proof. For the first condition, let $B \in \mathfrak{B}$ be arbitrary. If $B = \mathbb{S}_\omega^{(0)}$ or $\mathbb{S}_\omega$ then $B \in \tau_\omega$ (satisfied by the definition of τ_ω). Assuming that,

$B = \mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\}$ *or* $\mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\}$ for any $a \in A_\omega \backslash \{\omega\}$, which has a multiplicative inverse in A_ω, then $B \in \tau_\omega$, because $\mathbb{S}_\omega^{(0)} \subset B$, and the elements $+a and - a$ in B its multiplicative inverse $+a^{-1} and - a^{-1}$ respectively, exists in B. Thus $\mathfrak{B} \subseteq \tau_\omega$. For the second condition, let $sign(a)a \in \mathbb{S}_\omega$ be arbitrary. Let U be any open neighborhood of $sign(a)a$ in $\mathbb{S}_\omega$. Then we have three cases:

Case 1: If $sign(a) = \cdot$, then there exists $B = \mathbb{S}_\omega^{(0)} \in \mathfrak{B}$, such that $\cdot a \in B \subseteq U$, because the smallest open neighborhood in $\mathbb{S}_\omega$ containing $\cdot a$ is $\mathbb{S}_\omega^{(0)}$.

Case 2: If $sign(a) = +,$ where $a \neq \omega$ (If $a = \omega$, then we have $+\omega = -\omega = \cdot\omega$, this is Case 1),

Subcase 2.1: If a has a multiplicative inverse in A_ω, then there exists $B = \mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\} \in \mathfrak{B}$, such that $+a \in B \subseteq U$, because the smallest open neighborhood in $\mathbb{S}_\omega$ containing $+a$ is $\mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\}$.

Subcase 2.2: If a has no multiplicative inverse in A_ω, then there exists $B = \mathbb{S}_\omega$, such that $+a \in B \subseteq U$, because the smallest open neighborhood in $\mathbb{S}_\omega$ containing $+a$ is $\mathbb{S}_\omega$.

Case 3: If $sign(a) = -,$ where $a \neq \omega$.

Subcase 3.1: If a has a multiplicative inverse in A_ω, then there exists $B = \mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\} \in \mathfrak{B}$, such that $-a \in B \subseteq U$, because the smallest open neighborhood in $\mathbb{S}_\omega$ containing $-a$ is $\mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\}$.

Subcase 3.2: If a has no multiplicative inverse in A_ω, then there exists $B = \mathbb{S}_\omega$, such that $-a \in B \subseteq U$, because the smallest open neighborhood in $\mathbb{S}_\omega$ containing $-a$ is $\mathbb{S}_\omega$.

Therefore, $\mathfrak{B}$ is a base for the symmetrized omega topological space $(\mathbb{S}_\omega, \tau_\omega)$.

Corollary 19. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *has a base,*

$$\mathfrak{B} = \left\{ \mathbb{S}_\omega^{(0)}, \mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\}, \mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\} : a \in A_\omega \backslash \{\omega\} \right\}. \tag{21}$$

Corollary 20. *Let* $\varnothing \neq U \subseteq A_\omega$*, then* $U \in \tau_\omega$ *if and only if for each* $sign(a)a \in U$*, there exists basic open set* $B \in \mathfrak{B}$*, such that* $sign(a)a \in B \subseteq U$.

Proposition 21. *If* A_ω *has a finite number of elements, which have a multiplicative inverses, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is second countable.*

Proof. Suppose that $a_1, a_2, \cdots, a_m$, where $m \in \mathbb{Z}^+$ are the finite number of elements in A_ω, which have a multiplicative inverses. Then.

$\mathfrak{B} = \left\{ \mathbb{S}_\omega, \mathbb{S}_\omega^{(0)}, \mathbb{S}_\omega^{(0)} \cup \{+a_1, +a_1^{-1}\}, \mathbb{S}_\omega^{(0)} \cup \{-a_1, -a_1^{-1}\}, \cdots, \mathbb{S}_\omega^{(0)} \cup \{+a_m, +a_m^{-1}\}, \mathbb{S}_\omega^{(0)} \cup \{-a_m, -a_m^{-1}\} \right\}$ is a countable base for $(\mathbb{S}_\omega, \tau_\omega)$.

