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Using Transition Invariants for Reachability Analysis of Petri Nets

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1. Introduction

Petri nets are an important formal paradigm for modeling and analysis of discrete event systems. The related areas of application of Petri nets include deadlock avoidance and prevention, supervisory control, forbidden state detection, different aspects of flexible manufacturing systems, and many others (Zhou & DiCesare, 1993; Holloway et al., 1997; Boel et al., 1995). Quite often, given a discrete-event system, the designer is interested in determining whether the system can transit from an initial state to another, target state as a result of some operations from a predefined set. In terms of Petri nets, the answer to this question is obtained as a solution of a *reachability problem*.
The reachability problem in Petri nets is formulated as follows: for any Petri net PN, with an initial marking M^0, and for some other marking M, determine whether the relation $M \in R(PN, M^0)$ is true, where $R(PN, M^0)$ is the reachability set of PN for its initial marking M^0 (Murata, 1989). The decidability of the reachability problem has been proved for a number of restricted classes of Petri nets, and there are efficient algorithms for such classes as acyclic Petri nets, marked graphs, and others (Kodama & Murata, 1988; Caprotti et al., 1995; Kostin, 1997).
It has been shown that the reachability problem is decidable for generalized Petri nets as well (Mayr, 1984). The fundamental contribution of the paper (Mayr, 1984) is in proving that the reachability problem for generalized Petri nets is decidable. However, being highly important theoretically, the practical use of the algorithm described in that paper is limited. Actually, the algorithm creates a series of so called regular constrained refined graphs, each of which is a generalization of the basic coverability tree. As the author admits, the first refined graph would enumerate the whole reachability set of the given Petri net.
In practice, two different approaches are used most often to determine the reachability of a marking in Petri nets. The first approach is based on the creation and investigation of a complete or reduced reachability graph. The main drawback of this approach is a *state explosion problem*. A closely related technique is the use of *stubborn sets*. The main purpose of the stubborn sets technique is to choose, for each marking of the net, a set of transitions to fire that is large enough to preserve some desired information about the Petri net, but is as small as possible to get a significant reduction of the resulting reachability graph (Varpaaniemi, 1998). Unfortunately, generation of minimal or reduced reachability graphs in finite state systems is known to be an NP-hard problem (Peled, 1993). If Petri net has no

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specific properties like a symmetry or reversibility, the corresponding reduced reachbility graph will have almost the same size as that of the full reachability graph (Schmidt, 2000).
The second approach is based on methods of linear algebra. Given a *pure* Petri net (i.e. a net without self-loops), with sets of transitions T and places P, its structure is represented unambiguously by the *incidence matrix*

$$D = [d(t_i, p_j)] = [d_{ij}], \quad i = 1, 2, \dots, m = |T|, \quad j = 1, 2, \dots, n = |P|, \tag{1}$$

where $d(t_i, p_j) = Post(p_j, t_i) - Pre(p_j, t_i)$, Pre and $Post$ are the input and output functions of the Petri net, with $Pre(p, t) = v$ if there is a directed arc from p to t with the weight v, and $Post(p, t) = v$ if there is an arc from t to p with the weight v. Note that, in this matrix, rows correspond to transitions and columns correspond to places (Murata, 1989).
It is known that a necessary condition for reachability of marking M from some other marking M^0 is the existence of a nonnegative integer solution of the matrix equation

$$M = M^0 + FD \tag{2}$$

relative to F, where $F = [f_1, f_2, \dots, f_m]$ is a nonnegative integer firing count vector. Note that in this chapter, all vectors are considered as row vectors. In particular, markings of PN will be expressed as $(1 \times n)$ vectors, so that we can write

$$M^0 = [m_1^0, m_2^0, \dots, m_n^0] \quad \text{and} \quad M = [m_1, m_2, \dots, m_n], \tag{3}$$

where the ith entry in vectors M^0 and M denotes the number of tokens in place $p_i \in P$.
Equation (2), proposed in (Murata, 1977), is called the *fundamental equation* of Petri net. It is of the paramount importance for the investigation of the structural and behavioral properties of Petri nets with methods of linear algebra.
With the use of linear algebra, rachability analysis is usually carried out in two stages. At the first stage, by solving the equation (2) or its related integer programming form, firing count vectors are obtained. At the second stage, the computed firing count vectors are used in an attempt to determine *legal firing sequences* that transform initial marking M^0 into target marking M.
Unfortunately, the existence of a nonnegative integer solution of equation (2) is not a sufficient condition for reachability of marking M from M^0 (Murata, 1989). That is, it is quite possible that, in a given Petri net, no legal firing sequences exist for the valid firing count vectors. In general, the equation (2) can have *infinite* number of nonnegative integer solutions. Some of these solutions can correspond to legal firing sequences, while others fail (Peterson, 1981). Thus, there is a challenging problem to select working firing count vectors.
In (Kostin, 2003), with the use of linear algebra, a method was proposed to restrict the number of firing count vectors to be tried for the determination of legal firing sequences, without the loss of reachability information. The method is applicable for reachability analysis of a particular class of place/transition Petri nets having no transition invariants, or T-invariants. Algebraically, T-invariants of a Petri net with incidence matrix D are nonnegative integer $(1 \times m)$ vectors F such that $FD = 0$ (Memmi & Roucairol, 1980).
According to the scheme proposed in (Kostin, 2003), given a Petri net with an initial and a target markings, a so called *complemented* Petri net is created that consists of the given Petri net and an additional, *complementary* transition with some input and output places of the original Petri net, which are uniquely determined by the initial and target markings. Then

the reachability problem is reduced to computation and investigation of T-invariants of the complemented Petri net. The main result of that paper is that legal firing sequences, if they exist, can be found using only those T-invariants of the complemented Petri net in which the complementary transition fires only once. It was shown that this set is finite. This chapter generalizes the approach described in (Kostin, 2003) for arbitrary place/transition nets, including Petri nets with T-invariants. The existence of T-invariants in the original Petri nets considerably complicates the reachability analysis. In contrast with the scheme in (Kostin, 2003), where the number of T-invariants of any complemented Petri net that are sufficient for performing the reachability analysis is proven to be finite, in the generalized scheme the set of T-invariants for investigation is theoretically infinite. Nevertheless, as will be shown in this chapter, it is always possible to effectively limit this set without the loss of reachability information and then to use T-invariants from this finite set for reachability analysis.

This chapter is an extended version of the author's article published in Lecture Notes in Computer Science (Kostin, 2006). The use of the material of that article is done with kind permission of Springer Science and Business Media. The rest of the chapter is organized as follows. In Section 2, notation and basic statements used in the chapter are given. Section 3 explains how to compute so called minimal singular T-invariants of the complemented Petri net. In Section 4, a relation graph of T-invariants is introduced. Section 5 describes realization of T-invariants with borrowing of tokens. In Section 6, a scheme for linear combining of T-invarians is given. Section 7 illustrates the scheme by two examples. The most important points in sections are put down as proven statements. Some of the proofs are just skeletons or, for simple statements, omitted altogether.

2. Notation and basic statements

We adopt here the notation and basic statements from (Kostin, 2003). It is assumed without losing generality that Petri nets are *pure*, i.e. they have no self-loops. As was stated in the previous section, the structure of any pure Petri net is unambiguously represented by the incidence matrix (1).

Let M^0 be an initial marking and M be some other marking of given Petri net *PN*. If we are interested in reachability of M from M^0 then marking M will be called the *target* marking. It is assumed, throughout the chapter, that $M^0 \neq M$.

If marking M is reachable from marking M^0 in a Petri net *PN*, then there exists at least one sequence of markings $\mu = M^0 M^1 \ldots M^r$ with $M^r = M$, and a legal firing sequence $\tau = t_{i_1} t_{i_2} \ldots t_{i_r}$, with the two sequences related by the state equation $M^k = M^{k-1} + e[i_k]_m D,$ $k = 1, 2, \ldots, r$. Here $e[i_k]_m$ is an $(1 \times m)$ *control vector*, in which $m - 1$ entries are zero and the i_kth entry is one, indicating that a transition t_{i_k} fires at step k. Sequences μ and τ can be combined in one mixed sequence of interrelated markings and firing transitions that is called a *reachability path* from marking M^0 to marking M^r:

$$M^0 \xrightarrow{t_{i_1}} M^2 \xrightarrow{t_{i_2}} \ldots \xrightarrow{t_{i_r}} M^r . \tag{4}$$

Its determination is the main problem of reachability analysis. As was stated in Section 1, with linear algebra methods, this analysis is usually carried out in two stages. At the first stage, it is important to limit the number of firing count vectors, without the loss of reachability information. In the proposed approach, this stage is done with the use of T-invariants of so called complemented Petri net which is a simple extension of the original net .