Proposition 22. *The symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is first countable.*

Proof. Let $sign(a)a \in \mathbb{S}_\omega$ be arbitrary. Then we have three cases:

Case 1: If $sign(a) = \cdot$, then $\mathfrak{B}(\cdot a) = \left\{ \mathbb{S}_\omega^{(0)} \right\}$ is a countable local base at $\cdot a$.

Case 2: If $sign(a) = +,$ where $a \neq \omega$ (If $a = \omega$, then $+\omega = -\omega = \cdot\omega$, this is Case 1),

Subcase 2.1: If a has a multiplicative inverse in A_ω, then $\mathfrak{B}(+a) = \left\{ \mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\} \right\}$ is a countable local base at $+a$.

Subcase 2.2: If a has no multiplicative inverse in A_ω, then $\mathfrak{B}(+a) = \{\mathbb{S}_\omega\}$ is a countable local base at $+a$.

Case 3: If $sign(a) = -,$ where $a \neq \omega$,

Subcase 3.1: If a has a multiplicative inverse in A_ω, then $\mathfrak{B}(-a) = \left\{ \mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\} \right\}$ is a countable local base at $-a$.

Subcase 3.2: If a has no multiplicative inverse in A_ω, then $\mathfrak{B}(-a) = \{\mathbb{S}_\omega\}$ is a countable local base at $-a$. Hence, for any $sign(a)a \in \mathbb{S}_\omega$, there exists a countable local base at $sign(a)a$.

Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is first countable.

Proposition 23. *The symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is separable.*

Proof. There exists $\{\cdot\omega\} = \left\{\overline{(\omega, \omega)}\right\} \subseteq \mathbb{S}_\omega$, such that for any $U \in \tau_\omega$, we have $U \cap \{\cdot\omega\} \neq \emptyset$, because any open set in $(\mathbb{S}_\omega, \tau_\omega)$ must be containing $\mathbb{S}_\omega^{(0)}$, and $\cdot\omega \in \mathbb{S}_\omega^{(0)}$. Then $\{\cdot\omega\}$ is countable dense subset of $\mathbb{S}_\omega$. Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is separable.

Let us recall this definition.

Definition 24. A topological space X is said to be *hyperconnected space* if every non-empty open set of X is dense in X or there exists no disjoint non-empty open sets in X.

Proposition 25. *The symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is hyperconnected.*

Proof. If $\mathbb{S}_\omega$ is singleton, then it is hyperconnected. Suppose that $\mathbb{S}_\omega$, which has more than one element. Since any nonempty open set in $\mathbb{S}_\omega$ is containing $\mathbb{S}_\omega^{(0)}$, then $\mathbb{S}_\omega$ has no disjoint nonempty open sets. Hence, $(\mathbb{S}_\omega, \tau_\omega)$ is hyperconnected.

Since any hyperconnected space is connected and locally connected, then we conclude the following corollaries.

Corollary 26. *The symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is connected.*

Corollary 27. *The symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is locally connected.*

Proposition 28. *Let* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group, has more than one element. Then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is not* T_0.

Proof. If $A_\omega = \{\omega = e\}$, then $\mathbb{S}_\omega = \{\cdot\omega\}$ is singleton, we are done (because some of omega algebra, has $\omega = e$). Suppose that A_ω has more than one element. Let $a \neq \omega$, then there exist $\cdot a \neq \cdot\omega$ in $\mathbb{S}_\omega$. Let U be any open set in $\mathbb{S}_\omega$, containing either $\cdot a$ or $\cdot\omega$, by the definition of τ_ω we have $\mathbb{S}_\omega^{(0)} \subseteq U$, but $\cdot\omega$, $\cdot\, a \in \mathbb{S}_\omega^{(0)}$. Then there is no open set containing only $\cdot\omega$ or $\cdot a$. Hence, $(\mathbb{S}_\omega, \tau_\omega)$ is not T_0.

Proposition 29. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group, has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is not regular.*

Proof. There exists $K = \mathbb{S}_\omega \backslash \mathbb{S}_\omega^{(0)}$ is a closed subset of $\mathbb{S}_\omega$ and there exists $a \neq \omega$, such that $\cdot a \notin K$. We cannot separate $\cdot a$, and K by any open sets (because any open sets in $\mathbb{S}_\omega$ is containing $\mathbb{S}_\omega^{(0)}$, where $\cdot a \in \mathbb{S}_\omega^{(0)}$). Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is not regular.

Proposition 30. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group, has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is not normal.*

Proof. If $A_\omega = \{\omega\}$, then $\mathbb{S}_\omega = \{\cdot\omega\}$ is singleton, we are done (because some of omega algebra, we have $\omega = e$). Suppose that A_ω has more than one element. Let $a \in A_\omega \backslash \{\omega\}$. Then we have two cases:

Case 1: If $a = e$, then we have $K = \{+e\}$, and $H = \{-e\}$ are two disjoint closed subsets of $\mathbb{S}_\omega$, such that we cannot separate them by any open sets (because any nonempty open sets in $\mathbb{S}_\omega$ is containing $\mathbb{S}_\omega^{(0)}$).

Case 2: If $a \neq e$, then we have $K = \{+a, +a^{-1}\}$, and $H = \{-a, -a^{-1}\}$ are two disjoint closed subsets of $\mathbb{S}_\omega$, such that we cannot separate them by any open sets (because any nonempty open sets in $\mathbb{S}_\omega$ is containing $\mathbb{S}_\omega^{(0)}$). Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is not normal.

Proposition 31. *If* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is normal.*

Proof. Suppose that, V be any non-empty closed subset of $\mathbb{S}_\omega$. Then $+a \in V$. Suppose not, $+a \notin V$, then $+a \in \mathbb{S}_\omega \backslash V$. By the definition of τ_ω, $\mathbb{S}_\omega \backslash V$ is not open, thus a contradiction. Hence, $+a$ belong to any non-empty closed subsets of $\mathbb{S}_\omega$.

Let K and H be any two disjoint closed subsets of $\mathbb{S}_\omega$. Then K or H is equal $\varnothing$. If $K = \varnothing$, then there exists $U = \varnothing$ and $V = \mathbb{S}_\omega$ are two disjoint open sets in $\mathbb{S}_\omega$ containing K and H, respectively. If $H = \varnothing$, then there exists $U = \varnothing$ and $V = \mathbb{S}_\omega$ are two disjoint open sets in $\mathbb{S}_\omega$ containing H and K, respectively. Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is normal.

Proposition 32. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group and* A *is uncountable infinite set, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is not compact (Lindelöf).*

Proof. There exists $\left\{\mathbb{S}_\omega^{(0)}, \mathbb{S}_\omega^{(0)} \cup \{+a, +a^{-1}\}, \mathbb{S}_\omega^{(0)} \cup \{-a, -a^{-1}\} : a \in A_\omega \backslash \{\omega\}\right\}$, which is an open cover of $\mathbb{S}_\omega$, and has no finite (countable) subcover of $\mathbb{S}_\omega$.

Proposition 33. *Let* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse. Then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is compact.*

Proof. Let $\{C_\alpha : \alpha \in \Lambda\}$ be any open cover of $\mathbb{S}_\omega$. Since $+a \in \mathbb{S}_\omega$, then for some $\beta \in \Lambda$, there exists C_β containing $+a$. But $C_\beta = \mathbb{S}_\omega$, because $\mathbb{S}_\omega$ is the only open set containing $+a$. Hence, $\{C_\beta\}$ is a finite subcover of $\{C_\alpha : \alpha \in \Lambda\}$, which cover $\mathbb{S}_\omega$. Therefore $(\mathbb{S}_\omega, \tau_\omega)$ is a compact space.

Since any compact space is Lindelöf and countably compact, then we conclude the following corollaries.

Corollary 34. *If* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is Lindelöf.*

Corollary 35. *If* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is countably compact.*

Remark 36. Since every nonempty open sets in $(\mathbb{S}_\omega, \tau_\omega)$ contains $\mathbb{S}_\omega^{(0)}$. Then the closure of any nonempty open sets is equal $\mathbb{S}_\omega$.

4. Some of the fundamental properties for different examples on symmetrized omega topology

In this section, we give two different examples of symmetrized omega topologies. The examples are from an ordered infinite set.