Definition 1. For any Petri net *PN* with incidence matrix *D* specified by (1), and initial and target markings M^0 and M represented by vectors (3), there exists a unique *complemented* Petri net PN_c that has the same set of places *P* as *PN*, the set of transitions $T_c = T \cup \{t_{m+1}\}$, and is described structurally by the incidence matrix

$$D_c = \begin{bmatrix} D \\ \Delta M \end{bmatrix}, \tag{5}$$

where t_{m+1} is an additional, *complementary* transition, and $\Delta M = M^0 - M = [\Delta m_1, \Delta m_2, \ldots, \Delta m_n]$, with $\Delta m_i = m_i^0 - m_i$, $i = 1, 2, \ldots, n$ (Kostin, 2003). ♦

Using the right side of equation (2) with marking M instead of M^0, control vector $e[m + 1]_{m+1}$ instead of F and incidence matrix D_c instead of D, one can obtain

$$M + e[m + 1]_{m+1} D_c = M + \Delta M = M^0 . \tag{6}$$

That is, a *single* firing of the complementary transition in marking M of PN_c results in marking M^0.

It is known that the reproducibility of a firing sequence in a Petri net indicates the existence of a T-invariant (Memmi & Roucairol, 1980). Thus the following statement holds.

Statement 1. Given a Petri net *PN* with an initial marking M^0, a necessary condition for reachability of some other marking M is the existence of a T-invariant of the complemented Petri net PN_c, with a *single* firing of the complementary transition.

Denote by $F_c = [f_1, f_2, \ldots, f_m, f_{m+1}]$ a firing count vector of the complemented Petri net PN_c. Now Statement 1 may be reformulated as follows: given a Petri net *PN* with the incidence matrix *D* and an initial marking M^0, a necessary (but generally not sufficient) condition for some other marking M to be reachable from M^0 is the existence of an integer solution of the matrix equation

$$F_c D_c = 0 \tag{7}$$

relative to F_c, such that $F_c \geq 0$ and $f_{m+1} = 1$. Here D_c is the incidence matrix of PN_c as defined by (5). ♦

In sequel, each T-invariant of the complemented Petri net PN_c having the last entry $f_{m+1} = 1$ will be called a *singular complementary* T-invariant.

The importance of Statement 1 is that the reachability analysis of the *original* Petri net *PN* can be reduced to the computation and investigation of T-invariants of the complemented Petri net PN_c. One advantage of this reduction is the existence of efficient techniques for the calculation of T-invariants (Alaiwan, 1985; Krukeberg & Jaxy, 1987; Silva & Colom, 1991; Takano et al., 2001). Algorithms for the calculation of T-invariants are implemented in many Petri net software tools such as INA (Roch & Starke, 2001); GreatSPN (Chiola et al., 1995), TimeNET (German et al., 1995), and QPN (Bause & Kemper, 1994), to mention only a few.

It is known that, in any Petri net with T-invariants, there are *minimal-support T-invariants* which can be used as generators of all T-invariants of the given net (Memmi & Roucairol, 1980; Murata, 1989). Let $\Phi = \{F_1, F_2, \ldots, F_s\}$ be the set of minimal-support T-invariants of some Petri net consisting of $m = |T|$ transitions, where $F_i = [f_{i1}, f_{i2}, \ldots, f_{im}] \underset{\neq}{>} 0$, and s is the number of minimal-support T-invariants. We use here, for a vector X, a denotation $X \underset{\neq}{>} 0$ if $X \geq 0$ and $x_i \neq 0$ for some ith entry of X. Each $F_i \in \Phi$ specifies a nonempty subset of transitions $\|F_i\| \subseteq T$ such that $t_j \in \|F_i\|$ if and only if $f_{ij} > 0$, with $\|F_i\| \not\subset \|F_k\|$ and $\|F_k\| \not\subset \|F_i\|$ for every pair of distinct indices i, $k = 1, 2, \ldots, s$. Here $\|F_i\|$ represents the *minimal support* of T-invariant F_i .

Statement 2. In any Petri net the number of minimal-support T-invariants is finite (Kostin, 2003).

Statement 3. For any Petri net PN, its complemented net PN_c includes all T-invariants of PN (Kostin, 2003).

Statement 4. For every reachability path from an initial marking M^0 to a target marking M of a given Petri net PN, there exists a T-invariant $F = [f_1, f_2, \ldots, f_m, f_{m+1}]$ of the corresponding complemented Petri net PN_c, with $f_{m+1} = 1$. That is, F is a singular complementary T-invariant of PN_c.

Let

$$M^0 \xrightarrow{t_{i_1}} M^1 \xrightarrow{t_{i_2}} \ldots \xrightarrow{t_{i_k}} M^k = M \tag{8}$$

be some reachability path from M^0 to M in given Petri net PN, such that $M^j \neq M$ and $t_{i_j} \neq t_{m+1}$ for $j = 1, 2, \ldots, k$-1. Using this path, create an *expanded* reachability path

$$M^0 \xrightarrow{t_{i_1}} M^1 \xrightarrow{t_{i_2}} \ldots \xrightarrow{t_{i_k}} M^k \xrightarrow{t_{i_{k+1}}} M^{k+1} = M^0 . \tag{9}$$

Since $M^k = M$, marking M^k can be transformed, according to (6), into marking M^0 by a single firing of the complementary transition $t_{i_{k+1}} = t_{m+1}$. Consider now the firing count vector corresponding to the reachability path (9):

$$F = [f_1, f_2, \ldots, f_m, f_{m+1}], \tag{10}$$

where f_i is the number of times transition t_i appears in the sequence $t_{i_1} t_{i_2} \ldots t_{i_{k+1}}$, with $f_{m+1} = 1$. Since, in the reachability path (9), initial marking M^0 is transformed back into M^0, the corresponding firing count vector (10) is a T-invariant. Further, since the last entry in this vector $f_{m+1} = 1$, the vector is a singular complementary T-invariant of the complemented Petri net PN_c. ♦

Note that the reverse of Statement 4 is generally not true. That is, the existence of a singular complementary T-invariant does not guarantee that there exists a corresponding reachability path.

Corollary 1. For any Petri net, with given initial and target markings M^0 and M respectively, all existing reachability paths from M^0 to M are the paths that can be created on the set of singular complementary T-invariants. This corollary is a generalization of the corresponding result for T-invariant-less Petri nets obtained in (Kostin, 2003). It means that, to perform reachability analysis of a Petri net, it is sufficient to search for reachability paths only on the set of singular complementary T-invariants. ♦

The set implied by Corollary 1 is infinite in general and includes singular minimal-support complementary T-invariants and all linear combinations of minimal-support T-invariants that yield the last entry $f_{m+1} = 1$. As will be shown, it is sufficient to consider in this set, without losing reachability information, only a finite subset.

Let

$$\Phi_c = \{F_1, F_2, \ldots, F_w\} \tag{11}$$

be a set of all minimal-support T-invariants of PN_c, where

$$F_j = [f_{j1}, f_{j2}, \ldots, f_{jm}, f_{j,m+1}] \underset{\neq}{\geq} 0, \tag{12}$$

with $j = 1, 2, \ldots, w$. Notice that, according to the basic property of a T-invariant, each entry in vector F_j may be only a nonnegative integer (Memmi & Roucairol, 1980).

Now, depending on the value of the last entry, the minimal-support T-invariants of set Φ_c can be classified into the following three *disjoint* groups:

$$\{F_j \mid f_{j,m+1} = 0,\ j \in I_w\}, \tag{13}$$

$$\{F_j \mid f_{j,m+1} = 1,\ j \in I_w\}, \tag{14}$$

$$\{F_j \mid f_{j,m+1} > 1,\ j \in I_w\}. \tag{15}$$

where $I_w = \{1, 2, \ldots, w\}$ is the indexing set of Φ_c. According to Statement 2, each of these groups is finite. Depending on the Petri net and its initial and target markings, some or even all these three groups can be empty.

Without the last, $(m+1)$th entry, T-invariants of group (13), by Statement 3, are minimal-support T-invariants of the original Petri net PN. We will call members of group (13) *non-complementary* minimal-support T-invariants of the complemented Petri net PN_c. Group (14) consists of singular complementary T-invariants. Finally, members of group (15) are nonsingular complementary T-invariants in which the complementary transition fires more than once. Together, members of groups (14) and (15) are called minimal-support *complementary* T-invariants of PN_c.

3. Computing minimal singular T-invariants of a complemented Petri net

By Corollary 1, the search for all reachability paths from initial marking M^0 to target marking M in a given Petri net can be carried out only on singular T-invariants of the corresponding complemented Petri net. These include, first of all, minimal-support T-invariants of group (14). However, these are not the only singular T-invariants of the complemented Petri net. Indeed, linear combinations of minimal-support T-invariants of groups (13), (14), and (15) can yield additional singular T-invariants. The number of such combinations is infinite in general. In this section, we will show that there exists a finite set of *minimal* singular T-invariants of the complemented Petri net. Then an approach to the computation of such a set will be described. In Section 6, it will be shown how the

computed minimal singular T-invariants can be combined with non-complementary T-invariants of group (13) to produce new, non-minimal singular T-invariants.
Consider a linearly-combined T-invariant

$$F = [f_1, f_2, \ldots, f_m, f_{m+1}] = \sum_{j=1}^{w} k_j F_j \tag{16}$$

with rational coefficients k_j, where F_j are minimal-support T-invariants of groups (13), (14) and (15), and w is the number of elements in the three groups. In agreement with Corollary 1, we are looking only for those combined T-invariants F which yield $f_{m+1} = 1$. Thus, the following constraint must hold for each linear combination F in (16):

$$f_{m+1} = \sum_{j=1}^{w} k_j f_{j,m+1} = 1. \tag{17}$$

With $k_j \geq 0$, the product $k_j F_j$ in (16) can be considered as a contribution of firings of transitions of T-invariant F_j to firings of transitions of the combined T-invariant F. On the other hand, a negative coefficient k_j in (16) may be interpreted as a reverse, or backward firing of transitions, corresponding to T-invariant F_j, and this is *not legal* in the normal semantics of Petri nets. Thus, for T-invariants of groups (14) and (15), taking into account (17), their coefficients k_j must be in the following range:

$$0 \leq k_j \leq 1. \tag{18}$$

That is, for groups (14) and (15), in which $f_{j,m+1} \geq 1$, to satisfy (17) the following inequality must hold:

$$\sum k_j \leq 1. \tag{19}$$

However, coefficients k_j for T-invariants of group (13) in (16) may have arbitrary (non-negative) values without affecting the constraint (17). As a particular case, these T-invariants can be combined in (16) with coefficients $k_j \leq 1$. The case when T-invariants of group (13) can be included into combination (16) with arbitrary large coefficients is considered in Section 6.
The linearly-combined T-invariants (16), with the constraints (17), (18) and (19), are called *minimal singular* T-invariants of the complemented Petri net. As a subset, they include all minimal-support T-invariants of group (14).
Minimal singular T-invariants of the complemented Petri net can be computed in the following way. Rewrite (16) as a system of linear algebraic equations

$$\Psi K^T = F^T, \tag{20}$$

where Ψ is a matrix of size $((m + 1) \times w)$ whose columns are transposed minimal-support T-invariants F_j from groups (13), (14) and (15), $K = [k_1, k_2, \ldots, k_w]$, and F is vector (16), with $f_{m+1} = 1$.
In the system (20), not only coefficient vector K, but also entries f_i of F, for $i = 1, 2, \ldots, m$, are not known. We will show, however, that the number of different integer-valued vectors F with $f_{m+1} = 1$ is finite. Then we will explain how to compute the valid vectors in (20). The word "valid" means here that, in addition to the requirement $f_{m+1} = 1$, all coefficients k_j in

(16) satisfy the constrains (18) and (19). Taking into account (17) and (18), one can deduce that

$$0 \le f_i \le \max_j (f_{ji}), \; j = 1, 2, ..., w; \; i = 1, 2, ..., m, \tag{21}$$

where entries f_i are integer-valued components of vector F in (16).
One can see now that the number of different integer-valued vectors F in the system (20) is

$$N = \prod_{i=1}^{m} [\max_j (f_{ji}) + 1]. \tag{22}$$

This number includes one vector F with all zero entries except the last one, and all minimal-support T-invariants of group (14). Among the remaining vectors F, there can be additional singular T-invariants. They can be computed in the following way.
Assume that, in the system (20), K is a vector of unknowns. Then Ψ can be considered as a coefficient matrix, so that the augmented matrix of the system (20) is $U = \Psi \mid F^T$. It is known that, by elementary row operations, each matrix can be transformed to an upper trapezoidal form (Goldberg, 1991). In particular, for the augmented matrix U the result of its transformation $U^{\sim}$ can be written as follows:

$$U^{\sim} = \begin{bmatrix} * & * & * & * & \dots & * & y_1 \\ 0 & * & * & * & \dots & * & y_2 \\ 0 & 0 & * & * & \dots & * & y_3 \\ \circ & \circ & \circ & \circ & \dots & \circ & \circ \\ \circ & \circ & \circ & \circ & \dots & \circ & y_m \\ \circ & \circ & \circ & \circ & \dots & \circ & y_{m+1} \end{bmatrix}, \tag{23}$$

where the symbol '*' stands for some value (this value is not zero if the symbol is the first in the row), the symbol '∘' is a place holder, and $y_i = y_i(f_1, f_2, \dots, f_m, f_{m+1})$ is some linear function of its arguments, $i = 1, 2, \dots, m+1$. Each row in $U^{\sim}$ consists of $w + 1$ elements.
For the system (20) to be consistent, the following equation must hold for each ith row of matrix $U^{\sim}$ with all w leading elements equal to zero (Goldberg, 1991):

$$y_i (f_1, f_2, \dots, f_m, f_{m+1}) = 0. \tag{24}$$

Collecting now all equations (24), we obtain a derived system of linear algebraic equations

$$\begin{aligned} & y_{j_1}(f_1, f_2, ..., f_m, f_{m+1}) = 0 \\ & y_{j_2}(f_1, f_2, ..., f_m, f_{m+1}) = 0 \\ & \dots\dots\dots\dots\dots\dots \\ & y_{j_k}(f_1, f_2, ..., f_m, f_{m+1}) = 0 \end{aligned} \tag{25}$$

where $k \le m$ and $f_{m+1} = 1$.

Integer solutions of this system relative to $f_1, f_2, \ldots, f_m$ can be found using existing algorithms for integer systems of linear equations (Howell, 1971; Springer, 1986). With the constraints (21), the system has a finite number of solutions or no solutions at all. Note that, with nonempty group (14), for all its members $[f_{j1}, f_{j2}, \ldots, f_{jm},\ 1]$, the system (25) has solutions at least for the trivial linear combinations

$$F = [f_1, f_2, \ldots, f_m,\ 1] = [f_{j1}, f_{j2}, \ldots, f_{jm},\ 1], \tag{26}$$

since each vector (26) is the solution of (20), for which vector K has some entry $k_j = 1$, with all other coefficient entries equal to zero.
To illustrate this method, consider a Petri net of 6 transitions and 6 places having the incidence matrix

$$D = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

with the initial and target markings $M^0 = [2, 0, 0, 0, 0, 0]$ and $M = [0, 0, 0, 0, 0, 2]$, respectively. The corresponding complemented Petri net has two minimal-support T-invariants $F_1 = [0, 0, 2, 2, 2, 0, 1]$ and $F_2 = [2, 2, 0, 0, 0, 2, 1]$. Both are *singular* T-invariants (that is, they have $f_{m+1} = f_7 = 1$). We will try to determine whether there are some other minimal singular T-invariants. For this example, with $w = 2$, the augmented matrix of the system (20) and its upper trapezoidal form are

$$\begin{bmatrix} 0 & 2 & f_1 \\ 0 & 2 & f_2 \\ 2 & 0 & f_3 \\ 2 & 0 & f_4 \\ 2 & 0 & f_5 \\ 0 & 2 & f_6 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & f_1 \\ 0 & 0 & f_1 - f_2 \\ 0 & 0 & f_3 - f_4 \\ 0 & 0 & f_4 - f_5 \\ 0 & 0 & f_1 - f_6 \\ 0 & 0 & f_1 + f_3 - 2 \end{bmatrix}.$$