Example 37. By Example 4, we set $W = \{0, 1, 2, 3, \cdots\}$. Then $(\mathbb{S}_{-\infty}, \tau_{-\infty})$, which is topological space, where $\mathbb{S}_{-\infty} = \left(\frac{\mathcal{P}_{-\infty}}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{-\infty}, \overline{0}\right)$ be a symmetrized $-\infty-$algebra over the abelian group $\mathbb{Z} \times \mathbb{Z}$ and $\mathcal{P}_{-\infty} = W_{-\infty} \times W_{-\infty}$. Let $a \in W \backslash \{0\}$ be arbitrary. Then $+a^{-1}$ and $-a^{-1}$ are not exists in $\mathbb{S}_{-\infty}$, where $+a^{-1}$ and $-a^{-1}$ are the multiplicative inverses of $+a$ and $-a$ in $\mathbb{S}_{-\infty}$ respectively (because a in $W_{-\infty}$ has no multiplicative inverse). If $a = 0$, then $+0^{-1} = +0$ and $-0^{-1} = -0$ (because the multiplicative inverse of 0 in $W_{-\infty}$ is 0, that is $0^{-1} = 0$). Hence,

$$\tau_{-\infty} = \left\{\mathbb{S}_{-\infty}, \varnothing, \mathbb{S}_{-\infty}^{(0)}, \mathbb{S}_{-\infty}^{(0)} \cup \{+0\}, \mathbb{S}_{-\infty}^{(0)} \cup \{-0\}, \mathbb{S}_{-\infty}^{(0)} \cup \{+0, -0\}\right\}. \tag{22}$$

A direct check shows that $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is a topological space.

Proposition 38. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is Second countable.*

Proof. There exists only one element $0 \in W_{-\infty}$, which has a multiplicative inverse, then by Proposition 21, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is second countable.

Since any second countable space is first countable and separable, then we conclude the following corollaries.

Corollary 39. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is first countable.*

Corollary 40. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is separable.*

Proposition 41. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is not* T_0.

Proof. There exists $+2 \neq +3$ in $\mathbb{S}_{-\infty}$. Let U be any open set, which either containing $+2$ or $+3$. However, there exists only one open set $U = \mathbb{S}_{-\infty}$ containing $+2, +3$. Hence, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is not T_0.

Proposition 42. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is not regular.*

Proof. There exists a closed set $K = \mathbb{S}_{-\infty} \setminus \left(\mathbb{S}_{-\infty}^{(0)} \cup \{+0\}\right)$ and $+0 \notin K$, such that $+0$ and K cannot separate by any two disjoint open sets. Hence, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is not regular.

Proposition 43. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is normal.*

Proof. There exists an element $2 \in W_{-\infty} \setminus \{-\infty\}$, which has no multiplicative inverse, then by Proposition 31, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is a normal space.

Proposition 44. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is hyperconnected.*

Proof. Since any nonempty open set in $\mathbb{S}_{-\infty}$ is containing $\mathbb{S}_{-\infty}^{(0)}$, then $\mathbb{S}_{-\infty}$ has no disjoint nonempty open sets. Hence, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is hyperconnected.

Since any hyperconnected space is connected and locally connected, then we conclude the following corollaries.

Corollary 45. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is connected.*

Corollary 46. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is locally connected.*

Proposition 47. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is compact.*

Proof. There exists an element $2 \in W_{-\infty} \setminus \{-\infty\}$, which has no multiplicative inverse. Hence by Proposition 33, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is compact.

Since any compact space is Lindelöf and countably compact, then we conclude the following corollaries.

Corollary 48. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is countably compact.*

Corollary 49. *The symmetrized omega topological space* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *is Lindelöf.*

Example 50. In the ring $(\mathbb{R}, +, \cdot)$, we have $(\mathbb{R}, +)$ is an additive submonoid of an abelian group $(\mathbb{R}, +)$. Let $\omega = -\infty$, $a \oplus b = \max(a, b)$ and $a \otimes b = a + b, \forall a, b \in \mathbb{R}$. Then $\mathbb{R}_{-\infty} = (\mathbb{R}_{-\infty}, \oplus, \otimes, -\infty, 0)$ is $-\infty-$ algebra over the ring $(\mathbb{R}, +, \cdot)$. We have $\mathbb{S}_{-\infty} = \left(\frac{\mathcal{P}_{-\infty}}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{-\infty}, \overline{0}\right)$ be a symmetrized $-\infty-$algebra over the abelian group $\mathbb{R} \times \mathbb{R}$ and $\mathcal{P}_{-\infty} = \mathbb{R}_{-\infty} \times \mathbb{R}_{-\infty}$. Then, using the same proof as that Proposition 13. Therefore, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is a topological space.