Thus, the system (25) is

$$\begin{aligned} f_1 - f_2 &= 0 \\ f_3 - f_4 &= 0 \\ f_4 - f_5 &= 0 \\ f_1 - f_6 &= 0 \\ f_1 + f_3 &= 2 \end{aligned}$$

With the constraints $0 \le f_1, f_2, f_3, f_4, f_5, f_6 \le 2$, this system has the following three nonnegative integer solutions: [0, 0, 2, 2, 2, 0, 1], [2, 2, 0, 0, 0, 2, 1] and [1, 1, 1, 1, 1, 1, 1]. Clearly, the fist two solutions are minimal-support T-invariants F_1 and F_2, and the third solution is a minimal singular T-invariant that is the linear combination $F_3 = 0.5F_1 + 0.5F_2$. Neither F_1 nor F_2 are realizable in given initial marking. However, their linearly combined T-invariant F_3 is realizable. One legal firing sequence is $t_3\, t_1\, t_2\, t_4\, t_5\, t_6\, t_7$.

4. Relation graph of T-invariants

In general, each singular T-invariant should be tested for the creation of a reachability path (or a legal firing sequence) not only *alone*, but also in different linear combinations with non-complementary T-invariants (13), since these T-invariants can "help" the singular T-invariant to become realizable in given initial marking M^0 and to eventually provide a reachability path from M^0 to a target marking M. As will be shown in this section, in general not all non-complementary T-invariants can affect realization of the given singular T-invariant.

Definition 2. Let F be a T-invariant of a Petri net, with the support $\|F\|$. Then

$$P(F) = \{p_j \mid t_i \in \|F\|,\ d_{ij} \neq 0\} \tag{27}$$

is a set of places of this Petri net *affected* by F when it becomes realizable in some marking. Here, d_{ij} is an element of the incidence matrix of the Petri net as specified by (1). ♦

Statement 5. Let F_1 and F_2 be some T-invariants of a Petri net, and let P_1 and P_2 be sets of places affected by F_1 and F_2 respectively. If $P_1 \cap P_2 = \varnothing$, then T-invariants F_1 and F_2 have no *direct* effect on the realizability of each other.

Assume that, contrary to the statement, F_1 can directly affect the realizability of F_2. This is possible only if F_1, during its realization, will change the number of tokens in some places affected by F_2. This can happen only if $P_1 \cap P_2 \neq \varnothing$. The contradiction proves the statement.♦

Even if $P_1 \cap P_2 = \varnothing$, T-invariants F_1 and F_2 can *indirectly* affect the realizability of each other through other T-invariants having common affected places with F_1 and F_2.

Corollary 2. Let $F_{nc}^1, F_{nc}^2, \ldots, F_{nc}^k$ be some non-complementary T-invariants of a complemented Petri net, with sets of places $P_{nc}^1, P_{nc}^2, \ldots, P_{nc}^k$ affected by these T-invariants, respectively. Let further F_c be a singular complementary T-invariant of this Petri net, with the set of affected places P_c. Denote by $P_{nc} = \bigcup_{i=1}^{k} P_{nc}^i$ a set of places of this Petri net affected by mentioned *non-complementary* T-invariants.

If $P_c \cap P_{nc} = \varnothing$, then realization of any linear combination of T-invariants $F_{nc}^1, F_{nc}^2, \ldots, F_{nc}^k$ has no effect on realization of F_c. Therefore these T-invariants may be excluded from consideration in the reachability analysis with T-invariant F_c in given Petri net.♦

To represent formally the effects of different T-invariants on each other in a Petri net, it is instructive to introduce into consideration a *relation graph* of T-invariants. Nodes in this graph are T-invariants. Two nodes corresponding to T-invariants F_i and F_j are connected by a non-oriented edge if $P(F_i) \cap P(F_j) \neq \varnothing$, and the corresponding T-invariants F_i and F_j are called *directly connected* T-invariants.

For a Petri net, such a graph generally consists of a number of connected components. A connected component may include complementary and non-complementary T-invariants, or only one type of T-invariants. We say that two T-invariants F_i and F_j can affect realizability of each other if they belong to the same connected component, even if $P(F_i) \cap P(F_j) = \varnothing$. On the other hand, if F_i and F_j belong to different connected components, they can not affect each other in no way, directly or indirectly.
The algorithm for determining all connected components of a graph is well known (Goodrich, 2002). In our problem, the algorithm will determine a connected component consisting of nodes representing a given singular T-invariant and non-complementary T-invariants. For this purpose, the algorithm will use the incidence matrix of the *original* Petri net and the array of T-invariants.

5. Realization of T-invariants with borrowing of tokens

In this section, the meaning of the help provided by one T-invariant to another one to become realizable is explained. Let p be a place affected by two T-invariants F_i and F_j in a given Petri net. Assume that, in a given initial marking of the net, F_i is realizable, but F_j can become realizable if place p accumulates r_j tokens during realization of T-invariant F_i. Suppose further that, at some intermediate step during realization of F_i, r_i tokens will be created in place p. If $r_i \geq r_j$ then, by temporary borrowing of r_j tokens in place p, T-invariant F_j becomes realizable and, at the end of its realization, will return the borrowed tokens to place p, so that T-invariant F_i can complete its started realization.
With $r_i < r_j$, T-invariant F_j cannot borrow the necessary number of tokens in place p. However, if T-invariant F_i, after creation of r_i tokens in p at some step of its first realization, can start a new realization before the completion of the first one, then additional r_i tokens will be created in place p, so that this place will now accumulate $2r_i$ tokens. In general, if F_i can start z realizations before the completion of the previous ones, then place p will accumulate zr_i tokens. If, for some z, $zr_i \geq r_j$ then, after borrowing r_j tokens in p, T-invariant F_j becomes realizable. After the completion of its realization, all tokens borrowed by F_j will be returned to place p, and T-invariant F_i can complete all its started realizations.

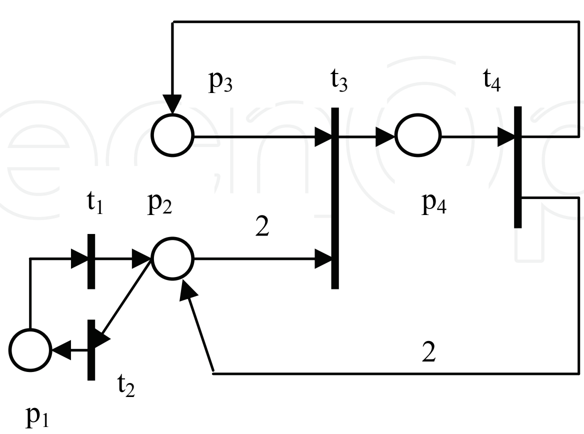


Fig. 1. Illustration of borrowing of tokens by a T-invariant.

Borrowing of tokens by a T-invariant is illustrated with a Petri net shown in Fig. 1, with arcs (p_2, t_3) and (t_4, p_2) having multiplicity 2. This net has two minimal-support T-invariants $F_1 = [1, 1, 0, 0]$ and $F_2 = [0, 0, 1, 1]$. In the initial marking $M^0 = [2, 0, 1, 0]$, F_1 is realizable, but F_2

becomes realizable only if it can borrow two tokens in place p_2, affected by the both T-invariants. These two tokens will be created here after T-invariant F_1 starts two realizations by firing transition t_1 two times. Afterwards, F_2 becomes realizable by borrowing two tokens in p_2. Then, after firing t_3 and t_4, the borrowed tokens reappear in p_2, and F_1 can complete its two started realizations. The corresponding sequence of transition firings for this example is $t_1t_1t_3t_4t_2t_2$.

To represent the relationship between *connected* T-invariants, when some non-realizable T-invariants can become realizable in given initial marking of a Petri net by borrowing tokens in places affected by other T-invariants, we will introduce a two-dimensional borrowing matrix G. In this matrix, rows correspond to T-invariants and columns correspond to places of the given Petri net. Formally, for a group of connected T-invariants,

$$G = [g_{ij}],\ i = 1, 2, \ldots, s;\ j = 1, 2, \ldots, n, \tag{28}$$

where s is the number of connected T-invariants in the group and n is the number of places in the net. The elements of matrix G are integers and have the following meaning. If $g_{ij} > 0$ then, for its realization, T-invariant F_i needs to borrow g_{ij} tokens in place p_j affected by some other T-invariant of the considered group. If $g_{ij} < 0$ then T-invariant F_i, at some intermediate step of its *single* realization, creates $|g_{ij}|$ tokens in place p_j. Finally, $g_{ij} = 0$ means that F_i does not affect place p_j.

As an example, matrix G for minimal-support T-invariants of the Petri net shown in Fig. 1 is:

	p_1	p_2	p_3	p_4
F_1	-1	-1	0	0
F_2	0	2	-1	-1

One can see from this matrix that the number of tokens created in place p_2 during a single realization of F_1 is 1 and is not sufficient for F_2 to borrow two tokens. In this example borrowing is possible if T-invariant F_1 starts two *interleaved* realizations. The maximal number of realizations that can be started by F_1 depends on the initial marking of place p_1. In particular, if this place initially contains only one token, then F_1 is still realizable, but it will never create, during its realizations, more than one token in p_2.

For a group of connected T-invariants of a complemented Petri net, the borrowing matrix can be created with the use of the incidence matrix of the given original Petri net. Due to a relative simplicity of the underlying procedure and to space limitation, the details of this procedure are omitted.

6. Combining a singular complementary T-invariant with non-complementary T-invariants

Denote by F_c a singular T-invariant of some complemented Petri net. It can be a member of group (14) or a minimal T-invariant calculated as was described in Section 3. Clearly, if group (13) is not empty, then the following linear combination

$$F = F_c + \sum k_j F_{nc}^j, \tag{29}$$

with coefficients $k_j \geq 0$, is also a singular T-invariant, if components of F are nonnegative integers. Here F_{nc}^{j} is a T-invariant of group (13). According to Corollary 2, it is sufficient to include in (29) only those T-invariants from (13) that belong to the same group of connected T-invariants together with F_c.
The expression (29) implies that the singular T-invariant F_c in general should be tested for the determination of a reachability path not only alone, but also in different linear combinations with non-complementary T-invariants (13), since these T-invariants can "help" the non-realizable T-invariant F_c to become realizable in given initial marking M^0 and to eventually provide a reachability path from M^0 to a target marking M of the given Petri net.
Without loosing generality, we assume that coefficients k_j in (29) are nonnegative integers. Indeed, if a singular T-invariant F_c is realizable with some non-integer values of coefficients k_j in (29), then it will remain realizable when these coefficient values are replaced by the nearest integer values not less than k_j. The case when $k_j \leq 1$ was considered in Section 3.
With integer coefficients $k_j > 1$, the product $k_j F_{nc}^{j}$ in (29) corresponds to a multiple realization of T-invariant F_{nc}^{j}. A multiple realization is a series of k_j sequential or interleaved single realizations. Interleaved realizations of a T-invariant, if they are possible, can have a different effect on place marking in comparison with sequential realizations. Consider, for example, a simple Petri net consisting of two transitions t_1, t_2 and one place p that is the output place for t_1 and the input place for t_2. This Petri net has a T-invariant $F = [1, 1]$ realizable in any initial marking of p. In particular, with the zero initial marking, place p will never have more than one token if single realizations of F are strictly sequential as in $t_1t_2t_1t_2t_1t_2$. However, if single realizations of F are interleaved, place p can accumulate an arbitrary large number of tokens at some intermediate step.
In general, the number of valid combinations (29) is infinite. This section describes how to limit the values of coefficients k_j in (29) without the loss of reachability information using the concept of structural boundedness of Petri nets.
It is known (Murata, 1989) that a Petri net is *structurally bounded* if and only if there exists a $(1 \times n)$ vector $Y = [y_1, y_2, \ldots, y_n]$ of positive integers, such that

$$D\,Y^T \leq 0, \tag{30}$$

where D is the $(m \times n)$ incidence matrix of the Petri net with m transitions and n places.
A Petri net is said to be *not structurally bounded* if and only if there exists a $(1 \times m)$ vector of (nonnegative) integers $X = [x_1, x_2, \ldots, x_m] \underset{\neq}{>} 0$, such that

$$D^T X^T = \Delta M^T \tag{31}$$

for some $\Delta M \underset{\neq}{>} 0$, where m is the number of transitions in the Petri net, and ΔM is a $(1 \times n)$ vector of marking increments as a result of firing of all transitions corresponding to vector X.

In a structurally unbounded Petri net, at least one place is structurally unbounded. A place p_i in such a Petri net is said to be *structurally unbounded* if and only if there exists a ($1 \times m$) vector $X \underset{\neq}{>} 0$ of nonnegative integers, such that

$$D^T X^T = \Delta M^T \tag{32}$$

for some $\Delta m_i > 0$ in $\Delta M = (\Delta m_1, \Delta m_2, ..., \Delta m_i, ..., \Delta m_n) \underset{\neq}{>} 0$.

The structural unboundedness can be tested separately for each place p_i of the Petri net, by setting an appropriate integer $\Delta m_i > 0$ and $\Delta m_j = 0$ for all $j \neq i$ in (32) and then trying to solve the system (32). The test may be done also simultaneously for a few desired places or even for all places of the net.

It is known that, according to Minkowski-Farkas' lemma (Kuhn & Tucker, 1956), one of the systems (30) or (31) has solutions. For our problem, we do not need to know all solutions of (30) or (31). Rather, it is sufficient to find only one, "minimal" solution of (30) or (31).

The minimal solutions of (30) or (31) can be found as solutions of integer linear programming (ILP) problems. For the system (30), the corresponding ILP problem can be formulated as follows:

$$\text{minimize} \quad a = \sum_{i=1}^{n} y_i, \tag{33}$$

$$\text{subject to:} \quad DY^T \leq 0, \; y_i \geq 1, \; i = 1, 2, ..., n.$$

For the system (31), the corresponding ILP problem is:

$$\text{minimize} \quad b = \sum_{i=1}^{m} x_i, \tag{34}$$

$$\text{subject to:} \quad D^T X^T \underset{\neq}{>} 0, \; \sum_{i=1}^{m} x_i \geq 1, \; x_i \geq 0, \; i = 1, 2, ..., m.$$

The property of structural boundedness can be considered also for subnets of a Petri net. We are interested in this property only for the subnets corresponding to non-complementary T-invariants F_{nc}^j in (29). For a non-complementary T-invariant F_{nc}^j, the related subnet consists of transitions of the support $\| F_{nc}^j \|$ and places $P(F_{nc}^j)$ affected by F_{nc}^j. The expressions (30) - (34) remain valid for the subnet corresponding to F_{nc}^j with the following restrictions: in the incidence matrix D rows are taken for transitions corresponding to nonzero entries in F_{nc}^j, and columns are taken for places affected by F_{nc}^j.

Let us consider initially the case when the subnet corresponding to F_{nc}^j is not structurally bounded and describe how to determine coefficients k_j for non-complementary T-invariants F_{nc}^j in the linear combination (29). If F_{nc}^j and F_c belong to different connected

components of the graph of relation of T-invariants then F_{nc}^{j} should be ignored at all, by setting $k_j = 0$ in (29).
If F_{nc}^{j} and F_c belong to the same connected component of the graph of relation of T-invariants then the subnet corresponding to F_{nc}^{j} will have common places with the subnets corresponding to F_c or other non-complementary T-invariants belonging to the same connected component. Thus, F_{nc}^{j} can affect realizability of F_c, directly or indirectly, and therefore should be included in (29) with $k_j > 0$.
Suppose for definiteness that T-invariant F_{nc}^{j} has the support $\{t_1, t_2, \ldots, t_l\}$, $l \le m$, and the set of affected places

$$\{p_1, p_2, \ldots, p_q\}, \; q \le n, \tag{35}$$

where m and n are the numbers of transitions and places in the original (non-complemented) Petri net. Assume that F_c, to become realizable, needs to borrow $n_i > 0$, $i = 1, 2, \ldots, h$, tokens at least in places

$$\{p_1, p_2, \ldots, p_h\}, \quad h \le q, \tag{36}$$

that belong to the set (35) and in which F_{nc}^{j} can create tokens during its realization. Then, to facilitate the realizability of F_c, F_{nc}^{j} should be included in the linear combination (29) with a positive integer coefficient k_j determined by applying the following steps.