Remark 51. The symmetrized omega topological space $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is first countable, separable, hyperconnected, connected and locally connected and does not satisfy any of these T_0, regular, normal, Lindelöf and compact.

Example 52. In the ring $(\mathbb{R}, +, \cdot)$, we have $(\mathbb{R}, +)$ is an additive submonoid of an abelian group $(\mathbb{R}, +)$. Let $\omega = +\infty$, $a \oplus b = \min(a, b)$ and $a \otimes b = a + b, \forall a, b \in \mathbb{R}$. Then, $\mathbb{R}_{+\infty} = (\mathbb{R}_{+\infty}, \oplus, \otimes, +\infty, 0)$ is $+\infty-$ algebra over the ring $(\mathbb{R}, +, \cdot)$. We have $\mathbb{S}_{+\infty} = \left(\frac{\mathcal{P}_{+\infty}}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{+\infty}, \overline{0}\right)$ be a symmetrized $+\infty-$algebra over the abelian group $\mathbb{R} \times \mathbb{R}$ and $\mathcal{P}_{+\infty} = \mathbb{R}_{+\infty} \times \mathbb{R}_{+\infty}$. Then, using the same proof as that Proposition 13. Therefore, $(\mathbb{S}_{+\infty}, \tau_{+\infty})$ is a topological space.

Proposition 53. *The symmetrized omega topological spaces* $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ *and* $(\mathbb{S}_{+\infty}, \tau_{+\infty})$ *are homeomorphic.*

Proof. There exists a map $h : (\mathbb{S}_{-\infty}, \tau_{-\infty}) \to (\mathbb{S}_{+\infty}, \tau_{+\infty})$ is defined by:

$$h(sign(a)a) = \begin{cases} sign(a)a & if \quad a \in \mathbb{R} \\ sign(-\infty)(+\infty) & if \quad sign(a)a = sign(-\infty)(-\infty) \end{cases}; \tag{23}$$

Let $sign(a)a, sign(b)b \in \mathbb{S}_{-\infty}$ be arbitrary. Let $h(sign(a)a) = h(sign(b)b)$, then $sign(a)a = sign(b)b$. Hence, h is an injective. Let $sign(a)a \in \mathbb{S}_{+\infty}$ is arbitrary, then there exists a $sign(a)a \in \mathbb{S}_{-\infty}$, such that $h(sign(a)a) = sign(a)a$. Hence, h is surjective.

Let $B \in \tau_{+\infty}$ be any basic open set. By Proposition 18, we have
$\mathfrak{B} = \left\{ \mathbb{S}_{-\infty}^{(0)}, \mathbb{S}_{-\infty}^{(0)} \cup \{+a, +a^{-1}\}, \mathbb{S}_{-\infty}^{(0)} \cup \{-a, -a^{-1}\} : a \in \mathbb{R} \right\}$ and
$\mathfrak{B} = \left\{ \mathbb{S}_{+\infty}^{(0)}, \mathbb{S}_{+\infty}^{(0)} \cup \{+a, +a^{-1}\}, \mathbb{S}_{+\infty}^{(0)} \cup \{-a, -a^{-1}\} : a \in \mathbb{R} \right\}$ are a base for $\mathbb{R}_{-\infty}$ and $\mathbb{R}_{+\infty}$, respectively.

To prove that h is continuous, we have three cases:

Case 1: If $B = \mathbb{S}_{+\infty}^{(0)}$, then $h^{-1}(B) = h^{-1}\left(\mathbb{S}_{+\infty}^{(0)}\right) = \mathbb{S}_{-\infty}^{(0)} \in \tau_{-\infty}$.

Case 2: If $B = \mathbb{S}_{+\infty}^{(0)} \cup \{+a, +a^{-1}\}$, then $h^{-1}(B) = h^{-1}\left(\mathbb{S}_{+\infty}^{(0)} \cup \{+a, +a^{-1}\}\right) = \mathbb{S}_{-\infty}^{(0)} \cup \{+a, +a^{-1}\} \in \tau_{-\infty}$.