1. Try to solve an ILP problem:

$$\text{minimize} \quad b = \sum\nolimits_{i=1}^{l} x_i, \tag{37}$$

$$\text{subject to:} \quad D^T X^T \ge \Delta M^T, \; \sum\nolimits_{i=1}^{l} x_i \ge 1, \; x_i \ge 0,$$

   where $\Delta M = [\Delta m_1, \Delta m_2, .., \Delta m_h, \Delta m_{h+1}, \ldots, \Delta m_q] = [n_1, n_2, \ldots., n_h, 0, \ldots, 0]$ is a vector of the desired numbers of tokens which are expected to be created in places (36) as a result of one or more realizations of F_{nc}^{j}, l is the number of transitions in the subnet corresponding to F_{nc}^{j}, and q is the number of places affected by F_{nc}^{j}. In the matrix multiplication, only those rows and columns of D are used which correspond to the support of F_{nc}^{j} and to places affected by F_{nc}^{j}.

2. If, for the specified vector ΔM, the problem (37) has a solution $X^* = [x_1^*, x_2^*, \ldots, x_l^*]$, then components of X^* represent the total numbers of firings of respective transitions *sufficient* to accumulate the desired number of tokens in places of set (36) in a few realizations of F_{nc}^{j}, and ratio $\left\lceil \dfrac{x_i^*}{f_i^j} \right\rceil$ is the number of realizations of F_{nc}^{j} to provide the necessary number of firings of transition t_i, $i = 1, 2, \ldots, l$. In this case,

$$k_j = \max(\left\lceil \frac{x_i^*}{f_i^j} \right\rceil \mid i = 1,\ 2,\ ...,\ l). \tag{38}$$

3. If, on the other hand, the problem (37) has no feasible solution then it means that at least one of places in set (36) p_i is structurally bounded and can not accumulate the desired number of tokens Δm_i in multiple realizations of F_{nc}^j. In this case, using (32), determine all structurally unbounded places in set (36). Since, as is assumed, the subnet for F_{nc}^j is not structurally bounded, there is at least one structurally unbounded place in this subnet.
4. Solve the ILP problem (37) simultaneously for all structurally unbounded places found at the previous step, to obtain a solution vector X^*. That is, in solving (37), vector ΔM should have nonzero entries $\Delta m_i = n_i$ only for structurally unbounded places. According to Minkowski-Farkas' lemma, this solution always exists. Then coefficient k_j is determined by the use of expression (38).

In case, when the subnet for F_{nc}^j is found to be structurally bounded, then the number of tokens in each of its places is bounded. However, this bound generally depends on realizations of other, connected T-invariants and is not known in advance. For such a subnet, coefficient k_j can be evaluated with the use of the borrowing matrix (28) computed for F_c and all its connected non-complementary T-invariants, including F_{nc}^j. Let, in this matrix, c and j be indexes of rows corresponding to F_{nc}^j and F_c, respectively. Then it is *sufficient* to include F_{nc}^j in the linear combination (29) with coefficient k_j computed with the use of the expression

$$k_j = \sum \left\lceil \frac{g_{ci}}{|\, g_{ji} \,|} \right\rceil, \tag{39}$$

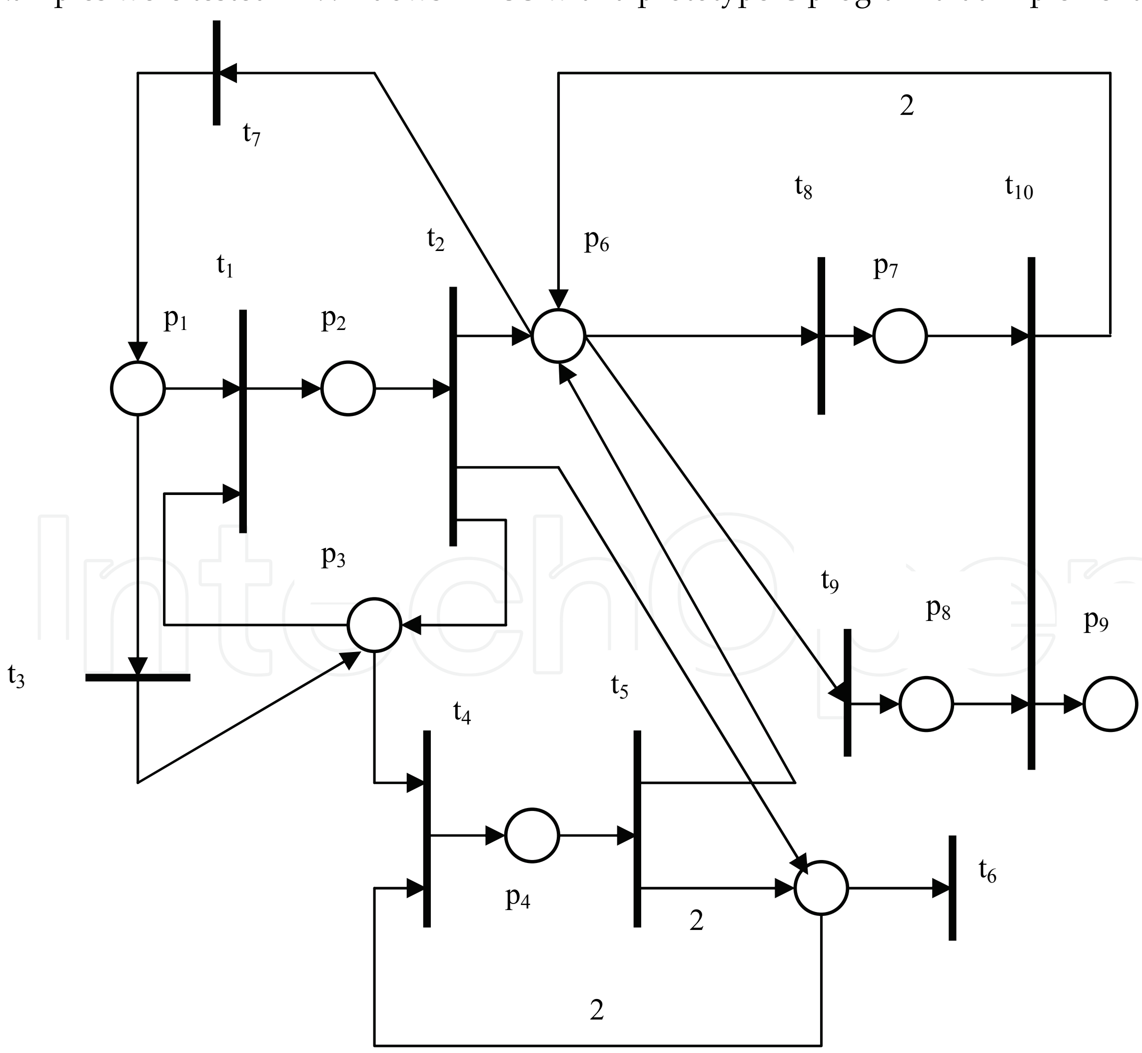
where g_{ci} and g_{ji} are entries in the borrowing matrix, and the sum is computed for all pairs $g_{ci} > 0$ and $g_{ji} < 0$. Indeed, with this coefficient, the sufficient number of interleaved realizations of F_{nc}^j are *allowed* to accumulate the required numbers of tokens in places which are common for F_c and F_{nc}^j and in which T-invariant F_c can borrow them during its realization.

However, the possibility of realizations of F_{nc}^j depends on marking of places in its subnet. For example, in the Petri net of Fig. 1, T-invariant F_2 can become realizable only with the help of T-invariant F_1 for which the corresponding subnet is structurally bounded. The borrowing matrix for this example has only one pair of non-zero entries $g_{12} = -1$ (for F_1) and $g_{22} = 2$ (for F_2). Thus, using (39), one can obtain $k_1 = 2$. That is, two interleaved realizations of F_1 are sufficient to create two tokens in place p_2 to make F_2 realizable. But this is possible only if place p_1 holds initially at least two tokens. If this place holds one token, F_1 is

sequentially realizable but it can never create two tokens in p_2 to facilitate the realizability of F_2.
In general, coefficient k_j calculated as was described for the two cases can result in a larger number of realizations of T-invariant F_{nc}^{j} than is actually necessary. The reason is that other T-invariants in (29) can also create tokens in places (36) and contribute to the realizability of F_c.
After computing coefficients k_j in (29), an appropriate method can be applied to find a reachability path (or a legal firing sequence) for the combined T-invariant F if such a path exists. The task here is the following. For a Petri net with given initial and target markings M^0 and M and a combined T-invariant F, find a legal firing transition sequence. To find a legal firing transition sequence, or reachabiity path as defined in (4), known computational techniques can be used (Kostin, 2003; Taoka et al., 2003; Watanabe, 2000; Huang & Murata, 1998).

7. Examples

This section illustrates the proposed reachability analysis scheme by two examples. The examples were tested in Windows XP OS with a prototype C program that implemented



$$D = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \end{bmatrix}$$

Fig. 2. Petri net of Example 1 and its incidence matrix.

almost all steps of the scheme, with the major exception of the sub-algorithm for solving an ILP problem. To solve this problem, the interactive system QS was used (Chang & Sullivan, 1996). For the first example, Fig. 2 shows a Petri net consisting of m = 10 transitions and n = 9 places, with its incidence matrix (recall that rows correspond to transitions), and the initial and target markings M^0 = [2, 0, 0, 0, 0, 0, 0, 0, 0] and M = [2, 0, 0, 0, 0, 0, 0, 0, 1], respectively. To get the complemented Petri net, the algorithm appends a row ΔM = M^0 – M = [0, 0, 0, 0, 0, 0, 0, 0, -1] to the original incidence matrix. Minimal-support T-invariants of the corresponding complemented Petri net are two non-complementary T-invariants F_1 = [0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0] and F_2 = [1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0], and one singular complementary T-invariant F_3 = [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1], with the sets of affected places {p_1, p_3, p_4, p_5, p_6}, {p_1, p_2, p_3, p_5, p_6} and {p_6, p_7, p_8, p_9}, respectively. Thus, all these T-invariants are connected and should be considered together. The borrowing matrix G for this example contains the following data:

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
F_1	-1	0	-1	-1	2	-1	0	0	0
F_2	-1	-1	1	0	-1	-1	0	0	0
F_3	0	0	0	0	0	2	-1	-1	-1

Thus, each of these T-invariants can become realizable if it borrows tokens in some of common affected places. Specifically, F_1 needs to borrow two tokens in place p_5, F_2 needs to borrow one token in place p_3, and F_3 borrows two tokens in place p_6. Note that a token borrowed by F_2 in place p_3 can be produced by F_1 in a single realization. In its turn, F_2 is capable, in a single realization, to lend one token to F_1, instead of necessary two tokens. Therefore, F_1 and F_2 can help each other to become realizable. Together, they are capable to produce 2 tokens in place p_6 to be borrowed by F_3.

The desired number of tokens in p_5 can be accumulated if the subnet corresponding to F_2 is not structurally bounded. To check this, the ILP problem (33) for F_2 is solved, in the following form:

$$\text{minimize } a = y_1 + y_2 + y_3 + y_5 + y_6,$$

$$\text{subject to: } -y_1 + y_2 - y_3 \le 0,\ -y_2 + y_3 + y_5 + y_6 \le 0,\ -y_5 \le 0,\ y_1 - y_6 \le 0,\ y_1, y_2, y_3, y_5, y_6 \ge 1.$$

This ILP problem has no feasible solution. Thus, the subnet corresponding to F_2 is not structurally bounded, so that at least one of its affected places is not structurally bounded. We are interested in accumulating two tokens in p_5, so that $\Delta M = [0, 0, 0, 2, 0]$. Therefore, now the ILP problem (37) should be attempted, in the following form:

$$\text{minimize } b = x_1 + x_2 + x_6 + x_7,$$

$$\text{subject to: } -x_1 + x_7 \ge 0,\ x_1 - x_2 \ge 0,\ -x_1 + x_2 \ge 0,\ x_2 - x_6 \ge 2,\ x_2 - x_7 \ge 0,$$

$$x_1 + x_2 + x_6 + x_7 \ge 1.$$

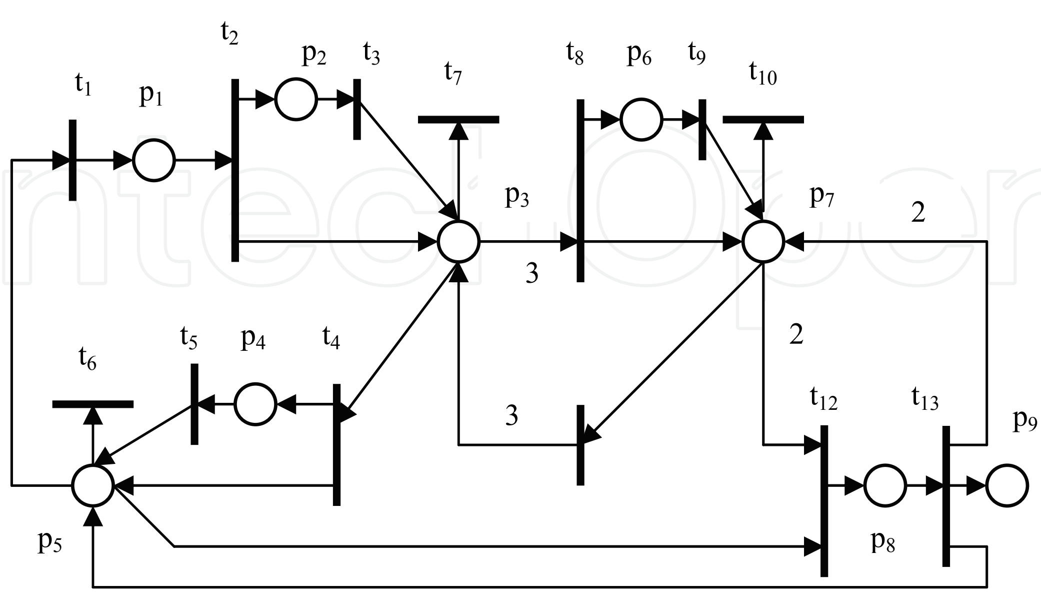
This ILP problem has the optimal (minimal) solution

$$X^* = [x_1^*,\ x_2^*,\ x_6^*,\ x_7^*] = [2,\ 2\ ,2,\ 0].$$

Now, using (38), one can find that

$$k_2 = \max(\left\lceil \frac{x_i^*}{f_{2i}} \right\rceil \mid\ i = 1,\ 2,\ 6,\ 7) = 2.$$

Since T-invariant F_2 borrows only one token in place p_3 and this token can be created during a single realization of F_1, it is sufficient to have $k_1 = 1$. Thus, the combined T-invariant (29), with $F_c = F_3$, is $F = F_1 + 2F_2 + F_3 = [2, 2, 1, 1, 1, 2, 3, 1, 1, 1, 1]$. For this T-invariant, a legal firing sequence can be found consisting of 15 firing transitions $t_3t_1t_2t_7t_1t_2t_4t_5t_6t_6t_8t_9t_{10}t_7t_7$ and transforming M^0 into M. This is the shortest sequence although there exist other sequences of the same length. Using the computed sequence, the corresponding reachability path (4) from M^0 to M can be easily found.



$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 1 \end{bmatrix}.$$

Fig. 3. Petri net of Example 2 and its incidence matrix.