Case 3: If $B = \mathbb{S}_{+\infty}^{(0)} \cup \{-a, -a^{-1}\}$, then $h^{-1}(B) = h^{-1}\left(\mathbb{S}_{+\infty}^{(0)} \cup \{-a, -a^{-1}\}\right) = \mathbb{S}_{-\infty}^{(0)} \cup \{-a, -a^{-1}\} \in \tau_{-\infty}$. Hence, h is continuous.

To prove that h^{-1} is continuous, we have three cases: (since h is one to one and onto, then $\left(h^{-1}\right)^{-1}(B) = h(B)$).

Case 1: If $B = \mathbb{S}_{-\infty}^{(0)}$, then $\left(h^{-1}\right)^{-1}(B) = h(B) = h\left(\mathbb{S}_{-\infty}^{(0)}\right) = \mathbb{S}_{+\infty}^{(0)} \in \tau_{+\infty}$.

Case 2: If $B = \mathbb{S}_{-\infty}^{(0)} \cup \{+a, +a^{-1}\}$, then $\left(h^{-1}\right)^{-1}(B) = h(B) = h\left(\mathbb{S}_{-\infty}^{(0)} \cup \{+a, +a^{-1}\}\right) = \mathbb{S}_{+\infty}^{(0)} \cup \{+a, +a^{-1}\} \in \tau_{+\infty}$.

Case 3: If $B = \mathbb{S}_{-\infty}^{(0)} \cup \{-a, -a^{-1}\}$, then $\left(h^{-1}\right)^{-1}(B) = h(B) = h\left(\mathbb{S}_{-\infty}^{(0)} \cup \{-a, -a^{-1}\}\right) = \mathbb{S}_{+\infty}^{(0)} \cup \{-a, -a^{-1}\} \in \tau_{+\infty}$. Hence h^{-1} is continuous (which means h is open).

Therefore, h is homeomorphism, then $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ and $(\mathbb{S}_{+\infty}, \tau_{+\infty})$ are homeomorphic.

5. Symmetrized omega topology and other properties

Recall that a subset A of a space X is said to be *regularly-open* or an *open domain* if it is the interior of its own closure (see [7]). A set A is said to be a *regularly-closed* or a *closed domain* if its complement is an open domain. A subset A of a space X is called a *π-closed* if it is a finite intersection of closed domain sets (see [8]). A subset A is called a *π-open* if its complement is a π-closed. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X such that $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}'$ is called the coarser topology than $\mathcal{T}$, and $\mathcal{T}$ is called the finer. A space X is *π-normal* [9] if any pair of disjoint closed subsets A and B of X, one of which is π-closed, can be separated by two disjoint open subsets. A space X is *almost-normal* [9] if any pair of disjoint closed subsets A and B of X, one of which is a closed domain, can be separated by two disjoint open subsets. A space X is *mildly normal* [10] if any pair of disjoint closed domain subsets A and B of X can be separated by two disjoint open subsets. A space $(X, \mathcal{T})$ is *epi-mildly normal* [11] if there exists a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_2 and mildly normal space. A space $(X, \mathcal{T})$ is *epi-almost normal* [12] if there exists a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_2 and almost normal space.

Theorem 54. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is π-normal.*

Proof. Since the only π-closed sets are the ground set S_ω and the empty set, then (S_ω, τ_ω) is a π-normal.

It is clear from the definitions that

$$normal \Rightarrow \pi - normal \Rightarrow almost\ normal \Rightarrow mildly\ normal. \tag{24}$$

By (24) and Theorem 54, we conclude the following Corollaries.

Corollary 55. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is almost normal.*

Corollary 56. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is mildly normal.*

If $(A_\omega \backslash \{\omega\}, \otimes)$ be a group has more than one element, then (S_ω, τ_ω) is not T_0(see Proposition 28), we have the following Propositions:

Proposition 57. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is not Epi-mildly Normal.*

Proof. Suppose that, $(\mathbb{S}_\omega, \tau_\omega)$ is Epi-mildly Normal. Then there exists a coarser topology $\mathcal{T}'$ on $\mathbb{S}_\omega$ such that $(\mathbb{S}_\omega, \mathcal{T}')$ is T_2 and mildly normal space. Hence $(\mathbb{S}_\omega, \tau_\omega)$ is T_2, thus a contradiction. Then $(\mathbb{S}_\omega, \tau_\omega)$ is not Epi-mildly Normal.

Proposition 58. *If* $(A_\omega \backslash \{\omega\}, \otimes)$ *be a group has more than one element, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is not Epi-almost Normal.*

Proof. Using the same proof of Proposition 57.

Definition 59. Let X be a space. Then:

1. A space X is called a *C-normal* if there exist a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [13].

2. A space X is called a CC-normal if there exists a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. [14].

3. A space X is called an L-normal if there exist a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each lindelöf subspace $A \subseteq X$, [15].

4. A space X is called an S- normal if there exist a normal space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each separable subspace $A \subseteq X$, [16].

5. A space X is called a C-paracompact if there exist a paracompact space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [17].

6. A space X is called a C_2-paracompact if there exist a Hausdorff paracompact space Y and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [17].

Proposition 60. *If* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is C-normal.*

Proof. By Proposition 31, $\mathbb{S}_\omega$ is a normal space. Then there exist $Y = \mathbb{S}_\omega$ is a normal space and the identity function $id : \mathbb{S}_\omega \to \mathbb{S}_\omega$ is bijective. Let C be any compact subset of $(\mathbb{S}_\omega, \tau_\omega)$. Then the restriction function $id\restriction_C : C \to f(C)$ is a homeomorphism. Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is a $C-$normal.

Since any normal space is CC-normal, L-normal and S-normal, just by taking $X = Y$ and f to be the identity function. Hence, we conclude the following Propositions.

Proposition 61. *If* $a \in A_\omega \backslash \{\omega\}$ *has no multiplicative inverse, then the symmetrized omega topological space* $(\mathbb{S}_\omega, \tau_\omega)$ *is CC-normal.*

Proof. Using the same proof of Proposition 60.

Proposition 62. *If $a \in A_\omega \backslash \{\omega\}$ has no multiplicative inverse, then the symmetrized omega topological space $(\mathbb{S}_\omega, \tau_\omega)$ is L-normal.*

Proof. Using the same proof of Proposition 60.

Proposition 63. *If $a \in A_\omega \backslash \{\omega\}$ has no multiplicative inverse, then the symmetrized omega topological space $(\mathbb{S}_\omega, \tau_\omega)$ is S-normal.*

Proof. Using the same proof of Proposition 60.

Example 64. By Example 37, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is C-normal, CC-normal, L-normal and S-normal.

Theorem 65. *If $(A_\omega \backslash \{\omega\}, \otimes)$ be a group has more than one element, then the symmetrized omega topological space $(\mathbb{S}_\omega, \tau_\omega)$ is not S-normal.*

Proof. From the proposition any separable S-normal must be normal (see [16]) and since $(\mathbb{S}_\omega, \tau_\omega)$ is separable and not normal (see Propositions 30, 23, respectively), then $(\mathbb{S}_\omega, \tau_\omega)$ is not S-normal.

Example 66. By Example 50, $(\mathbb{S}_{-\infty}, \tau_{-\infty})$ is not a S-normal.

Theorem 67. *The symmetrized omega topological space $(\mathbb{S}_\omega, \tau_\omega)$ is not C_2-paracompact.*

Proof. Since any C_2-paracompact Fre′chet space is Hausdorff (see [17]) and $(\mathbb{S}_\omega, \tau_\omega)$ is First countable and not a Hausdorff space, $(\mathbb{S}_\omega, \tau_\omega)$ cannot be C_2-paracompact.

Theorem 68. *Let $a \in A_\omega \backslash \{\omega\}$ has no multiplicative inverse. Then the symmetrized omega topological space $(\mathbb{S}_\omega, \tau_\omega)$ is not C-paracompact.*

Proof. Assume that $(\mathbb{S}_\omega, \tau_\omega)$ is C-paracompact. Let Y be a paracompact space and $f : \mathbb{S}_\omega \to Y$ be bijective such that the restriction $f\restriction_C : C \to f(C)$ is a homeomorphism for all compact subspace C of $(\mathbb{S}_\omega, \tau_\omega)$. Hence, $\mathbb{S}_\omega \equiv Y$, since $\mathbb{S}_\omega$ is compact (see Proposition 33). However, $\mathbb{S}_\omega$ is paracompact, thus a contradiction. Because any paracompact space is Hausdorff space and $\mathbb{S}_\omega$ is not a Hausdorff space. Therefore, $(\mathbb{S}_\omega, \tau_\omega)$ is not a C-paracompact.

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