Fig. 3 shows the second example of a Petri net, consisting of m = 13 transitions and n = 9 places. With the initial and target markings M^0 = [1, 0, 0, 0, 0, 0, 0, 0, 0] and M = [1, 0, 0, 0, 0, 0, 0, 0, 1], there are seven minimal-support T-invariants in the corresponding complemented Petri net: six non-complementary T-invariants F_1 = [1, 1, 1, 2, 2, 3, 0, 0, 0, 0, 0, 0, 0, 0], F_2 = [2, 2, 2, 1, 1, 0, 3, 0, 0, 0, 0, 0, 0, 0], F_3 = [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0], F_4 = [0, 0, 0, 0, 0, 0, 3, 1, 1, 0, 2, 0, 0, 0], F_5 = [0, 0, 0, 3, 3, 6, 0, 1, 1, 0, 2, 0, 0, 0], F_6 = [2, 2, 2, 1, 1, 0, 0, 1, 1, 2, 0, 0, 0, 0], and one singular complementary T-invariant F_7 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1], with the sets of affected places {p_1, p_2, p_3, p_4, p_5}, {p_1, p_2, p_3, p_4, p_5}, {p_3, p_6, p_7}, {p_3, p_6, p_7}, {p_3, p_4, p_5, p_6, p_7}, {p_1, p_2, p_3, p_4, p_5, p_6, p_7}, and {p_5, p_7, p_8, p_9}, respectively.

Thus, all these T-invariants are connected. Linear combinations of F_7 with non-complementary T-invariants, according to Section 3, yield four additional minimal singular T-invariants F_8 = [1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1], F_9 = [2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 1, 1, 1], F_{10} = [0, 0, 0, 2, 2, 4, 1, 1, 1, 0, 2, 1, 1, 1] and F_{11} = [0, 0, 0, 1, 1, 2, 2, 1, 1, 0, 2, 1, 1, 1].

For reachability analysis, consider the singular complementary T-invariant F_7. For F_7 and its connected non-complementary T-invariants, the borrowing matrix G contains the following data:

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
F_1	-1	-1	-2	-1	-2	0	0	0	0
F_2	-1	-1	-2	-1	-2	0	0	0	0
F_3	0	0	3	0	0	-1	-2	0	0
F_4	0	0	6	0	0	-1	-2	0	0
F_5	0	0	6	-1	-2	-1	-2	0	0
F_6	-1	-1	-2	-1	-2	-1	-2	0	0
F_7	0	0	0	0	1	0	2	-1	-1

Thus, T-invariant F_7 can become realizable if only it borrows tokens. Specifically, F_7 needs to borrow one token in place p_5 and two tokens in place p_7. The necessary number of tokens in the both places can be produced by realizable T-invariant F_6 alone. Indeed, F_6 creates, in a single realization, two tokens in place p_5 and two tokens in place p_7. However, at this point we cannot say that there exist a state of the Petri net in which places p_5 and p_7 hold at least one and two tokens, respectively. To learn this possibility, it is necessary initially to test the structural boundedness of the subnet corresponding to F_6, by attempting to solve the ILP problem (33), in the following form:

$$\text{minimize } a = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7,$$

$$\text{subject to: } y_1 - y_5 \le 0,\ -y_1 + y_2 + y_3 \le 0,\ -y_2 + y_3 \le 0,\ -y_3 + y_4 + y_5 \le 0,\ -y_4 + y_5 \le 0,$$

$$-3y_3 + y_6 + y_7 \le 0,\ -y_6 + y_7 \le 0,\ -y_7 \le 0,\ y_1, y_2, y_3, y_4, y_5, y_6, y_7 \ge 1.$$

This ILP problem has no feasible solution. Thus, the subnet corresponding to F_6 is not structurally bounded, so that at least one of its affected places is not structurally bounded. We are interested in having at least one token in p_5 and at least two tokens in p_7, so that $\Delta M = [0, 0, 0, 0, 1, 0, 2]$. Therefore, now it is necessary to try to solve the ILP problem (37), in the following form:

$$\text{minimize } b = x_1 + x_2 + x_3 + x_4 + x_5 + x_8 + x_9 + x_{10},$$

$$\text{subject to: } x_1 - x_2 \ge 0,\ x_2 - x_3 \ge 0,\ x_2 + x_3 - x_4 - 3x_8 \ge 0,\ x_4 - x_5 \ge 0,\ -x_1 + x_4 + x_5 \ge 1,$$

$$x_8 - x_9 \ge 0,\ x_8 + x_9 - x_{10} \ge 2,\ x_1 + x_2 + x_3 + x_4 + x_5 + x_8 + x_9 + x_{10} \ge 1.$$

This ILP problem has the optimal (minimal) solution

$$X^* = [x_1^*, x_2^*,\ x_3^*,\ x_4^*,\ x_5^*,\ x_8^*,\ x_9^*,\ x_{10}^*] = [3,\ 3,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0].$$

Now, using (38), we can find that

$$k_6 = \max(\left\lceil \frac{x_i^*}{f_{6i}} \right\rceil \mid\ i = 1,\ 2,\ ...,\ 5,\ 8,\ 9,\ 10) = 2.$$

Thus, the combined complementary T-invariant (29), with $F_c = F_7$, is $F = 2F_6 + F_7 = [4, 4, 4, 2, 2, 0, 0, 2, 2, 4, 0, 1, 1, 1]$. For F, a legal firing sequence can be found consisting of 26 transition firings and transforming M^0 into M. This is not the shortest sequence. The shortest sequence exists for the decremented value of coefficient $k_6 = 1$ and consists of 14 transition firings $t_2t_3t_4t_1t_2t_3t_5t_8t_9t_{12}t_{13}t_1t_{10}t_{10}$.

Although non-complementary T-invariants F_1 and F_2 are realizable as well, they are not appropriate to be combined with F_7 to create a realizable combined complementary T-invariant since they cannot produce tokens in place p_7. The necessary number of tokens could be produced in p_7 also by F_5 but it needs itself to borrow six tokens in place p_3.

In this way, one can proceed with the remaining singular T-invariants F_8, F_9, F_{10}, and F_{11}. Calculating coefficient $k_6 = 2$ and combining each of these T-invariants with F_6, it will be possible to successfully find the corresponding legal firing sequences and, if necessary, reachability paths. In all cases, coefficient k_6 can be decremented to one, to get the shortest legal firing sequence.

This example shows that, in general, it is not necessary to compute coefficients k_j for all T-invariants F_{nc}^{j} in (29). The reachability test can be done as soon as coefficient k_j is computed for the first F_{nc}^{j}. If this test fails, then coefficient k_j is computed for the next F_{nc}^{j}, until the reachability test is successful or all connected non-complementary T-invariants in (29) are considered.

8. Conclusion

A new approach to reachability analysis in general Petri nets is proposed, formally described, and illustrated by examples tested with a prototype program. For a given original Petri net, the reachability analysis is reduced to the computation and investigation of T-invariants of the complemented Petri net consisting of the original Petri net and an additional, complementary transition with input and output arcs depending on the given initial and target markings. It is shown that, without the loss of reachability information, one can carry out reachability analysis using only a finite number of T-invariants.
We did not address, in this chapter, complexity aspects of the proposed approach to reachability analysis. Complexity of some problems of Petri nets, including the reachability problem, was investigated elsewhere (Jones et al., 1977). Most of the running time in the proposed reachability analysis scheme will be spent in computing minimal-support T-invariants and their linear combinations, solving ILP problems, and trying to find legal firing sequences for the computed T-invariants. This can be done with the use of existing methods (Watanabe, 2000; Yamauchi & Watanabe, 1998; Huang & Murata, 1998).

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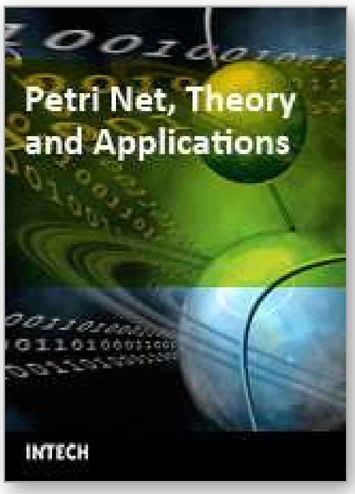

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Although many other models of concurrent and distributed systems have been de- veloped since the introduction in 1964 Petri nets are still an essential model for concurrent systems with respect to both the theory and the applications. The main attraction of Petri nets is the way in which the basic aspects of concurrent systems are captured both conceptually and mathematically. The intuitively appealing graphical notation makes Petri nets the model of choice in many applications. The natural way in which Petri nets allow one to formally capture many of the basic notions and issues of concurrent systems has contributed greatly to the development of a rich theory of concurrent systems based on Petri nets. This book brings together reputable researchers from all over the world in order to provide a comprehensive coverage of advanced and modern topics not yet reflected by other books. The book consists of 23 chapters written by 53 authors from 12 different countries.

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